The Fiscal Theory of the Price Level with a Bubble*

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Abstract

This paper extends the unique price level determination of the Fiscal Theory of the Price Level (FTPL) to environments with a bubble. The expanded FTPL equation incorporates a bubble term to explain why countries with persistently negative primary surpluses can have a positively valued government debt and currency. Appropriate off-equilibrium taxation and "fiscal space" can ensure a unique equilibrium in which the bubble stays on government debt rather than on other assets, like crypto assets. In this case, the government enjoys an exorbitant privilege and is able to continuously roll over its debt.

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1 Introduction

Different monetary theories emphasize different roles of money and different equilibrium equations to determine the price level. The Fiscal Theory of the Price Level (FTPL) stresses the role of broad money, inclusive of nominal government debt, as a store of value and links determination of the price level to the sustainability of government debt. Its key equation equates the real market value of debt, i.e., the nominal debt level, $\mathcal{B}_t$, divided by the price level, $\mathcal{P}_t$, to the fundamental value. The fundamental value of government debt is the expected discounted present value (PV) of the stream of future primary surpluses – the difference between the government revenues and expenditures excluding interest payments. According to the FTPL, the price level adjusts to achieve equality between the market value and the fundamental value, such that debt remains sustainable.

In this paper, we expand the FTPL, so that it even applies in settings with bubbles. In such environments, the key debt valuation equation emphasized by the FTPL may no longer hold because government debt can have a bubble component. Instead, a more general equation takes its place:

$$\frac{\mathcal{B}_t}{\mathcal{P}_t} = \mathbb{E}_t[PV(\text{primary surpluses})] + \text{bubble}. \tag{1}$$

Including the typically ignored bubble term is important to reconcile the theory with empirical debt valuation puzzles. For the United States, Jiang et al. (2019) suggest that the price of U.S. Treasury debt significantly exceeds the present value of primary surpluses. For Japan, the standard debt valuation equation without a bubble term appears to be even more at odds with the data. The fact that since the 1960s Japan's primary surpluses were mostly negative and with no positive future primary surplus in sight does not square well with the bubble-less valuation equation.[1]

Theoretically, bubbles can exist whenever the real interest rate is persistently below the growth rate of the economy, i.e., whenever $r \leq g$. It is well known that this can be the case in overlapping generations models (Samuelson 1958), models of perpetual youth (Blanchard 1985), and incomplete market models with uninsurable idiosyncratic risk à la Bewley (1980). In this paper, we spell out the details of the FTPL with a bubble in a simple illustrative model with uninsurable idiosyncratic risk based on Brunnermeier and Sannikov (2016a,b) in which $r \leq g$ arises naturally as a precautionary savings demand depresses r.

Our key contribution is to expand the FTPL uniqueness reasoning to make it compatible

[1]This empirical evidence is, of course, not just an issue for the FTPL. The same valuation equation also holds in conventional theories, but is there interpreted as an "intertemporal government budget constraint". We adopt here the interpretation of a valuation equation in line with the FTPL. In fact, in the presence of bubbles, it is unclear how exactly this equation constrains government policy.

with the possibility of bubbles, which present a significant challenge to the conventional FTPL argument. The existing FTPL literature is merely concerned with multiplicity of the price level that arises from nominal indeterminacy. In a setting with bubbles, two additional layers of multiplicity emerge: First, equilibria with and without bubbles may coexist. Second, when bubbles do exist, they may be attached to different assets in different equilibria. A bubble on government debt is merely one possibility. In other equilibria, other assets may have bubble components, such as corporate bonds or crypto assets. Because a potential bubble component affects the real value of nominal government debt, all three layers of multiplicity might interact and affect the price level and inflation via equation (1).

We show how appropriate fiscal policy can not only uniquely determine the price level but also select the unique bubble equilibrium in which the bubble is attached to government debt. The key idea is to adjust policy *off-equilibrium*, should the bubble on government debt burst or deflate. Specifically, we propose a threshold policy whereby the government commits to tax hikes that prop up the value of its debt if it were ever to fall below a given threshold value. The resulting capital gains bondholders would experience in such an event make it optimal for them to hold on to the bonds in the first place, so that *in equilibrium*, the value of debt never falls below the threshold.

Ruling out no-bubble equilibria merely requires that the government commits to a minimal threshold that would trigger taxation after an extreme drop in the real value of its debt. Such a fiscal rule rules out not only the no-bubble equilibrium, but also equilibria in which the bubble deflates over time.

A threshold policy is, in principle, also suitable to ensure that the bubble is associated with government debt instead of any other asset. However, accomplishing this additional goal is more challenging and can impose additional restrictions on suitable thresholds that trigger taxation. The threshold required depends on the issuance rate of bonds relative to that of the other asset. If the issuance rate of bonds is lower, the bubble on the other asset has to grow faster in any alternative equilibrium, ultimately crowding out the bubble on the government bond. Hence, any threshold that imposes an effective lower bound on the value of government debt can ultimately eliminate these equilibria. The same policies that rule out the no-bubble equilibrium also achieve this additional goal. In contrast, if the issuance rate of bonds is at least as large as that of the other asset, too low thresholds do not rule out all equilibria with bubbles on other assets. The reason is that in these alternative equilibria the bubble on government debt never fully deflates and hence may not fall below the threshold. To ensure that the bubble is fully attached to the government debt in this case, the government has to commit to raise primary surpluses immediately, if the debt value were to drop (off-equilibrium) only marginally below the full bubble value.

The fact that the proposed threshold policy promises off-equilibrium actions that are never observed in equilibrium raises the issue of credibility. In the context of our model, we formally analyze credibility of the off-equilibrium backing by relaxing the assumption of perfect commitment. That analysis yields a sharp prediction. When fiscal policy cannot commit to off-equilibrium tax hikes, it is still able to eliminate both the no bubble equilibrium and all bubbles on assets whose supply grows at a faster rate than the government's bond growth rate along the desired equilibrium path. But it is no longer able to eliminate bubbles on assets whose supply grows at a slower rate. The reason is that ruling out the former type of equilibria requires future governments to react only if the value of government debt becomes very small. Because government bonds serve as a safe asset that allows agents to partially overcome incomplete market frictions, such a future government will find it optimal to provide additional government debt even if it has to be backed with taxes.

We also discuss alternative policies that are specifically directed at preventing certain bubbles on assets other than government debt. Such policies include insolvency laws that rule out private Ponzi schemes and holding restrictions or taxation of specific assets. When most or all assets in the economy are liabilities of entities subject to insolvency laws, these laws are generally effective in ruling out alternative bubble equilibria. Crypto assets such as bitcoin are not affected by insolvency laws, and hence additional regulation of these assets may be required to prevent bubbles on them and to preserve debt sustainability when credibility of off-equilibrium taxation is in question. Our paper thus provides a new rationale for the regulation of crypto assets.

Besides raising the question of uniqueness and bubble fragility, a public debt bubble has also beneficial implications for debt sustainability: the debt bubble represents a fiscal resource. By "printing" bonds, the government imposes an inflation tax that reduces the return on the bonds. Since government bonds are a bubble, the government in a sense "mines a bubble" to generate seigniorage revenue. This seigniorage revenue can be used to finance government expenditures without ever having to raise extra taxes. There is a limit to bubble mining seigniorage, however. As more aggressive bubble mining reduces the attractiveness of government bonds as a store of value, private agents try to substitute away into other assets or reduce their total savings. As with traditional inflation taxes, bubble mining can erode the tax base. A Laffer curve emerges.

We also study optimal debt issuance policy. A positive rate of bubble mining with perpetually negative primary fiscal surpluses can be the optimal policy prescription, since bubble mining discourages bond holdings and boosts physical capital investments and thereby economic growth. Importantly, the optimal debt issuance policy only corrects for pecuniary externalities but it never reacts to the size of or need for public expenditures. The main takeaway is that welfare-maximizing policy should rely on taxes, not bubble mining, as the marginal funding source for (additional) public expenditures.

Literature. This paper contributes to the FTPL literature and its antecedent on the importance of fiscal arrangements in monetary economies (e.g. Sargent and Wallace, 1981). Classic references for the FTPL are Leeper (1991), Sims (1994), and Woodford (1995). For more comprehensive treatments see Leeper and Leith (2016) and Cochrane (2021). That literature considers bubble-free environments. An exception is Bassetto and Cui (2018) who study the validity of the FTPL in low interest rate environments.[2] While they conclude that the FTPL breaks down, we show that more sophisticated fiscal rules can ensure a unique price level.

Our paper is also related to an extensive literature on rational bubbles (e.g. Caballero and Krishnamurthy 2006, Farhi and Tirole 2012, Martin and Ventura 2012, 2016, Miao and Wang 2012, 2018, Asriyan et al. 2021). Recent survey papers include Miao (2014) and Martin and Ventura (2018).[3] A common theme in this literature is the existence of bubbles on assets issued by private agents and how these bubbles alleviate financial frictions. In contrast, our paper emphasizes bubbles on government debt and how such bubbles generate fiscal space. Closest to our discussion of equilibrium uniqueness is Asriyan et al. (2021) who show that monetary policy can select a unique equilibrium path for bubbly money and the price level out of a given pre-selected set of equilibria that are constrained by a "market psychology" that determines the evolution of private bubbles. By connecting the uniqueness question with the FTPL, we derive results that are considerably more far-reaching: using off-equilibrium fiscal backing, fiscal policy can select a unique equilibrium out of the set of all possible equilibria.

Our uniqueness argument also relates to papers that seek to rule out hyperinflationary equilibria in models of fiat money. Wallace (1981), Obstfeld and Rogoff (1983), and Tirole (1985) show how policy interventions such as partial commodity backing or reserve requirements can ensure equilibrium uniqueness in models with money in the utility function or with bubbly money, respectively. The fiscal strategy we propose can accomplish the same, but is considerably broader in scope, as we show that it can also rule out equilibria with bubbles on other assets than government debt.[4] This broader scope connects our paper with work on multi-currency environments such as Kareken and Wallace (1981) or Sargent and Smith (1997). In addition of allowing for alternative bubbly assets, we also investigate when policies that ensure uniqueness remain credible under imperfect commitment.

[2]Like Bassetto and Cui (2018), Farmer and Zabczyk (2020) also study the FTPL in an OLG model and conclude that the FTPL is unable to resolve equilibrium multiplicity. However, their result is based on indeterminancy in the underlying real model that is not directly related to either bubble multiplicity or indeterminacy of nominal valuations.

[3]This paper abstracts from aggregate risk and then bubbles can exist if the risk-free rate r is below the economic growth rate g, consistent with the empirical evidence. However, $r < g$ does not necessarily imply the existence of bubbles in all models, like in models with aggregate (disaster) risk and complete markets (e.g. Bohn 1995, Barro 2021, Mehrotra and Sergeyev 2021).

[4]In Wallace (1981) and Obstfeld and Rogoff (1983), money is by assumption the only possible store of value and medium of exchange, respectively. Tirole (1985) does allow for bubbles on alternative assets, but his reserve requirement does not ensure uniqueness (it only eliminates asymptotically bubble-free equilibria).

The possibility to run perpetual deficits through bubble mining relates our paper to the literature on debt rollovers in OLG models (Diamond 1965, Ball et al. 1998, Blanchard and Weil 2001, Blanchard 2019). In an influential recent contribution, Blanchard (2019) concludes that public debt may have no fiscal cost. Brumm et al. (2021) dispute Blanchard (2019)'s conclusion by presenting four settings in which public debt expansion is not the ideal policy to overcome the fundamental frictions causing the low interest rate. Methodologically, this literature does not established a link to bubbles, but considers the dynamic stability properties of the debt-to-gdp ratio. In contrast, our paper emphasizes that the perpetual deficits are possible precisely when there is a public debt bubble. In our analytically tractable example, we provide the conditions for the possibility of public debt bubbles, delineate the limits of bubble mining, and characterize optimal bubble mining policy. Our FTPL focus also allows us to make progress on the question of how policy can prevent coordination on adverse alternative equilibria, an issue that Blanchard (2019) recognizes as important, but ultimately ignores.

Our characterization of optimal bubble mining relates to literature on the optimal quantity of debt in Aiyagari (1994) models, without (Aiyagari and McGrattan, 1998) and with bubbles (Domeij and Ellingsen, 2018). These papers study quantitative numerical solutions while we provide analytical characterizations. More loosely related are Aguiar et al. (2023), who show how debt expansions in Aiyagari (1994)-type models can be used to generate robust Pareto improvements, and Angeletos et al. (2021) and Sims (2022), who study optimal taxation and debt smoothing à la Barro (1979) when government debt enters the utility function. Unlike our paper, these papers are not concerned with resolving equilibrium multiplicity or establishing a link between a socially optimal positive quantity of debt and the ability of the government to commit to off-equilibrium tax backing.

Since circulation of a previous draft, some recent papers have taken up and extended the core insights from our paper. Like this paper, Reis (2021) emphasizes the bubble as a fiscal resource that has implications for debt sustainability, but his focus is on the interaction with other policies while we focus on FTPL aspects and optimal bubble mining. Brunnermeier et al. (2021a) develop a safe asset theory of government debt based on a model related to ours but with aggregate risk. Kocherlakota (2021) studies a bubble on government debt caused by tail risk. Kocherlakota (2022) and Li and Merkel (2020) study monetary and fiscal policy in New Keynesian models with government debt bubbles and show that monetary policy may be inferior to fiscal policy in stabilizing inflation and the output gap. The uniqueness argument based on fiscal backing made in this paper serves as a theoretical underpinning for the (often implicit) equilibrium selection made in all of these papers.

2 Model

There are several model structures in which rational bubbles can exist and thus the bubble term can emerge in the debt valuation equation (1). We illustrate this in a simple example based on a streamlined variant of Brunnermeier and Sannikov (2016a) without banks.[5]

In our model, bubbles can emerge due to incomplete idiosyncratic risk sharing. Government debt may circulate as a bubble because bond trading allows agents to self-insure against idiosyncratic shocks. We further include an alternative asset that is intrinsically worthless but may also circulate as a bubble. This asset competes with the government bond as a safe store of value.

While, in our view, incomplete idiosyncratic risk sharing is a plausible mechanism to generate bubbles, we emphasize that this modeling choice is not crucial for our key results. The main insights from our paper should equally apply to other environments in which a bubble term in equation (1) is possible.[6]

In this section, we briefly outline the model elements and present the solution. Additional formal details and derivations are presented in Appendix A.1.

Environment. There is a continuum of households indexed by $i \in [0,1]$. All households have identical logarithmic preferences

$$V_0^i := \mathbb{E}\left[\int_0^\infty e^{-\rho t} \log c_t^i dt\right]$$

with discount rate ρ.

Each agent operates one firm that produces an output flow $ak_t^i dt$ of a perishable output good, where k_t^i is the (physical) capital input chosen by the firm. Absent market transactions of capital, capital of firm i evolves according to

$$\frac{dk_t^i}{k_t^i} = \left(\Phi\left(\iota_t^i\right) - \delta\right) dt + \tilde{\sigma} d\tilde{Z}_t^i,$$

where $\iota_t^i k_t^i dt$ are physical investment expenditures of firm i (in output goods), Φ is a concave function that captures adjustment costs in capital accumulation, δ is the depreciation rate, and $\tilde{Z}^i$ is an agent-specific Brownian motion that is i.i.d. across agents i. $\tilde{Z}^i$ introduces firm-specific idiosyncratic risk. To obtain simple closed-form expressions, we choose the functional form

[5] The model version without banks has previously been analyzed in Brunnermeier and Sannikov (2016b) and Di Tella (2020). These papers frame the model as a model of money. Here, we add fiscal policy and reinterpret money as bonds. The bond interpretation is also adopted in the safe asset framework of Brunnermeier et al. (2021a).

[6] A previous version of this paper also contained a second example based on the perpetual youth model. To make space, we have removed this model.

$\Phi(\iota) = \frac{1}{\phi} \log(1 + \phi\iota)$ with adjustment cost parameter $\phi \geq 0$ for the investment technology.

The key friction in the model is that agents are not able to share idiosyncratic risk. While they are allowed to trade physical capital and risk-free assets, they cannot write financial contracts contingent on individual $\tilde{Z}^i$ histories. As a consequence, all agents have to bear the idiosyncratic risk inherent in their physical capital holdings.

Besides households, there is a government that funds government spending, imposes taxes, and issues nominal bonds. The government has an exogenous need for real spending $\mathfrak{g}K_t dt$, where $\mathfrak{g}$ is a model parameter and $K_t := \int k_t^i di$ denotes the aggregate capital stock. The government levies proportional output taxes (subsidies, if negative) τ_t on households. Outstanding government debt has a nominal face value of $\mathcal{B}_t$ and pays nominal interest i_t. $\mathcal{B}_t$ follows a continuous process $d\mathcal{B}_t = \mu_t^{\mathcal{B}} \mathcal{B}_t dt$, where the growth rate $\mu_t^{\mathcal{B}}$ is a policy choice of the government. The government chooses the policy instruments τ_t, i_t, $\mu_t^{\mathcal{B}}$ as functions of histories of prices taking $\mathfrak{g}$ as given and subject to the nominal budget constraint[7]

$$\overbrace{\underbrace{\left(\mu_t^{\mathcal{B}} - i_t\right)}_{=:\breve{\mu}_t^{\mathcal{B}}} \mathcal{B}_t}^{\text{bond growth in excess of payouts}} + \overbrace{\mathcal{P}_t \underbrace{(\tau_t a - \mathfrak{g})}_{=:s_t} K_t}^{\text{(nominal) primary surpluses}} = 0, \tag{2}$$

where $\mathcal{P}_t$ denotes the price level. $\breve{\mu}_t^{\mathcal{B}}$ can be interpreted as the rate at which the government dilutes the claim of existing bond owners to future primary surpluses.

Unlike capital, government bonds are free of idiosyncratic risk and therefore represent a safe store of value for households. We do not assume that government bonds are special in this regard but allow for the possibility of a competing safe store of value. Specifically, households have access to an additional (non-perishable) asset that is intrinsically worthless and exists in limited supply[8] – for concreteness called cryptocoin. To keep matters simple, we assume that the nominal supply of cryptocoins, $\mathcal{C}_t$, grows at a constant exogenous rate,

$$d\mathcal{C}_t = \breve{\mu}^{\mathcal{C}} \mathcal{C}_t dt,$$

where $\breve{\mu}^{\mathcal{C}} \geq 0$ is a model parameter. We furthermore assume that, if $\breve{\mu}^{\mathcal{C}} > 0$, newly generated cryptocoins are produced by capital. Under this assumption, seigniorage from cryptocoin growth accrues to capital owners symmetrically to how a higher dilution rate $\breve{\mu}_t^{\mathcal{B}}$ of government bonds leads to lower capital taxes (compare equation (2)). The quantity of cryptocoins at $t = 0$, $\mathcal{C}_0$, is initially owned by households according to some exogenously given distribution. Like

[7] Letting policy depend on histories of endogenous price paths is common in the FTPL literature to discuss what happens off-equilibrium and important for our selection results in Section 4.

[8] While we limit attention to a single additional asset, this is without loss of generality. We could always combine the portfolio of all such assets in the economy into a single one.

capital and government bonds, households can trade cyptocoins on Walrasian asset markets.

The model is closed by the aggregate resource constraint

$$C_t + \mathfrak{g} K_t + I_t = a K_t, \tag{3}$$

where $C_t := \int c_t^i di$ is aggregate consumption and $I_t := \int \iota_t^i k_t^i di$ is aggregate investment.

Price Processes and Returns. At each date, agents can exchange four goods on Walrasian markets: the output good, capital, bonds, and cryptocoins. Three variables are sufficient to characterize all relative prices. The first is q_t^K, which denotes the price of a single unit of physical capital in terms of the output good. As a second variable, we could use nominal price level $\mathcal{P}_t$, which is the price of the output good in units of nominal bonds. However, it turns out to be more convenient to use the transformation $q_t^B := \frac{\mathcal{B}_t/\mathcal{P}_t}{K_t}$, which is the ratio of the real value of total government debt to total capital in the economy.[9] As a third variable, we denote similarly by q_t^C the ratio of the real value of all cryptocoins to total capital.

Households can trade three assets, bonds, cryptocoins, and capital. We denote the real returns on these assets by $dr_t^{\mathcal{B}}$, $dr_t^{\mathcal{C}}$, and $dr_t^{K,i}(\iota_t^i)$, respectively. The return on capital depends on the household's own choice ι_t^i of the physical investment rate. Explicit expressions for the three returns are relegated to the appendix. Here, we merely emphasize that $dr_t^{K,i}(\iota_t^i)$ is risky due to idiosyncratic capital risk, whereas $dr_t^{\mathcal{B}}$ and $dr_t^{\mathcal{C}}$ are both risk-free.

Household Problem. Let n_t^i denote the net worth of agent i, which consists of capital, bond, and cryptocoin holdings. Denote by $\theta_t^{B,i}$, $\theta_t^{C,i}$ the shares of total net worth the agent invests into bonds and cryptocoins, respectively. Because agents can freely adjust portfolios at all times, the household problem can be formulated in terms of the single state variable n_t^i. Net worth evolves according to

$$\frac{dn_t^i}{n_t^i} = -\underbrace{\frac{c_t^i}{n_t^i}}_{=:\hat{c}_t^i} dt + dr_t^{K,i}\left(\iota_t^i\right) + \theta_t^{B,i}\left(dr_t^{\mathcal{B}} - dr_t^{K,i}\left(\iota_t^i\right)\right) + \theta_t^{C,i}\left(dr_t^{\mathcal{C}} - dr_t^{K,i}\left(\iota_t^i\right)\right). \tag{4}$$

The household takes the initial net worth n_0^i and the returns $dr_t^{\mathcal{B}}$, $dr_t^{\mathcal{C}}$, and $dr_t^{K,i}(\cdot)$ as given[10] and chooses the consumption-wealth ratio $\{\hat{c}_t^i\}_{t\geq 0}$, real investment $\{\iota_t^i\}_{t\geq 0}$, and the portfolio shares $\{\theta_t^{B,i}\}_{t\geq 0}$ and $\{\theta_t^{C,i}\}_{t\geq 0}$ to maximize utility V_0^i subject to the net worth evolution (4) and a standard solvency constraint $n_t^i \geq 0$ that precludes Ponzi schemes.

[9] It is more convenient to work with this normalized version of the inverse price level $1/\mathcal{P}_t$ because the latter depends on the scale of the economy and the nominal quantity of outstanding bonds in equilibrium, whereas q_t^B does not.

[10] For the capital return, the function $dr_t^{K,i}(\cdot)$ is taken as given but the household understands how choosing ι_t^i affects the ultimate return $dr_t^{K,i}(\iota_t^i)$.

Equilibrium. Informally, a competitive equilibrium is a set of time paths for prices, government policies, and allocations such that all households solve their decision problem given prices and government policies and markets clear.[11] The market clearing conditions are[12]

$$\hat{c}_t(q_t^B + q_t^C + q_t^K) + \mathfrak{g} + \iota_t = a \qquad \text{goods market clearing}$$
$$\theta_t^B(q_t^B + q_t^C + q_t^K) = q_t^B \qquad \text{bond market clearing}$$
$$\theta_t^C(q_t^B + q_t^C + q_t^K) = q_t^C \qquad \text{cryptocoin market clearing}$$

An admissible government policy rule is, loosely speaking, a rule that maps histories of prices into values of the policy instruments τ_t, i_t, and $\mu_t^{\mathcal{B}}$ such that the government budget constraint (2) is satisfied after all histories. Some care must be taken in how to interpret the constraint in the case $q_t^B = 0$. In this case, it is not feasible for the government to fund a negative primary surplus, $s_t = \tau_t a - \mathfrak{g} < 0$, because no finite amount of new bond issuance will collect any real resources. However, it is still feasible to generate a positive primary surplus, $s_t > 0$ because the government's taxation power does not cease to exist when bonds become worthless.

We provide a formal equilibrium definition in Appendix A.1.1.

Model Solution. We state and solve the HJB equation of households in Appendix A.1.2. The first-order conditions for the four choices can be written as

$$c_t^i = \rho n_t^i, \qquad \text{permanent income consumption}$$
$$q_t^K = \frac{1}{\Phi'\left(\iota_t^i\right)}, \qquad \text{Tobin's } q$$
$$\frac{\mathbb{E}_t[dr_t^K(\iota_t^i)]}{dt} - r_t^f = \left(1 - \theta_t^{B,i} - \theta_t^{C,i}\right)\tilde{\sigma}^2, \qquad \text{Merton portfolio for capital}$$
$$\frac{dr_t^{\mathcal{B}}}{dt} = \frac{dr_t^{\mathcal{C}}}{dt} = r_t^f, \qquad \text{no arbitrage}$$

where r_t^f denotes the risk-free rate.

The first condition is the familiar log-utility consumption rule. The agent optimally consumes a constant fraction of total net worth at all times. The second condition equates the market value of an installed unit of capital to the marginal cost of physical investment. The third condition equates the excess return on capital with the required risk premium for bearing

[11] Throughout this paper, we limit attention to equilibria that are deterministic and feature absolutely continuous price paths. This is not crucial for any of our results. However, considering non-time-continuous price paths and equilibria driven by sunspot noise leads to additional technical complications that make the mathematical arguments considerably more involved without generating additional economic insights.

[12] To be precise, the equations stated here are only the correct clearing conditions if all agents make the same choices. This is indeed the case in equilibrium as shown in Appendix A.1.2.

idiosyncratic risk. The fourth condition states that bonds and cryptocoins must both earn the risk-free rate.

By combining these optimal choice conditions with market clearing, the model can be fully solved up to two dynamic equations for the values q_t^B of bonds and q_t^C of cryptocoins. Instead of using q_t^B and q_t^C directly, we normalize the values by total net worth, as this leads to simpler equations. Specifically, we use the notation

$$\vartheta_t^B := \frac{q_t^B}{q_t^B + q_t^C + q_t^K}, \qquad \vartheta_t^C := \frac{q_t^C}{q_t^B + q_t^C + q_t^K},$$

for the shares of total wealth due to bonds and cryptocoins, respectively.

Proposition 1. *In any equilibrium,*

$$\iota_t = \frac{(1-\vartheta_t)(a-\mathfrak{g}) - \rho}{1-\vartheta_t + \phi\rho}, \qquad q_t^K = (1-\vartheta_t)\frac{1+\phi(a-\mathfrak{g})}{1-\vartheta_t+\phi\rho},$$
$$q_t^B = \vartheta_t^B \frac{1+\phi(a-\mathfrak{g})}{1-\vartheta_t+\phi\rho}, \qquad q_t^C = \vartheta_t^C \frac{1+\phi(a-\mathfrak{g})}{1-\vartheta_t+\phi\rho},$$

where $\vartheta_t = \vartheta_t^B + \vartheta_t^C$ is the share of total wealth due to safe assets. Furthermore, ϑ_t^B and ϑ_t^C satisfy the equations

$$\vartheta_t^B = \int_t^\infty e^{-\rho(s-t)}\left(\tilde{\sigma}_s^c - \breve{\mu}_s^{\mathcal{B}}\right)\vartheta_s^B ds, \tag{5}$$
$$\vartheta_t^C = \int_t^\infty e^{-\rho(s-t)}\left(\tilde{\sigma}_s^c - \breve{\mu}^{\mathcal{C}}\right)\vartheta_s^C ds, \tag{6}$$

where $\tilde{\sigma}_t^c = (1-\vartheta_t)\tilde{\sigma}$ is the (common) idiosyncratic consumption growth volatility faced by all households.

In particular, the first part of this proposition implies that asset values, physical investment, and the consumption allocation[13] only depend on the combined safe asset share ϑ_t but not on the composition of the safe asset portfolio, i.e. the split into ϑ_t^B and ϑ_t^C.

Equations (5) and (6) represent valuation equations that relate the values of bonds and cryptocoins to two future flows. Equation (5) relates ϑ_t^B positively to the future path $\left(\tilde{\sigma}_t^c\right)^2$ of the residual idiosyncratic consumption growth variance faced by households. This is a measure of the value of the self-insurance "services" provided by safe bonds to households. The equation also relates ϑ_t^B negatively to the future path of $\breve{\mu}_t^{\mathcal{B}}$, which measures the dilution of the claims of existing bond holders through the issuance of new bonds in excess of what is required to fund nominal interest payments. Equation (6) is the analogous condition for ϑ_t^C. Cryptocoins

[13]The consumption allocation is implied by asset values because log utility agents consume a fraction ρ of wealth.

provide the same self-insurance "services" as government bonds, whereas their dilution rate may differ.

Steady-State Equilibria. We now focus on steady-state equilibria with constant paths for the policy variables $\breve{\mu}^{\mathcal{B}}$ and τ and for the asset prices q^B, q^C, and q^K – and hence constant ϑ^B and ϑ^C. In these equilibria, equations (5) and (6) simplify to

$$\vartheta^B = \frac{(\tilde{\sigma}^c)^2 - \breve{\mu}^{\mathcal{B}}}{\rho}\vartheta^B, \qquad \vartheta^C = \frac{(\tilde{\sigma}^c)^2 - \breve{\mu}^{\mathcal{C}}}{\rho}\vartheta^C \tag{7}$$

For any given value for $\breve{\mu}^{\mathcal{B}}$, there are four types of possible steady state equilibria:[14]

1. There is always a non-monetary, no-bubble steady state in which both government bonds and cryptocoins are worthless, $q^B = q^C = 0$.[15]

2. There is at most one "monetary steady state" in which government bonds have a positive value and cryptocoins are worthless, $q^B > 0$, $q^C = 0$. In this equilibrium, if it exists,

$$\iota = \frac{\sqrt{\rho + \breve{\mu}^{\mathcal{B}}}\,(a - \mathfrak{g}) - \rho\tilde{\sigma}}{\sqrt{\rho + \breve{\mu}^{\mathcal{B}}} + \phi\rho\tilde{\sigma}}, \quad q^B = \frac{\left(\tilde{\sigma} - \sqrt{\rho + \breve{\mu}^{\mathcal{B}}}\right)(1 + \phi\,(a - \mathfrak{g}))}{\sqrt{\rho + \breve{\mu}^{\mathcal{B}}} + \phi\rho\tilde{\sigma}}, \quad q^K = \frac{\sqrt{\rho + \breve{\mu}^{\mathcal{B}}}\,(1 + \phi\,(a - \mathfrak{g}))}{\sqrt{\rho + \breve{\mu}^{\mathcal{B}}} + \phi\rho\tilde{\sigma}}.$$

   These formulas describe a valid equilibrium if the value of capital and the value of bonds are both positive. This is the case if and only if idiosyncratic risk is sufficiently large,

$$\tilde{\sigma}^2 > \rho + \breve{\mu}^{\mathcal{B}}. \tag{8}$$

   In this equilibrium, government debt may or may not have a bubble component depending on the sign of $\breve{\mu}^{\mathcal{B}}$. We return to this issue in Section 3.

3. Symmetrically, there is at most one "crypto bubble steady state" in which cryptocoins have a positive value and government bonds are worthless, $q^C > 0$, $q^B = 0$. This equilibrium is isomorphic to the second type except that cryptocoins take the role of government bonds.[16] This equilibrium exists if and only if $\tilde{\sigma}^2 > \rho + \breve{\mu}^{\mathcal{C}}$. In this equilibrium, cryptocoins always must have a bubble component because they have a positive value despite being intrinsically worthless.

4. Finally, there may be steady state equilibria in which both government bonds and cryptocoins coexist with a positive value. From equations (7) we observe that this is only possible if $\breve{\mu}^{\mathcal{B}} = \breve{\mu}^{\mathcal{C}} =: \breve{\mu}$. Economically, this makes sense as only then government bonds and cryptocoins are diluted at the same rate, so that a constant exchange rate is consistent

[14] The exercise here is to fix $\breve{\mu}^{\mathcal{B}}$ and let s be implicitly defined by the government budget constraint.
[15] In this equilibrium, the price level is infinite, $\mathcal{P} = \infty$, and the government does not raise any primary surplus.
[16] In particular, the same equations for ι and q^K hold but with $\breve{\mu}^{\mathcal{B}}$ replaced by $\breve{\mu}^{\mathcal{C}}$.

with no arbitrage. As before, these equilibria exist if idiosyncratic risk is sufficiently large, $\tilde{\sigma}^2 > \rho + \breve{\mu}$. If this is the case, there is a continuum of stationary steady state equilibria with the same total safe asset value

$$q^B + q^C = \frac{\left(\tilde{\sigma} - \sqrt{\rho + \breve{\mu}}\right)\left(1 + \phi\left(a - \mathfrak{g}\right)\right)}{\sqrt{\rho + \breve{\mu}} + \phi\rho\tilde{\sigma}},$$

but an undetermined split into government bonds (q^B) and cryptocoins (q^C). This is reminiscent of Kareken and Wallace (1981) exchange rate indeterminacy.

We remark that the previous discussion only focuses on steady-state equilibria. There can be other, nonstationary, equilibria in which both cryptocoins and government bonds coexist with a positive value. A policy rule that holds $\breve{\mu}^{\mathcal{B}}$ constant after every price history can therefore be consistent with multiple competitive equilibria. In Section 4 we show how a simple off-equilibrium modification to such a policy rule can select the monetary steady state as the unique equilibrium whenever it exists.

3 Transversality Conditions and Existence of Bubbles

Using the notation of our model, the debt valuation equation (1) with a bubble takes the following form:[17]

$$\frac{\mathcal{B}_t}{\mathcal{P}_t} = q_t^B K_t = \mathbb{E}_t\left[\int_t^\infty \frac{\xi_s^i}{\xi_t^i} s_s K_s ds\right] + \lim_{T\to\infty} \mathbb{E}_t\left[\frac{\xi_T^i}{\xi_t^i} q_T^B K_T\right]. \tag{9}$$

Here, $\xi_t^i := e^{-\rho t}\frac{1}{c_t^i}$ denotes the SDF process of household i.[18] Relative to the conventional valuation equation, the limit in the second term may not vanish because government debt can have a bubble component. More generally, we say for any asset that it has a bubble component if its market value exceeds its fundamental value. We define the fundamental value as the discounted present value of the asset's cash flows where cash flows are discounted using the SDF ξ^i generated by the marginal utility of the marginal holder of the asset. Note that for the total public debt stock – as opposed to individual bonds – the fundamental value is precisely the present value of primary surpluses.

A bubble component is possible in our model because government debt serves as a store of value that is free of idiosyncratic risk and thus allows agents to self-insure against their risk

[17]We provide a generic derivation of this equation in Appendix A.2.

[18]Because primary surpluses and the value of total debt are free of idiosyncratic risk, all agents i agree on the present values in equation (9).

exposures. In this section, we discuss why the private-sector transversality condition may not rule out the existence of a bubble despite the infinite lifespan of all agents. The key insight is that a bubble on government debt can exist because agents do not buy and hold government bonds, but optimally trade them. Such trading makes their individual bond portfolios look very different from the aggregate bond stock.[19] We remark that symmetric arguments also apply for cryptocoins. However, for simplicity, we discuss here only government debt bubbles in the monetary steady state with $q^C \equiv 0$.

Let $b_t^i := \theta_t^{B,i} n_t^i$ be the real value of agent i's bond holdings. For each individual agent, a transversality condition for bond holdings is necessary for an optimal choice:

$$\lim_{T\to\infty} \mathbb{E}\left[\xi_T^i b_T^i\right] = 0.$$

This transversality condition appears to suggest that it should not be possible to have a nonzero bubble term in the debt valuation equation (9). However, this argument overlooks that individual bond wealth b_T^i that enters the transversality condition differs from the aggregate value of bonds $q_T^B K_T$ that enters the valuation equation. The aggregate bond stock $q_T^B K_T$ evolves deterministically, yet individual bond wealth b_T^i is optimally chosen to be stochastic because agents constantly rebalance their portfolios in response to idiosyncratic shocks. Agents thus discount b_T^i at a risk-adjusted rate that takes into account their idiosyncratic risk. As idiosyncratic risk cancels out in the aggregate, when valuing a fixed fraction of the outstanding bond stock, as in the debt valuation equation (9), the relevant discount rate from the perspective of all agents is instead the risk-free rate.

Formally, we have $c_t^i = \rho n_t^i$ and $n_t^i \geq b_t^i$ (because capital wealth is positive), so that

$$\mathbb{E}\left[\xi_T^i b_T^i\right] = e^{-\rho T}\frac{1}{\rho}\mathbb{E}\left[\frac{b_T^i}{n_T^i}\right] \leq \frac{1}{\rho}e^{-\rho T} \to 0 \qquad (T\to\infty)$$

and thus the individual transversality condition is clearly satisfied in any of the steady-state equilibria determined in Section 2. Yet, when determining agent i's time-0 valuation of the entire government bond stock at time T, we obtain (up to a scaling constant)

$$\mathbb{E}\left[\xi_T^i \int b_T^j dj\right] = \mathbb{E}\left[\xi_T^i q^B K_T\right] \propto e^{-r^f T} q^B K_T = e^{\left(g-r^f\right)T} q^B K_0,$$

where $g := \Phi(\iota) - \delta$ is the (steady-state) output growth rate. The latter expression does not converge to zero, if $r^f \leq g$. In this case, a bubble is possible in equation (9).

The difference in the second equation is the presence of the dj-integral. This integral av-

[19]While we discuss here the specific case of our model, this insight holds generally for any rational bubble model. In the absence of equilibrium trades, individual transversality conditions are sufficient to rule out bubbles.

erages out idiosyncratic shocks and changes the risk characteristics relative to the individual bond portfolios in the integrand. All individual bond portfolios b_T^i have idiosyncratic fluctuations that are negatively correlated with agent i's SDF ξ_T^i. The effective discount rate in the individual transversality condition therefore contains a covariance term (risk premium) that raises the discount rate above r^f. For the total bond stock, idiosyncratic risk averages out and discounting happens at the risk-free rate.

Nothing in the model prevents the bubble existence condition $r^f \leq g$. Indeed, the growth rate of the economy equals the capital growth rate and the risk-free rate equals the return on bonds. In the monetary steady state,

$$r^f = g - \breve{\mu}^{\mathcal{B}}. \tag{10}$$

Consequently, $r^f \leq g$, if and only if $\breve{\mu}^{\mathcal{B}} \geq 0$. By the government budget constraint (2), $\breve{\mu}^{\mathcal{B}} \geq 0$ if and only if primary surpluses are nonpositive. In addition, a nonnegative value of $\breve{\mu}^{\mathcal{B}}$ is consistent with the existence condition (8) of the monetary steady state if idiosyncratic risk is sufficiently large relative to the time preference rate, $\tilde{\sigma}^2 \geq \rho$. In this case, households' precautionary motive generates a sufficiently strong savings demand to sustain a bubble on government bonds.

The possibility of $r^f \leq g$ is also not merely a theoretical curiosity. Historically, real interest rates on government bonds of advanced economies have mostly been below the growth rate. Even Abel et al. (1989), who are often cited as providing evidence against the existence of rational bubbles, report that the safe interest rate r^f is smaller than g. With the more recent decline in r^f, as stressed by Blanchard (2019), the evidence for $r^f < g$ has become more clear-cut. See also Geerolf (2013) and Lian et al. (2020).

4 Price Level Determination and Equilibrium Uniqueness

A core theme in the FTPL literature is the question of price level determinacy, i.e. whether across the set of possible equilibria, there is a unique prediction for the price level. In many model environments, including ours, price level determinacy requires the equilibrium itself to be unique.

Uniqueness is not a purely technical issue but of economic importance. Price level determinacy has always been a concern in monetary economics, ultimately because an indeterminate price level casts doubt on the ability of government policy to control inflation. In the presence of a bubble, the uniqueness question becomes an even bigger concern. If there is multiplicity, then a bubble on government debt is an inherently fragile arrangement. Markets could coordinate any time on a different equilibrium, the government would lose the bubble on its debt and would either have to replace its value with a larger fundamental value by raising surpluses or accept an inflationary collapse of its currency (or default). Exploiting a bubble by mining it could then be a very risky proposition because it would expose the government's debt to sunspot revaluations that would have to be met with large and sudden fiscal corrections. In contrast, if there is a policy that can select a unique equilibrium with a bubble, then a bubble can in fact be a stable arrangement that is not threatened by shifts in market beliefs.

In this section, we show how fiscal policy can both determine the price level and ensure a stable bubble value that remains attached to government debt. We first briefly revisit the conventional FTPL arguments and discuss how the possible presence of a bubble complicates the situation. We subsequently provide a formal discussion in the context of our model. We ultimately conclude that fiscal price level determination as studied in the FTPL literature still works in the presence of a bubble, albeit the details of how to implement such a policy have to be adapted.

4.1 Fiscal Policy as an Equilibrium Selection Device in the Previous Literature

Price Level Determination in the FTPL without Bubbles. In the standard FTPL, price level determination is often explained by starting from the key equation (1) (without a bubble) and interpreting it as an asset pricing equation. A holder of the total stock of government debt who absorbs all new issuances in the future receives as a cash flow in each period precisely the stream of primary surpluses. Like the value of a stock is determined by the present value of its future dividends, the value of government debt should thus be determined by the present value of its future cash flows, the primary surpluses. Because the nominal price level is the relative price between nominal bonds and consumption goods, this is the correct "asset price" that must adjust to clear the bond market.[20] If fiscal policy ensures that the present value of primary surpluses is unique, then precisely one value of debt and thus one price level is consistent with the (bubble-free) debt valuation equation (1), which is the main reason why this is the key equation of the theory. The simplest way to ensure a unique present value is by assuming that policy commits to a fully exogenous sequence of primary surpluses. This assumption is commonly made in the FTPL literature, but it is by no means essential for the FTPL to work.

How Bubbles Challenge this Intuition. In the presence of a bubble, the previous intuition breaks down because the size of the bubble is not determined by the present value identity itself. There is, however, an alternative intuition about the economic mechanism behind the FTPL that centers on goods market clearing and wealth effects and that remains fully operational when

[20] To be precise, the price level is the relative price between a maturing bond and consumption goods, while there are additional bond prices for longer-term bonds that depend on the term structure of nominal rates.

bubbles are possible.[21] A larger real value of government bonds, holding taxes constant, means bonds represent more net wealth for the private sector, which increases consumption demand through a wealth effect. The equilibrium price level is the price level at which consumption demand equals consumption supply.

Still, this mechanism can generally not ensure a unique prediction for the price level in an environment with bubbles. Goods market clearing only determines the size of aggregate net wealth consisting of the pre-tax value of capital and the *aggregate bubble*, the sum of all bubble values in the economy. Even if the value of capital is given, there is no economic force that suggests that the residual, the aggregate bubble, should be attached to government debt. There could be bubble components on other assets, so that government debt has a smaller bubble component and a lower value. In addition, also the value of capital wealth is typically not given because discount rates depend on the presence and size of bubbles. Goods market clearing therefore at best imposes an upper bound on the real value of government debt, but it does not pin it down uniquely.

Based on this reasoning, one may be tempted to conclude that in an environment with bubbles, fiscal policy is generally unable to select a unique equilibrium and thereby pin down the price level. This is the conclusion of Bassetto and Cui (2018) in the context of a dynamically inefficient OLG model. They study constant tax policies that are not contingent on the price level and conclude that "the FTPL breaks down in [their] OLG economy" (p. 13).[22] In contrast to their conclusion, we show in the remainder of this section how fiscal price level determination can succeed even in the presence of bubbles when the government's tax policy is made contingent on the price level. Our analysis highlights the importance of contingent policy that raises positive surpluses at least off-equilibrium to deliver fiscal price level determination.[23]

Our argument proceeds in three steps. We first consider positive surplus policies that eliminate all bubbles, including on government debt, and thus restore the standard fiscal theory intuition. By construction, such policies are unable to deliver uniqueness in the presence of an equilibrium bubble. However, we show in a second and third step that the elimination of all bubbles can be used as an "off-equilibrium threat" to select a unique bubble equilibrium. The second step focuses on the special case that a bubble can at most be attached to government bonds. In this case, uniqueness requires merely the elimination of all asymptotically bubble-free equilibria. Our fiscal strategy is then very similar to the ones proposed by Wallace (1981) and, in particular, Obstfeld and Rogoff (1983) to eliminate hyperinflationary equilibria in monetary economies. These strategies are specifically designed to rule out asymptotically non-monetary

[21] This alternative intuition is in fact the original economic story told by Woodford (1995).

[22] Bassetto and Cui (2018) do not discuss the possibility of bubbles explicitly. But it is our interpretation that their result is ultimately due to bubble multiplicity in their framework.

[23] A contingent policy specification is also more desirable, because an uncontingent commitment to primary deficits, as analyzed by Bassetto and Cui (2018), is not feasible in the sense of Bassetto (2005).

equilibria. However, we do not stop there. In a third step, we show under which conditions the same policy specification can still ensure uniqueness even when bubbles can also emerge on other assets than government debt. This is a significant extension because, with alternative bubbly assets, merely ruling out asymptotically bubble-free equilibria is no longer sufficient.

4.2 Fiscal Policy as an Equilibrium Selection Device in Our Model

Positive Surplus Policies and Elimination of All Bubbles. We start by observing that wealth effects limit the total value of private-sector net wealth by an upper bound proportional to available resources. Any bubble (on any asset) represents net wealth, so that the aggregate bubble value cannot persistently outgrow the economy. As a consequence, the expected long-horizon growth rate of the bubble component on any asset cannot exceed the growth rate of the economy, g, and therefore no bubble on any asset can exist in an equilibrium that features $r^f > g$ in the long run.

More precisely, a sufficient condition for the absence of bubbles is

$$\lim_{T\to\infty} \mathbb{E}\left[\xi_T Y_T\right] = 0, \tag{11}$$

where Y_T denotes aggregate output at time T. If output growth and the risk-free rate are constant, then this condition is equivalent to $r^f > g$.

In the monetary steady state of our model, a positive surplus-capital ratio $s > 0$ necessarily implies $r^f > g$. This follows immediately from the risk-free rate equation (10) and the government budget constraint (2). This steady state must therefore be bubble-free. We now argue that a similar conclusion applies more generally: if the government commits to a positive surplus policy by choosing[24]

$$\tau_t = \frac{s + \mathfrak{g}}{a}, \qquad \breve{\mu}_t^{\mathcal{B}} = -\frac{s}{q_t^{\mathcal{B}}}, \tag{12}$$

any equilibrium consistent with this policy specification must satisfy condition (11) and, thus, no bubbles can exist.

Intuitively, if $r^f \leq g$, at least on average, then the present value of primary surpluses would be infinite and government debt would have infinite value.[25] Indeed, we have the inequality

$$\frac{\mathcal{B}_0}{\mathcal{P}_0} \geq \mathbb{E}\left[\int_0^\infty \xi_t s K_t dt\right] = \frac{s}{a}\mathbb{E}\left[\int_0^\infty \xi_t Y_t dt\right].$$

If condition (11) is violated, then the integral on the right does not converge and thus $\mathcal{B}_0/\mathcal{P}_0 =$

[24]This is always consistent with the government's flow budget constraint (2) and thus a feasible policy specification after any price history.

[25]A possible bubble component can only increase the value of debt.

∞. However, then also total net wealth must be infinite in contradiction to our previous argument that wealth effects bound the ratio of total net wealth to output from above.[26] Therefore, condition (11) must be satisfied in any equilibrium under this policy.

The previous argument implies that a positive surplus policy is inconsistent with bubbles. In Appendix A.3.2 we establish that then the equilibrium is indeed unique.

Proposition 2. *If government policy is specified by* (12) *with* $s \in (0, a - \mathfrak{g}]$,[27] *then there is a unique equilibrium. That equilibrium coincides with a monetary steady state and the initial price level* $\mathcal{P}_0$ *is uniquely determined.*

The key takeaway from this proposition is that the conventional FTPL without bubbles remains valid in an environment where bubbles are possible, if the government commits to positive primary surpluses at all times. Such a commitment also destroys the possibility of bubbles and therefore restores the conventional intuition.

We remark, however, on the following subtlety in the previous argument. For the argument to work, it is crucial that the process of taxation to back government debt generates net wealth for the private sector. Otherwise, there would be no reason why the total value of debt has to be bounded. Instead, any positive wealth effect from higher debt would simply be offset by a negative wealth effect from the additional tax liability. This is not the case in our models because Ricardian equivalence does not apply to proportional output taxes.[28] Backing of debt with surpluses would fail to select a unique bubble-free equilibrium if the government used taxes that are consistent with Ricardian equivalence and thus failed to generate net wealth.[29]

Uniqueness with an Equilibrium Bubble. We now discuss how government policy can select a unique equilibrium with a bubble *under the additional assumption that a bubble can only be on government debt*, i.e. cryptocoins are not available, $q_t^C = 0$. Even under this assumption, the equilibrium is not necessarily unique as the debt bubble can take multiple values.

Specifically, we are interested in the selection of a steady state equilibrium with a bubble. By Proposition 2 any such equilibrium must be associated with a non-positive primary surplus. However, except for the special case of zero surpluses, the government cannot simply choose a constant surplus policy rule of the form (12) with $s \leq 0$. This is not a feasible policy rule because the government cannot commit to funding deficits with bond issuance even after histories in which bonds are worthless.[30]

[26] Total net wealth at $t = 0$ is $\mathcal{B}_0/\mathcal{P}_0 + q_0^K K_0$ (and possibly larger if other assets have bubble components). While higher taxes depress the second component, $q_0^K K_0$, this component cannot become negative. Hence, an infinitely large value of debt cannot be offset by an infinitely large present value of future taxes.

[27] We impose the additional upper bound $a - \mathfrak{g}$ to ensure that taxes τ_t do not exceed 1.

[28] Individual tax liabilities are contingent on individual idiosyncratic shock histories, so that individuals discount their future tax liability at a higher rate than the return on bonds.

[29] In our model, this would be the case if surpluses were raised exclusively by imposing lump-sum taxes.

[30] Feasibility considerations aside, such a policy would also fail to attain uniqueness because of the Laffer curve

An alternative is to consider a policy rule that fixes $\check{\mu}^{\mathcal{B}} \geq 0$ at a constant level and adjusts surpluses s_t such that the government budget constraint (2) holds after any price history. Under the assumption (8), we have shown in Section 2 that this policy is consistent with two steady-state equilibria: a monetary steady state with (scaled) bond value $q^{B*} > 0$ and a non-monetary steady state in which $q^B = 0$. In addition, there is a continuum of nonstationary equilibria in complete analogy to the well-known situation in OLG models.

To derive economic intuition for the structure of these equilibria, note that a larger value of government bonds tends to raise the real interest rate. This is the case because a higher bond wealth allow households to better self-insure against idiosyncratic risk and thereby weakens the precautionary motive. If the initial bond value was $q_0^B > q^{B*}$, the interest rate would therefore be higher than in the monetary steady state. Such a higher (required) interest rate must be associated with a higher rate of appreciation of q_t^B, as otherwise households would no longer be willing to hold bonds.[31] In an equilibrium with $q_0^B > q^{B*}$, agents would have to expect an even larger bond value q_t^B in the future. Clearly, the same reasoning then also holds for any future period. Hence, in such an equilibrium, bond valuations would have to growth without bounds, which is inconsistent with market clearing (and wealth effects).

Symmetrically, if $0 < q_0^B < q^{B*}$, the interest rate would have to be lower than in the monetary steady state. At the lower interest rate, households would increase their bond demand unless they expect q_t^B to depreciate over time. However, unlike in the case $q_0^B > q^{B*}$, a decaying path for q_t^B is indeed consistent with an equilibrium. All such equilibrium paths have the property that q_t^B converges asymptotically to 0, the non-monetary steady state.

While intuitive and usually correct, the previous arguments do not apply universally for all model parameters. The reason is that movements in endogenous physical investment induced by changes in the bond value can sometimes have opposing effects on the interest rate. However, this issue can easily be fixed by normalizing the value of bonds differently and, instead of using q_t^B, considering the share ϑ_t^B of bonds as a proportion of total wealth. In Appendix A.3.1, we show that any equilibrium must be associated with a path for ϑ_t^B that is contained in the interval $[0, 1]$ and solves an ordinary differential equation (ODE) of the form

$$\dot{\vartheta}_t^B = f\left(\vartheta_t^B\right)\vartheta_t^B + \check{\mu}_t^{\mathcal{B}}\vartheta_t^B \tag{13}$$

with some function f.[32] This equation captures the same economic intuition as previously described. But it is mathematically more convenient: the key feature is that the function f is *always* strictly increasing in ϑ_t^B.

discussed in Section 5. This is essentially the observation of Bassetto and Cui (2018).

[31] This is the case because a fixed $\check{\mu}^{\mathcal{B}}$ keeps the dilution rate of bonds, the effective "dividend yield", constant.

[32] Equation (13) can be obtained by taking the time derivative in equation (5).

Equation (13) verifies the previous intuitive argument formally. With constant $\breve{\mu}^{\mathcal{B}} \geq 0$, the right-hand side of equation (13) is of course zero for $\vartheta_t^B = \vartheta^{B*}$, the steady-state value associated with the monetary steady state. Because f is strictly increasing, ϑ^{B*} is an unstable steady state of the forward evolution implied by equation (13). All solutions with larger or smaller ϑ_t^B drift away from ϑ^{B*}. The larger solutions eventually cross 1 at which capital values become negative. These are purely mathematical solutions that do not correspond to equilibrium paths. The smaller solutions decay and converge asymptotically to the stable steady state $\vartheta = 0$. Hence, there is a continuum of solution paths that remain inside $[0, 1]$. This continuum can be indexed by the initial value $\vartheta_0^B \in \left[0, \vartheta^{B*}\right]$. All but one of these equilibria converge to 0.

Convergence to 0 of all equilibrium paths with the exception of the monetary steady state suggests a simple strategy for selecting the latter. Suppose a policy maker is somehow able to convince households that the equilibrium value of ϑ^B could never fall below a positive threshold $\underline{\vartheta} > 0$. Then a value of government debt below the monetary steady state is no longer consistent with any equilibrium path.

These considerations suggest the following off-equilibrium modification of the fiscal policy rule to achieve equilibrium uniqueness: fix an arbitrary threshold $0 < \underline{\vartheta} \leq \vartheta^*$ and, whenever ϑ_t falls below $\underline{\vartheta}$, switch from a constant debt growth rule (constant $\breve{\mu}^{\mathcal{B}}$) to a positive surplus rule as discussed previously for as long as $\vartheta_t < \underline{\vartheta}$. This works because, as we have shown previously, the positive surplus rule implies a unique equilibrium with a positive value of debt. Furthermore, an expectation of the regime change cannot become self-fulfilling: the value of debt under the positive surplus rule is so large that $\vartheta_t > \vartheta^* \geq \underline{\vartheta}$ and thus the threshold criterion would be violated if the positive surplus policy was permanently in place.

Formally, this threshold policy is given

$$\tau_t = \begin{cases} \frac{-\breve{\mu}^{\mathcal{B}} q_t^B + \mathfrak{g}}{a}, & \vartheta_t \geq \underline{\vartheta} \\ \frac{s+\mathfrak{g}}{a}, & \vartheta_t < \underline{\vartheta} \end{cases}, \qquad \breve{\mu}_t^{\mathcal{B}} = \begin{cases} \breve{\mu}^{\mathcal{B}}, & \vartheta_t \geq \underline{\vartheta} \\ -\frac{s}{q_t^B}, & \vartheta_t < \underline{\vartheta} \end{cases}, \tag{14}$$

where $\breve{\mu}^{\mathcal{B}} \geq 0$ and $s > 0$ are given constants. As we show in detail in Appendix A.3.3, the threshold rule (14) implies that the right-hand side of ODE (13) becomes even more negative as ϑ_t^B falls below $\underline{\vartheta}$ and, most importantly, remains negative even for $\vartheta_t^B = 0$. As a consequence, any mathematical solution with initial condition $\vartheta_0^B < \vartheta^{B*}$ would eventually lead to $\vartheta_t^B < 0$, which is inconsistent with an equilibrium. Households could only be convinced that bond valuations are so low despite the government's commitment to positive primary surpluses if they expected future bond values to become negative. Such a belief cannot be rationally held because it is inconsistent with free disposal of bonds.

Proposition 3. *Suppose the government follows the threshold policy (14) with threshold* $\underline{\vartheta} \in (0, \vartheta^{B*}]$

and surplus $s > 0$. Then there is a unique equilibrium among equilibria that satisfy $q^C \equiv 0$. This equilibrium does not depend on $\underline{\vartheta}$ or s and satisfies $\vartheta_t^B = \vartheta^{B*}$ for all t.

We emphasize that our policy rule modifies fiscal policy only off-equilibrium. Along the equilibrium path, the government is free to choose any rate of bubble mining $\breve{\mu}^{\mathcal{B}}$, effectively selecting the monetary steady state.

Bubbles on Other Assets. We now remove the assumption that only government debt can have a bubble by considering also equilibria in which cryptocoins have positive value, $q_t^C > 0$. Our main result is that the same threshold policy as studied previously can rule out bubbles on other assets, too, if either the supply of these assets grows sufficiently fast or the taxation threshold $\underline{\vartheta}$ is tight, i.e. $\underline{\vartheta} = \vartheta^{B*}$.

It is again convenient to use the fractions ϑ_t^B and ϑ_t^C of total wealth that is due to government bonds and cryptocoins, respectively, to analyze equilibrium dynamics. Recall also that $\vartheta_t = \vartheta_t^B + \vartheta_t^C$ denotes the wealth share due to all safe assets combined. When cryptocoins can have positive value, equilibrium dynamics are described by two ODEs in analogy to equation (13):

$$\dot{\vartheta}_t^B = f\left(\vartheta_t^B + \vartheta_t^C\right)\vartheta_t^B + \breve{\mu}_t^{\mathcal{B}}\vartheta_t^B, \tag{15}$$

$$\dot{\vartheta}_t^C = f\left(\vartheta_t^B + \vartheta_t^C\right)\vartheta_t^C + \breve{\mu}^{\mathcal{C}}\vartheta_t^C, \tag{16}$$

where f is the same increasing function as before.

Now, $f(\vartheta_t)$ depends on the total value of safe assets because both government bonds and cryptocoins equally allow households to self-insure against idiosyncratic risk. A higher value of either lowers the precautionary motive and raises the interest rate. This observation is important. Consider again a policy that keeps $\breve{\mu}^{\mathcal{B}}$ constant and denote by ϑ^{B*} the associated monetary steady state value in the steady state with $\vartheta^C = 0$. If the value of government bonds is below this steady-state value, $\vartheta_t^B < \vartheta^{B*}$, it no longer follows that also $\vartheta_t < \vartheta^{B*}$. Instead, a sufficiently large cryptocoin bubble can fill the gap. A higher value of safe assets raises the required interest rate, so that we can no longer conclude that agents must expect ϑ^B to decay. This opens up the possibility for equilibria in which bonds retain a positive value asymptotically, yet their value is smaller than in the desired monetary steady state. The simplest example of such equilibria is the continuum of steady state equilibria in which which bond and cryptocoin bubbles coexist in the special case $\breve{\mu}^{\mathcal{B}} = \breve{\mu}^{\mathcal{C}}$.

Outside of the special case $\breve{\mu}^{\mathcal{B}} = \breve{\mu}^{\mathcal{C}}$, the set of equilibria for a constant $\breve{\mu}^{\mathcal{B}}$ policy is illustrated in the left column of Figure 1. That figure plots phase diagrams for dynamics of ϑ_t^B and ϑ_t^C implied by ODEs (15) and (16).

In the top left panel, the growth rate of cryptocoins exceeds the dilution rate of government bonds. There are three steady state equilibria depicted by the colored circles. Additional non-stationary equilibria can start either on the blue solid line or below it. The latter type of equilibria converge asymptotically to the locally stable no-bubble steady state at the original (orange circle). The former type of equilibria travels along the blue line downward over time and converges asymptotically to a saddle-path stable steady state in which only cryptocoins retain value (blue circle). The only equilibrium in which bonds retain value asymptotically is the – locally unstable – monetary steady state (red circle). Hence, in this case there are no other equilibria in which bonds retain a positive value asymptotically, just as in the case without cryptocoins.

The bottom left panel, in contrast, depicts the same situation in the case that the growth rate of cryptocoins is smaller than the dilution rate of government bonds. In this case, the stability properties of the monetary and the pure cryptocoin steady states flip (hence the change of color). All saddle-path stable non-stationary equilibria that originate on the blue line now travel upward over time towards the monetary steady state. Therefore, all these equilibria also have the property that bonds retain their value even asymptotically.

The economic difference between the top left and the bottom left panel lies in the no-arbitrage condition between bonds and cryptocoins. If cryptocoins are diluted at a larger rate than bonds, $\breve{\mu}^{\mathcal{C}} > \breve{\mu}^{\mathcal{B}}$, then households are only willing to hold cryptocoins if they expect them to appreciate in value relative to government bonds over time at rate $\breve{\mu}^{\mathcal{C}} - \breve{\mu}^{\mathcal{B}}$. Because the aggregate bubble is bounded (relative to the size of the economy), this is only possible if ϑ_t^B shrinks over time. Asymptotically, the cryptocoin bubble dominates the economy. Conversely, symmetry considerations imply that the exact opposite conclusion must hold if cryptocoins are diluted at a lower rate than bonds, $\breve{\mu}^{\mathcal{C}} < \breve{\mu}^{\mathcal{B}}$. In this case, bonds must appreciate over time relative to cryptocoins in any equilibrium. For such equilibrium paths, the bond bubble can be very small initially, yet it always remains strictly bounded away from zero.

These considerations suggest that a threshold policy of the type (14) may still succeed to select the monetary steady state, even for an arbitrarily small taxation threshold $\underline{\vartheta} > 0$, when cryptocoins grow sufficiently fast ($\breve{\mu}^{\mathcal{C}} > \breve{\mu}^{\mathcal{B}}$). If the threshold policy is still capable of eliminating any equilibria in which ϑ_t^B falls below $\underline{\vartheta}$, then the dynamics in the top left panel of Figure 1 suggest that only the monetary steady state remains. Indeed, the structure of equation (15) is sufficiently similar to that of equation (13) that we can still show that there cannot be an equilibrium path for which ϑ_t^B falls below the threshold. The economic intuition is as before: if ϑ^B ever fell below $\underline{\vartheta}$, this low valuation of bonds would only be consistent with the positive surplus policy if agents expected ϑ^B to fall even below 0 in finite time. Such an expectation is not consistent with an equilibrium. For the case $\breve{\mu}^{\mathcal{C}} > \breve{\mu}^{\mathcal{B}}$, the dynamics under a threshold policy are depicted in the top right panel of Figure 1. In this case, only the monetary steady state remains a valid equilibrium.

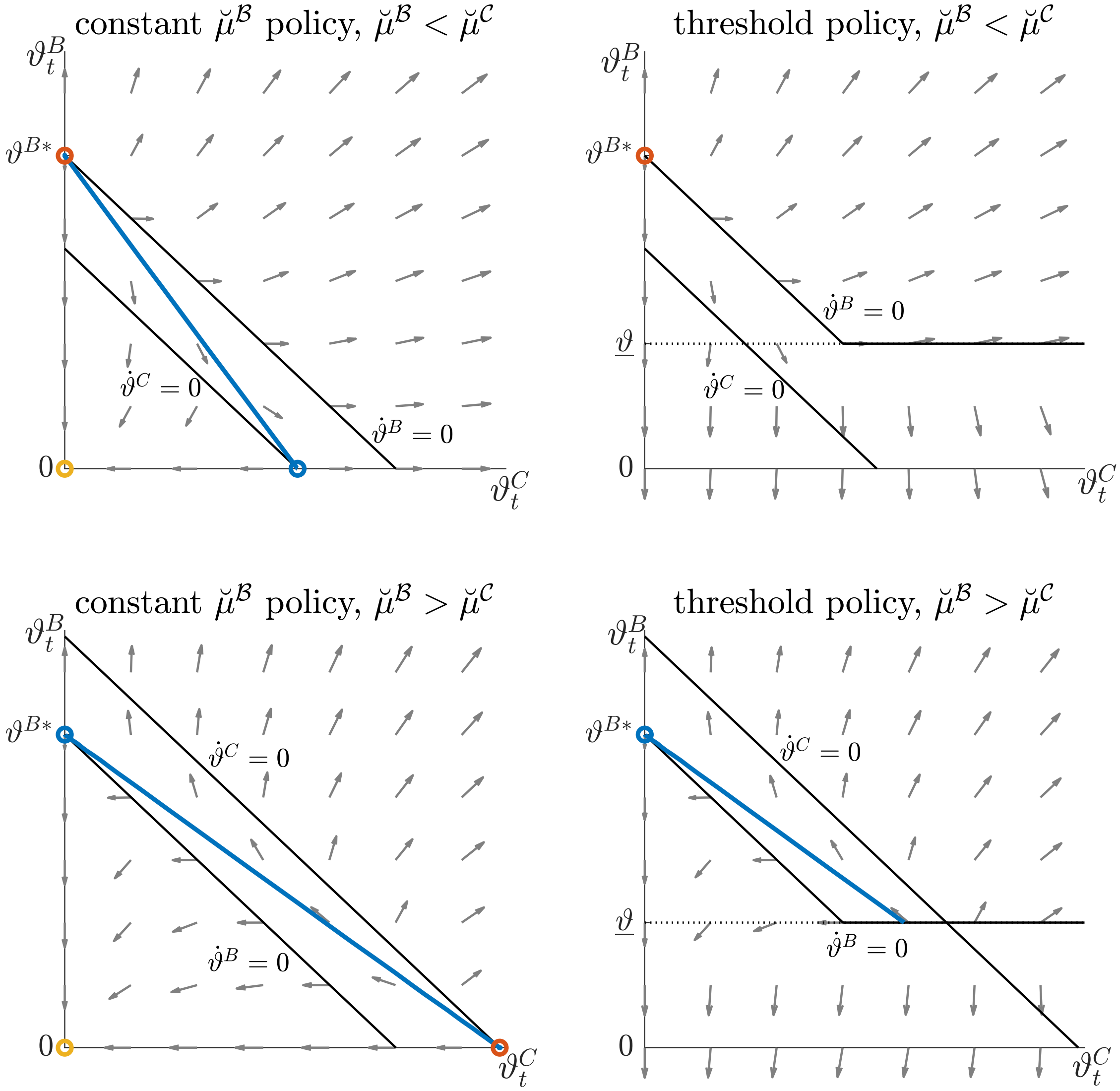

Figure 1: Phase diagrams illustrating typical dynamics of ϑ^B (vertical axes) and ϑ^C (horizontal axes) under a constant $\breve{\mu}^{\mathcal{B}}$ policy (left column) and a threshold policy (right column), respectively. The top row illustrates the case of fast cryptocoin growth, $\breve{\mu}^{\mathcal{B}} < \breve{\mu}^{\mathcal{C}}$. The bottom row the case of slow cryptocoin growth, $\breve{\mu}^{\mathcal{B}} > \breve{\mu}^{\mathcal{C}}$. Gray vector fields indicate the direction of change over time, black solid lines the regions where either ϑ_t^B or ϑ_t^C is locally constant. Possible equilibria with the exception of those converging to $(0,0)$ asymptotically are depicted by the colored circles and the blue solid lines. Circles depict steady state equilibria, their colors encode stability properties: red: locally unstable, yellow: locally stable, blue: saddle-path stable. The blue solid lines depict saddle-path stable equilibrium paths toward the blue steady state.
Under a constant $\breve{\mu}^{\mathcal{B}}$ policy (left column), the bubble-free steady state $(0,0)$ always exists and is a point of attraction for all starting points below the blue line. Under a threshold policy (right column), these trajectories cross the horizontal axis in finite time and are no longer valid equilibria.
In the case $\breve{\mu}^{\mathcal{B}} < \breve{\mu}^{\mathcal{C}}$ (top row), the desired monetary steady state is unstable and the threshold policy eliminates all other equilibria as well. In the case $\breve{\mu}^{\mathcal{B}} > \breve{\mu}^{\mathcal{C}}$ (bottom row), the monetary steady state is the limit point of the saddle-path stable trajectory. The parts of the trajectory above the threshold remain valid equilibria even under a threshold policy. Only if $\underline{\vartheta} = \vartheta^{B*}$, the saddle path reduces to the unique steady state.

The situation is, unfortunately, less favorable for the policy if $\breve{\mu}^{\mathcal{C}} \leq \breve{\mu}^{\mathcal{B}}$. Even in this case, the policy is still able to rule out equilibria for which ϑ_t^B falls below the threshold $\underline{\vartheta}$. But, as we have discussed before, there are many equilibria that feature an asymptotically large bond wealth share ϑ_t^B. The bottom right panel of Figure 1 depicts this situation. Under the threshold policy, the two steady states with no bond bubble and all equilibria that converge asymptotically to $(0, 0)$ are inconsistent with the policy. But there is still a continuum of equilibria for any initial point on the blue line. This is the same blue line as without a threshold policy (bottom left panel), but only points above the threshold remain valid equilibrium solutions.

This leads us to the conjecture that a single threshold policy still succeeds in selecting a unique equilibrium, the one with a tight threshold, $\underline{\vartheta} = \vartheta^{B*}$. Then, there is no "space" left for a bubble on cryptocoins. For any lower threshold $\underline{\vartheta} < \vartheta^{B*}$, the policy fails to select a unique equilibrium if $\breve{\mu}^{\mathcal{C}} \leq \breve{\mu}^{\mathcal{B}}$. In total, we obtain the following proposition.

Proposition 4. *Under the threshold policy (14) (with $s > 0$), all equilibria have the property that $\vartheta_t^B \geq \underline{\vartheta}$ for all t. If $\underline{\vartheta} = \vartheta^{B*}$ or $\breve{\mu}^{\mathcal{C}} > \breve{\mu}^{\mathcal{B}}$, the equilibrium is unique and satisfies $\vartheta_t^B = \vartheta_t = \vartheta^{B*}$ for all t.*

We remark that the distinction between $\breve{\mu}^{\mathcal{C}} > \breve{\mu}^{\mathcal{B}}$ and $\breve{\mu}^{\mathcal{C}} \leq \breve{\mu}^{\mathcal{B}}$ affects the nature of the off-equilibrium backing considerably. In the former case, any threshold $\underline{\vartheta}$ works. Any promise of arbitrarily small surpluses arbitrarily far in the future is sufficient to rule out all equilibria but the one featuring a stationary public debt bubble. In the latter case, only $\underline{\vartheta} = \vartheta^{B*}$ works. The fiscal authority must start raising surpluses immediately in response to any drop in the value of debt below its equilibrium value.

We conclude this subsection with a remark regarding the connection with Obstfeld and Rogoff (1983). That paper shows how an off-equilibrium commodity backing of money can rule out hyperinflationary equilibria in models of money as a medium of exchange. The important difference relative to our proposed policy is that their policy requires the government to hold the stock of commodity backing *in equilibrium*. In contrast, in our approach, government debt remains fully unbacked along the equilibrium path. The tax backing is a strict off-equilibrium phenomenon.

In their setting, which is closer to the situation analyzed in Proposition 3, this may not appear to be a major issue. Like the threshold $\underline{\vartheta}$ in Proposition 3, the fraction of commodity backing in the Obstfeld and Rogoff (1983) setting can be arbitrarily small. This is a consequence of assigning the money role to a specific asset and not allowing for any competing moneys, just like Proposition 3 does not allow for competing bubbles. However, once we do allow for competing bubbles, as in Proposition 4, the backing threshold $\underline{\vartheta}$ may have to be "large" (in the case $\breve{\mu}^{\mathcal{C}} \leq \breve{\mu}^{\mathcal{B}}$). In this case, the equivalent Obstfeld and Rogoff (1983) policy would reduce to full commodity backing and thereby *eliminate all bubbles in equilibrium*, just like the positive surplus policy in Proposition 2.

4.3 Discussion

Beyond our Model. We have used a specific model to discuss how a suitable specification of off-equilibrium fiscal backing can select a unique equilibrium in the presence of bubbles and determine the price level. We wish to emphasize, however, that the key ideas remain valid beyond this specific model.

First, we have in fact verified Propositions 3 and 4 also for one additional model. A previous version of this paper has featured a second example based on a perpetual youth model. Both propositions hold identically also for that model with identical proofs.

Second, we conjecture that the essence of the economic argument generalizes to other settings. Proposition 3 follows immediately from the fact that ϑ^{B*} is an unstable steady state in the forward evolution described by ODE (13). In a different model, as long as the value of government debt or scaled version thereof (such as ϑ_t^B) necessarily satisfies an equation like this with strictly increasing f in any equilibrium, a variant of Proposition 3 will hold. Similarly, Proposition 4 ultimately only requires that ODEs (15) and (16) with strictly increasing f must hold in any equilibrium.

Economically, we have traced the monotonicity of f to the fact that a larger bubble tends to raise the equilibrium interest rate. This fact is true in any other rational bubble model we are aware of. In rational bubble models, a lack of certain trading opportunities depresses the interest rate below the growth rate. This is what creates room for bubbles in the first place. Trade in the bubble mitigates the original trading friction and typically more so, the larger is the size of the aggregate bubble. Therefore, larger bubbles tend to raise interest rates.

One sense in which our analysis appears more restrictive is that f does not depend on additional state variables of the model. However, such additional state variables do not necessarily represent a challenge to our argument. For one, if the state variables evolve independently from the value of government debt or other bubbly assets, such as in the case of exogenous variables, our analysis easily generalizes. Second, even if the state variables do depend on the value of government debt, they only represent a challenge to our argument if the feedback between the value of debt and the state variable stabilizes the dynamics of the hypothetical expected future value of debt required to justify a certain value today. While theoretically possible, it is unclear why such a stabilizing feedback should only occur when bubbles are present but not under positive surplus policies that ensure a positive fundamental value of government debt. Therefore, such a situation is most likely to be encountered in models that feature multiplicity even in the absence of bubbles. In such a framework, no one would expect the FTPL to select a unique equilibrium in the first place.

The Monetary Approach to Price Level Determination. While the FTPL emphasizes fiscal price level determination, the more conventional approach to resolve price level indeterminacy is the "monetary" one based on active Taylor rules. This approach is less suitable for our purposes for two reasons.

First, active Taylor rules only locally determine the price level but not globally. Other equilibria are typically consistent with a Taylor rule but are ruled out by ad-hoc requirements, e.g. boundedness of inflation, that are not derived from economic conditions (e.g. Atkeson et al. 2010, Cochrane 2011). The FTPL, in contrast, can determine the price level globally. While this is an issue unrelated to the presence of bubbles, it is particularly severe when there can also be multiplicity due to bubbles. Local determinacy merely means that there are no other equilibria arbitrarily close to a given reference equilibrium. But with bubbles, alternative equilibria typically involve a bubble that bursts or jumps to another asset.[33] These equilibria therefore naturally feature outcomes that are "far away" from a reference equilibrium along some dimensions. Ruling out that private agents are able to coordinate their actions on such alternative equilibria precludes such bubble bursts and jumps by assumption. Surely, the sustainability of bubbly government debt should rest on a more solid foundation than such an assumption.

Second, and more importantly, the conventional "monetary" selection approach still comes with fiscal requirements: it assumes that the fiscal authority faces equation (1) without a bubble as an intertemporal budget constraint that restricts the set of admissible policies and forces the fiscal authority to adjust surpluses "passively" in the background such that this constraint holds for all prices. But such a constraint on fiscal policy precludes public debt bubbles from the outset and is thus unsuitable for studying them. If we instead allow for the more general equation (1) with bubbles, it is unclear in which sense this equation can be interpreted as an intertemporal budget constraint and, consequently, how a more generalized notion of "passive" fiscal policy would need to be defined to make the monetary approach workable. Adopting a FTPL perspective avoids this difficulty.

The Nature of Equilibrium Selection. In line with a common tradition in macroeconomics, and, in particular, monetary economics, we have pursued the question of unique implementation of an equilibrium by restricting attention to uniqueness of competitive equilibria starting at time zero. This approach to unique implementation has been criticized by Kocherlakota et al. (1999) in the context of the FTPL, and by Atkeson et al. (2010) more generally. The main concern raised by Atkeson et al. (2010) is that this type of analysis allows for a trivial type of unique implementation that they call implementation via non-existence (p. 59): "specify policies so that no competitive equilibrium exists after deviation histories."

[33] Due to the perfect foresight nature of our analysis, such "bursts" and "jumps" are to be interpreted as differences in time-zero valuations across equilibria.

While we agree in principle with the sentiment that uniqueness via non-existence is uninteresting economically, we wish to emphasize that this implementation strategy is not underlying the uniqueness arguments of either the traditional FTPL or our paper. For the traditional FTPL, Bassetto (2002) recasts the standard model into an explicit dynamic game and validates the key predictions of the FTPL in this setting.

While also our model could be recast as an explicit game or, at least, formulated in the language of "sophisticated equilibrium" inspired by game theory that was proposed by Atkeson et al. (2010), we have decided not to pursue this approach in this paper. Such a reformulation would significantly complicate the notation requirements and confound the basic economic ideas discussed in this section. However, we briefly remark here why such a reformulation would likely confirm the viability of our threshold policies to select a unique equilibrium.

Under the specific policies considered in this paper, we could restart time at a later date $t_0 > 0$ and obtain the exact same results as in Propositions 2, 3, and 4. The reason is that private agent decision rules are purely forward looking and the specific government policy rule we consider only depends on current prices but not on the history of past prices and allocations. Therefore, there is always a unique (competitive) continuation equilibrium after any time-t_0 history, not just after the equilibrium history from $t = 0$ to $t = t_0$. The off-equilibrium "threat" to back government debt with taxes does not select a unique equilibrium because no continuation equilibrium exists after an off-equilibrium history. It selects a unique equilibrium because the higher payout yield associated with positive primary surpluses makes bonds more attractive, so that individual private households find it optimal to deviate and demand more bonds.

Despite the different context, the off-equilibrium prescription for fiscal policy in our setting is therefore quite similar to the one emphasized by Bassetto (2002) in the context of the standard FTPL. In that paper, the government raises additional taxes in off-equilibrium contingencies in which private agents are not willing to purchase sufficient bonds required to fund a planned deficit. These additional surpluses generate gains to bond holders and make bonds more attractive in the first place.[34]

5 Mining the Bubble

In this section, we show how the government can mine a bubble, i.e. finance government expenditures without ever raising taxes for them. We also discuss limits to bubble mining, under which circumstances bubble mining is inflationary, and optimal bubble mining policy.

Throughout this section, we restrict attention to equilibria in which cryptocoins are worthless, $q_t^C = 0$, and bonds remain asymptotically valuable. These assumptions are justified by the

[34] We wish to thank an anonymous referee for pointing out this connection.

results of Section 4.

5.1 The Bubble as a Fiscal Resource

Recall that the primary surplus in the monetary steady state is given by $sK_t = (\tau_t a - \mathfrak{g})K_t$ where $s = -\breve{\mu}^{\mathcal{B}} q^B$ (by the government budget constraint (2)). Because capital grows at a constant rate g in steady state, we can write the debt valuation equation (9) as

$$q^B K_0 = \lim_{T\to\infty} \left(\underbrace{\int_0^T e^{-(r^f - g)t} sK_0 dt}_{=:PVS_{0,T}} + e^{-(r^f-g)T} q^B K_0 \right).$$

Provided $q^B > 0$, equation (10) implies precisely three cases:

1. $s > 0, \breve{\mu}^{\mathcal{B}} < 0$: then $r^f > g$, $PVS_{0,\infty} > 0$ and a bubble cannot exist. This is the "conventional" situation commonly considered in the literature.
2. $s = \breve{\mu}^{\mathcal{B}} = 0$: then $r^f = g$, $PVS_{0,\infty} = 0$ and there is a finite positive bubble whose value exactly equals $q^B K_0$ and grows over time at the growth rate/risk-free rate.
3. $s < 0, \breve{\mu}^{\mathcal{B}} > 0$: then $r^f < g$ and thus the integral $PVS_{0,T}$ converges to $-\infty$ as $T \to \infty$. Yet, q^B is still positive, which is only possible if there is an offsetting infinite positive bubble. These infinite values do not violate any no-arbitrage condition and are also not otherwise economically problematic, since the bubble cannot be traded separately from the claim to surpluses. Both are necessarily bundled in the form of government bonds. As long as $\frac{\mathcal{B}_t}{\mathcal{P}_t} = q^B K_t$ is determined and finite in equilibrium, the model remains economically and mathematically sensible despite the infinite values in the decomposition of the value of government bonds.

In all three cases, the (possible) presence of a bubble represents a fiscal resource that grants the government some extra leeway. Clearly in case 3, the government can run a perpetual deficit, "mine the bubble" and never has to raise taxes to fully fund all government expenditures. In case 2, the existence of the bubble is beneficial, because the value of government debt is positive – allowing agents to self-insure against risk – despite the fact that the present value of primary surpluses is zero. Even in case 1, government debt is more sustainable since an unexpected drop of primary surpluses to zero results in a bubble instead of a total collapse of the value of debt.

5.2 The Bubble Mining Laffer Curve

In case 3 above, the bubble can become arbitrarily large. Does this mean that the government faces no budget constraint and can expand spending without limits? The answer must of course be no as real resources are still finite. Considering present value relationships can be misleading when $r^f \leq g$. Instead, it is instructive to look at flow quantities.

Specifically, primary deficits per unit of capital are given by

$$-s = \breve{\mu}^{\mathcal{B}} q^B.$$

The first factor, $\breve{\mu}^{\mathcal{B}}$, represents the rate of bubble mining: revenue raised by bond issuance that is not distributed to bond holders in the form of interest payments. If it is positive, the claim of old bond holders is diluted by the issuance of new bonds, i.e., a positive $\breve{\mu}^{\mathcal{B}}$ effectively represents a tax on existing bond holders.

The second factor, q^B, is the tax base, the real value of existing debt (per unit of capital). If this was unaffected by $\breve{\mu}^{\mathcal{B}}$, then the government could indeed generate arbitrarily large deficits by bubble mining. However, private agents react do the dilution of their claims by reducing bond demand. Thus, the tax base q^B reacts negatively to an increase in $\breve{\mu}^{\mathcal{B}}$. A standard Laffer curve intuition emerges.

We can see the reaction of q^B to $\breve{\mu}^{\mathcal{B}}$ explicitly from our closed-form solution in the monetary steady state:

$$q^B = \frac{\left(\tilde{\sigma} - \sqrt{\rho + \breve{\mu}^{\mathcal{B}}}\right)\left(1 + \phi\left(a - \mathfrak{g}\right)\right)}{\sqrt{\rho + \breve{\mu}^{\mathcal{B}}} + \phi\rho\tilde{\sigma}}.$$

This equation reveals two possible reasons how higher deficits may decrease q^B. First, there is a direct effect from increasing $\breve{\mu}^{\mathcal{B}}$. This emerges because higher debt growth makes bond savings less attractive, reduces bond demand, and thereby lowers the fraction ϑ^B of wealth that originates from bond wealth. If additional deficits are used to lower the output tax rate τ, as we have assumed throughout, this is the only effect. However, if additional deficits were instead used to fund government spending by raising $\mathfrak{g}$, q^B would decrease further due to the presence of the term $a - \mathfrak{g}$. This second effect is a consequence of the resource constraint (3): when the government claims a larger share of output, consumption has to decline, which lowers all asset values symmetrically.[35]

[35] This intuition breaks down for $\phi = 0$, as then agents can convert existing capital goods freely into consumption goods and instead the growth rate declines.

5.3 Bubble Mining and Inflation

Is bubble mining inflationary? Not necessarily. Among steady-state policies the answer depends on how the government mines the bubble, by issuing more debt or paying less interest, and on the impact of policy on economic growth.

Specifically, by the Fisher equation, inflation in our models is

$$\pi = i - r^f = i + \breve{\mu}^{\mathcal{B}} - g.$$

For a given nominal interest rate i, there is a direct inflationary effect from an increase in bubble mining $\breve{\mu}^{\mathcal{B}}$. Higher bubble mining at a given interest rate requires the government to grow its debt at a larger rate $\mu^{\mathcal{B}}$. When the growth rate is exogenously given, then this is the only effect. But in general, there could be an additional indirect effect that operates through the growth rate g. This is the case in our model: bubble mining decreases the attractiveness of bonds, making the agents want to hold more physical capital, which stimulates real investment and increases the steady-state growth rate g. This latter effect tends to be deflationary.

When the growth rate is endogenous, an increase in $\breve{\mu}^{\mathcal{B}}$ may therefore in principle lower the $\breve{\mu}^{\mathcal{B}} - g$ term and thus inflation. However, this is unlikely to be the case for any realistic calibration of our model: the effect on growth g is largest without capital adjustment costs ($\phi = 0$) and then

$$\frac{dg}{d\breve{\mu}^{\mathcal{B}}} = \frac{d\iota}{d\breve{\mu}^{\mathcal{B}}} = \frac{1}{2}\frac{\rho}{\rho + \breve{\mu}^{\mathcal{B}}}\frac{1}{1-\vartheta}.$$

For $\breve{\mu}^{\mathcal{B}} \geq 0$, this derivative can only be larger than 1 if $\vartheta > 1/2$, that is if the majority of private wealth is bond wealth. Despite the recent rise in the levels of public debt throughout advanced economies, this condition is unlikely to be satisfied in the foreseeable future. The most plausible situation is therefore the one in which the direct effect dominates the indirect growth effect. Thus, for a fixed nominal interest rate i, an increase in bubble mining is inflationary.

The government can also offset the inflationary effect of bubble mining further by lowering the interest rate i. This is possible whenever there is no binding lower bound on nominal interest rates. If i fully offsets the rise in $\breve{\mu}^{\mathcal{B}}$, so that $i + \breve{\mu}^{\mathcal{B}} = \mu^{\mathcal{B}}$ is unaffected, then only the indirect deflationary effect due to higher growth remains. By using the policy tools of debt growth and interest rate in the right proportion, the government can increase bubble mining in an inflation-neutral way.

Note, however, that the previous discussion solely centers on the steady-state inflation rate as a result of a steady-state level of bubble mining $\breve{\mu}^{\mathcal{B}}$. If the government was to announce more aggressive bubble mining going forward, government debt would become less attractive and its real value would have to fall, as we have seen in our Laffer curve discussion. This is brought about in equilibrium by an inflationary upward jump in the price level.[36]

5.4 Optimal Bubble Mining

Even if bubble mining is possible, is it ever socially optimal for the government to engage in mining? In this subsection, we characterize the optimal policy and draw two key conclusions:

First, a bubble facilitates trade in response to idiosyncratic shocks and mining the bubble inhibits these beneficial trades. Optimal policy therefore only calls for a positive rate of bubble mining, $\breve{\mu}^{\mathcal{B}} > 0$, if pecuniary externalities generate an equilibrium bubble that is "too large". Such a situation can arise in our model because the bubble crowds out real investment ι_t. The optimal policy balances a trade-off between growth and risk sharing and may call for a positive rate of bubble mining, $\breve{\mu}^{\mathcal{B}} > 0$, if idiosyncratic risk is sufficiently large.

Second, the optimal degree of bubble mining is independent of the government spending need $\mathfrak{g}$. This implies that, under the optimal policy, any additional government spending is optimally funded by raising taxes, not by bubble mining.

Formally, expected utility of an agent with initial wealth share $\eta_0^i := n_0^i / \left((q_0^B + q_0^K) K_0 \right)$ is[37]

$$\mathbb{E}\left[\int_0^\infty e^{-\rho t} \log c_t^i dt\right] = \frac{\log \eta_0^i + \log K_0}{\rho}$$
$$+\mathbb{E}\left[\int_0^\infty e^{-\rho t} \left(\underbrace{\log\left(\frac{\rho\left(1+\phi\left(a-\mathfrak{g}\right)\right)}{1-\vartheta_t+\phi\rho}\right)}_{=\log(a-\mathfrak{g}-\iota_t)} + \underbrace{\frac{1}{\phi\rho}\log\left(\frac{\left(1-\vartheta_t\right)\left(1+\phi\left(a-\mathfrak{g}\right)\right)}{1-\vartheta_t+\phi\rho}\right) - \frac{\delta}{\rho}}_{=\frac{(\Phi(\iota_t)-\delta)}{\rho}} - \underbrace{\frac{\left(1-\vartheta_t\right)^2\tilde{\sigma}^2}{2\rho}}_{=\frac{(1-\vartheta_t)^2\tilde{\sigma}^2}{2\rho}}\right) dt\right]. \tag{17}$$

For arbitrary Pareto weights, a social planner would like to manipulate the safe asset wealth share ϑ_t period by period to maximize the integrand in the second line.[38] The first term in the integrand is utility from consumption $a - \mathfrak{g} - \iota_t$, which is increasing in ϑ_t because a higher ϑ_t depresses investment and leaves more resources for consumption. The second term is proportional to the endogenous component $\Phi(\iota_t)$ of the growth rate, which is decreasing in ϑ_t. The last term represents the reduction of utility due to idiosyncratic risk. Higher ϑ_t reduces residual consumption risk $(1-\vartheta_t)\tilde{\sigma}$ and thereby increases this term.[39]

[36]If we were to add price stickiness to the model, this initial price level jump would translate into a transition period of larger inflation. See e.g. Li and Merkel (2020) for a closely related model with sticky prices.

[37]We provide a derivation of this equation in Appendix A.4.

[38]This is the case because the second line is the same for all agents i, whereas the first line depends only on initial conditions that cannot be affected by bubble mining policy.

[39]Representing the objective in this way highlights similarities to the classic analysis of the optimal quantity of debt by Aiyagari and McGrattan (1998). In their framework, a larger value of government debt increases liquidity by effectively relaxing borrowing constraints, but reduces the quantity of capital. Here, a larger debt wealth share

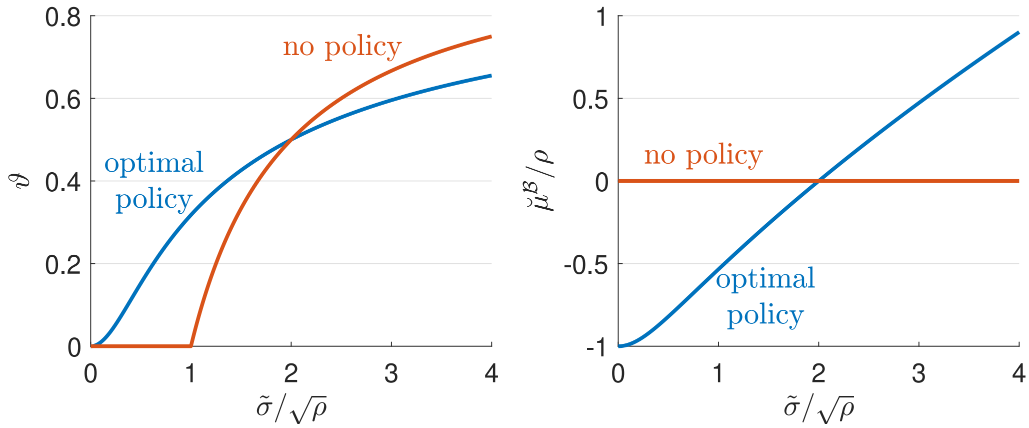

Figure 2: Optimal policy versus no policy ($\breve{\mu}^{\mathcal{B}} = 0$) for $\phi = 0$ as a function of $\tilde{\sigma}/\sqrt{\rho}$. The left panel depicts the bond wealth share $\vartheta = \vartheta^{\mathcal{B}}$, the right panel the associated bubble mining policy $\breve{\mu}^{\mathcal{B}}$ normalized by the time-preference rate ρ.

While ϑ_t is not a policy instrument, in the selected equilibrium it equals the bond wealth share $\vartheta_t^{\mathcal{B}}$. The government can effectively control the latter by adjusting $\breve{\mu}_t^{\mathcal{B}}$. We show in the appendix that there is a unique optimal solution ϑ^* for ϑ_t, which is time-invariant, depends only on ρ, $\tilde{\sigma}$, and ϕ, and is strictly increasing in idiosyncratic risk $\tilde{\sigma}$. Figure 2 depicts this optimal bond wealth share ϑ and the bubble mining rate $\breve{\mu}^{\mathcal{B}}$ required to implement it as a function of idiosyncratic risk.[40] It also compares the optimal policy to the competitive equilibrium without policy intervention ($\breve{\mu}^{\mathcal{B}} = 0$). Relative to that benchmark, optimal policy backs the value of government debt by primary surpluses (negative $\breve{\mu}^{\mathcal{B}}$) if risk is low. In these cases, the bubble created by market forces is too small (for $\tilde{\sigma} > \sqrt{\rho}$) or even absent (for $\tilde{\sigma} \leq \sqrt{\rho}$) and risk-sharing is suboptimal. If risk is high, market forces generate a bubble that is too large. Optimal policy then runs deficits (positive $\breve{\mu}^{\mathcal{B}}$) and funds government expenditures out of the bubble to encourage higher real investment and growth.

Market forces may fail to generate a bubble that achieves the optimal trade-off between growth and risk sharing. Inefficiencies are possible due to pecuniary externalities with respect to agents' portfolio choices because agents take returns as given when making these choices, yet their collective choice affects the risk-free rate and risk-premium on capital.[41] On the one hand, a greater portfolio allocation to bonds discourages real investment ι in the economy, which in turn affects the real return on all assets through the growth term in the risk-free rate. This force tends to generate too much bond demand, a too high ϑ and thus under-investment in capital.

directly improves risk sharing (even in the absence of borrowing constraints) but reduces the growth rate of capital and output.

[40]The figure assumes no capital adjustment cost, $\phi = 0$. It looks qualitatively identical for $\phi \in (0, \infty)$.

[41]These pecuniary externalities have been previously identified by Brunnermeier and Sannikov (2016b) and Di Tella (2020) in closely related frameworks.

On the other hand, a greater allocation to bonds increases the total value of bonds and thus reduces the residual (proportional) idiosyncratic consumption risk $(1-\vartheta)\tilde{\sigma}$ that each agent has to bear.[42] This in turn affects asset returns through the precautionary motive in the risk-free rate and through the risk premium on capital. This second force tends to generate too little bond demand, a too low ϑ and thus over-investment in capital.

It is instructive, however, that the optimal value ϑ^* is independent of the government spending need $\mathfrak{g}$. In addition, also the optimal degree of bubble mining $\breve{\mu}^{\mathcal{B}}$ required to implement ϑ^* is independent of $\mathfrak{g}$. While the government could increase $\breve{\mu}^{\mathcal{B}}$ in response to an (unanticipated) increase in $\mathfrak{g}$ in order to fund the additional spending, this is never optimal. The optimal policy should rely on taxes as the marginal funding source for additional government spending.

The reason for this result is that when government spending $\mathfrak{g}$ increases, the government must transfer a larger fraction of current output away from the private sector to itself. Taxing current output is the most direct way of achieving this and does not distort the portfolio choice between capital and bonds. In contrast, funding additional spending by increasing primary deficits and bubble mining dilutes the bubble at a faster rate and thereby distorts agents' portfolio choice in favor of larger capital holdings. Because the pecuniary externalities just discussed do not depend on either the level of government spending $\mathfrak{g}K_t$ or total output left for private uses $(a-\mathfrak{g})K_t$, the optimal portfolio distortion induced by $\breve{\mu}^{\mathcal{B}}$ is also independent of these quantities.[43]

6 Extensions and Additional Considerations

In this section, we discuss three extensions to the uniqueness analysis from Section 4: (i) imperfect government commitment, (ii) aggregate shocks and imperfect shock observability by the government, (iii) a richer structure of alternative bubbles than "cryptocoins" that includes the possibility of bubbles attached to assets issued by private households. With regard to the first two extensions, which we analyze formally, the key takeaway is that equilibrium selection with fiscal policy remains feasible and credible if the desired dilution rate $\breve{\mu}^{\mathcal{B}}$ is below the growth rate of cryptocoins $\breve{\mu}^{\mathcal{C}}$ but may otherwise break down (in case (i)) or require modifications (in case (ii)). With regard to the last extension, we argue verbally how more generic bubbles can be

[42] Specifically, higher bond prices benefit the bond-selling agents: those who suffered idiosyncratic losses and who have higher marginal utilities, on average.

[43] The size of the pecuniary externalities does depends on the aggregate consumption-wealth ratio which equals the time preference rate ρ in our model with log utility. Admittedly, this is a somewhat knife-edge case that only holds for unit EIS. For general EIS, the aggregate consumption-wealth ratio depends on the growth rate of the economy, which in turn is increasing in output left for private uses $a-\mathfrak{g}$ per unit of capital. Nevertheless, our result represents an important benchmark case and the broader point that optimal bubble mining only adjusts to correct pecuniary externalities remains valid also for EIS $\neq 1$.

reinterpreted as cryptocoin bubbles but with slightly more general assumptions regarding the dilution rate $\breve{\mu}^{\mathcal{C}}$ and the production of cryptocoins than we have made in this paper. For this reason, we conjecture that our results could be adapted to such settings.

We conclude the section with a discussion of alternative policies that may complement the fiscal selection strategy or substitute for it. A common theme is that several different alternative strategies limit the scope of possible assets that could play the role of cryptocoins or make them less attractive by reducing the effective difference $\breve{\mu}^{\mathcal{B}} - \breve{\mu}^{\mathcal{C}}$.

6.1 Imperfect Commitment

The propositions in Section 4 hold under the assumption that the government can perfectly commit to a threshold policy rule as in equation (14). We now ask whether off-equilibrium fiscal backing can remain credible if the government has a social welfare objective and imperfect commitment power.

We modify the setup of our model as follows to capture imperfect commitment. We replace the single infinite-horizon government by a sequence of governments, each with a finite term of office $T > 0$. Government $j \in \{0, 1, \dots\}$ chooses the policy variables $i_t, \mu_t^{\mathcal{B}}, \tau_t$ subject to the budget constraint (2) for periods $t \in [jT, (j+1)T)$, but takes the policy choices at other times (made by other governments $j' \neq j$) as given. All governments maximize the same social welfare function which is given by some weighted average of individual agent utilities as stated in equation (17).

We describe the details of the modified setup in Appendix A.5. Effectively, each government j takes prices $q^B_{(j+1)T}, q^C_{(j+1)T}, q^K_{(j+1)T}$ at the end of its tenure as given and chooses an optimal (Ramsey) policy path over $[jT, (j+1)T)$ that implies price paths for q_t^B, q_t^C, q_t^K in the competitive equilibrium of our model over that time interval.[44] In the language of dynamic games, this means that we are focusing on Markov perfect equilibria in the game played by the sequence of governments. Specifically, an *equilibrium in this policy game* consists of a set of price paths $\{q_t^B, q_t^C, q_t^K\}_{t=0}^{\infty}$ and policies $\{i_t, \mu_t^B, \tau_t\}_{t=0}^{\infty}$ such that (1) the sequences are part of a competitive equilibrium of our model and (2) for each j, the policy sequence restricted to the interval $[jT, (j+1)T)$ is optimal for government j conditional on terminal prices $q^B_{(j+1)T}, q^C_{(j+1)T}, q^K_{(j+1)T}$.

We assume that model parameters are such that, in the situation of Section 5.4, a positive rate of bubble mining is optimal and denote that rate by $\breve{\mu}^*$. As in Section 5.4, we denote by ϑ^* the optimal value of ϑ corresponding to that policy. With these assumptions and definitions, we can formulate our main result, which characterizes the possible values of government bonds in the equilibria of the policy game:

[44] Once terminal prices over the finite time interval have been fixed, there is no scope for multiplicity anymore. Each policy path over the interval is associated with a unique equilibrium.

Proposition 5. *Suppose $\breve{\mu}^* \geq 0$. Any equilibrium of the policy game features a positive aggregate bubble with $\vartheta_t \geq \vartheta^*$. In addition:*

(i) If $\breve{\mu}^{\mathcal{C}} > \breve{\mu}^*$, the equilibrium is unique and satisfies $\vartheta_t = \vartheta_t^B = \vartheta^*$, $\vartheta_t^C = 0$.

(ii) If $\breve{\mu}^{\mathcal{C}} \leq \breve{\mu}^*$, there exists an equilibrium for any initial value $\vartheta_0^B \in [0, \vartheta^*]$.

We prove this result in Appendix A.5. There are two key takeaways from this proposition. First, even under imperfect commitment, fiscal policy always eliminates the no bubble equilibrium. Second, equilibria with bubbles on cryptocoins can only be ruled out in case (i), i.e. if the growth rate of cryptocoins exceeds the optimal bubble mining rate $\breve{\mu}^*$ in the equilibrium with a stationary public debt bubble (and no other bubbles). Otherwise, equilibria with arbitarily small public debt bubbles cannot be ruled out. Selecting a unique equilibrium under imperfect commitment fails in precisely the cases in which the switching threshold $\underline{\vartheta}$ has to be tight in the results under perfect commitment presented in the previous subsection.

To gain intuition for the result, we first explain why commitment is not required to eliminate the no bubble equilibrium. In many standard models, there is a basic time-inconsistency problem of nominal debt: a government would prefer to inflate away pre-existing nominal liabilities in order to avoid current and future distortionary taxes necessary to repay the debt. Key to this argument is that the taxes required to support additional debt are welfare-reducing. But this is *not* the problem here. Even though debt is funded by distortionary capital taxes that reduce growth, a larger value of government debt can nevertheless be beneficial because it improves risk sharing (compare Section 5.4). If the total value of safe assets such as government bonds is sufficiently small, each finite-horizon government has incentives to raise taxes during its own term of office to create more safe assets and improve risk sharing. Commitment power is thus not needed to eliminate equilibria in which the total value of safe assets is too small. This explains why the no bubble equilibrium can be ruled out even under limited commitment.

Importantly, it is the total value of safe assets ($(q_t^B + q_t^C)K_t$) not the value of public debt ($q_t^B K_t$) that is relevant for risk sharing. For a sufficiently large cryptocoin bubble, safe assets are abundant and the marginal welfare impact of additional (tax-funded) debt becomes negative. In this case, the standard intuition is restored: a government would prefer not to honor the full outstanding real value of its nominal liabilities.

The question is therefore whether, in equilibrium, a sufficiently large cryptocoin bubble can provide a substitute for government debt as a safe asset. The answer is yes if the growth rate of cryptocoins is sufficiently low, but it is no otherwise.[45] If $\breve{\mu}^{\mathcal{C}} > \breve{\mu}^*$, cryptocoins are diluted at a faster rate than the optimal bubble mining rate, so that, unless cryptocoins are worthless,

[45]Even if the answer is yes, this does not mean that the equilibria in which cryptocoins have value are equally desirable. Except for the knife-edge case $\breve{\mu}^{\mathcal{C}} = \breve{\mu}^*$, the equilibrium bubble is then too large.

the aggregate bubble must be suboptimally small, at least in some future period. In such a future period, the government in charge would have incentives to raise taxes in order to create additional safe assets (through government debt). These incentives to tax by some future government implement a form of off-equilibrium fiscal backing that eliminates cryptocoin bubbles in the first place.

6.2 Aggregate Shocks and Imperfect Observability

A different concern with our baseline analysis from Section 4 is that, even if the government can commit, the only threshold policy that selects a unique equilibrium in the case $\breve{\mu}^{\mathcal{C}} \leq \breve{\mu}^{\mathcal{B}}$ is the one with $\underline{\vartheta} = \vartheta^{B*}$. That is, the government must start raising surpluses immediately in response to a drop in the value of debt. Such a policy can only be implemented if the government knows the precise level of the equilibrium value ϑ^{B*} at all times. This is not an issue in our setting with deterministic aggregates where ϑ^{B*} is a time-invariant constant. But it could become an issue in the presence of aggregate shocks that stochastically shift the value of ϑ^{B*} along the equilibrium path.

In this subsection, we discuss this issue in the context of a minimal model extension that generates stochastic asset price variation by introducing a two-state Markov process for the idiosyncratic shock volatility $\tilde{\sigma}$, i.e. we replace this parameter with a time-varying stochastic process $\tilde{\sigma}_t$ that may take values in $\{\tilde{\sigma}^l, \tilde{\sigma}^h\}$ with $\tilde{\sigma}^l < \tilde{\sigma}^h$.[46] We relegate the formal details of the analysis to Appendix A.6.

First, we note that all our previous results remain valid in the context of this extension if the government can observe the underlying state $\tilde{\sigma}_t$. If the government exogenously fixes $\breve{\mu}^{\mathcal{B}}_t = \breve{\mu}^{\mathcal{B}}(\tilde{\sigma}_t)$, there is at most one stationary equilibrium in which cryptocoins are worthless and bonds have a positive value. Assuming that parameters are such that this equilibrium exists, we denote by ϑ^{B*}_t the associated equilibrium process for ϑ^B_t and by $\vartheta^{B*,l}$ and $\vartheta^{B*,h}$ the realizations of ϑ^{B*}_t in the two states. To avoid the need for case distinctions, we also assume $0 < \vartheta^{B*,l} < \vartheta^{B*,h}$.[47] A result analogous to Proposition 4 holds (compare Proposition A.1): if the government chooses a threshold policy with state-contingent taxation threshold $\underline{\vartheta}_t := \vartheta^{B*}_t$, there is a unique equilibrium, cryptocoins have no value, and $\vartheta_t = \vartheta^B_t = \vartheta^{B*}_t$. The intuition for this result is precisely the same as in Section 4.

The situation becomes more interesting if we assume that only private agents observe $\tilde{\sigma}_t$ but the government does not – and has also insufficient information about endogenous variables to

[46]We restrict attention to a two-state process for simplicity. Our arguments would equally apply for a more general finite-state process.

[47]This is the most relevant case. E.g., both a constant $\breve{\mu}^{\mathcal{B}}$ policy and the optimal policy analyzed in Section 5.4 would lead to this outcome.

infer it. To connect the analysis to the results in Section 4, we still require that the government is able to observe or infer ϑ_t^B. Both requirements are satisfied, for example, under the following assumptions: (1) the government observes the value of its own liabilities (e.g. by observing the price level $\mathcal{P}_t$ and the outstanding stock of debt $\mathcal{B}_t$), (2) the government observes the value of all other assets $((q_t^C + q_t^K)K_t)$ but is unable to determine whether these assets are backed by productive capital or a bubble, (3) the capital stock evolves exogenously ($\phi \to \infty$).[48]

When the government does not observe the state ($\tilde{\sigma}_t$), it cannot determine with certainty whether a cryptocoin bubble exists. We analyze this situation in two steps. We first investigate in Appendix A.6.4 whether threshold policies can still be used to select the equilibrium that features $\vartheta_t = \vartheta_t^B = \vartheta_t^{B*}$ at all times. We conclude that this is only sometimes possible. Second, we go further in Appendix A.6.5 and devise a richer fiscal strategy that succeeds in selecting the desired equilibrium except for a knife-edge case.[49] This richer strategy is based on the threshold policies analyzed in the first step. It adds an additional trigger after which the government permanently switches from one such threshold policy to a different one. Here, we summarize the main economic points.

With regard to threshold policies, the same arguments as in Section 4 apply: no equilibrium can exist for which ϑ_t^B falls below the threshold. However, when the state is unobservable, the threshold $\underline{\vartheta}$ may not depend (explicitly) on the $\tilde{\sigma}_t$-state. We consider the special case that $\underline{\vartheta}$ is constant over time. There are then two natural choices for $\underline{\vartheta}$, a "high" threshold $\underline{\vartheta} = \vartheta^{B*,h}$ and a "low" threshold $\underline{\vartheta} = \vartheta^{B*,l}$.

Under a threshold policy with a "high" threshold, $\underline{\vartheta} = \vartheta^{B*,h}$, the tax backing is no longer confined to off-equilibrium contingencies. Instead, the government raises positive primary surpluses in equilibrium whenever $\tilde{\sigma}_t = \tilde{\sigma}^l$.[50] One would expect that such a policy always selects a unique equilibrium. We show in Proposition A.2 that this is indeed the case. However, this unique equilibrium is only the desired equilibrium with $\vartheta^B = \vartheta^{B*}$, if (1) in the latter equilibrium, the government generates positive primary surpluses in the low-risk state and (2) the constant surplus-capital ratio s below the threshold is chosen consistent with the desired equilibrium surplus in the low state. In this special case, the fact that the government must raise primary surpluses in equilibrium under this policy does not prevent implementation of the desired equilibrium because the government would want to raise primary surpluses in the latter equilibrium anyway. In all other cases, the unique equilibrium under the "high" threshold

[48] Assumption (3) is required so that observation of total wealth is insufficient to determine the split of aggregate resources into consumption and investment, which in turn, through Tobin's q condition, allows the government to infer q_t^K.

[49] The knife-edge case is $\breve{\mu}^{\mathcal{B},l} = \breve{\mu}^{\mathcal{C}} + \lambda^h$, where $\breve{\mu}^{\mathcal{B},l}$ is the desired bubble mining rate in the low-risk state and λ^h is the transition rate from the high-risk to the low-risk state.

[50] Nevertheless, there can still be a bubble in equilibrium if primary deficits are sufficiently large in the high-risk state.

policy leads to a higher than desired bond value in the low-risk state, $\vartheta^{B,l} > \vartheta^{B*,l}$.[51]

The threshold policy with a "low" threshold, $\underline{\vartheta} = \vartheta^{B*,l}$, in turn, remains always consistent with the desired equilibrium but may fail to implement it uniquely under some circumstances. Specifically, this policy only rules out equilibria for which the bond wealth share falls below $\vartheta^{B*,l}$ with positive probability. Whether more than one (continuation) equilibrium with this property exists depends on the specific situation. We draw three conclusions from this specific lower bound on ϑ_t^B:

First, if the cryptocoin dilution rate $\breve{\mu}^{\mathcal{C}}$ is sufficiently large, then, once again, only the desired equilibrium survives. All other potential solution paths would require agents to expect that the bond wealth share ϑ^B shrinks over time and eventually falls below the lower bound $\vartheta^{B*,l}$.

Second, regardless of the cryptocoin dilution rate, this specific threshold policy involves a tight threshold policy in the sense of Proposition 4 *conditionally on being in the low-risk state*. Hence, there is no longer any space left for cryptocoin bubbles in the low-risk state. Any such bubble must become worthless whenever $\tilde{\sigma}_t = \tilde{\sigma}^l$. But because a bubble on a given asset that has burst completely can never re-appear,[52] this implies that the policy in fact succeeds in implementing the desired equilibrium uniquely once the system has entered the low state for the first time. However, cryptocoin bubbles may exist at the initial time if the state at time zero is $\tilde{\sigma}_0 = \tilde{\sigma}^h$. Such bubbles must necessarily burst once the system enters the low-risk state for the first time.

Third, because cryptocoin bubbles must be expected to burst after the first transition into the low-risk state, they become also harder to sustain at the initial time in the high-risk state. The reason is that households are only willing to hold cryptocoins if they are compensated by higher capital gains conditional on the bubble not bursting. This has the same economic effects as a higher effective dilution rate $\breve{\mu}^{\mathcal{C}}$ in the deterministic model. In Proposition A.3 we provide a precise condition for when the policy can select a unique equilibrium. This condition is strictly weaker than the condition $\breve{\mu}^{\mathcal{B},h} < \breve{\mu}^{\mathcal{C}}$ that would be obtained if there was no possibility for $\tilde{\sigma}_t$ to jump to the low-risk state $\tilde{\sigma}^l$ in the future (here $\breve{\mu}^{\mathcal{B},h}$ is the desired bubble mining rate in the high-risk state). Still if $\breve{\mu}^{\mathcal{B},h}$ is sufficiently large, then cryptocoin bubbles may not be ruled out.

To sum up, the two types of threshold strategies can still select the desired equilibrium uniquely in many special cases, but they fail to do so in general. In Appendix A.6.5, we therefore devise a richer strategy that generically succeeds in selecting the desired equilibrium uniquely (except for one very special parameter restriction). This richer strategy combines the threshold policies with "low" and "high" thresholds in a fully history-contingent way. Specifically, it

[51]We assume in Appendix A.6.4 that the government adjusts $\breve{\mu}^{\mathcal{B}}$ above the threshold upward to still achieve its target level for ϑ^B in the high state.

[52]Otherwise, the expected future value of cryptocoins would be positive and so would be the current value in the low-risk state itself.

follows the "low" threshold strategy, that is always consistent with the desired equilibrium, so long as only price histories consistent with the desired equilibrium have been observed. Off-equilibrium, if the government ever observes a price history inconsistent with the desired equilibrium, it switches permanently to the "high" threshold rule, which always selects a unique equilibrium, but not necessarily the desired one.

The key insight for why combining the two threshold policies in this way can be successful is because these policies have complementary benefits. The "low" threshold policy is always consistent with the desired equilibrium but leaves room for alternative equilibria in the (initial) high-risk state. These alternative equilibria involve values for ϑ_t^B different from $\vartheta^{B*,h}$ in that state. In contrast, the "high" threshold policy selects a unique equilibrium that features a value of government debt in line with the desired equilibrium ($\vartheta_t^B = \vartheta^{B*,h}$) in the high-risk state. "Threatening" to switch to this unique equilibrium if any other value of debt was to be observed eliminates the alternative equilibria.

To be more specific, suppose first that the government was always to follow the threshold policy with a "low" threshold. Under this policy, only the desired equilibrium would have the feature that ϑ_t^B assumes the two values $\vartheta^{B*,l}$ or $\vartheta^{B*,h}$ along the equilibrium path at all times.[53] All other equilibria would result in a value of ϑ_t^B in the (initial) high-risk state that is not in $\{\vartheta^{B*,l}, \vartheta^{B*,h}\}$. Next, suppose the government was to switch immediately to a "high" threshold strategy forever if it ever observed a value different from $\vartheta^{B*,l}$ or $\vartheta^{B*,h}$. Then, the unique continuation equilibrium after the switch would require $\vartheta_t^B = \vartheta^{B*,h}$ in the high-risk state. Because the switch only happens if $\vartheta_t^B \neq \vartheta^{B*,h}$ right before the switching time, such a switch would generate capital gains or losses to bond holders at an infinite rate around the switching time. Households anticipating these capital gains would alter their bond demands right before the switching time such that the bond market cannot clear at any value other than $\vartheta_t^B = \vartheta^{B*,h}$. Hence, the additional (off-equilibrium) threat to switch to a policy with a more stringent taxation threshold can achieve uniqueness even in the initial high-risk state.

In Appendix A.6.5 we provide a formal definition of the switching policy described in the previous paragraphs. Proposition A.4 proves that then, indeed, there is a unique equilibrium.

6.3 Bubbles on Private Assets and Ponzi Schemes

So far, we have attached the bubble on other assets than government debt to cryptocoins, a separate intrinsically worthless asset in exogenous supply. Here, we argue that there is no economic difference if instead we consider one of the following two alternative arrangements.

First, bubbles could be attached to long-lived liabilities issued by households in the model.

[53] To be precise: this holds if $\check{\mu}^{\mathcal{B},l} \neq \check{\mu}^{\mathcal{C}} + \lambda^h$. Except for the knife-edge case of equality, the relative valuation of bonds and cryptocoins necessarily must drift upward or downward over time by a no arbitrage condition.

These liabilities neither need to be intrinsically worthless, nor in exogenous supply. If, for example, a private household issued a perpetuity with a positive fundamental value, there could be an additional bubble component attached to it and the household may be able to create additional bubbles when issuing more perpetuities. Whether this is possible is ultimately a matter of coordination of market beliefs and thus depends on the equilibrium selection. If other agents are only willing to buy such a bond at a price that does not exceed the present value of future coupon payments, then bubble creation fails and the agent has to pay back in present value exactly what she has borrowed. However, when rational bubbles are possible, then other agents could coordinate on an equilibrium in which they are willing to pay more for the bond than the present value of coupon payments in the expectation that they can pass it on to others at a high price in the future.

However, we can in such cases always logically split the assets into a bubble-free asset and an intrinsically worthless asset and simply select an equilibrium in which the two are always effectively held as a bundle. Therefore, any policy that rules out intrinsically worthless bubbles ultimately also rules out bubbles on assets issued by private agents.

The difference to our model with cryptocoins is therefore not fundamental. It merely lies in the auxiliary assumptions that we have made to keep the model tractable and make the analysis more transparent. Specifically, there is no reason why bubbles attached to other assets should grow at a constant rate $\breve{\mu}^{\mathcal{C}}$ or the ability to create them should be tied to an agent's capital holdings. But there is also no reason to expect that generalizing either assumption will invalidate the economic arguments we have presented.

Second, the benefits of issuing a bubbly asset can be generated in an alternative way without issuing long-lived liabilities. We have defined a bubble as a situation in which the market value of an asset exceeds its fundamental value and this works only for long-lived assets. However, the economic equivalent of issuing a bubbly asset can also be achieved through a Ponzi scheme, a chain of debt issuance that is perpetually grown and rolled over such that the present value of time-T debt liabilities does not converge to zero as the horizon T approaches infinity. Unlike issuing a long-lived asset with a bubble, each individual debt claim in this chain can have finite maturity and be priced according to its fundamental value and thus not have a bubble component. Yet, when the totality of all debt claims is considered as a bundle, the Ponzi scheme represents a bubble because the present value of payouts to debt holders falls short of the total value of debt issued. An agent able to run a Ponzi scheme can effectively mine this bubble by growing such "Ponzi debt" at a faster rate.

Formally, the ability of private households to run Ponzi schemes would require that markets do not enforce a strict no Ponzi condition on individual agents as we have assumed so far by imposing on each agent a solvency constraint $n_t^i \geq 0$. This constraint has been set ad-hoc and not been justified by economic arguments. Indeed, in an economy that allows for bubbles, strict no Ponzi conditions or solvency constraints are no necessary requirement of equilibrium. Specifically, the transversality conditions of all other agents than agent i does not preclude that agent i runs a Ponzi scheme for precisely the same reason why these conditions do not preclude a bubble (compare Section 3).

In Appendix A.7 we therefore present a generalized equilibrium definition in which each agent only faces a less stringent solvency constraint $n_t^i \geq \underline{n}_t^i$ with a lower bound $\underline{n}_t^i \leq 0$ that binds only asymptotically.[54] An agent is able to run a (limited) Ponzi scheme in equilibrium if $\lim_{T\to\infty} \mathbb{E}_0[\xi_T^i \underline{n}_T^i] < 0$. We show that there are generalized equilibria that permit agents to engage in such Ponzi schemes, whenever bubbles can exist in our baseline model without Ponzi schemes. As a specific example, we construct equilibria with private Ponzi schemes that are equivalent to equilibria with cryptocoin bubbles and $\breve{\mu}^{\mathcal{C}} = 0$.

In a sense, such Ponzi scheme equilibria are always equivalent to certain bubble equilibria in economies in which agents face strict solvency constraints. To see this, suppose that agent i faces a generalized solvency constraint $n_t^i \geq \underline{n}_t^i$. To simplify matters, further assume that $\underline{n}_t^i$ is deterministic.[55] Then that agent's transversality condition becomes $\mathbb{E}_0[\xi_T^i n_T^i] \to \lim_{T\to\infty} \mathbb{E}_0[\xi_T^i \underline{n}_T^i] < 0$. Denote by $n_0^{p,i}$ the negative of the right-hand side, which is the present value of bubble mining or "Ponzi wealth" that the market permits the agent in a given equilibrium. The equilibrium allocation is then equivalent to the one of a model in which the agent faces a strict solvency constraint $n_t^i \geq 0$, but is permitted to issues a long-lived bubble asset of value $n_0^{p,i}$ at time 0, so that $n_0^{p,i}$ is included in the agent's measured net worth n_0^i.

Because of this equivalence between equilibria with private Ponzi schemes and private bubble issuance, policies that rule out bubbles on private assets also eliminate equilibria featuring private Ponzi schemes. This is in particular true for the off-equilibrium fiscal policy analyzed in this paper.

Importantly, we remark that private agents could not resort to a similar equilibrium selection policy as the government to force the bubble onto their liabilities. While the government can use taxation to raise surpluses that grow proportionally with the economy, private households in our models are unable to generate resources that grow in lockstep with the economy and thus provide off-equilibrium backing to their liabilities in the same way as the government. The reason is that private agents face idiosyncratic risk while the government taxes everyone and thereby diversifies idiosyncratic risk away. For any given agent, there are some states in

[54] This is the sense in which the solvency constraint imposes merely a "borrowing constraint at $t = \infty$", i.e. a no-Ponzi condition in analogy to the "natural borrowing constraint" $\underline{n}_t = 0$.

[55] If $\underline{n}_t^i$ contains idiosyncratic risk, this may encode idiosyncratic expansions or contractions of the permissible Ponzi scheme which alters the agent's overall risk exposure and changes portfolio behavior relative to our baseline model. However, we conjecture that such a Ponzi scheme equilibrium is still equivalent to a suitable equilibrium with idiosyncratic bubble creation opportunities for bubbles on private assets.

which this agent has experienced many negative shocks and become arbitrarily poor relative to the size of the economy.

6.4 Alternative Policies

Fiscal policy is not the only policy that can rule out bubbles on other assets. We briefly discuss here how other policies, insolvency law, restrictions on certain assets, and financial repression, can facilitate the equilibrium selection.

Insolvency Law. Institutional rules such as insolvency laws can effectively impose no Ponzi conditions in the form of strict solvency constraints on private agents through the legal system. If effective, such rules can rule out all Ponzi schemes run by private agents, so that at most the government is able to run a Ponzi scheme.

While effective to eliminate Ponzi schemes, such rules do not directly rule out bubbles on long-lived assets. However, because most assets in reality are not long-lived, eliminating Ponzi schemes goes a long way in narrowing down the possibilities for alternative bubbles.

In addition, insolvency law can indirectly affect the viability of bubbles on long-lived assets that are liabilities of firms subject to insolvency law such as stocks. If the claims of holders of these assets are wiped out in bankruptcy, then the institutional environment triggers a forced bubble burst in any bankruptcy event.[56] To the extent that all firms face some background bankruptcy risk, bubbles on their long-lived liabilities are harder to sustain because their required return conditional on no bankruptcy must rise to compensate investors for the bubble burst in bankruptcy. This is effectively very similar to a larger dilution rate due to supply growth of the asset, i.e. like a higher $\breve{\mu}^{\mathcal{C}}$ in our model. Thus off-equilibrium fiscal backing is more credible against such bubbles.

Restrictions on Specific Assets. Specific asset classes (of long-lived assets) can also be targeted directly by legal restrictions to prevent bubbles on them. This is particularly relevant for cryptoassets that are not liabilities of entities affected by insolvency law and thus not affected by considerations in the previous paragraph.

For example, the government could impose a tax on holding such asset. In our model, a tax on cryptocoins whose tax revenues are used to lower output taxes would have an identical effect as a higher exogenous growth rate $\breve{\mu}^{\mathcal{C}}$. A sufficiently high tax can therefore drive the effective $\breve{\mu}^{\mathcal{C}}$ above $\breve{\mu}^{*}$ and thereby ensure that off-equilibrium fiscal backing remains credible.

Alternatively, a government could also impose trading restrictions on long-lived assets that could support a bubble. Because the ultimate economic value of a bubble results from enabling

[56] Specifically, it is important that a bankruptcy also inhibits the continued trading of the fundamentally worthless claims post-bankruptcy. This is, e.g., the case if the claims only exist in digital accounts and are deleted in bankruptcy.

beneficial trades (compare Section 3), bubbles on illiquid assets are less likely or even outright impossible.

Financial Repression. Instead of making bubbles on other specific assets more difficult, financial repression tools such as reserve and liquidity requirements seek to support the demand for government liabilities relative to other assets. Formally, such policies work like monetary frictions, e.g. a cash-in-advance constraint, but for bonds, and open up a spread, or convenience yield, Δi between bonds and both the (illiquid) risk-free rate r^f and the return on cryptocoins. If conducted on a sufficiently large scale, such policies can even drive r^f above g and eliminate bubbles, yet keep the government's funding costs $r^f - \Delta i$ low, so that the government still enjoys the same advantages as with a bubble.[57]

7 Conclusion and Lessons for Debt Sustainability Analysis

This paper integrates the typically ignored bubble term in the FTPL, which is necessary to explain low inflation in countries with persistently negative primary surpluses. We conclude with some lessons for debt sustainability analysis. Applying these lessons to assess debt sustainability of specific countries appears an interesting avenue for future research.

The traditional concern of debt sustainability analysis is the ability of a government to generate the future primary surpluses that are necessary to back its outstanding debt obligations (in equilibrium). A public debt bubble opens up the possibility that debt may be sustainable even in the absence of such future surpluses. Whether it is, requires an analysis of the bubble. There are two aspects to be considered.

First, debt sustainability analysis should attempt to quantify the fiscal space created by the bubble (in equilibrium). This can be done, for example, by determining the maximum debt level that can be supported in the absence of any primary surpluses or by mapping out the Laffer curve. To do so, one has to determine not only the size of the bubble for the policy regime in place, but also how the bubble reacts to changes in policy. An example of such an exercise is carried out by Brunnermeier et al. (2021a) who extend our model, calibrate it to U.S. data, and quantify the Laffer curve.

Second, debt sustainability analysis should assess stability of the bubble by considering the defenses in place to prevent coordination to other equilibria. The primary defense discussed in this paper is (contingent) fiscal policy. Just like traditional debt sustainability analysis, assessing the stability of the bubble thus starts with an assessment of the government's capacity to generate future primary surpluses. However, the emphasis shifts from the capacity to raise

[57] See e.g. Di Tella (2020) and Merkel (2020) for frameworks similar to our model in which monetary frictions may crowd out bubbles.

actual future surpluses in equilibrium to backup capacity, the mere ability and credibility to do so off-equilibrium. Backup capacity may be limited, for example, by actual limits to taxation, by imperfect commitment, or by political frictions. In addition to analyzing backup fiscal capacity, the stability of the public debt bubble can be assessed by identifying other potential assets to which the bubble could jump and the policies in place to prevent such bubble jumps (e.g. insolvency law, regulation of crypto assets). Going beyond the closed economy framework of this paper, this analysis should also consider foreign assets such as foreign government debt as competing stores of value that may carry a bubble. Some important policy considerations in the international case are explored in Brunnermeier et al. (2021b).

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A Appendix

A.1 Omitted Details in Section 2

In this section, we present additional formal details for the model setup and solution that have been omitted from the main text.

A.1.1 Additional Details about the Model Setup

Return Expressions. We denote by $\mu_t^{q,B} := \dot{q}_t^B / q_t^B$, $\mu_t^{q,C} := \dot{q}_t^C / q_t^C$ and $\mu_t^{q,K} := \dot{q}_t^K / q_t^K$ the instantaneous growth rates of q_t^B, q_t^C and q_t^K, respectively.[58]

The (real) return on bonds is

$$dr_t^{\mathcal{B}} = i_t dt + \frac{d(1/\mathcal{P}_t)}{1/\mathcal{P}_t} = \left(-\breve{\mu}_t^{\mathcal{B}} + \mu_t^K + \mu_t^{q,B}\right) dt, \tag{18}$$

where the second equality follows immediately from $1/\mathcal{P}_t = q_t^B K_t / \mathcal{B}_t$ and the definition of $\breve{\mu}_t^{\mathcal{B}}$, $\breve{\mu}_t^{\mathcal{B}} = \mu_t^{\mathcal{B}} - i_t$. $\mu_t^K := \dot{K}_t / K_t$ denotes the growth rate of aggregate capital.

The return on cryptocoins is defined analogously. The value of a single unit of cryptocoins is $q_t^C K_t / \mathcal{C}_t$. Consequently, the return on cryptocoins is

$$dr_t^{\mathcal{C}} = \frac{d(q_t^C K_t / \mathcal{C}_t)}{q_t^C K_t / \mathcal{C}_t} = \left(-\breve{\mu}^{\mathcal{C}} + \mu_t^K + \mu_t^{q,C}\right) dt.$$

The return on agent i's capital, conditional on choosing an investment rate ι_t^i, is

$$dr_t^{K,i}\left(\iota_t^i\right) = \left(\frac{(1-\tau_t)\,a - \iota_t^i + \breve{\mu}^{\mathcal{C}} q_t^C}{q_t^K} + \Phi\left(\iota_t^i\right) - \delta + \mu_t^{q,K}\right) dt + \tilde{\sigma} d\tilde{Z}_t^i. \tag{19}$$

The expected capital return consists of the after-tax dividend yield, $\frac{(1-\tau_t)a - \iota_t^i + \breve{\mu}^{\mathcal{C}} q_t^C}{q_t^K}$, and the capital gains rate, $\Phi\left(\iota_t^i\right) - \delta + \mu_t^{q,K}$. Capital returns are risky due to the presence of idiosyncratic risk $\tilde{\sigma} d\tilde{Z}_t^i$.

Equilibrium Definition. Before we provide a formal definition, we make two remarks that allow us to reduce notation and simplify the exposition. First, all households face essentially the same decision problem and individual net worth n_t^i does not affect optimal choices. We may

[58]These rates are only well-defined if $q_t^B, q_t^C, q_t^K > 0$. As $q_t^B = 0$ is a possibility in our model, we use in this case the convention that $\mu_t^{q,B} = 0$ if also $\dot{q}_t^B = 0$ and $\mu_t^{q,B} = \infty$ if $\dot{q}_t^B > 0$. The remaining case $q_t^B = 0$, $\dot{q}_t^B < 0$ will never be relevant. A similar convention applies to $\mu_t^{q,C}$.

therefore limit attention to symmetric equilibria in which all households make the same choices. Second, while the aggregate capital stock K_t, the stocks of nominal bonds $\mathcal{B}_t$ and cryptocoins $\mathcal{C}_t$, and the cross-sectional wealth distribution are natural state variables of the economic system, we can define equilibrium in terms of appropriately descaled aggregates and do not need to keep track of these variables explicitly. Given any such equilibrium for descaled aggregates and initial conditions for K_0, $\mathcal{B}_0$, $\mathcal{C}_0$, and the cross-sectional asset distribution, these objects can be recovered ex post.[59]

Definition A.1 (Competitive Equilibrium)**.** A (symmetric) *competitive equilibrium* (in descaled aggregates) consists of absolutely continuous time paths

$$[0, \infty) \to \mathbb{R}^9, t \mapsto (\breve{\mu}_t^{\mathcal{B}}, \tau_t, q_t^B, q_t^C, q_t^K, \hat{c}_t, \iota_t, \theta_t^B, \theta_t^C)$$

for government policy $(\breve{\mu}_t^{\mathcal{B}}, \tau_t)$, asset prices (q_t^B, q_t^C, q_t^K) and household choices $(\hat{c}_t, \iota_t, \theta_t^B, \theta_t^C)$ such that

(i) $\breve{\mu}_t^{\mathcal{B}}$ and τ_t satisfy the government budget constraint given prices (for all $t \geq 0$)[60]

$$\breve{\mu}_t^{\mathcal{B}} q_t^B + \tau_t a - \mathfrak{g} = 0;$$

(ii) prices are nonnegative, $q_t^B, q_t^C, q_t^K \geq 0$ (for all $t \geq 0$);[61]

(iii) for all agents i, $\hat{c}^i = \hat{c}$, $\iota^i = \iota$, $\theta^{B,i} = \theta^B$, and $\theta^{C,i} = \theta^C$ solve the household problem for any initial $n_0^i > 0$ given the returns $dr_t^{\mathcal{B}}$, $dr_t^{\mathcal{C}}$ and $dr^{K,i}(\cdot)$ implied by prices and government policies;

(iv) all markets clear (for all $t \geq 0$):[62]

$$\begin{aligned} \hat{c}_t(q_t^B + q_t^C + q_t^K) + \mathfrak{g} + \iota_t &= a && \text{goods market clearing} \\ \theta_t^B(q_t^B + q_t^C + q_t^K) &= q_t^B && \text{bond market clearing} \\ \theta_t^C(q_t^B + q_t^C + q_t^K) &= q_t^C && \text{cryptocoin market clearing} \end{aligned}$$

[59]Note also that the following definition only includes the difference $\breve{\mu}_t^{\mathcal{B}} = \mu_t^{\mathcal{B}} - i_t$ between nominal bond growth and nominal interest rates into the equilibrium definition. To determine the law of motion of $\mathcal{B}_t$ (and $\mathcal{P}_t$), one therefore also needs to specify either i_t or $\mu_t^{\mathcal{B}}$ individually.

[60]This equation follows immediately from equation (2) and the definition of q_t^B.

[61]This requirement can be derived from a free disposal assumption (in the case of capital, disposal also removes the associated tax liability).

[62]In all equations, note that $q_t^B + q_t^C + q_t^K = \int n_t^i di / K_t$ represents aggregate net worth per unit of capital. The goods market clearing condition is effectively the resource constraint (3) divided by K_t. The bond market clearing condition uses that real bond supply is $\mathcal{B}_t / \mathcal{P}_t = q_t^B K_t$. The cryptocoin market clearing condition is analogous. The capital market clearing condition is omitted as that market clears by Walras' law.

The previous definition includes government policy *paths* as part of the equilibrium definition. In the main text, we are concerned with government policy *rules* that specify policy in reaction to observed aggregate histories. To keep notation minimal, we only consider policies that ignore all histories but those of asset prices q^B, q^C, and q^K. Policy rules are feasible if they satisfy the government flow budget constraint (2) after any price history. We must take special care in the case $\mathcal{P}_t = \infty \Leftrightarrow q_t^B = 0$. In this case, it is not feasible for the government to fund a negative primary surplus, $s_t = \tau_t a - \mathfrak{g} < 0$, because no finite amount of new bond issuance in excess of interest payments $\breve{\mu}_t^{\mathcal{B}} < \infty$ will collect any real resources. However, it is still feasible to generate a positive primary surplus, $s_t > 0$ because the government's taxation power does not cease to exist when bonds become worthless. In this case, the nominal budget constraint (2) implies that the government attempts to repurchase bonds at an infinite rate, $\breve{\mu}_t^{\mathcal{B}} = -\infty$. We therefore need to allow for this possibility in our formal definition of a feasible policy rule. It is simplest to define first a feasible rule for primary surpluses s_t and then define τ_t and $\breve{\mu}_t^{\mathcal{B}}$, such that they are consistent with this surplus choice and the government budget constraint.

Definition A.2 (Policy Rules)**.** A *feasible surplus rule at time t* is a function

$$s_t : \left([0,\infty)^{[0,t]}\right)^3 \to \mathbb{R}$$

such that for all nonnegative histories of present and past prices $\{q_{t'}^B, q_{t'}^C, q_{t'}^K\}_{t'\leq t}$, the following condition holds

$$q_t^B = 0 \Rightarrow s_t\left(\{q_{t'}^B, q_{t'}^C, q_{t'}^K\}_{t'\leq t}\right) \geq 0$$

A *feasible surplus rule* is a time path $\{s_t\}_{t\geq 0}$ of feasible primary surplus rules s_t for all times t.

For a feasible surplus rule s, the *associated policy rules* for taxes τ and debt dilution $\breve{\mu}^{\mathcal{B}}$ are defined by

$$\tau_t\left(\{q_{t'}^B, q_{t'}^C, q_{t'}^K\}_{t'\leq t}\right) := \frac{s\left(\{q_{t'}^B, q_{t'}^C, q_{t'}^K\}_{t'\leq t}\right) + \mathfrak{g}}{a},$$

$$\breve{\mu}_t^{\mathcal{B}}\left(\{q_{t'}^B, q_{t'}^C, q_{t'}^K\}_{t'\leq t}\right) \begin{cases} := \frac{s\left(\{q_{t'}^B, q_{t'}^C, q_{t'}^K\}_{t'\leq t}\right)}{q_t^B}, & q_t^B > 0 \\ := -\infty, & q_t^B = 0, s\left(\{q_{t'}^B, q_{t'}^C, q_{t'}^K\}_{t'\leq t}\right) > 0 \\ \in \mathbb{R}, & q_t^B = s\left(\{q_{t'}^B, q_{t'}^C, q_{t'}^K\}_{t'\leq t}\right) = 0 \end{cases} \cdot$$

In principle, a policy is therefore fully specified by a surplus rule up the special case in which bonds are worthless and surpluses are not positive. This incomplete definition in the latter case is economically irrelevant, however. In the main text, we nevertheless often prefer to specify also the $\breve{\mu}^{\mathcal{B}}$-component of policy explicitly for additional clarity.

Definition A.3 (Equilibrium Consistent with a Policy)**.** Let s be a given feasible surplus rule. We say that a competitive equilibrium $(\mu^{\mathcal{B}*}, \tau^*, q^{B*}, q^{C*}, q^{K*}, \hat{c}^*, \iota^*, \theta^{B*}, \theta^{C*})$ is *consistent* with s if for all $t \geq 0$

$$\breve{\mu}_t^{\mathcal{B}*} = \breve{\mu}_t^{\mathcal{B}} \left(\{q_{t'}^{B*}, q_{t'}^{C*}, q_{t'}^{K*}\}_{t' \leq t} \right), \qquad \tau_t^* = \tau_t \left(\{q_{t'}^{B*}, q_{t'}^{C*}, q_{t'}^{K*}\}_{t' \leq t} \right),$$

where τ and $\breve{\mu}^{\mathcal{B}}$ are the policy rules for taxes and debt dilution associated with s

A.1.2 Model Solution

Solution to the Household Problem. The HJB equation for the household problem is problem is[63]

$$\rho V_t \left(n^i\right) - \partial_t V_t \left(n^i\right) = \max_{\hat{c}^i, \theta^{B,i}, \theta^{C,i}, \iota^i} \Bigg\{ \log\left(\hat{c}^i n^i\right) + V_t'\left(n^i\right) n^i \left(-\hat{c}^i + \theta^{B,i} \frac{dr_t^{\mathcal{B}}}{dt} + \theta^{C,i} \frac{dr_t^{\mathcal{C}}}{dt} + \left(1 - \theta^{B,i} - \theta^{C,i}\right) \frac{\mathbb{E}_t \left[dr_t^{K,i}\left(\iota^i\right)\right]}{dt} \right) + \frac{1}{2} V_t''\left(n^i\right)\left(n^i\right)^2 \left(1 - \theta^{B,i} - \theta^{C,i}\right)^2 \tilde{\sigma}^2 \Bigg\}.$$

This is a standard consumption-portfolio-choice problem, so we conjecture a functional form $V_t\left(n^i\right) = \alpha_t + \frac{1}{\rho} \log n_t^i$ for the value function, where α_t depends on (aggregate) investment opportunities, but not on individual net worth n^i.

Substituting this guess into the HJB equation and simplifying yields

$$\rho \alpha_t - \dot{\alpha}_t = \max_{\hat{c}^i} \left(\log \hat{c}^i - \frac{\hat{c}^i}{\rho} \right) + \frac{1}{\rho} \max_{\theta^{B,i}, \theta^{C,i}, \iota^i} \left(\theta^{B,i} \frac{dr_t^{\mathcal{B}}}{dt} + \theta^{C,i} \frac{dr_t^{\mathcal{C}}}{dt} + \left(1 - \theta^{B,i} - \theta^{C,i}\right) \frac{\mathbb{E}_t \left[dr_t^{K,i}\left(\iota^i\right)\right]}{dt} - \frac{\left(1 - \theta^{B,i} - \theta^{C,i}\right)^2 \tilde{\sigma}^2}{2} \right). \tag{20}$$

The first-order conditions for the maximization with respect to c^i, ι^i, $\theta^{B,i}$, and $\theta^{C,i}$ are

$$0 = \frac{1}{\hat{c}_t^i} - \frac{1}{\rho},$$

[63] Here, we have used the government budget constraint (2) to eliminate $\tau_t a$ in the return on capital.

$$0 = \frac{1 - \theta_t^{B,i} - \theta_t^{C,i}}{\rho} \frac{d}{d\iota_t^i} \frac{\mathbb{E}_t \left[dr_t^{K,i} \left(\iota_t^i \right) \right]}{dt}$$

$$0 = \frac{dr_t^{\mathcal{B}}}{dt} - \frac{\mathbb{E}_t \left[dr_t^{K,i} \left(\iota_t^i \right) \right]}{dt} + \left(1 - \theta_t^{B,i} - \theta_t^{C,i} \right) \tilde{\sigma}^2$$

$$0 = \frac{dr_t^{\mathcal{C}}}{dt} - \frac{\mathbb{E}_t \left[dr_t^{K,i} \left(\iota^i \right) \right]}{dt} + \left(1 - \theta_t^{B,i} - \theta_t^{C,i} \right) \tilde{\sigma}^2$$

The first condition is equivalent to the permanent income consumption rule stated in the main text. The second condition reduces to

$$\frac{d}{d\iota^i} \frac{\mathbb{E}_t \left[dr_t^{K,i} \left(\iota^i \right) \right]}{dt} = 0$$

because $1 - \theta^{B,i} - \theta^{C,i}$ cannot be zero for all agents simultaneously in equilibrium by market clearing. Substituting in the return expression (19) and rearranging yields

$$\frac{1}{q_t^K} = \Phi'(\iota_t^i),$$

which is equivalent to the Tobin's q condition stated in the main text.

By subtracting the last two first-order conditions from each other, we obtain

$$\frac{dr_t^{\mathcal{B}}}{dt} = \frac{dr_t^{\mathcal{C}}}{dt},$$

which is the no-arbitrage condition stated in the main text. Because both returns are risk-free, they also equal the risk-free rate r_t^f. Substituting this into one of the last two first-order conditions and rearranging yields the Merton portfolio condition for capital stated in the main text.

Note that these conditions also verify that all agents make the same choices up to indifference,[64] an assumption made in Definition A.1.

Expressing q^B, q^C q^K, ι in Terms of ϑ^B and ϑ^C. Combining the aggregate resource constraint (3) with the optimal consumption rule (aggregated over all agents) relates total wealth to total

[64]Clearly $\hat{c}_t^i = \rho$ for all i, ι_t^i is equalized because all face the same price q_t^K, then as a consequence all face the same expected return on capital; individual $\theta_t^{B,i}$, $\theta_t^{C,i}$ might differ due to indifference but we can assume without loss of generality that they are not i-dependent.

consumption in each period,

$$q_t^B + q_t^C + q_t^K = \frac{1}{\rho} C_t / K_t = \frac{a - \mathfrak{g} - \iota_t}{\rho}.$$

Divide both equations by q_t^K, use $1 - \vartheta_t = \frac{q_t^K}{q_t^B + q_t^C + q_t^k}$ on the left-hand side and $q_t^K = \frac{1}{\Phi'(\iota_t)} = 1 + \phi \iota_t$ on the right hand side to obtain an equation that relates ϑ_t to the investment rate ι_t:

$$\frac{1}{1 - \vartheta_t} = \frac{a - \mathfrak{g} - \iota_t}{1 + \phi \iota_t}.$$

Solving for ι_t yields

$$\iota_t = \frac{(1 - \vartheta_t)(a - \mathfrak{g}) - \rho}{1 - \vartheta_t + \phi \rho}.$$

Substituting this equation into $q_t^K = 1 + \phi \iota_t$ and $q_t^B + q_t^C = \frac{\vartheta_t}{1 - \vartheta_t} q_t^K$ implies

$$q_t^B + q_t^C = \vartheta_t \frac{1 + \phi (a - \mathfrak{g})}{1 - \vartheta_t + \phi \rho},$$
$$q_t^K = (1 - \vartheta_t) \frac{1 + \phi (a - \mathfrak{g})}{1 - \vartheta_t + \phi \rho}.$$

Finally, $q_t^B = \vartheta_t^B / \vartheta_t (q_t^B + q_t^C)$ and $q_t^C = \vartheta_t^C / \vartheta_t (q_t^B + q_t^C)$ by definition of $\vartheta_t^B, \vartheta_t^C, \vartheta_t$. Hence,

$$q_t^B = \vartheta_t^B \frac{1 + \phi (a - \mathfrak{g})}{1 - \vartheta_t + \phi \rho}, \qquad q_t^C = \vartheta_t^C \frac{1 + \phi (a - \mathfrak{g})}{1 - \vartheta_t + \phi \rho}$$

This also proves the first part of Proposition 1.

Derivation of Equations (5) and (6). Bond market clearing and the fact that all households choose the same $\theta_t^{B,i}$, $\theta_t^{C,i}$ imply $\theta_t^{B,i} = \vartheta_t^B$ and $\theta_t^{C,i} = \vartheta_t^C$. Hence, the Merton portfolio choice condition can be written as (recall $\vartheta_t = \vartheta_t^B + \vartheta_t^C$)

$$\frac{\mathbb{E}_t[dr_t^K(\iota_t)]}{dt} - r_t^f = (1 - \vartheta_t) \tilde{\sigma}^2 = \frac{1}{1 - \vartheta_t} (\tilde{\sigma}_t^c)^2,$$

where the last equality holds because $\tilde{\sigma}_t^c = (1 - \vartheta_t)\tilde{\sigma}$ is the idiosyncratic net worth volatility in equilibrium as can be easily observed from equation (4). Next, plug in the explicit return expression for the expected return on capital from equation (19):

$$\frac{(1 - \tau_t) a - \iota_t + \breve{\mu}^C q_t^C}{q_t^K} + \Phi(\iota_t) - \delta + \mu_t^{q,K} - r_t^f = \frac{1}{1 - \vartheta_t} (\tilde{\sigma}_t^c)^2,$$

Using $\tau_t a = \mathfrak{g} - \breve{\mu}_t^{\mathcal{B}} q_t^B$ (from the government budget constraint (2)), $a - \mathfrak{g} - \iota_t = \rho(q_t^B + q_t^C + q_t^K)$ (by goods market clearing), and $\mu_t^K = \Phi(\iota_t) - \delta$, this can be written as

$$\frac{\rho}{1-\vartheta_t} + \frac{\breve{\mu}_t^{\mathcal{B}} \vartheta_t^B + \breve{\mu}^{\mathcal{C}} \vartheta_t^C}{1-\vartheta_t} + \mu_t^K + \mu_t^{q,K} - r_t^f = \frac{1}{1-\vartheta_t} \left(\tilde{\sigma}_t^c\right)^2, \tag{21}$$

If bonds have positive value in equilibrium, then $\frac{dr_t^{\mathcal{B}}}{dt}$ is well-defined and must equal r_t^f. Making this substitution in equation (21) and also replacing $\frac{dr_t^{\mathcal{B}}}{dt}$ with the explicit expression from equation (18) yields

$$\frac{\rho + \breve{\mu}_t^{\mathcal{B}}}{1-\vartheta_t} + \frac{\left(\breve{\mu}^{\mathcal{C}} - \breve{\mu}_t^{\mathcal{B}}\right) \vartheta_t^C}{1-\vartheta_t} + \mu_t^{q,K} - \mu_t^{q,B} = \frac{1}{1-\vartheta_t} \left(\tilde{\sigma}_t^c\right)^2$$

Finally, note that $\left(\breve{\mu}^{\mathcal{C}} - \breve{\mu}_t^{\mathcal{B}}\right) \vartheta_t^C = \left(\mu_t^{q,C} - \mu_t^{q,B}\right) \vartheta_t^C$. If cryptocoins have positive value, $\vartheta_t^C > 0$, this holds by the no arbitrage condition between bonds and cryptocoins. Otherwise, this holds trivially because the factor $\vartheta_t^C = 0$ appears on both sides. Substituting this relationship into the previous equation and multiplying everything by $1 - \vartheta_t$ yields

$$\rho + \breve{\mu}_t^{\mathcal{B}} + \underbrace{\vartheta_t^B \mu_t^{q,B} + \vartheta_t^C \mu_t^{q,C} + (1-\vartheta_t)\mu_t^{q,K} - \mu_t^{q,B}}_{=\dot{\vartheta}_t^B / \vartheta_t^B} = \left(\tilde{\sigma}_t^c\right)^2$$

Rearranging this equation yields the ODE

$$\dot{\vartheta}_t^B = \left(\rho + \breve{\mu}_t^{\mathcal{B}} - \left(\tilde{\sigma}_t^c\right)^2\right) \vartheta_t^B. \tag{22}$$

While the previous derivation has assumed that bonds have positive value ($\vartheta_t^B > 0$), equation (22) remains valid even if they do not, provided $\breve{\mu}_t^{\mathcal{B}}$ remains finite (we discuss the case that it does not below): if $\vartheta_t^B = 0$, then it must also be that $\dot{\vartheta}_t^B = 0$ as otherwise the return $\frac{dr_t^{\mathcal{B}}}{dt}$ would become infinite in absolute value, allowing agents to earn an infinite risk-free return, either by investing into bonds or by short-selling them (depending on the sign). This is clearly impossible in any equilibrium.[65]

A reasoning that is fully symmetric to the previous one lead also to an ODE for the fraction of cryptocoin wealth ϑ_t^C:

$$\dot{\vartheta}_t^C = \left(\rho + \breve{\mu}^{\mathcal{C}} - \left(\tilde{\sigma}_t^c\right)^2\right) \vartheta_t^C. \tag{23}$$

Again, this equation must hold in any equilibrium regardless of whether cryptocoins have a

[65] In fact, the equation even remains valid in the case that $\breve{\mu}_t^{\mathcal{B}}$ is not finite if the product $\breve{\mu}_t^{\mathcal{B}} \vartheta_t^B$ is suitably interpreted. We ignore this for now but return to this issue in Appendix A.3.1

positive value.[66]

Equations (22) and (23) are the differential versions of the equations (5) and (6) that are stated in Proposition 1. We present in the following only a derivation of equation (5). The derivation of equation (6) is analogous:

Taking the time derivative of $e^{-\rho t}\vartheta_t^B$ yields

$$d(e^{-\rho t}\vartheta_t^B) = \left(-\rho e^{-\rho t}\vartheta_t^B + e^{-\rho t}\dot{\vartheta}_t^B\right) dt$$
$$= e^{-\rho t}\left(\breve{\mu}_t^{\mathcal{B}} - (\tilde{\sigma}_t^c)^2\right)\vartheta_t^B dt,$$

where the second line uses equation (22). Integrating both sides of the previous equation over all $s \in [t, T]$ yields

$$e^{-\rho T}\vartheta_T^B - e^{-\rho t}\vartheta_t^B = \int_t^T e^{-\rho s}\left(\breve{\mu}_s^{\mathcal{B}} - (\tilde{\sigma}_s^c)^2\right)\vartheta_s^B ds.$$

Rearranging and taking the limit $T \to \infty$ implies equation (5).

This concludes the proof of the second part of Proposition 1.

Steady-State Equilibria. Clearly, the equations stated in formula (7) are the steady-state versions of ODEs (22) and (23) (they can be obtained by setting $\dot{\vartheta}_t^B = \dot{\vartheta}_t^C = 0$). The first equation is satisfied if either of the following two conditions is satisfied:

$$\vartheta^B = 0 \quad \text{or} \quad (\tilde{\sigma}^c)^2 = \rho + \breve{\mu}^{\mathcal{B}}.$$

The second equation is satisfied if either of the following two conditions is satisfied:

$$\vartheta^C = 0 \quad \text{or} \quad (\tilde{\sigma}^c)^2 = \rho + \breve{\mu}^{\mathcal{C}}.$$

In addition, $\tilde{\sigma}^c = (1-\vartheta)\tilde{\sigma}$ and only a mathematical solution with $\vartheta = \vartheta^B + \vartheta^C \leq 1$ is a valid equilibrium solution as otherwise the equilibrium condition $q^K \geq 0$ is violated (compare Definition A.1). These considerations lead to the four cases for steady state equilibria discussed in the main text.

A.2 Derivation of the Debt Valuation Equation (Equation (9))

We derive here the debt valuation equation with a bubble in a generic partial equilibrium setting that is much more general than our model. We specialize below to the variant that holds in our model. The derivation of the debt valuation equation typically starts with the

[66] Here, there is no issue of potentially infinite $\breve{\mu}^{\mathcal{C}}$, so this equation holds in fact without qualifications on how to interpret it.

government flow budget constraint, which is, in a generic setting, given by

$$\left(\mu_t^{\mathcal{B}}\mathcal{B}_t + \mathcal{P}_t T_t\right) dt = \left(i_t \mathcal{B}_t + \mathcal{P}_t G_t\right) dt,$$

where $\mathcal{B}_t$ is the nominal face value of outstanding government bonds, $\mu_t^{\mathcal{B}}$ is its growth rate, $\mathcal{P}_t$ is the price level, T_t are (real) taxes, G_t is (real) government spending, and i_t is the nominal interest rates paid on bonds.[67]

Denote by ξ_t the real stochastic discount factor (SDF) process that prices government bonds. Multiplying the budget constraint by the nominal SDF $\xi_t/\mathcal{P}_t$ and rearranging yields

$$\left(\mu_t^{\mathcal{B}} - i_t\right) \frac{\xi_t}{\mathcal{P}_t}\mathcal{B}_t dt = -\xi_t \left(T_t - G_t\right) dt. \tag{24}$$

Next, Ito's product rule implies

$$d\left(\frac{\xi_t}{\mathcal{P}_t}\mathcal{B}_t\right) = \left(\mu_t^{\mathcal{B}} - i_t\right) \frac{\xi_t}{\mathcal{P}_t}\mathcal{B}_t dt + \frac{\xi_t}{\mathcal{P}_t}\mathcal{B}_t \left(\frac{d\left(\xi_t/\mathcal{P}_t\right)}{\xi_t/\mathcal{P}_t} + i_t dt\right).$$

Solving this equation for $\left(\mu_t^{\mathcal{B}} - i_t\right) \frac{\xi_t}{\mathcal{P}_t}\mathcal{B}_t dt$ and substituting the result into equation (24) yields (after rearranging)

$$d\left(\frac{\xi_t}{\mathcal{P}_t}\mathcal{B}_t\right) = -\xi_t \mathcal{P}_t \left(T_t - G_t\right) dt + \xi_t \frac{\mathcal{B}_t}{\mathcal{P}_t} \left(\frac{d\left(\xi_t/\mathcal{P}_t\right)}{\xi_t/\mathcal{P}_t} + i_t dt\right),$$

or in integral form

$$\xi_T \frac{\mathcal{B}_T}{\mathcal{P}_T} - \xi_t \frac{\mathcal{B}_t}{\mathcal{P}_t} = -\int_t^T \xi_s \left(T_s - G_s\right) ds + \int_t^T \xi_s \frac{\mathcal{B}_s}{\mathcal{P}_s} \left(\frac{d\left(\xi_s/\mathcal{P}_s\right)}{\xi_s/\mathcal{P}_s} + i_s dt\right).$$

Up to this point, we have merely rearranged and integrated the government budget constraint. To derive the debt valuation equation, the literature proceeds by using two equilibrium conditions. First, if the nominal SDF $\xi/\mathcal{P}$ prices the government bonds, then its expected rate of change must be the negative of the nominal interest rate. Then, the last stochastic integral on the right must be a martingale and disappears when taking conditional time-t expectations $\mathbb{E}_t[\cdot]$. Second, a private-sector transversality condition is invoked to eliminate a terminal value of government debt when passing to the limit $T \to \infty$. We perform the first operation, but are interested in environments where transversality conditions cannot rule out a nonzero discounted terminal value. When taking the limit $T \to \infty$, we therefore arrive at the more general

[67] Here we abstract from long-term bonds and the possibility of taxes, spending, and adjustments in $\mathcal{B}$ that are not absolutely continuous over time (e.g., lumpy adjustments in response to a Poisson shock).

equation:

$$\frac{\mathcal{B}_t}{\mathcal{P}_t} = \mathbb{E}_t\left[\int_t^\infty \frac{\xi_s}{\xi_t}\left(T_s - G_s\right)ds\right] + \lim_{T\to\infty}\mathbb{E}_t\left[\frac{\xi_T}{\xi_t}\frac{\mathcal{B}_T}{\mathcal{P}_T}\right].$$

To get to equation (9) stated in the main text for our model, we note that the generic SDF ξ_t has to be replaced with the SDF ξ_t^i of some agent in the model (they are all marginal in government bonds, so the identity of i does not matter), primary surpluses are $T_t - G_t = s_t K_t$, and the real value of total debt is $\mathcal{B}_t/\mathcal{P}_t = q_t^B K_t$. With these replacements, the previous equation becomes equation (9).

A.3 Proofs of the Uniqueness Propositions Presented in Section 4

In this appendix, we present the left-out technical steps and proofs necessary to establish the uniqueness results stated in Section 4. In line with Definition A.1, we remark that the equilibria we consider always feature deterministic and absolutely continuous price paths. In addition, we impose as a (purely technical) regularity condition on government policy that the ratio $\bar{s}_t := \frac{s_t}{q_t^B + q_t^C + q_t^K}$ of primary surpluses to total wealth must be a bounded and measurable function $t \mapsto \bar{s}_t$ along any equilibrium path. The precise nature of this additional assumption is of no relevance for any of the proofs below. But some regularity condition is required for all mathematical objects to be well-defined.

A.3.1 The Equilibrium ODEs

We start with a technical lemma that reduces the study of uniqueness of model equilibria to the study of uniqueness of solution paths $t \mapsto \vartheta_t^B, \vartheta_t^C$ to certain ODEs that satisfy some additional requirements. Specifically, we have shown in Appendix A.1.2 that any model equilibrium is necessarily associated with time paths for ϑ_t^B and ϑ_t^C that satisfy the ODEs (22) and (23) – except in the former case for $\vartheta_t^B = 0$ unless the product $\vartheta_t^B \breve{\mu}_t^{\mathcal{B}}$ is "suitably interpreted", compare footnote 65. Here, we revisit the case $\vartheta_t^B = 0$ with more care as a precise mathematical formulation is important for the uniqueness results. The following lemma gives such a precise formulation.

Lemma A.1. *Any model equilibrium is associated with absolutely continuous functions $\vartheta^B, \vartheta^C : [0, \infty) \to \mathbb{R}$ for the time paths of ϑ^B and ϑ^C such that*

(i) $\vartheta_t^B, \vartheta_t^C \geq 0$ *for all* $t \geq 0$;

(ii) $\vartheta_t^B + \vartheta_t^C \leq 1$ *for all* $t \geq 0$;

(iii) there is a (bounded and measurable) function $[0, \infty) \to \mathbb{R}, t \mapsto \bar{s}_t$ satisfying $\bar{s}_t \geq 0$ whenever

$\vartheta_t^B = 0$ *such that* ϑ^B *and* ϑ^C *solve the two ODEs*

$$\dot{\vartheta}_t^B = f\left(\vartheta_t^B + \vartheta_t^C\right)\vartheta_t^B - \bar{s}_t \tag{25}$$

$$\dot{\vartheta}_t^C = f\left(\vartheta_t^B + \vartheta_t^C\right)\vartheta_t^C + \breve{\mu}^C\vartheta_t^C \tag{26}$$

where

$$f : \mathbb{R} \to \mathbb{R}, x \mapsto \rho - (1-x)^2\tilde{\sigma}^2.$$

Proof. The requirement of absolutely continuous time paths follows immediately from Definition A.1. Also by that definition, an equilibrium must be associated with positive asset prices, hence $\vartheta_t^B, \vartheta_t^C, 1 - \vartheta_t^B - \vartheta_t^C \geq 0$ for all $t \geq 0$. This yields properties (i) and (ii).

ODE (26) is precisely ODE (23), which is a necessary condition for an equilibrium by the arguments made in Appendix A.1.2. Similarly, whenever $\vartheta_t^B > 0$, the arguments made in that appendix also imply ODE (22), which is equivalent to ODE (25) for the specific choice $\bar{s}_t := -\breve{\mu}_t^{\mathcal{B}}\vartheta_t^B$.

In the remaining case that $\vartheta_t^B = 0$, a feasible policy according to Definition A.2 can assume one of two cases. The first case is $s_t = 0$, in which case any arbitrary finite $\breve{\mu}_t^{\mathcal{B}}$ is consistent with the government budget constraint. In this case, ODE (22) remains mathematically well-defined and valid. Therefore, ODE (25) also holds in this case with the definition $\bar{s}_t := -\breve{\mu}_t^{\mathcal{B}}\vartheta_t^B = 0$. The second case is $s_t > 0$, in which case one easily checks that all steps in the derivation of ODE (22) still work if $\breve{\mu}^{\mathcal{B}}\vartheta_t^B$ is replaced with $-\bar{s}_t := -s_t/(q_t^B + q_t^C + q_t^K)$ throughout. Hence, also in this case ODE (25) holds and $\bar{s}_t > 0$. In total, $\bar{s}_t$ must be nonnegative in the case $\vartheta_t^B = 0$.

Finally, boundedness and measurability of $\bar{s}_t$ are the additional technical requirements discussed in the beginning of this appendix. □

We remark that the proof also reveals that the quantity $\bar{s}_t$ corresponds to the ratio of primary surpluses to total wealth in equilibrium. Whenever $\vartheta_t^B > 0$, the identity $\bar{s}_t = -\breve{\mu}_t^{\mathcal{B}}\vartheta_t^B$ holds. Note also that if the government chooses a constant surplus $s > 0$, the associated $\bar{s}_t$ is given by

$$\bar{s}_t = sh(\vartheta_t^B + \vartheta_t^C), \tag{27}$$

where

$$h : \mathbb{R} \to \mathbb{R}, x \mapsto \frac{1 - x + \phi\rho}{1 + \phi(a - \mathfrak{g})}.$$

We remark further that the following proofs (as well as the uniqueness proofs under imperfect commitment in Appendix A.5 and with aggregate shocks in Appendix A.6) do not depend on the precise definitions of f and h but only on the following properties:[68]

Fact A.1. f is continuous and strictly increasing on $[0,1]$, $f(0) < 0$, and $f(1) = \rho > 0$.

Fact A.2. h is continuous, weakly decreasing, strictly positive for all $x \in [0,1)$, and satisfies

$$(a - \mathfrak{g})h(1) \leq \rho.$$

We emphasize that the exact same proofs would remain valid in any other model, provided there are ODE representations as in Lemma A.1 with a function f satisfying Fact A.1 and a constant surplus policy can be described as in equation (27) with a function h satisfying Fact A.2.[69]

A.3.2 Omitted Steps in Proof of Proposition 2

We have already shown in the main text that the policy (12) with $s > 0$ is inconsistent with equilibrium bubbles on any asset. It is therefore without loss of generality (w.l.o.g.) to complete the proof of Proposition 2 under the assumption $q_t^C = \vartheta_t^C \equiv 0$.

We first show that there is a unique solution path ϑ_t^B to ODE (25) that is consistent with the policy specification (12) and remains within the interval $[0,1]$ at all times (a requirement of Lemma A.1). Under policy (12), ODE (25) can be written as

$$\dot{\vartheta}_t^B = f(\vartheta_t^B)\vartheta_t^B - sh(\vartheta_t^B) \tag{28}$$

where f and h have properties as in Facts A.1 and A.2. In particular, f is strictly increasing and h is positive and weakly decreasing.

It is sufficient to show that the right-hand side of equation (28) crosses zero at precisely one value for $\vartheta_t^B \in [0,1]$, is negative below that value and positive above it.[70] If $f(\vartheta_t^B) < 0$ and $\vartheta_t^B \geq 0$, then the right-hand side of (28) is negative. In addition, monotonicity of f implies that also $f(\vartheta) < 0$ for any other $\vartheta \in [0, \vartheta_t^B]$, so that the right-hand side remains negative for lower values of ϑ_t^B. Instead, if $f(\vartheta_t^B) \geq 0$, then the monotonicity properties of f and h imply that the right-hand side of (28) is strictly increasing in ϑ_t^B, so that there can be at most one point at which the right-hand vanishes. Finally, there must be also at least one such point by the intermediate value theorem: the right-hand side of (28) is continuous, negative for $\vartheta_t = 0$

[68] For f, these properties are only true if $\tilde{\sigma}^2 > \rho$, which is the requirement for bubbles in our model to be possible.

[69] This is, for example, the case for the perpetual youth model that we have considered in a previous version of this paper.

[70] Then, all mathematical solutions that start at a different value than that steady state value drift off to values below 0 or above 1 in finite time and can thus not correspond to valid equilibria.

(because h is positive) and nonnegative for $\vartheta_t = 1$,[71]. In addition, this unique value ϑ^{B*} must satisfy $\vartheta^{B*} > 0$.

To conclude the proof, note that the unique solution ϑ^{B*} indeed corresponds to a model equilibrium. It is associated with the monetary steady state equilibrium discussed in Section 2 for the parameter choice $\breve{\mu}^{\mathcal{B}} = -\frac{sh(\vartheta^{B*})}{\vartheta^{B*}}$.

A.3.3 Proof of Proposition 3

We know already that there is a unique equilibrium with $\vartheta_t^B = \vartheta^{B*}$ and $\vartheta_t^C = 0$ at all times, the monetary steady state equilibrium discussed in Setion 2. It is therefore sufficient to show that, under the threshold policy (14), $\vartheta_t^B = \vartheta^{B*}$ in any equilibrium such that $q^C \equiv 0 \Leftrightarrow \vartheta^C \equiv 0$.

By Lemma A.1, we need to show that $\vartheta_t^B = \vartheta^{B*}$ is the only solution to ODE (25) that is consistent with the specified threshold policy and contained in the interval $[0,1]$ under the assumption that $\vartheta^C \equiv 0$. Under the threshold policy (14), ODE (25) can be written as

$$\dot{\vartheta}_t^B = \begin{cases} \left(f\left(\vartheta_t^B\right) + \breve{\mu}^{\mathcal{B}}\right)\vartheta_t^B, & \vartheta_t^B \geq \underline{\vartheta} \\ f\left(\vartheta_t^B\right)\vartheta_t^B - sh\left(\vartheta_t^B\right), & \vartheta_t^B < \underline{\vartheta} \end{cases},$$

where f and h satisfy Facts A.1 and A.2.

By definition, $\vartheta^B = \vartheta^{B*}$ is the unique solution to the equation

$$f(\vartheta^B) + \breve{\mu}^{\mathcal{B}} = 0.$$

Because the left-hand side of this equation is strictly increasing in ϑ^B, it must be negative for any $\vartheta^B < \vartheta^{B*}$ and positive for any $\vartheta^B > \vartheta^{B*}$. In addition, because $\breve{\mu}^{\mathcal{B}} \geq 0$ by assumption, we can also conclude that $f(\vartheta^B) < 0$ if $\vartheta^B < \vartheta^{B*}$.

The previous considerations together with $sh(\vartheta_t^B) > 0$ allow us to make the following conclusions about the ODE stated previously: (1) $\vartheta_t^B = \vartheta^{B*} \Rightarrow \dot{\vartheta}_t^B = 0$, (2) $\vartheta_t^B \in (\vartheta^{B*}, 1] \Rightarrow \dot{\vartheta}_t^B > 0$, (3) $\vartheta_t^B \in [0, \vartheta^{B*}) \Rightarrow \dot{\vartheta}_t^B < 0$. Conclusion (1) implies that $\vartheta_t^B = \vartheta^{B*}$ is a solution that always remains inside the interval $[0,1]$. Conclusion (2) implies that any solution that is ever above ϑ^{B*} must be larger than 1 within a finite time and can thus not be contained in $[0,1]$. Conclusion (3) implies, symmetrically, that any solution that is ever below ϑ^{B*} must turn negative within

[71] If $\vartheta_t^B = 1$, then by Facts A.1 and A.2, $f(\vartheta_t^B) \geq \rho$ and

$$sh(\vartheta_t^B) \leq (a - \mathfrak{g})h(1) \leq \rho,$$

where the first inequality follows from the assumption $s \leq a - \mathfrak{g}$. Combining these results implies that the right-hand side of (28) must be nonnegative.

a finite time and can thus also not be contained in $[0,1]$. Consequently, $\vartheta_t^B = \vartheta^{B*}$ is the unique solution that satisfies $0 \leq \vartheta_t^B \leq 1$ for all t.

A.3.4 Proof of Proposition 4

The proof of Proposition 4 follows the logic outlined in the main text and depicted in Figure 1. However, while the economic logic presented in the main text is sound, there are some subtle technical difficulties associated with the proof that the threshold policy indeed succeeds in establishing a lower bound for the equilibrium fraction of wealth ϑ_t^B that is due to government bonds. We relegate these technical points to Appendix A.3.5 and here simply state the result as a lemma:

Lemma A.2. *In any equilibrium consistent with the threshold policy (14), $\vartheta_t^B \geq \underline{\vartheta}$ for all t.*

With the help of Lemma A.2, the proof of Proposition 4 is relatively straightforward. Before presenting it, we prove another simple lemma that establishes also an upper bound on ϑ_t^B:

Lemma A.3. *In any equilibrium consistent with the threshold policy (14), $\vartheta_t^B \leq \vartheta^{B*}$ for all t.*

Proof. We prove the assertion by contradiction. Suppose otherwise that $\vartheta^{B*} < \vartheta_t^B$ for some time t. Then also $\vartheta_t \geq \vartheta_t^B > \vartheta^{B*}$ and thus $\dot{\vartheta}_t^B = \left(f(\vartheta_t) + \breve{\mu}^{\mathcal{B}}\right)\vartheta_t^B > 0$. Such a solution would have to exceed 1 in finite time, contradicting Lemma A.1 □

Proof of Proposition 4. The first part of Proposition 4 follows immediately from Lemma A.2.

For the second part, we consider the cases $\underline{\vartheta} = \vartheta^{B*}$ and $\breve{\mu}^{\mathcal{C}} > \breve{\mu}^{\mathcal{B}}$ separately:

1. In the case $\underline{\vartheta} = \vartheta^{B*}$, Lemma A.3 implies the inequality chain

$$\vartheta^{B*} = \underline{\vartheta} \leq \vartheta_t^B \leq \vartheta^{B*}$$

   for all $t \geq 0$ in any equilibrium. Clearly, this can only be the case if $\vartheta_t^B = \vartheta^{B*}$ at all times. Then, in particular, $\dot{\vartheta}_t^B = 0$ and ODE (25) also implies $f(\vartheta_t) + \breve{\mu}^{\mathcal{B}} = 0$ and, hence, $\vartheta_t = \vartheta^{B*}$. It is also clear that there is unique equilibrium that satisfies $\vartheta_t^B = \vartheta_t = \vartheta^{B*}$ for all $t \geq 0$ (the monetary steady-state equilibrium without cryptocoin bubbles).

2. Now consider the case $\breve{\mu}^{\mathcal{C}} > \breve{\mu}^{\mathcal{B}}$. The time derivative of the ratio $\vartheta_t^C / \vartheta_t^B$ along any equilibrium path[72] is given by

$$\frac{d(\vartheta_t^C / \vartheta_t^B)}{dt} = \left(f(\vartheta_t) + \breve{\mu}^{\mathcal{C}}\right)\frac{\vartheta_t^C}{\vartheta_t^B} - \left(f(\vartheta_t) + \breve{\mu}^{\mathcal{B}}\right)\frac{\vartheta_t^C}{\vartheta_t^B} = \left(\breve{\mu}^{\mathcal{C}} - \breve{\mu}^{\mathcal{B}}\right)\frac{\vartheta_t^C}{\vartheta_t^B}$$

[72] Note that this ratio is always well-defined because $\vartheta_t^B \geq \underline{\vartheta} > 0$ in any equilibrium.

and hence

$$\vartheta_t^C = \vartheta_0^C \frac{\vartheta_t^B}{\vartheta_0^B} e^{(\breve{\mu}^{\mathcal{C}} - \breve{\mu}^{\mathcal{B}})t} \geq \vartheta_0^C \frac{\underline{\vartheta}}{\vartheta_0^B} e^{(\breve{\mu}^{\mathcal{C}} - \breve{\mu}^{\mathcal{B}})t}.$$

Because $\breve{\mu}^{\mathcal{C}} - \breve{\mu}^{\mathcal{B}} > 0$, the term on the right-hand side is unbounded unless $\vartheta_0^C = 0$. As ϑ_t^C must remain bounded in any equilibrium, $\vartheta_0^C = 0$ is the only possibility. But this implies $\vartheta_t^C = 0$ for all t. Hence, the assertion reduces to the conclusions of Proposition 3.

This completes the proof of Proposition 4.

□

A.3.5 Proof of Lemma A.2

We start by remarking that for the threshold policy (14), the ODE (25) can be written as

$$\dot{\vartheta}_t^B = \begin{cases} \left(f\left(\vartheta_t^B + \vartheta_t^C\right) + \breve{\mu}^{\mathcal{B}} \right) \vartheta_t^B, & \vartheta_t^B \geq \underline{\vartheta} \\ f\left(\vartheta_t^B + \vartheta_t^C\right) \vartheta_t^B - sh\left(\vartheta_t^B + \vartheta_t^C\right), & \vartheta_t^B < \underline{\vartheta} \end{cases}. \tag{29}$$

The proof of the lemma is split in a series of additional lemmas that exclude certain paths for ϑ_t^B and ϑ_t^C as valid equilibrium paths. In the proofs, we repeatedly make us of the following simple fact about ODEs: if the right-hand side of an ODE for a function x_t is continuous in x_t, strictly negative for all $x_t \in [0, \underline{x})$, and strictly positive for all $x_t \in (\overline{x}, 1]$, then all solution paths with $x_t < \underline{x}$ for some t fall below 0 in finite time and all solution paths with $x_t > \bar{x}$ for some t rise above 1 in finite time.[73]

We start with a lemma that excludes values for the total safe asset share $\vartheta_t = \vartheta_t^B + \vartheta_t^C$ that are "too large" under the condition that $\vartheta_t^B \geq \underline{\vartheta}$. In the following, let ϑ^0 denote the unique solution to the equation $f(\vartheta) = 0$.

Lemma A.4. *Let $(\vartheta_t^B, \vartheta_t^C)$ be a solution to ODEs (26) and (29) satisfying $\vartheta_t^B, \vartheta_t^C \geq 0$ for all $t \geq 0$. If there is a t_0 such that $\vartheta_{t_0}^B + \vartheta_{t_0}^C > \vartheta^0$ and $\vartheta_{t_0}^B \geq \underline{\vartheta}$, then there is also a $t_1 < \infty$ such that $\vartheta_{t_1}^B + \vartheta_{t_1}^C > 1$.*

Proof. Suppose that $\vartheta_t := \vartheta_t^B + \vartheta_t^C \geq \vartheta^0$, $\vartheta_t^B \geq \underline{\vartheta}$, and $\vartheta_t \leq 1$. For any such time t, equation (29) implies

$$\dot{\vartheta}_t^B = \underbrace{\left(f(\vartheta_t) + \breve{\mu}^{\mathcal{B}} \right)}_{>0} \vartheta_t^B > 0$$

[73] The proof of this simple fact is omitted, but the basic idea is that, close to the boundaries, $\dot{x}_t$ is bounded away from zero and thus boundaries must be reached (and crossed) in finite time as opposed to asymptotically.

and equation (26) implies

$$\dot{\vartheta}_t^C = \underbrace{\left(f(\vartheta_t) + \breve{\mu}^{\mathcal{C}}\right)}_{>0} \vartheta_t^C \geq 0,$$

so that also $\dot{\vartheta}_t = \dot{\vartheta}_t^B + \dot{\vartheta}_t^C > 0$. Hence, if the first two inequalities are satisfied for some time t_0, then they must also be satisfied for all subsequent times $t \geq t_0$. In addition, for such times t, because $\dot{\vartheta}_t$ is strictly positive so long as $\vartheta_t \leq 1$, it must be the case that ϑ_t crosses 1 in finite time, i.e. there is some $t_1 < \infty$ such that $\vartheta_{t_1} > 1$. □

The next lemma is the counterpart of the previous lemma for the case $\vartheta_t^B < \underline{\vartheta}$. In this case, the situation is more difficult because positive surpluses tend to be associated with decaying ϑ_t whereas large ϑ_t by itself tends to be associated with further growth in ϑ_t. The additional condition $\dot{\vartheta}_{t_0} > 0$ in the following lemma is required to resolve this tension.

Lemma A.5. *Let $(\vartheta_t^B, \vartheta_t^C)$ be a solution to ODEs (26) and (29) satisfying $\vartheta_t^B, \vartheta_t^C \geq 0$ for all $t \geq 0$. If there is a t_0 such that $\vartheta_{t_0}^B + \vartheta_{t_0}^C \geq \vartheta^0$, $\vartheta_{t_0}^B < \underline{\vartheta}$, and $\dot{\vartheta}_{t_0}^B + \dot{\vartheta}_{t_0}^C > 0$, then there is also a $t_1 < \infty$ such that $\vartheta_{t_1}^B + \vartheta_{t_1}^C > 1$.*

Proof. We show that for all $t \geq t_0$ such that $\vartheta_t := \vartheta_t^B + \vartheta_t^C \leq 1$, necessarily

$$\dot{\vartheta}_t \geq \dot{\vartheta}_{t_0} > 0.$$

It follows then that ϑ_t must exceed 1 in finite time.

Combining equations (26) and (29) implies (for all t)

$$\dot{\vartheta}_t \geq \underbrace{f\left(\vartheta_t\right)\vartheta_t - sh\left(\vartheta_t\right)}_{=:F(\vartheta_t)} + \breve{\mu}^{\mathcal{C}} \vartheta_t^C.$$

To see why this holds, note that in the case $\vartheta_t^B < \underline{\vartheta}$ it holds with equality. This is particularly true at $t = t_0$. Because $\breve{\mu}^{\mathcal{B}} \vartheta_t^B > -sh\left(\vartheta_t\right)$, the inequality version must then also hold in the case $\vartheta_t^B \geq \underline{\vartheta}$.

Note that F is a strictly increasing function on the interval $[\vartheta^0, 1]$, so if $1 \geq \vartheta_t \geq \vartheta_{t_0}$, then $F(\vartheta_t) \geq F(\vartheta_{t_0})$. In addition, if $1 \geq \vartheta_t \geq \vartheta_{t_0} \geq \vartheta^0$, then $f\left(\vartheta_t\right) + \breve{\mu}^{\mathcal{C}} \geq 0$ and thus ϑ_t^C is nondecreasing over time by equation (26).

From these considerations, it is straightforward to establish for all $t \geq t_0$ such that $\vartheta_t \leq 1$

$$\dot{\vartheta}_t \geq F(\vartheta_t) + \breve{\mu}^{\mathcal{C}} \vartheta_t^C \geq F(\vartheta_{t_0}) + \breve{\mu}^{\mathcal{C}} \vartheta_{t_0}^C = \dot{\vartheta}_{t_0} > 0.$$

This completes the proof of the lemma.

□

The final lemma deals with solutions for which ϑ remains below 1 at all times. Under this additional restriction, it can be shown that when ϑ_t^B falls below the threshold $\underline{\vartheta}$, it must further decay in all future periods with a derivative that is bounded away from 0:

Lemma A.6. *There is $\varepsilon > 0$ such that $\vartheta_t^B < \underline{\vartheta}$ implies $\dot{\vartheta}_t^B \leq -\varepsilon$ for any solution $(\vartheta_t^B, \vartheta_t^C)$ to ODEs (26) and (29) that satisfies $\vartheta_t^B, \vartheta_t^C \geq 0$ and $\vartheta_t^B + \vartheta_t^C \leq 1$ for all $t \geq 0$.*

Proof. By Facts A.1 and A.2, f and h are continuous. Consequently, the function $\delta \mapsto f(\vartheta^0 + \delta)\underline{\vartheta} - sh(\vartheta^0 + \delta)$ is also continuous. Because $s > 0$, h is strictly positive, and $f(\vartheta^0) = 0$, this function is negative for $\delta = 0$. By continuity, we can find some $\delta > 0$ such that it is still negative. Then, by construction

$$\varepsilon_1 := sh(\vartheta^0 + \delta) - f(\vartheta^0 + \delta)\underline{\vartheta} > 0$$

Furthermore, define

$$\varepsilon_2 := f(\vartheta^0 + \delta)\delta > 0$$

We claim that $\varepsilon := \min\{\varepsilon_1, \varepsilon_2\}$ has the desired property. To prove it, suppose that $\vartheta_t^B < \underline{\vartheta}$ and distinguish two cases (in the following, $\vartheta_t := \vartheta_t^B + \vartheta_t^C$):

1. $\vartheta_t \leq \vartheta^0 + \delta$:
   In this case, equation (29) implies

$$\begin{aligned}\dot{\vartheta}_t^B &= f(\vartheta_t)\vartheta_t^B - sh(\vartheta_t)\\ &\leq f(\vartheta^0 + \delta)\vartheta_t^B - sh(\vartheta^0 + \delta)\\ &\leq f(\vartheta^0 + \delta)\underline{\vartheta} - sh(\vartheta^0 + \delta) = -\varepsilon_1 \leq -\varepsilon.\end{aligned}$$

   Here, the first inequality follows from the fact that f is (strictly) increasing, h is (weakly) decreasing (Facts A.1 and A.2), and $\vartheta_t^B \geq 0$ by assumption. The second inequality follows from $\vartheta_t^B < \underline{\vartheta}$ and $f(\vartheta^0 + \delta) > 0$.

2. $\vartheta_t > \vartheta^0 + \delta$:
   We first remark that it must be the case that $\dot{\vartheta}_t \leq 0$. Otherwise, the assumptions of Lemma A.5 are satisfied, whose conclusion is inconsistent with the requirement that ϑ remains (weakly) below 1 at all times. Next, using $\vartheta_t \geq \vartheta^0 + \delta$, $\vartheta_t^B < \underline{\vartheta} \leq \vartheta^{B*} \leq \vartheta^0$, we can conclude that $f(\vartheta_t) \geq f(\vartheta^0 + \delta) \geq 0$ (because f is increasing) and $\vartheta_t - \vartheta_t^B \geq \delta$. Consequently,

$$f(\vartheta_t)(\vartheta_t - \vartheta_t^B) \geq f(\vartheta_t)\delta \geq f(\vartheta^0 + \delta)\delta = \varepsilon_2.$$

We can then use the previous inequality and $\dot{\vartheta}_t \leq 0$ to bound $\dot{\vartheta}_t^B$, which is by equation (29) given by

$$\begin{aligned}\dot{\vartheta}_t^B &= f(\vartheta_t)\vartheta_t^B - sh(\vartheta_t) = f(\vartheta_t)\vartheta_t - f(\vartheta_t)(\vartheta_t - \vartheta_t^B) - sh(\vartheta_t) \\ &\leq f(\vartheta_t)\vartheta_t - sh(\vartheta_t) - \varepsilon_2 \\ &\leq f(\vartheta_t)\vartheta_t + \breve{\mu}^C\vartheta_t^C - sh(\vartheta_t) - \varepsilon_2 \\ &= \dot{\vartheta}_t - \varepsilon_2 \leq -\varepsilon_2 \leq -\varepsilon\end{aligned}$$

Here, the inequality in the third line follows from $\breve{\mu}^C\vartheta_t^C \geq 0$ and the inequalities in the last line from $\dot{\vartheta}_t \leq 0$ and $\varepsilon_2 \geq \varepsilon$.

Hence, in either case we can conclude $\dot{\vartheta}_t^B \leq -\varepsilon$. □

With the help of the previous lemma, it is straightforward to prove Lemma A.2:

Proof of Lemma A.2. Take any equilibrium consistent with the threshold policy and let $(\vartheta_t^B, \vartheta_t^C)$ be the associated equilibrium paths for ϑ_t^B and ϑ_t^C. By Lemma A.1, $(\vartheta_t^B, \vartheta_t^C)$ must solve the ODEs (26) and (29) and satisfy the additional properties $\vartheta_t^B, \vartheta_t^C \geq 0$ and $\vartheta_t^B + \vartheta_t^C \leq 1$ for all $t \geq 0$. We show that the assumption $\vartheta_{t_0}^B < \underline{\vartheta}$ for any time t_0 leads to a contradiction.

Indeed, if $\vartheta_{t_0}^B < \underline{\vartheta}$, then Lemma A.6 implies that $\vartheta_t^B < \underline{\vartheta}$ for all $t \geq t_0$ and, hence, again by the same lemma, there is some $\varepsilon > 0$ such that $\dot{\vartheta}_t^B \leq -\varepsilon$ for all $t \geq t_0$. The previous inequality implies (for $t \geq t_0$)

$$\vartheta_t^B \leq \vartheta_{t_0}^B - \varepsilon(t - t_0)$$

The right-hand side of this inequality turns negative for finite $t_1 > t_0$, so that necessarily also $\vartheta_{t_1}^B < 0$, in contradiction to $\vartheta_t^B \geq 0$ for all $t \geq 0$. □

A.4 Omitted Details in Section 5.4

Derivation of Equation (17). Because all agents consume the same constant fraction ρ of their wealth, the consumption share c_t^i/C_t of agent i at time t must equal the agent's wealth share η_t^i. We can therefore write, using the aggregate resource constraint (3),

$$c_t^i = \eta_t^i C_t = \eta_t^i \left(a - \mathfrak{g} - \iota_t\right) K_t.$$

Thus, expected utility of agent i is given by

$$\mathbb{E}\left[\int_0^\infty e^{-\rho t} \log c_t^i dt\right] = \mathbb{E}\left[\int_0^\infty e^{-\rho t} \left(\log \eta_t^i + \log\left(a - \mathfrak{g} - \iota_t\right) + \log K_t\right) dt\right]. \tag{30}$$

To compute the integrals in equation (30), note that if

$$\frac{dx_t}{x_t} = \mu_t^x dt + \tilde{\sigma}_t^x d\tilde{Z}_t,$$

then

$$\mathbb{E}\left[\int_0^\infty e^{-\rho t}\log x_t dt\right] = \frac{\log x_0}{\rho} + \mathbb{E}\left[\int_0^\infty e^{-\rho t}\frac{\mu_t^x - \left(\tilde{\sigma}_t^x\right)^2/2}{\rho}dt\right]. \tag{31}$$

This follows from a simple calculation:

$$\begin{aligned}
\int_0^\infty e^{-\rho t}\left(\log x_t - \log x_0\right)dt &= \int_0^\infty e^{-\rho t}\int_0^t d\log x_s dt \\
&= \int_0^\infty e^{-\rho t}\left(\int_0^t \mu_s^x ds + \int_0^t \tilde{\sigma}_t^x d\tilde{Z}_s - \frac{1}{2}\int_0^t \left(\tilde{\sigma}_t^x\right)^2 ds\right)dt \\
&= \int_0^\infty \int_s^\infty e^{-\rho t}dt\left(\mu_s^x - \frac{1}{2}\left(\tilde{\sigma}_t^x\right)^2\right)ds + \int_0^\infty e^{-\rho t}\int_0^t \tilde{\sigma}_t^x d\tilde{Z}_s dt \\
&= \int_0^\infty e^{-\rho s}\frac{\mu_s^x - \left(\tilde{\sigma}_t^x\right)^2/2}{\rho}ds + \int_0^\infty e^{-\rho t}\int_0^t \tilde{\sigma}_t^x d\tilde{Z}_s dt.
\end{aligned}$$

When taking expectations, the second term disappears because it is a martingale. Thus, we obtain formula (31).

To apply formula (31), we need to determine $\frac{dK_t}{K_t}$ and $\frac{d\eta_t^i}{\eta_t^i}$. We know that

$$\frac{dK_t}{K_t} = \left(\Phi(\iota_t) - \delta\right)dt. \tag{32}$$

For η_t^i, we have (recall that $q_t^C \equiv 0$ and $\vartheta^B = \vartheta$)

$$\begin{aligned}
\frac{d\eta_t^i}{\eta_t^i} &= \frac{dn_t^i}{n_t^i} - \frac{d\bar{q}_t}{\bar{q}_t} - \frac{dK_t}{K_t} \\
&= \left(-\rho dt + dr_t^{\mathcal{B}} + (1-\vartheta_t)\left(dr_t^{K,i}(\iota_t) - dr_t^{\mathcal{B}}\right)\right) - \mu_t^{\bar{q}}dt - \left(\Phi(\iota_t) - \delta\right)dt \\
&= \left(-\rho - \breve{\mu}_t^{\mathcal{B}} + \mu_t^{\vartheta}\right)dt + (1-\vartheta_t)\left(\frac{a - \mathfrak{g} - \iota_t}{q_t^K} + \frac{\breve{\mu}_t^{\mathcal{B}} - \mu_t^{\vartheta}}{1-\vartheta_t}\right)dt + (1-\vartheta_t)\tilde{\sigma}d\tilde{Z}_t^i \\
&= \left(-\rho + (1-\vartheta_t)\frac{\rho}{1-\vartheta_t}\right)dt + (1-\vartheta_t)\tilde{\sigma}d\tilde{Z}_t^i \\
&= (1-\vartheta_t)\tilde{\sigma}d\tilde{Z}_t^i
\end{aligned} \tag{33}$$

where $\bar{q}_t := q_t^B + q_t^K$ and $\mu_t^{\bar{q}} := \frac{\dot{\bar{q}}_t}{\bar{q}_t}$. Here, the third line uses the return expressions and the government budget constraint (2) and the fourth line the aggregate resource constraint (3).

Equations (32) and (33) together with formula (31) allow us to compute the integrals in (30):

$$\mathbb{E}\left[\int_0^\infty e^{-\rho t}\log \eta_t^i dt\right] = \frac{\log \eta_0^i}{\rho} - \frac{1}{2\rho}\mathbb{E}\left[\int_0^\infty e^{-\rho t}\left(1-\vartheta_t\right)^2 \tilde{\sigma}^2 dt\right],$$

$$\mathbb{E}\left[\int_{t_0}^\infty e^{-\rho t}\log K_t dt\right] = \frac{\log K_0}{\rho} + \mathbb{E}\left[\int_0^\infty e^{-\rho t}\frac{\left(\Phi(\iota_t)-\delta\right)}{\rho}dt\right].$$

Consequently,

$$\mathbb{E}\left[\int_{t_0}^\infty e^{-\rho t}\log c_t^i dt\right] = \frac{\log \eta_0^i + \log K_0}{\rho}$$

$$+\,\mathbb{E}\left[\int_0^\infty e^{-\rho t}\left(\log\left(a-\mathfrak{g}-\iota_t\right) + \frac{\left(\Phi(\iota_t)-\delta\right)}{\rho} - \frac{\left(1-\vartheta_t\right)^2\tilde{\sigma}^2}{2\rho}\right)dt\right]$$

After substituting ι_t as a function of ϑ_t (as stated in Proposition 1) and the functional form $\Phi(\iota) = \frac{1}{\phi}\log\left(1+\phi\iota\right)$, we obtain equation (17).

Existence, Uniqueness and Properties of ϑ^*. Taking first order conditions for maximizing the time-t integrand in equation (17) with respect to ϑ_t implies

$$\left(1-\vartheta_t\right)^3\tilde{\sigma}^2 + \phi\rho\left(1-\vartheta_t\right)^2\tilde{\sigma}^2 + \rho\left(1-\vartheta_t\right) - \rho = 0. \tag{34}$$

This is a third-order polynomial equation in $1-\vartheta_t$ and has thus precisely three complex solutions. Because the coefficients on all monomials of positive order are nonnegative and the constant coefficient is negative, standard results on polynomial roots imply that precisely one of these complex solutions is real and that solution must be positive. Consequently, there is a unique real number $\vartheta^* < 1$ such that $1-\vartheta^*$ satisfies the first-order condition.[74] It is also easy to see that $\vartheta^* > 0$ as otherwise the positive-sign terms in equation (34) would exceed the negative-sign term in absolute value. Therefore, there is a unique optimal $\vartheta^* \in (0,1)$ that maximizes the time-t integrand in equation (17) with respect to ϑ_t. Because the coefficients in equation (34) just depend on the parameters $\tilde{\sigma}$, ρ and ϕ, so does ϑ^*. By the implicit function theorem, ϑ^* must be strictly increasing in $\tilde{\sigma}$.

A.5 Uniqueness under Imperfect Commitment

In this Appendix, we provide a more detailed formal description of the policy game outlined in Section 6.1 and prove Proposition 5.

[74] The objective is not generally concave, but it is quasiconcave (over the domain $[0,1]$), such that the first-order condition nevertheless always corresponds to a global maximum.

A.5.1 The Policy Game

We have described asset prices in the main text by the three variables q_t^B, q_t^C, q_t^K and policies by the three variables $i_t, \mu_t^{\mathcal{B}}, \tau_t$. Instead, our approach in Appendix A.3 was to reduce the study of equilibria to the analysis of the two ODEs (25) and (26) for ϑ_t^B and ϑ_t^C. These ODEs are affected by the government policy only through the one-dimensional policy variable $\bar{s}_t$ that has to satisfy the requirement $\bar{s}_t \geq 0$ if $\vartheta_t^B = 0$. It is relatively straightforward to show that, conversely, any solution to ODEs (25) and (26) that satisfies the requirements stated in Lemma A.1 corresponds to a model equilibrium.[75] As this saves on notation, we formulate here the policy game in terms of just the variables ϑ_t^B, ϑ_t^C, and $\bar{s}_t$ instead of working with the original variables as in the main text.

Problem of a Government. Our assumption of Markov perfect equilibria implies that governments do not react to the policy choices of previous governments because there are no state variables in the model. As a consequence, each government j takes the terminal asset values $\vartheta^B_{(j+1)T}, \vartheta^C_{(j+1)T}$ as given because actions of future governments (that will ultimately determine these values) are not directly impacted by government j's actions.

It is sufficient to describe the problem of government $j = 0$ in more detail. The problem of governments $j > 0$ is identical with the only difference that time has to be shifted to the right by jT time units.

Government $j = 0$ takes terminal asset values $\vartheta_T^B, \vartheta_T^C \geq 0$ with $\vartheta_T^B + \vartheta_T^C \leq 1$ as given and makes a policy choice for $\{\bar{s}_t\}_{t\in[0,T)}$ in order to control the price dynamics of ϑ^B and ϑ^C over $[0, T)$ in line with ODEs (25) and (26). As discussed in Appendix A.3.1, the policy $\{\bar{s}_t\}_{t\in[0,T)}$ must satisfy the constraint $\bar{s}_t \geq 0$ whenever $\vartheta_t^B = 0$. In addition, we only admit policies that lead to valuation paths satisfying $\vartheta_t^B + \vartheta_t^C \leq 1$ for all $t \in [0, T]$, as otherwise the resulting solution to ODEs (25) and (26) would not correspond to a valid competitive equilibrium of the underlying model.[76]

Because each path for $\bar{s}$ (that is bounded and measurable) implies a unique solution for ϑ^B and ϑ^C (given terminal values), there is no gain from requiring that a policy must be specified as a full function from observed histories of prices as we have previously done. Instead, it is sufficient to think about the government's problem as a Ramsey problem, i.e. choosing a time path of the policy variable that is actually followed conditional on terminal asset values being ϑ_T^B and ϑ_T^C.

[75] The proof consists in constructing all other model variables out of ϑ_t^B, ϑ_t^C, and $\bar{s}_t = s_t/(q_t^B + q_t^C + q_t^K)$ using Proposition 1 and the optimal choice conditions derived in Appendix A.1. Once this constructions have been completed, one has to verify all conditions in Definition A.1, which is not difficult but tedious. A previous version of this paper contained an explicit statement of this proof, but in the interest of space it has been removed.

[76] We do not need to impose the inequalities $\vartheta_t^B, \vartheta_t^C \geq 0$ explicitly. For any backward solution to the ODEs, these conditions are automatically satisfied at all times if they are satisfied at the terminal time $t = T$.

A strategy of government $j = 0$ is therefore a function that maps pairs of terminal asset values $(\vartheta_T^B, \vartheta_T^C)$ satisfying $\vartheta_T^B, \vartheta_T^C \geq 0$ and $\vartheta_t^B + \vartheta_t^C \leq 1$ into a time path $\{\bar{s}_t\}_{t\in[0,T)}$ for the policy variable. Given a strategy, ODEs (25) and (26) then provide a mapping from pairs $(\vartheta_T^B, \vartheta_T^C)$ into time paths $\{\vartheta_t^B, \vartheta_t^C\}_{t\in[0,T)}$ of asset values over the interval $[0, T)$.

The government desires to maximize a social welfare function as in Section 5.4:

$$\mathbb{W} := \int_0^1 \lambda(i)\mathbb{E}\left[\int_0^\infty e^{-\rho t}\log c_t^i dt\right] di,$$

where $\{\lambda(i)\}_{i\in[0,1]}$ is a given distribution of (nonnegative) welfare weights. As we have argued in Section 5.4, individual utility can be written in the form (17) that separates individual and aggregate variables, so that the former (and the welfare weights) are not relevant for a planner's optimal choices. In addition, the government in question here has no way of influencing prices after period T. Consequently, we can assume that government $j = 0$ maximizes the simpler objective

$$\mathcal{W}\left(\{\vartheta_t\}_{t\in[0,T]}\right) := \int_0^T e^{-\rho T} \underbrace{\left(\log\left(\frac{\rho\left(1+\phi\left(a-\mathfrak{g}\right)\right)}{1-\vartheta_t+\phi\rho}\right) + \frac{1}{\phi\rho}\log\left(\frac{\left(1-\vartheta_t\right)\left(1+\phi\left(a-\mathfrak{g}\right)\right)}{1-\vartheta_t+\phi\rho}\right) - \frac{\delta}{\rho} - \frac{\left(1-\vartheta_t\right)^2\tilde{\sigma}^2}{2\rho}\right)}_{=:\Psi(\vartheta_t)} dt.$$

where $\vartheta_t := \vartheta_t^B + \vartheta_t^C$.

We remark that Ψ is a strictly quasiconcave function with a unique global maximizer. As in Section 5.4, we denote this maximizer by ϑ^* in the following.

Equilibrium. An equilibrium in the policy game consists of absolutely continuous valuation paths $\{\vartheta_t^B, \vartheta_t^C\}_{t\in[0,\infty)}$ and a bounded and measurable government policy path $\{\bar{s}_t\}_{t\in[0,\infty)}$ such that for all $j = 0, 1, \ldots$, $\{\bar{s}_t\}_{t\in[jT,(j+1)T)}$ is the optimal policy chosen by government j given terminal valuations $(\vartheta_{(j+1)T}^B, \vartheta_{(j+1)T})$ and $\{\vartheta_t^B, \vartheta_t^C\}_{t\in[jT,(j+1)T}$ is the valuation path over $[jT, (j+1)T)$ implied by that optimal policy.

Note that, this definition automatically implies that $\{\vartheta_t^B, \vartheta_t^C\}_{t\in[0,\infty)}$ corresponds to a solution to ODEs (25) and (26) for the policy choice $\{\bar{s}_t\}_{t\in[0,\infty)}$. By the remark made in the beginning of this appendix, these solution paths identify a competitive equilibrium in the sense of Definition A.1.

A.5.2 Proof of Proposition 5

We first show that any equilibrium in the policy game must feature $\vartheta_t \geq \vartheta^*$ at all times. The proof of this part is organized into a sequence of technical lemmas that provide key results about the associated ODEs (25) and (26) and the optimal choices of any individual government.

Lemma A.7. *Let* $(\vartheta_T^{B,i}, \vartheta_T^{C,i})$, $i = 1, 2$ *be two sets of terminal conditions and* $\{\bar{s}_t^i\}_{t\in[0,T)}$, $i = 1, 2$ *two policy paths over* $[0, T)$. *Let* $\{\vartheta_t^{B,i}, \vartheta_t^{C,i}\}_{t\in[0,T]}$ *be the implied solution paths to ODEs (25) and (26). Define* $\vartheta_t^i := \vartheta_t^{B,i} + \vartheta_t^{C,i}$. *If* $\vartheta_t^1 \leq \vartheta_t^2$ *for all* $t \in [0, T]$ *and* $\vartheta_T^{C,1} \geq \vartheta_T^{C,2}$, *then also* $\vartheta_t^{C,1} \geq \vartheta_t^{C,2}$ *for all* $t \in [0, T]$.

Proof. Let $\alpha(\vartheta) := f(\vartheta) + \breve{\mu}^C$. By Fact A.1, α is strictly increasing in $\vartheta \in (0, 1)$. ODE (26) implies

$$\dot{\vartheta}_t^{C,i} = \left(f(\vartheta_t) + \breve{\mu}^C\right) \vartheta_t^{C,i} = \alpha(\vartheta_t^i)\vartheta_t^{C,i}.$$

This has the solution

$$\vartheta_t^{C,i} = \vartheta_T^{C,i} \exp\left(-\int_t^T \alpha(\vartheta_s^i) ds\right). \tag{35}$$

If $\vartheta_t^1 \leq \vartheta_t^2$ for all t, then $\alpha(\vartheta_t^1) \leq \alpha(\vartheta_t^2)$ for all t, and so

$$\exp\left(-\int_t^T \alpha(\vartheta_s^1) ds\right) \geq \exp\left(-\int_t^T \alpha(\vartheta_s^2) ds\right).$$

Combining this inequality and the assumption $\vartheta_T^{C,1} \geq \vartheta_T^{C,2}$ with inequality (35) implies $\vartheta_t^{C,1} \geq \vartheta_t^{C,2}$ for all t. □

Lemma A.8. *Consider the problem of government* $j = 0$ *with terminal condition* $(\vartheta_T^B, \vartheta_T^C)$ *and let* $\{\vartheta_t\}_{t\in[0,T]}$, $\{\vartheta_t'\}_{t\in[0,T]}$ *be two absolutely continuous time paths such that* $\vartheta_T = \vartheta_T' = \vartheta_T^B + \vartheta_T^C$. *Suppose there is a feasible policy that implements* $\{\vartheta_t\}_{t\in[0,T]}$ *as a time path for* $\vartheta^B + \vartheta^C$, $\{\vartheta_t'\}_{t\in[0,T]}$ *has bounded derivative, and* $\vartheta_t' \geq \vartheta_t$ *for all* $t \in [0, T]$. *Then there is also a feasible policy that implements* $\{\vartheta_t'\}_{t\in[0,T]}$ *as a time path for* $\vartheta^B + \vartheta^C$.

Proof. First, we "backsolve" the ODEs for the policy $\bar{s}_t'$ required to generate the path ϑ_t'. Assuming $\vartheta^B + \vartheta^C$ follows indeed the path ϑ_t', ODE (26) can be solved backward from the terminal condition ϑ_T^C to obtain a time path for ϑ^C, denote it by $\vartheta_t^{C\prime}$. Given $\vartheta_t^{C\prime}$, the corresponding path for ϑ^B must be $\vartheta_t^{B\prime} := \vartheta_t' - \vartheta_t^{C\prime}$. Note that this definition automatically satisfies the terminal condition $\vartheta_T^{B\prime} = \vartheta_T^B$. Substituting this path into ODE (25) allows us to back out the associated policy $\bar{s}_t'$.[77]

We need to show that $\bar{s}'$ is a feasible policy choice. The policy $\bar{s}'$ is feasible if it satisfies the condition $\bar{s}_t' \geq 0$ whenever $\vartheta_t^{B\prime} = 0$. Inspecting ODE (25) reveals that this property is equivalent

[77] $\bar{s}_t'$ is always uniquely defined, except possibly on a set of measure zero. It is easy to see that $\bar{s}_t'$ must be bounded because ϑ_t' has bounded derivative by assumption and all other terms in equation (25) are also bounded.

to $\vartheta_t^{B\prime} \geq 0$ for all t and we choose to prove this equivalent property.[78]

Let ϑ_t^B denote the solution for ϑ^B associated with the feasible path ϑ_t. Then $\vartheta_t^B \geq 0$ for all t. By assumption, also $\vartheta_t \leq \vartheta_t'$ and thus by Lemma A.7, $\vartheta_t^C \geq \vartheta_t^{C\prime}$. Combining these facts yields for all $t \in [0, T]$ the inequality chain

$$0 \leq \vartheta_t^B = \vartheta_t - \vartheta_t^C \leq \vartheta_t' - \vartheta_t^{C\prime} = \vartheta_t^{B\prime}.$$

Thus, policy $\bar{s}'$ is indeed feasible. □

Lemma A.9. *Consider the problem of government $j = 0$ for a given terminal condition. Suppose $\{\vartheta_t^\circ\}_{t\in[0,T]}$ is an arbitrary absolutely continuous time path with bounded derivative that satisfies $\vartheta_T^\circ \leq \vartheta_T^B + \vartheta_T^C$ and $\vartheta_t^\circ \leq \vartheta^*$ for all t. Then the optimal policy choice implies $\vartheta_t^B + \vartheta_t^C \geq \vartheta_t^\circ$ for all $t \in [0, T]$.*

Proof. Let $\vartheta_t := \vartheta_t^B + \vartheta_t^C$ be the time path for $\vartheta^B + \vartheta^C$ implied by the optimal solution. Define $\vartheta_t' := \max\{\vartheta_t, \vartheta_t^\circ\}$. Because both ϑ and ϑ° are absolutely continuous with bounded derivative, so is also the pointwise maximum ϑ'. In addition, by construction $\vartheta_t' \geq \vartheta_t$ for all t and by assumption on the terminal values, $\vartheta_T' = \vartheta_T$. Thus, all assumptions of Lemma A.8 are satisfied and we can conclude that there is a feasible policy that implements the time path ϑ'.

We show next that this alternative policy generates at least as high welfare as the original plan leading to ϑ and strictly larger welfare if ϑ and ϑ' are different. Because ϑ is optimal by assumption, the latter cannot be the case and so it must be that $\vartheta_t = \vartheta_t' \geq \vartheta_t^\circ$ for all t.

To do so, we use that the relevant part of the government's objective can be written as

$$\mathcal{W}\left(\{\vartheta\}_{t\in[0,T]}\right) = \int_0^T e^{-\rho t}\Psi(\vartheta_t)dt$$

and Ψ is a strictly quasiconcave function with global maximizer ϑ^*. For each $t \in [0, T]$ we have two cases

(i) if $\vartheta_t \geq \vartheta_t^\circ$, then $\vartheta_t' = \vartheta_t$ and thus $\Psi(\vartheta_t') = \Psi(\vartheta_t)$;

(ii) if $\vartheta_t < \vartheta_t^\circ$, then $\vartheta^* \geq \vartheta_t' > \vartheta_t$ and thus $\Psi(\vartheta_t') > \Psi(\vartheta_t)$.

So in any case, $\Psi(\vartheta_t') \geq \Psi(\vartheta_t)$ for all $t \in [0, T]$. Thus ϑ' must yield at least as high welfare as ϑ. In addition, if case (ii) ever occurs, then $\Psi(\vartheta_t') > \Psi(\vartheta_t)$ must hold on a set of positive measure for t because all expressions are continuous in t. But that would imply $\mathcal{W}\left(\{\vartheta'\}_{t\in[0,T]}\right) > \mathcal{W}\left(\{\vartheta\}_{t\in[0,T]}\right)$. Therefore, ϑ' yields strictly higher welfare unless the two paths are identical at all times. □

[78] This is the case because $f(0) < 0$, so the only way ϑ^B can cross zero in a backward solution is if $\bar{s}_t < 0$ at a time when $\vartheta_t^B = 0$.

The previous lemma implies the following corollary that is key to the proof that $\vartheta_t \geq \vartheta^*$ in all equilibria in the policy game.

Corollary A.1. *Consider the problem of any government j for any terminal condition at time $(j+1)T$. Then the optimal solution to the government problem features $\vartheta^B_{jT} + \vartheta^C_{jT} \geq \vartheta^*$.*
In addition, if the terminal condition satisfies $\vartheta^B_{(j+1)T} + \vartheta^C_{(j+1)T} \geq \vartheta^*$, then the optimal solution even satisfies $\vartheta^B_t + \vartheta^C_t \geq \vartheta^*$ for all $t \in [jT, (j+1)T]$.

Proof. We show the assertion for $j = 0$. Because the problem of government $j > 0$ is identical to the problem of government $j = 0$ except for a time shift, the result for $j = 0$ immediately carries over to any $j \geq 0$. Throughout the proof, we write $\vartheta_t = \vartheta^B_t + \vartheta^C_t$.

We start by proving the additional statement in the special case $\vartheta_T \geq \vartheta^*$. This also proves the main assertion in this case. To see that this statement must hold, simply define $\vartheta^\circ_t := \vartheta^*$ and apply Lemma A.9 to conclude $\vartheta_t \geq \vartheta^\circ_t = \vartheta^*$ for all $t \in [0, T]$.

Next, suppose $\vartheta_T < \vartheta^*$. In this case, the time path ϑ° defined by

$$\vartheta^\circ_t := \vartheta^* + \frac{t}{T}\vartheta_T$$

satisfies the terminal condition and $\vartheta^\circ_t \leq \vartheta^*$ for all $t \in [0, T]$. We can thus again apply Lemma A.9 and conclude $\vartheta_t \geq \vartheta^\circ_t$. In particular, $\vartheta_0 \geq \vartheta^\circ_0 = \vartheta^*$. □

Corollary A.2. *Any equilibrium in the policy game features $\vartheta_t \geq \vartheta^*$ for all t.*

Proof. By the previous corollary, any government j always chooses a policy that implies $\vartheta_{jT} \geq \vartheta^*$. But because this holds for all j, this means that any government $j - 1$ for $j \geq 1$ faces a terminal condition with $\vartheta_{jT} \geq \vartheta^*$. Applying again the previous corollary, we can even conclude $\vartheta_t \geq \vartheta^*$ for all $t \in [(j-1)T, jT]$. As this must hold for all $j \geq 1$, we obtain $\vartheta_t \geq \vartheta^*$ for all $t \in [0, \infty)$. □

The previous result completes the proof of the first part of Proposition 5 that $\vartheta_t \geq \vartheta^*$.

We next characterize all equilibria in the policy game in which the inequality always holds with equality, $\vartheta_t = \vartheta^*$ for all t. In this case, we only need to check under which conditions there is a feasible infinite-horizon policy path $\{\bar{s}_t\}_{t\in[0,\infty)}$ and there are valuation paths $\{\vartheta^B_t, \vartheta^C_t\}_{t\in[0,\infty)}$ for this policy consistent with ODEs (25) and (26) and $\vartheta_t := \vartheta^B_t + \vartheta^C_t = \vartheta^*$ for all t. Whenever this is the case, this automatically represents an equilibrium of the policy game because this particular path for ϑ_t is the global maximizer of the objective of any government $j = 0, 1, \ldots$ and so no government has an incentive to deviate.

If $\vartheta_t = \vartheta^*$ for all t, then ODE (26) implies for ϑ_t^C

$$\dot{\vartheta}_t^C = (f(\vartheta^*) + \breve{\mu}^C)\vartheta_t^C =: \alpha \vartheta_t^C.$$

This is a linear ODE with the general solution

$$\vartheta_t^C = \vartheta_0^C e^{\alpha t},$$

where $\vartheta_0^C \in \mathbb{R}$ parameterizes the possible solutions. All solutions with $\vartheta_0^C < 0$ violate the nonnegativity requirement and are therefore not associated with equilibria in the policy game. To determine which solutions with $\vartheta_0^C \geq 0$ are possible in the policy game, we consider two cases separately:

(i) If $\breve{\mu}^C > \breve{\mu}^*$, then $\alpha > 0$ and thus any solution with $\vartheta_0^C > 0$ generates a ϑ^C path that grows without bounds. Eventually, such a solution implies $\vartheta_t^C > \vartheta^* = \vartheta_t$ and thus $\vartheta_t^B < 0$. Such a solution does not correspond to an equilibrium. Consequently, only the solution with $\vartheta_0^C = 0$, i.e. $\vartheta_t^C = 0$ for all $t \geq 0$ remains. This solution implies $\vartheta_t^B = \vartheta_t = \vartheta^*$ and this represents indeed a valid equilibrium in the policy game.

(ii) If $\breve{\mu}^C \leq \breve{\mu}^*$, then $\alpha \leq 0$. For any initial condition $\vartheta_0^C \in [0, \vartheta^*]$, the implied path satisfies $\vartheta_t^C \in [0, \vartheta^*]$ for all $t \geq 0$. This implies $\vartheta_t^B = \vartheta^* - \vartheta_t^C \in [0, \vartheta^*]$ for all $t \geq 0$. All of these solutions are valid solutions in the policy game.

To complete the proof of Proposition 5, we only need to show that in the case $\breve{\mu}^C > \breve{\mu}^*$ (case (i) above), there can be also no other equilibria in the policy game (so far we have only shown there can be none satisfying $\vartheta_t = \vartheta^*$). Such other equilibria must necessarily feature $\vartheta_t > \vartheta^*$ for some time t. But it is easy to see that this cannot be possible:

First, we show that there is no equilibrium with $\vartheta_t^C > 0$ for any $t \geq 0$. The proof is largely identical to the one in case (i) above. We know already that $\vartheta_t \geq \vartheta^*$ in any equilibrium and thus with α defined as previously,

$$\dot{\vartheta}_t^C \geq \alpha \vartheta_t^C.$$

Therefore, any solution must satisfy $\vartheta_{t_1}^C \geq \vartheta_{t_0}^C e^{\alpha(t_1 - t_0)}$ for any $t_0 < t_1$. If $\vartheta_{t_0}^C > 0$, then, because $\alpha > 0$, there is some t_1 such that $\vartheta_{t_1}^C > 1$, violating $\vartheta_{t_1}^B + \vartheta_{t_1}^C \leq 1$. Consequently, $\vartheta_{t_0}^C = 0$ is the only possibility. Because t_0 was arbitrary, any equilibrium must feature $\vartheta_t^C = 0$ for all $t \geq 0$. For the remainder of the proof, we therefore identify ϑ_t and ϑ_t^B and remark that ϑ_t is determined by ODE (25).

Second, restricting attention to the situation $\vartheta = \vartheta^B$, we show that $\vartheta_t > \vartheta^*$ at any time t is impossible in any equilibrium of the policy game. We proceed in two steps:

1. If any government j faces a terminal condition $\vartheta_{(j+1)T} = \vartheta^*$, then it is feasible and (strictly) optimal to implement $\vartheta_t = \vartheta^*$ for all $t \in [jT, (j+1)T]$. Feasibility follows from the previous discussion, optimality from the structure of the government's objective. Therefore, the observation $\vartheta_t > \vartheta^*$ for some t cannot result from the policy of any government that faces the terminal condition $\vartheta_{(j+1)T} = \vartheta^*$.

2. We show next that if any government j was to face a terminal condition $\vartheta_{(j+1)T} > \vartheta^*$, its optimal policy would still imply $\vartheta_{jT} = \vartheta^*$ at the beginning of its period of office, so that the previous government $j-1$ would face the terminal condition $\vartheta_{jT} = \vartheta^*$. But if this holds for all j, then a terminal condition $\vartheta_{(j+1)T} > \vartheta^*$ cannot actually be the outcome of any equilibrium.

   We prove this last result by contradiction and assume w.l.o.g. that $j = 0$. Suppose the optimal response of the government to a terminal condition $\vartheta_T > \vartheta^*$ was leading to a price path $\{\vartheta_t\}_{t\in[0,T]}$ such that $\vartheta_0 > \vartheta^*$. We construct a feasible alternative price path $\{\vartheta'_t\}_{t\in[0,T]}$ that generates higher welfare than $\{\vartheta_t\}_{t\in[0,T]}$ contradicting the assumption that the latter is the result of an optimal policy choice by the government.

   For the construction of $\{\vartheta'_t\}_{t\in[0,T]}$, let $\bar{s}_t$ denote the optimal policy chosen by the government. For $x > 0$, define $\bar{s}''_t := \bar{s}_t - x$. Consider the solution ϑ''_t to ODE (25) under this policy with the terminal condition $\vartheta''_T = \vartheta_T$. Clearly, that solution satisfies $\vartheta''_t < \vartheta_t$ for all $t < T$.[79] Also, if x is sufficiently large, ϑ''_t must cross ϑ^* in the interval $[0, T]$. We define the last crossing time,

$$t_0 := \sup\{t \in [0, T] \mid \vartheta''_t \leq \vartheta^*\}.$$

   By continuity of ϑ'' and $\vartheta'' = \vartheta_T$, $\vartheta''_{t_0} = \vartheta^*$ whenever $t_0 > 0$. Clearly t_0 is increasing in x and $t_0 \to T$ as $x \to \infty$. Consequently, for any given $\varepsilon > 0$, we can choose $x > 0$ sufficiently large such that $t_0 > T - \varepsilon$. Choose such x appropriately and define next

$$\bar{s}'_t := \begin{cases} \alpha\vartheta^*, & t < t_0 \\ \bar{s}''_t, & t \geq t_0 \end{cases},$$

   where α is defined as previously. Now let ϑ'_t be the solution to ODE (25) with terminal condition $\vartheta'_T = \vartheta_T$ and the policy $\bar{s}'$. Because the ODE coincides with the one for ϑ'' on $[t_0, T]$ and because the terminal conditions are identical, it must be that $\vartheta'_t = \vartheta''_t$ for all $t \in [t_0, T]$. Furthermore, $\vartheta'_{t_0} = \vartheta''_{t_0} = \vartheta^*$ and we have seen above that the policy $\bar{s} = \alpha\vartheta^*$ makes the right-hand side of ODE (25) vanish whenever $\vartheta_t = \vartheta^*$. Consequently, the solution ϑ' must satisfy $\vartheta'_t = \vartheta^*$ for all $t \leq t_0$.

[79] This is because the ODE's right-hand side for policy $\bar{s}''$ is strictly larger than that for $\bar{s}$ if evaluated at any given ϑ path. By ODE comparison logic, ϑ'' must thus fall faster than ϑ when moving backward in time.

The path ϑ' just constructed has the property $\vartheta'_t = \vartheta^*$ for all $t \leq T - \varepsilon$ and $\vartheta'_t \leq \bar{\vartheta} := \max_{t\in[0,T]} \vartheta_t \leq 1$ for all $t \in [T-\varepsilon, T]$. Consequently, the relevant part of the government's welfare objective under this path is

$$\begin{aligned}
\mathcal{W}\left(\{\vartheta'_t\}_{t\in[0,T]}\right) &= \int_0^T e^{-\rho t}\Psi\left(\vartheta'_t\right) dt \\
&= \int_0^{T-\varepsilon} e^{-\rho t}\Psi\left(\vartheta^*\right) dt + \int_{T-\varepsilon}^T e^{-\rho t}\Psi\left(\vartheta'_t\right) dt \\
&\geq \int_0^T e^{-\rho t}\Psi\left(\vartheta^*\right) dt - \int_{T-\varepsilon}^T e^{-\rho t}\left(\Psi\left(\bar{\vartheta}\right) - \Psi\left(\vartheta^*\right)\right) dt \\
&\geq \mathcal{W}^* - \varepsilon\left(\Psi\left(\bar{\vartheta}\right) - \Psi\left(\vartheta^*\right)\right),
\end{aligned}$$

where $\mathcal{W}^*$ denotes the global maximum of the government's objective.

On the one hand, as we have constructed such an alternative path ϑ' for any $\varepsilon > 0$, we can move the achievable welfare arbitrarily close to the global maximum $\mathcal{W}^*$ by choosing $\varepsilon > 0$ sufficiently small. On the other hand, welfare attained under the optimal policy, $\mathcal{W}\left(\{\vartheta_t\}_{t\in[0,T]}\right)$ must be strictly below $\mathcal{W}^*$ for the following reason: $\vartheta_0 > \vartheta^*$ and by continuity there must be numbers $\delta_0, \delta_1 > 0$ such that $\vartheta_t \geq \vartheta^* + \delta_0$ for all $t \leq \delta_1$. We therefore arrive at a contradiction.

A.6 Model Extension and Uniqueness Results with Aggregate Shocks

In this appendix we provide additional details that complement Section 6.2. We first outline the extended model with aggregate shocks and discuss how our uniqueness arguments from Section 4 and Appendix A.3 can be extended to this setting if the government is able to observe the underlying state. We then analyze threshold policies of the type discussed previously if the government cannot observe the state. Finally, we provide a more complex and sophisticated example policy that nevertheless achieves unique implementation of the desired equilibrium (except in a special case).

We remark that, in the interest of space, we keep this appendix more concise than previous appendices and occasionally skip over (purely technical) definitions and arguments that are in analogy to the ones presented in the deterministic model in this paper. From our presentation, it should be feasible for a reader to fill in the gaps and work out the remaining details. The space required to make them explicit, however, would easily exceed the number of pages required for Appendices A.1 and A.3 combined.

A.6.1 Model Setup and Solution

The model is identical to the baseline model presented in Section 2 with one difference: we replace the parameter $\tilde{\sigma}$ with a time-varying stochastic process $\tilde{\sigma}_t$ that may take values in $\{\tilde{\sigma}^l, \tilde{\sigma}^h\}$ with $\tilde{\sigma}^l < \tilde{\sigma}^h$. We assume that $\tilde{\sigma}_t$ follows a continuous-time Markov chain with transition rates $\lambda^l, \lambda^h > 0$, where λ^l is the transition rate from $\tilde{\sigma}^l$ to $\tilde{\sigma}^h$ and λ^h is the transition rate from $\tilde{\sigma}^h$ to $\tilde{\sigma}^l$.

State transitions in $\tilde{\sigma}_t$ can, in equilibrium, lead to jumps in asset prices q_t^B, q_t^C, and q_t^K. These jumps introduce aggregate risk into the returns that households face on the three assets. Otherwise, the household problem is identical to the one of the baseline model.

A competitive equilibrium is defined as in Definition A.1, except that time paths must be replaced with stochastic processes. We continue to assume that, except at state transition times, paths for all variables in the equilibrium definition are absolutely continuous in t.

The definition of a feasible surplus rule and the associated policy rules for τ and $\breve{\mu}^{\mathcal{B}}$ is in analogy to Definition A.2. The only difference is that, in principle, the surplus s_t at time t may not only depend on the the history $\{q_{t'}^B, q_{t'}^C, q_{t'}^K\}_{t' \leq t}$ of asset prices but also on the history of exogenous shocks.

Finally, an equilibrium consistent with a policy rule is then defined in complete analogy to Definition A.3.

We note that the model can be solved along the same lines as outlined in Section 2 and Appendix A.1.2. The HJB equation of households is largely identical, the additional jump risk terms only affect the first-order condition for portfolio choices $\theta_t^{B,i}$ and $\theta_t^{C,i}$. Because the remaining first-order conditions are unaffected, the first part of Proposition 1 holds in this model as well and the proof is identical word by word. The second part also holds if we add a conditional expectations operator $\mathbb{E}_t$ in front of the integrals in equations (5) and (6) and replace $\tilde{\sigma}_t^c = (1 - \vartheta_t)\tilde{\sigma}$ with the equation $\tilde{\sigma}_t^c = (1 - \vartheta_t)\tilde{\sigma}_t$. We briefly explain why this is the case below. Given that Proposition 1 essentially still holds in this model, the qualitative dynamics in this model can be fully understood by considering comparative statics with respect to $\tilde{\sigma}$ in our baseline model.

We now briefly explain why the dynamic equations (5) and (6) still holds even though there is aggregate jump risk (if an expectation operator is added). Specifically, note that these equations can be equivalently written in differential form as

$$\mathbb{E}_t[d\vartheta_t^B] = \left(\rho + \breve{\mu}_t^{\mathcal{B}} - (\tilde{\sigma}_t^c)^2\right) \vartheta_t^B dt, \tag{36}$$

$$\mathbb{E}_t[d\vartheta_t^C] = \left(\rho + \breve{\mu}^{\mathcal{C}} - (\tilde{\sigma}_t^c)^2\right) \vartheta_t^C dt. \tag{37}$$

Unlike the equations (22) and (23), which we have derived as the differential versions of equations (5) and (6) in the baseline model, the previous two equations are not ODEs but stochastic differential equations (SDEs). Specifically, because they only specify the dynamics of the expected rates of change $\mathbb{E}_t[d\vartheta_t^B]$ and $\mathbb{E}_t[d\vartheta_t^C]$ of the processes ϑ_t^B and ϑ_t^C, they are what is called backward SDEs (BSDEs). A solution to each of these equations consists of both a value, ϑ_t^B and ϑ_t^C, at time t and a (hypothetical) jump target, denoted by $\vartheta_t^{B,+}$ and $\vartheta_t^{C,+}$, to which the process would jump if a state transition was to occur at time t. We now motivate equation (36). The same reasoning explains why equation (37) must hold.

In Appendix A.1.2 we have derived ODE (22) by first deriving the portfolio choice condition of households with returns expressed in the usual consumption numeraire and then imposing a number of market clearing conditions and rearranging. While this is the most natural approach, there is actually a faster way to get to equation (22) directly. Following this approach avoids having to re-state the HJB equation and to follow all the steps in Appendix A.1.2 once again. For this faster way, we express returns for the portfolio choice condition in a different numeraire, in units of total wealth in the economy. In this numeraire, the return on bonds is simply $\check{\mu}_t^{\mathcal{B}} dt + d\vartheta_t^B/\vartheta_t^B$. The return on household i's total wealth portfolio is $\hat{c}_t^i dt + \tilde{\sigma}_t^n d\tilde{Z}_t^i$, where $\tilde{\sigma}_t^n$ is the idiosyncratic risk loading of the household's net worth. We know that $\tilde{\sigma}_t^n = \tilde{\sigma}_t^c$ because of the consumption rule $c_t^i = \rho n_t^i$ (which holds regardless of numeraire). The reason for this return is that, in equilibrium, all households make the same choices so that $n_t^i / \int n_t^j dj$ only moves because of idiosyncratic shocks. The previous two return expressions are valid regardless of whether there are jumps in asset prices or not. The portfolio choice condition for bonds, expressed in this way, is

$$\mathbb{E}_t \left[\check{\mu}_t^{\mathcal{B}} dt + d\vartheta_t^B / \vartheta_t^B - \left(\hat{c}_t^i dt + \tilde{\sigma}_t^c d\tilde{Z}_t^i \right) \right] = \text{required risk premium}$$

The required risk premium, in turn, when expressed in the total net worth numeraire, only includes a compensation for idiosyncratic risk in net worth. Why? Because of the consumption rule $c_t^i = \rho n_t^i$ (which holds regardless of the numeraire), the agent's consumption is always proportional to net worth and, in this numeraire, has the evolution $\tilde{\sigma}_t^n d\tilde{Z}_t^i = \tilde{\sigma}_t^c d\tilde{Z}_t^i$. There is no jump risk in consumption relative to total wealth because all agents experience the same relative appreciation or depreciation of their wealth (and consumption) in a state transition. Hence, the required risk premium is simply $-(\tilde{\sigma}_t^c)^2$. Substituting this into the previous portfolio choice condition and rearranging yields equation (36).

A.6.2 The Valuation ODEs

As in Appendix A.3.1, we can derive two ODEs for ϑ_t^B and ϑ_t^C as necessary conditions in any equilibrium. These ODEs take the place of ODEs (25) and (26) in a result analogous to Lemma A.1 in this augmented model.

We start with the BSDEs (36) and (37). We remark that, as in the deterministic model, equation (36) is not entirely precise in the special case $\vartheta_t^B = 0$. The precise conditions are the stochastic variants of ODEs (25) and (26):

$$\mathbb{E}_t[d\vartheta_t^B] = \left(f(\vartheta_t^B + \vartheta_t^C, \tilde{\sigma}_t)\vartheta_t^B - \bar{s}_t \right) dt,$$
$$\mathbb{E}_t[d\vartheta_t^C] = \left(f(\vartheta_t^B + \vartheta_t^C, \tilde{\sigma}_t) + \breve{\mu}_t^C \right) \vartheta_t^C dt,$$

where the function f is defined by

$$f(\vartheta, \tilde{\sigma}) = \rho - (1 - \vartheta)^2 \tilde{\sigma}^2$$

and satisfies, for any fixed second argument, the properties in Fact A.1. With the previous BSDEs in place of equations (25) and (26), Lemma A.1 continues to hold in this model variant with aggregate shocks.

Because stochastic variation only occurs at discrete state transition times, ϑ_t^B and ϑ_t^C follow "most of the time" deterministic time paths that can be equivalently described by ODEs like in earlier parts of the paper. To transform the two BSDEs into ODEs, we define the following notation. For any process x_t whose paths are continuously differentiable except at state transition times, define by $\dot{x}_t$ the time derivative *conditional on no state transition*. Recall also that we denote by x_t^+ the (hypothetical) jump target conditional on a state transition happening at time t. With this notation, $\mathbb{E}_t[dx_t]/dt = \dot{x}_t + \lambda_t(x_t^+ - x_t)$, where λ_t is the transition intensity at time t (i.e. $\lambda_t = \lambda^l 1_{\{\tilde{\sigma}_t = \tilde{\sigma}^l\}} + \lambda^h 1_{\{\tilde{\sigma}_t = \tilde{\sigma}^h\}}$).

Applying these definitions to the previous BSDEs for ϑ^B and ϑ^C yields the ODEs

$$\dot{\vartheta}_t^B = f(\vartheta_t^B + \vartheta_t^C, \tilde{\sigma}_t)\vartheta_t^B - \bar{s}_t - \lambda_t(\vartheta_t^{B,+} - \vartheta_t^B) \tag{38}$$
$$\dot{\vartheta}_t^C = \left(f(\vartheta_t^B + \vartheta_t^C, \tilde{\sigma}_t) + \breve{\mu}_t^C \right) \vartheta_t^C - \lambda_t(\vartheta_t^{C,+} - \vartheta_t^C) \tag{39}$$

These ODEs are necessary equilibrium conditions that have to hold (almost surely) for any equilibrium path.[80]

All uniqueness arguments in this appendix follow precisely the same strategy as in Ap-

[80] We remark that the ODEs are not an equivalent representation of the BSDEs as they lack explicit conditions for the determination of the jump targets $\vartheta^{B,+}$ and $\vartheta^{C,+}$. For our uniqueness proofs, such explicit conditions are not required.

pendix A.3. There is an added difficulty in how to bound the additional jump terms. We illustrate how this can be done explicitly in Appendix A.6.6. Similar arguments are required throughout to translate the proofs from Appendix A.3 to the present setting.

A.6.3 Threshold Policies when the State is Observable

We start by investigating threshold policies when the state $\tilde{\sigma}_t$ can be observed by the government. Specifically, we restrict attention to policies (in terms of $\bar{s}$) of the form

$$\bar{s}_t = \begin{cases} -\breve{\mu}^{\mathcal{B}}(\vartheta_t^B, \tilde{\sigma}_t)\vartheta_t^B, & \vartheta_t^B \geq \underline{\vartheta}(\tilde{\sigma}_t) \\ sh(\vartheta_t^B + \vartheta_t^C), & \vartheta_t^B < \underline{\vartheta}(\tilde{\sigma}_t) \end{cases}, \tag{40}$$

where h denotes the same function as in Appendix A.3 and $\breve{\mu}^{\mathcal{B}}(\cdot,\cdot)$ is a continuous function that is nonnegative everywhere and weakly increasing in its first argument.

Define $\vartheta^{B*,l}$, $\vartheta^{B*,h}$ as the solution to the two "steady state" equations

$$0 = \left(f(\vartheta^{B*,l}, \tilde{\sigma}^l) + \breve{\mu}^{\mathcal{B}}(\vartheta^{B*,l})\right)\vartheta^{B*,l} - \lambda^l(\vartheta^{B*,h} - \vartheta^{B*,l}), \tag{41}$$

$$0 = \left(f(\vartheta^{B*,h}, \tilde{\sigma}^h) + \breve{\mu}^{\mathcal{B}}(\vartheta^{B*,h})\right)\vartheta^{B*,h} - \lambda^h(\vartheta^{B*,l} - \vartheta^{B*,h}). \tag{42}$$

Using standard monotonicity arguments, it is easy to establish that at most one solution to these equations with $\vartheta^{B*,l}, \vartheta^{B*,h} \in (0,1)$ can exist. If $\tilde{\sigma}^l$, $\tilde{\sigma}^h$ are large enough, such a solution does indeed exist. We assume in the following always that this is the case. As stated in the main text, we also assume that parameters are such that $\vartheta^{B*,l} < \vartheta^{B*,h}$. In what follows, we denote by $\breve{\mu}^{\mathcal{B},l}$ and $\breve{\mu}^{\mathcal{B},h}$ the implied values for $\breve{\mu}_t^{\mathcal{B}}$ in the low and high state, respectively, that is

$$\breve{\mu}^{\mathcal{B},j} := \breve{\mu}^{\mathcal{B}}(\vartheta^{B*,j}, \tilde{\sigma}^j), \qquad j \in \{l, h\}.$$

Our key result is that, when the state is observable, so that any policy of the form (40) is feasible, then a conclusion in analogy to Proposition 4 holds:

Proposition A.1. *Under the threshold policy* (40) *(with* $s > 0$*), all equilibria have the property that* $\vartheta_t^B \geq \underline{\vartheta}_t$ *for all* t *almost surely. If* $\underline{\vartheta} = \vartheta^{B*}$ *or* $\breve{\mu}^{\mathcal{C}} > \max\{\breve{\mu}^{\mathcal{B},l}, \breve{\mu}^{\mathcal{B},h}\}$*, the equilibrium is unique and satisfies* $\vartheta_t^B = \vartheta_t = \vartheta_t^{B*}$ *for all* t *almost surely.*

The proof of this result is analogous to the one presented in Appendix A.3.4. Again, most of the technical details are established in the following lemmas that are analogous to Lemmas A.2 and A.3.

Lemma A.10. *In any equilibrium consistent with the threshold policy (40), $\vartheta_t^B \geq \underline{\vartheta}_t$ for all t almost surely.*

Lemma A.11. *In any equilibrium consistent with the threshold policy (40), $\vartheta_t^B \leq \vartheta_t^{B*}$ for all t almost surely.*
In addition, if $\vartheta_{t_0}^B = \vartheta_{t_0}^{B*}$ at some time t_0 (after some exogenous history), then, conditional on this event, necessarily $\vartheta_t^B = \vartheta_t^{B*}$ and $\vartheta_t^C = 0$ for all $t \geq t_0$.

We provide a proof of Lemma A.11 in Appendix A.6.6. The proof of the main statement follows the same logic as the proof of Lemma A.3 in Appendix A.3.4. However, unlike the short proof of the latter result, the proof in Appendix A.6.6 is considerably longer as additional technical considerations are necessary to deal with the jump terms. We therefore present the details of this proof explicitly to illustrate how to handle these additional terms.

We omit the proof of Lemma A.10. Its logic follows exactly the same lines as the proof of Lemma A.2 presented in Appendix A.3.5. When following those arguments, additional auxiliary considerations are necessary to deal with the jump terms. Those, in turn, work in the same way as in the proof of Lemma A.11 presented in Appendix A.6.6.

Proof of Proposition A.1. The first part of the proposition follows immediately from Lemma A.10. The second part in the case $\underline{\vartheta} = \vartheta^{B*}$ follows by combining Lemmas A.10 and A.11.

For the second part in the case $\breve{\mu}^{\mathcal{C}} > \max\{\breve{\mu}^{\mathcal{B},l}, \breve{\mu}^{\mathcal{B},h}\}$, the proof needs to be adapted. Note first that this inequality implies $\breve{\mu}^{\mathcal{C}} > \breve{\mu}_t^{\mathcal{B}}$ along any equilibrium path because $\breve{\mu}^{\mathcal{B}}(\cdot)$ is weakly increasing and $\vartheta_t^B \leq \vartheta^{B*}$ by Lemma A.11.

We use that equations (5) and (6) from Proposition 1 still hold in this model if a conditional expectation $\mathbb{E}_t$ is added to the right-hand sides (compare the discussion in Section A.6.1). It is easy to see that $\vartheta_t^C > 0$ in alternative equilibria, at least in some states. Suppose there is such an alternative equilibrium with $\vartheta_t^C > 0$. Then due to $\vartheta_t^C < 1$ and $\vartheta_t^B \geq \underline{\vartheta} > 0$, the ratio $x_t := \vartheta_t^C / \vartheta_t^B$ must be bounded above. Let $\bar{x}$ be the supremum and suppose it is attained at t_0 (after some exogenous history). Then

$$
\begin{aligned}
\vartheta_{t_0}^C &= \mathbb{E}_{t_0} \int_{t_0}^{\infty} e^{-\rho(t-t_0)} \left(\tilde{\sigma}_t^c - \breve{\mu}^{\mathcal{C}}\right) \vartheta_t^C dt \\
&< \mathbb{E}_{t_0} \int_{t_0}^{\infty} e^{-\rho(t-t_0)} \left(\tilde{\sigma}_t^c - \breve{\mu}_t^{\mathcal{B}}\right) \vartheta_t^C dt \\
&= \mathbb{E}_{t_0} \int_{t_0}^{\infty} e^{-\rho(t-t_0)} \left(\tilde{\sigma}_t^c - \breve{\mu}_t^{\mathcal{B}}\right) \vartheta_t^B x_t dt \\
&\leq \bar{x}\mathbb{E}_{t_0} \int_{t_0}^{\infty} e^{-\rho(t-t_0)} \left(\tilde{\sigma}_t^c - \breve{\mu}_t^{\mathcal{B}}\right) \vartheta_t^B x_t dt = \bar{x}\vartheta_{t_0}^B
\end{aligned}
$$

and hence

$$x_{t_0} = \frac{\vartheta^C_{t_0}}{\vartheta^B_{t_0}} < \bar{x}$$

in contradiction that the supremum is attained at t_0.[81] If the supremum is not attained anywhere, an additional ε-argument can be used to arrive at the same conclusion. □

A.6.4 Threshold Policies when the State is Unobservable

When the state is unobservable, not every threshold policy of the type (40) is feasible. Specifically, $\underline{\vartheta}$ may no longer be a function of $\tilde{\sigma}_t$ but must be a constant. In addition, the function $\breve{\mu}^{\mathcal{B}}$ may only depend on ϑ^B but not on the second argument $\tilde{\sigma}_t$. Any threshold policy of the type (40) that satisfies these additional restrictions is still feasible when the state is unobservable.[82]

Definition A.4. We say that a threshold policy of the type (40) is *feasible for an unobserved state* if the functions $\breve{\mu}^{\mathcal{B}}$ and $\underline{\vartheta}$ do not depend (explicitly) on the argument $\tilde{\sigma}_t$.

If we choose such a feasible threshold policy, then Proposition A.1 continues to hold, even if the state is unobservable. This leads immediately to the following two conclusions:

1. If $\max\{\breve{\mu}^{\mathcal{B},l}, \breve{\mu}^{\mathcal{B},h}\} < \breve{\mu}^{\mathcal{C}}$, a threshold policy with any (arbitarily small) constant threshold $\underline{\vartheta} < \vartheta^{B*,l}$ can be used to uniquely implement the equilibrium that satisfies $\vartheta^B = \vartheta^{B*}$ and $\vartheta^C \equiv 0$. It is not necessary to observe the state and adapt the threshold in a state-contingent way.

   Note that when implementing this equilibrium, $\breve{\mu}^{\mathcal{B}}_t$ *in equilibrium* will typically depend on $\tilde{\sigma}_t$ (unless $\breve{\mu}^{\mathcal{B},l} = \breve{\mu}^{\mathcal{B},h}$). This is feasible because the observed equilibrium price ϑ^B_t reveals the state. The function $\breve{\mu}^{\mathcal{B}}$ does not need to depend explicitly on $\tilde{\sigma}_t$ to implement this but only on the observed equilibrium object ϑ^B_t.

2. If $\breve{\mu}^{\mathcal{B},l} < 0$, so that the government plans to run positive primary surpluses in the low-risk state, then a suitable threshold policy can be constructed with constant threshold $\underline{\vartheta} = \vartheta^{B*,h}$ that uniquely selects the equilibrium that satisfies $\vartheta^B = \vartheta^{B*}$ and $\vartheta^C \equiv 0$. The government simply chooses the constant surplus s below the threshold so that it equals the desired surplus in the low-risk state on the equilibrium path, $s = -\frac{\breve{\mu}^{\mathcal{B},l}\vartheta^{B*,l}}{h(\vartheta^{B*,l})}$.

In these cases, suitably designed threshold policies are therefore still sufficient to select the desired stationary equilibrium in which only government bonds have a positive value.

[81] Note that this argument only works if $\vartheta^C_{t_0} > 0$ and hence $\vartheta^C_t > 0$ on a set of positive measure for some $t \geq t_0$. Otherwise there is no strict inequality in the second line.

[82] Note that the unobservable ϑ^C_t still enters the function h, which may appear problematic. However, this is not problematic because the government does not choose $\bar{s}_t$ but the constant surplus s. The function h simply captures how the mathematically more convenient variable $\bar{s}_t$ adjusts in response in any equilibrium.

In the remaining cases, this may not be possible. We discuss in the following the most interesting remaining case in which both $\breve{\mu}^{\mathcal{B},l}$ and $\breve{\mu}^{\mathcal{B},h}$ are positive and at least as large as $\breve{\mu}^{\mathcal{C}}$. In this case, we were not able to find a threshold policy of the type (40) that is feasible for an unobserved state and always selects the desired equilibrium $\vartheta^B = \vartheta^{B*}$, $\vartheta^C = 0$ in all contingencies. In fact, we conjecture that, without additional parameter restrictions, no such policy exists. Instead, we discuss two natural choices of threshold policies that achieve the objective at least partially. The first choice imposes a "high" taxation threshold $\underline{\vartheta} = \vartheta^{B*,h}$ and succeeds in selecting a unique equilibrium, but at the cost of equilibrium taxation in the low-risk state which raises ϑ^B in that state above the desired target $\vartheta^{B*,l}$. The second choice imposes a "low" taxation threshold $\underline{\vartheta} = \vartheta^{B*,l}$ and therefore always remains consistent with the desired equilibrium. However, it does not always succeed in selecting this equilibrium uniquely.

We start with the first policy that chooses a "high" threshold $\underline{\vartheta} = \vartheta^{B*,h}$. Below this threshold, the government chooses a constant surplus-capital ratio of $s > 0$. We denote the resulting stationary equilibrium solution (under the condition $\vartheta^C \equiv 0$) for ϑ^B by ϑ^{B**}. We assume that the government adjusts $\breve{\mu}^{\mathcal{B}}$ in the high-risk state to a value above $\breve{\mu}^{\mathcal{B},h}$, so that the equilibrium value of ϑ^B remains at $\vartheta^{B*,h}$ in that state, i.e. $\vartheta^{B**,h} = \vartheta^{B*,h}$. The equilibrium value for $\vartheta^{B**,l}$ in the low-risk state is determined by the equation

$$0 = f(\vartheta^{B**,l}, \tilde{\sigma}^l)\vartheta^{B**,l} - sh(\vartheta^{B**,l}) - \lambda^l(\vartheta^{B*,h} - \vartheta^{B**,l}). \tag{43}$$

Comparing this with equation (41), it is easy to see that $\vartheta^{B**,l} > \vartheta^{B*,l}$. Hence, the government fails in implementing the desired equilibrium values for ϑ^B. However, it is easy to see from Proposition A.1 that the equilibrium is unique. We thus obtain the following proposition:

Proposition A.2. *Under the parameter assumptions made in the previous paragraphs and if the government chooses a threshold policy with a "high" threshold, then there is a unique equilibrium. This equilibrium features $\vartheta^B = \vartheta^{B**}$ and $\vartheta^C \equiv 0$, where $\vartheta^{B**,h} = \vartheta^{B*,h}$ and $\vartheta^{B**,l} > \vartheta^{B*,l}$ solves equation (43). $\vartheta^{B**,l}$ is strictly increasing in the surplus s.*

Next, consider a policy that chooses a "low" threshold $\underline{\vartheta} = \vartheta^{B*,l}$ and, above that threshold, picks a functional form for $\breve{\mu}^{\mathcal{B}}$ that is consistent with the desired equilibrium (e.g. the affine linear function that satisfies $\breve{\mu}^{\mathcal{B}}(\vartheta^{B*,l}) = \breve{\mu}^{\mathcal{B},l}$ and $\breve{\mu}^{\mathcal{B}}(\vartheta^{B*,h}) = \breve{\mu}^{\mathcal{B},h}$). Clearly, $\vartheta^B = \vartheta^{B*}$, $\vartheta^C \equiv 0$ is one equilibrium solution under this policy.

While this may not be the unique equilibrium solutions, we can rule out some alternative solution paths as equilibria. Specifically, Proposition A.1 implies that no equilibrium can exist for which $\vartheta_t^B < \underline{\vartheta} = \vartheta^{B*,l}$ with positive probability. Combining this with Lemma A.11 allows us to conclude even more:

Lemma A.12. *Suppose the parameter assumptions made in this subsection hold and the government* *chooses a threshold policy with a "low" threshold as just described. Let $\tau_0 \geq 0$ be the (random) time at which the low-risk state $\tilde{\sigma}^l$ is visited for the first time. Then $\vartheta_t^B = \vartheta_t^{B*}$ and $\vartheta_t^C = 0$ for all $t \geq \tau_0$ in any equilibrium.*

Proof. Proposition A.1 implies that $\vartheta_t^B \geq \vartheta^{B*,l}$ (a.s.) at all times in any equilibrium. Lemma A.11 also implies the opposite inequality $\vartheta_t^B \leq \vartheta^{B*,l}$ at all times t such that $\tilde{\sigma}_t = \tilde{\sigma}^l$. Consequently, in all equilibria and for any time $t_0 \geq 0$ (and any path), $\tilde{\sigma}_{t_0} = \tilde{\sigma}^l$ implies $\vartheta_{t_0}^B = \vartheta_{t_0}^{B*}$. By the additional statement in Lemma A.11, we can then conclude even that $\tilde{\sigma}_{t_0} = \tilde{\sigma}^l$ implies $\vartheta_t^B = \vartheta_t^{B*}$ and $\vartheta_t^C = 0$ for all $t \geq t_0$, that is after the full continuation path following time t_0. The assertion follows immediately by choosing for each path the smallest t_0 possible, i.e. the realization of τ_0 for that path. □

The immediate consequence from the previous lemma is that, while the "low" threshold policy may fail to achieve global uniqueness, continuation equilibria are unique once the low-risk state is visited for the first time. At this point, all potential cryptocoin bubbles must burst forever. A potential cryptocoin bubble can therefore only exist if the initial state is $\tilde{\sigma}^h$ before time τ_0.

A more subtle additional conclusion is that on the event $\{\tau_0 > 0\}$, even in the initial high state, a cryptocoin bubble is harder to sustain because it must compensate its holders for the anticipated eventual burst at time τ_0. Specifically, the ODEs (38) and (39) before time τ_0 become[83]

$$\dot{\vartheta}_t^B = \left(f(\vartheta_t^B + \vartheta_t^C, \tilde{\sigma}^h) + \lambda^h + \breve{\mu}^{\mathcal{B}}(\vartheta_t^B) - \lambda^h \frac{\vartheta^{B*,l}}{\vartheta_t^B} \right) \vartheta_t^B, \tag{44}$$

$$\dot{\vartheta}_t^C = \left(f(\vartheta_t^B + \vartheta_t^C, \tilde{\sigma}^h) + \lambda^h + \breve{\mu}^{\mathcal{C},h} \right) \vartheta_t^C. \tag{45}$$

This looks like our deterministic model without state transitions. The effective dilution rate of cryptocoins is constant and given by $\breve{\mu}^{\mathcal{C},h} + \lambda^h$. The effectively dilution rate of bonds is not constant but increasing in ϑ_t^B. The largest possible value is

$$\breve{\mu}^{\mathcal{B}}(\vartheta^{B*,h}) + \lambda^h - \lambda^h \frac{\vartheta^{B*,l}}{\vartheta^{B*,h}} = \breve{\mu}^{\mathcal{B},h} + \lambda^h - \lambda^h \frac{\vartheta^{B*,l}}{\vartheta^{B*,h}}.$$

Note that this is always smaller than $\breve{\mu}^{\mathcal{B},h} + \lambda^h$ because, unlike cryptocoins, bonds do not lose their full value at time τ_0. Therefore, even if $\breve{\mu}^{\mathcal{B},h} > \breve{\mu}^{\mathcal{C},h}$, as we have assumed, it may still be the case that the effective dilution rate of bonds is always below that of cryptocoins. In this case, arguments in complete analogy to the ones we have made in the proofs of Propositions 4 and A.1 imply that, whenever $\vartheta_0^C > 0$, a solution to the ODEs has the feature that $\vartheta_t^C / \vartheta_t^B$ must grow

[83] We have also slightly rearranged the equations and restricted attention to $\vartheta_t^B \geq \underline{\vartheta} = \vartheta^{B*,l}$ as we already know that any values below are not valid equilibrium solutions.

over time. This is only possible if ϑ_t^B falls below the threshold $\underline{\vartheta}$ in finite time. With positive probability, this finite time is before τ_0, so that this cannot happen on any equilibrium path. Hence, we have established the following proposition:

Proposition A.3. *Suppose the parameter assumptions made in this subsection hold and the government chooses a threshold policy with a "low" threshold as just described. Let $\tau_0 \geq 0$ be the (random) time at which the low-risk state $\tilde{\sigma}^l$ is visited for the first time. Then the following statements are true:*

(i) There is one equilibrium such that $\vartheta^B = \vartheta^{B*}$ and $\vartheta^C \equiv 0$ ("desired equilibrium").

(ii) All equilibria coincide with the desired equilibrium after time τ_0.

(iii) Under the additional condition

$$\breve{\mu}^{\mathcal{C}} + \lambda^h > \breve{\mu}^{\mathcal{B},h} + \lambda^h - \lambda^h \frac{\vartheta^{B*,l}}{\vartheta^{B*,h}}$$

all equilibria coincide with the desired equilibrium also before time τ_0.

A.6.5 A Policy that Delivers Uniqueness with Unobservable State

We continue to assume that parameter are such that $0 < \vartheta^{B*,l} < \vartheta^{B*,h}$, $\breve{\mu}^{\mathcal{B},l}, \breve{\mu}^{\mathcal{B},h} > 0$ and $\breve{\mu}^{\mathcal{B},j} \geq \breve{\mu}^{\mathcal{C}}$ for at least one $j \in \{l, h\}$. As we have discussed in the previous section, threshold policies may then fail to uniquely implement the desired equilibrium $\vartheta^B = \vartheta^{B*}$, $\vartheta^C \equiv 0$. Propositions A.2 and A.3 nevertheless show that such policies can partially achieve the desired goal. A "high" threshold policy as in Propositions A.2 can deliver uniqueness and the desired value for ϑ^B in the high-risk state. A "low" threshold policy as in Propositions A.3 is consistent with the desired equilibrium and delivers uniqueness in the low-risk state. However, it may fail to deliver uniqueness before the first occurrence of the low-risk state if $\breve{\mu}^{\mathcal{B},h} - \breve{\mu}^{\mathcal{C}}$ is too large.

In this section, we show how a richer strategy that combines both types of threshold policies can uniquely implement the desired equilibrium. The idea is to follow the "low" threshold policy as in Proposition A.3 whenever the history of the observed ϑ_t^B-path is not obviously inconsistent with the desired equilibrium but switch to the "high" threshold policy as in Propositions A.2 that delivers global uniqueness if an inconsistent observation ever occurs. As we show below in more detail, if the "low" threshold policy was followed forever, any alternative equilibrium would necessarily lead to values for ϑ_t^B outside the set $\{\vartheta^{B*,l}, \vartheta^{B*,h}\}$ eventually, i.e. values that are obviously inconsistent with the desired equilibrium. Hence, coordination on any alternative equilibrium would trigger a switch to the "high" threshold policy. But second, under the latter policy, the equilibrium is necessarily unique and involves $\vartheta^B = \vartheta^{B*,h}$ in the high-risk state. If $\vartheta^B \neq \vartheta^{B*,h}$ right before the switch, the switch would generate infinite capital gains or losses to bond holders, so that a switch in the high-risk state cannot become a self-fulfilling alternative equilibrium. With a small modification to the switching trigger, one can also ensure that the expectation of a switch in the low-risk state can never be self-fulfilling. Then, the only equilibrium that remains is the desired one in which the government never sees any reason to switch to the "high" threshold policy.

We now provide some formal details that clarify the previous arguments. Formally, the policy is a rule of the form (40) that is feasible for an unobserved state, except that $\underline{\vartheta}$ is not constant but depends on a switching indicator $\omega_t \in \{0, 1\}$ as follows:

$$\underline{\vartheta}(\omega_t) = (1 - \omega_t)\vartheta^{B*,l} + \omega_t \vartheta^{B*,h}.$$

ω_t, in turn, depends on the full history $\{\vartheta_s^B\}_{s \le t}$ observed up to time t. We define it below. As for all threshold policies considered in this paper, the parameter $s > 0$ may be chosen arbitrarily so long as it is positive. The function $\breve{\mu}^{\mathcal{B}}$ also depend on ω_t. For $\omega_t = 1$, only the value assumed at $\vartheta^B = \vartheta^{B*,h}$ matters. This must be chosen larger than $\breve{\mu}^{\mathcal{B},h}$ such that for the unique equilibrium solution ϑ^{B**} the condition $\vartheta^{B**,h} = \vartheta^{B*}$ holds, as in Proposition A.2. For $\omega_t = 0$, $\breve{\mu}^{\mathcal{B}}$ can be any continuous and weakly increasing function such that $\breve{\mu}^{\mathcal{B}}(\vartheta^{B*,l}) = \breve{\mu}^{\mathcal{B},l}$ and $\breve{\mu}^{\mathcal{B}}(\vartheta^{B*,h}) = \breve{\mu}^{\mathcal{B},h}$, as required in Proposition A.3. In addition, if $\vartheta^{B**,l} \neq \vartheta^{B*,h}$, we also require that $\breve{\mu}^{\mathcal{B}}(\vartheta^{B**,l}) \neq \breve{\mu}^{\mathcal{C}} + \lambda^h \frac{\vartheta^{B*,l}}{\vartheta^{B**,l}}$ for reasons that become apparent in the proof of the following proposition. Note that due to the condition $\vartheta^{B**,l} \neq \vartheta^{B*,l}$, this additional requirement is never in conflict with the previously stated properties of $\breve{\mu}^{\mathcal{B}}$ and can therefore always be achieved, possibly by perturbing a given $\breve{\mu}^{\mathcal{B}}$-function slightly around the point $\vartheta^{B**,l}$. Finally, the indicator ω_t is defined as

$$\omega_t = \begin{cases} 0, & \forall s \le t : \vartheta_s^B \in \{\vartheta^{B*,l}, \vartheta^{B*,h}, \vartheta^{B**,l}\} \\ 1, & \exists s \le t : \vartheta_s^B \notin \{\vartheta^{B*,l}, \vartheta^{B*,h}, \vartheta^{B**,l}\} \end{cases}.$$

Note that we also include $\vartheta^{B**,l}$ in the set of observations that prevent switching.[84]

The previous paragraph fully specifies the proposed policy rule up to irrelevant off-equilibrium specifications. We now prove that this policy rule indeed delivers uniqueness (and implements the desired equilibrium):

Proposition A.4. *Under the parameter assumptions of this section and the additional assumption* $\breve{\mu}^{\mathcal{B},l} \neq \breve{\mu}^{\mathcal{C},h} + \lambda^h$*, the policy rule just defined implements a unique equilibrium. The equilibrium satisfies* $\vartheta^B = \vartheta^{B*}$ *and* $\vartheta^C \equiv 0$ *as well as* $\omega \equiv 0$ *at all times almost surely.*

Proof. We first show that there cannot be an equilibrium in which ω ever switches to 1 (with

[84] Even though such an observation is typically obviously inconsistent with the desired equilibrium (unless by chance $\vartheta^{B**,l} = \vartheta^{B*,h}$), switching to $\omega_t = 1$ at such a value may lead to an alternative equilibrium, in which, after a transition to the low-risk state, ϑ^{B*} does not jump to $\vartheta^{B*,l}$ but to $\vartheta^{B**,l}$.

positive probability). To do so, consider any given equilibrium path and suppose ω switches to 1 at some time t_1. Proposition A.2 then implies that there is a unique continuation equilibrium for all $t \geq t_1$ that features $\vartheta_t^B = \vartheta_t^{B**}$. Note that by definition of ω, it cannot be that $\vartheta_{t_1-}^B$ (the left limit of values just before t_1) is in the set $\{\vartheta^{B**,l}, \vartheta^{B*,h}\} = \{\vartheta^{B**,l}, \vartheta^{B**,h}\}$. Hence, $\vartheta_{t_1}^B \neq \vartheta_{t_1-}^B$. If this holds with inequality $>$, then agents just before time t_1 have an unbounded bond demand because the capital gains rate becomes locally ∞. If this holds with inequality $<$, then agents just before time t_1 seek to take an unbounded short position in bonds because the capital gains rate becomes locally $-\infty$. In both cases, the bond market cannot clear so that this cannot occur on an equilibrium path (except perhaps on a set of measure zero).

The previous argument establishes $\omega \equiv 0$ in any equilibrium. We are therefore in the situation of Proposition A.3 with the additional requirement that any equilibrium consistent with that proposition must also satisfy $\vartheta_t^B \in \{\vartheta^{B*,l}, \vartheta^{B*,h}, \vartheta^{B**,l}\}$ at all times (almost surely) to prevent ω from switching to 1. We show next that $\vartheta^B = \vartheta^{B*}$, $\vartheta^C \equiv 0$ is the only possibility.

First of all, note that we may restrict attention to times before τ_0, where again τ_0 denotes the first time the low-risk state occurs. Because ϑ_t^B is continuous over time absent state transitions, it must then be constant and take on precisely one of the three values in $\{\vartheta^{B*,l}, \vartheta^{B*,h}, \vartheta^{B**,l}\}$ for $t \leq \tau_0$.[85] We show that only $\vartheta_t^B = \vartheta^{B*,h}$ is consistent with a constant solution to ODE (44) (and any solution ϑ_t^C to ODE (45)).

Clearly, $\vartheta_t^B = \vartheta^{B*,h}$, $\vartheta_t^C = 0$ is a constant solution that is consistent with ODEs (44) and (45).

Next, consider a constant solution $\vartheta_t^B = \vartheta^{B*,l}$ to ODE (44). This only works if ϑ_t^C is also constant and takes on a value $\hat{\vartheta}^C$ such that

$$0 = \dot{\vartheta}_t^B = \left(f(\vartheta^{B*,l} + \hat{\vartheta}^C, \tilde{\sigma}^h) + \breve{\mu}^{\mathcal{B},l} \right) \vartheta^{B*,l}.$$

We know that this value for $\hat{\vartheta}^C$ must be different from zero because $f(\vartheta^{B*,l}, \tilde{\sigma}^h) < f(\vartheta^{B*,l}, \tilde{\sigma}^l)$. But to obtain also a constant and nonzero ϑ^C, equation (45) requires further

$$0 = \dot{\vartheta}_t^C = \left(f(\vartheta^{B*,l} + \hat{\vartheta}^C, \tilde{\sigma}^h) + \lambda^h + \breve{\mu}^{\mathcal{C}} \right) \hat{\vartheta}^C.$$

Clearly, both equations can only hold simultaneously if $\breve{\mu}^{\mathcal{B},l} = \breve{\mu}^{\mathcal{C}} + \lambda^h$. Because we have excluded this parameter configuration by assumption, a constant solution $\vartheta_t^B = \vartheta^{B*,l}$ cannot exist.

An entirely analogous argument applies to a potential constant solution $\vartheta_t^B = \vartheta^{B**,l}$ in the case $\vartheta^{B**,l} \neq \vartheta^{B*,h}$ (otherwise, we are already done). Such a solution is only possible if $\breve{\mu}^{\mathcal{B}}(\vartheta^{B**,l}) = \breve{\mu}^{\mathcal{C}} + \lambda^h \frac{\vartheta^{B*,l}}{\vartheta^{B**,l}}$. Because we have excluded this possibility by choice of the $\breve{\mu}^{\mathcal{B}}$-function, a constant solution $\vartheta_t^B = \vartheta^{B**,l}$ is also not possible.

[85] On a given path. The values may, of course, be different for different paths, at least in principle.

□

A.6.6 Proof of Lemma A.11

Define recursively $\left(\bar{\vartheta}^{B,(k),j}\right)_{j\in\{l,h\},k=0,1,2,...}$ as follows: for $k=0$, let $\bar{\vartheta}^{B,(0),l}=\bar{\vartheta}^{B,(0),h}:=1$ and for any $k>1$, let $(\bar{\vartheta}^{B,(k),l},\bar{\vartheta}^{B,(k),h})$ be the unique solution to

$$0=f(\bar{\vartheta}^{B,(k),j},\tilde{\sigma}^{j})+\breve{\mu}^{\mathcal{B},j}+\lambda^{j}-\lambda^{j}\frac{\bar{\vartheta}^{B,(k-1),-j}}{\bar{\vartheta}^{B,(k),j}},\qquad j\in\{l,h\}. \tag{46}$$

Finally, define $\bar{\vartheta}_t^{B,(k)}:=\bar{\vartheta}^{B,(k),l}1_{\{\tilde{\sigma}_t=\tilde{\sigma}^l\}}+\bar{\vartheta}^{B,(k),h}1_{\{\tilde{\sigma}_t=\tilde{\sigma}^h\}}$.

We first establish a number of properties of these objects in a sequence of additional lemmas. We then apply these lemmas to proof Lemma A.11.

Lemma A.13. $\vartheta_t^{B*}\leq\bar{\vartheta}_t^{B,(k)}\leq\bar{\vartheta}_t^{B,(k-1)}$ *for all* $k\geq 1$.

Proof. We show both inequalities by induction. First note that $\bar{\vartheta}_t^{B,(0)}=1\geq\vartheta_t^{B*}$. Also note that the right-hand side of the equation for both j is strictly increasing in $\bar{\vartheta}^{B,(k),j}$ and strictly decreasing in $\bar{\vartheta}^{B,(k-1),-j}$ and the equation would be satisfied for the specific choice $\bar{\vartheta}^{B,(k),j}=\bar{\vartheta}^{B,(k-1),j}=\vartheta^{B*,j}$ (which is generally not the true value for $\vartheta^{B,(k),j}$ and $\vartheta^{B,(k-1),j}$, however).

$\vartheta_t^{B*}\leq\bar{\vartheta}_t^{B,(k)}$ follows then immediately by induction. Because $\bar{\vartheta}^{B,(k-1),-j}\geq\vartheta^{B*,-j}$, the solution $\bar{\vartheta}^{B,(k),j}$ cannot be smaller than the solution in the equation that replaces $\bar{\vartheta}^{B,(k-1),-j}$ with $\vartheta^{B*,-j}$ (which equals $\vartheta^{B*,j}$). Hence, also $\bar{\vartheta}^{B,(k),j}\geq\vartheta^{B*,j}$.

For the inequality $\bar{\vartheta}_t^{B,(k)}\leq\bar{\vartheta}_t^{B,(k-1)}$, note that it holds trivially for $k=1$. If it holds for k, then, again, we can use the monotonicity of the solution to the equations to conclude that it also holds for $k+1$. □

Lemma A.14. *Any solution that satisfies* $\vartheta_t^B\leq\bar{\vartheta}_t^{B,(k)}$ *for all* t *with probability* 1 *already satisfies* $\vartheta_t^B\leq\bar{\vartheta}_t^{B,(k+1)}$ *for all* t *with probability* 1.

Proof. Let ϑ_t^B be a solution that satisfies $\vartheta_t^B\leq\bar{\vartheta}_t^{B,(k)}$ for all t and let $\vartheta_t^{B,+}$ the associated jump target process. If $\tilde{\sigma}_t=\tilde{\sigma}^j$, then clearly also $\vartheta_t^{B,+}\leq\bar{\vartheta}^{B,(k),-j}$ as otherwise ϑ_t^B would jump above $\bar{\vartheta}_t^{B,(k)}$ with positive probability. Now, if $\vartheta_t^B>\bar{\vartheta}_t^{B,(k+1)}$ (still assuming $\tilde{\sigma}_t=\tilde{\sigma}^j$), then

$$\begin{aligned}\frac{\dot{\vartheta}_t^B}{\vartheta_t^B}&=\left(f(\vartheta_t^B+\vartheta_t^C,\tilde{\sigma}_t)+\breve{\mu}^{\mathcal{B}}(\tilde{\sigma}_t)+\lambda(\tilde{\sigma}_t)\right)-\lambda(\tilde{\sigma}_t)\frac{\vartheta_t^{B,+}}{\vartheta_t^B}\\&\geq\left(f(\vartheta_t^B,\tilde{\sigma}_t)+\breve{\mu}^{\mathcal{B}}(\tilde{\sigma}_t)+\lambda(\tilde{\sigma}_t)\right)-\lambda(\tilde{\sigma}_t)\frac{\bar{\vartheta}^{B,(k),-j}}{\vartheta_t^B}\end{aligned}$$

$$
\begin{aligned}
&> \left(f(\bar{\vartheta}^{B,(k+1),j}, \tilde{\sigma}^j) + \breve{\mu}^{\mathcal{B}}\left(\tilde{\sigma}^j\right) + \lambda\left(\tilde{\sigma}^j\right) \right) - \lambda\left(\tilde{\sigma}^j\right) \frac{\bar{\vartheta}^{B,(k),-j}}{\bar{\vartheta}^{B,(k+1),j}} \\
&= 0,
\end{aligned}
$$

where the last line follows by definition of $\bar{\vartheta}^{B,(k+1),j}$. Therefore, conditional on remaining in state j, ϑ_t^B must grow beyond $1 \geq \bar{\vartheta}^{B,(k)}$ in finite time contradicting the assumption that $\vartheta_t^B \leq \bar{\vartheta}_t^{B,(k)}$ for all t. □

Lemma A.15. $\bar{\vartheta}^B := \lim_{k\to\infty} \bar{\vartheta}^{B,(k)}$ *exists and any equilibrium solution satisfies* $\vartheta_t^B \leq \bar{\vartheta}^B$.

Proof. We establish first by induction that $\vartheta_t^B \leq \bar{\vartheta}^{B,(k)}$ for all equilibrium solutions. For $k = 0$, this is trivial. For the induction step from k to $k+1$, apply Lemma A.14: any equilibrium solution must satisfy $\vartheta^B \leq \bar{\vartheta}^{B,(k)}$, but then by that lemma it must already satisfy $\vartheta^B \leq \bar{\vartheta}^{B,(k+1)}$.

Next, Lemma A.13 implies that $\bar{\vartheta}^{B,(k+1)} \leq \bar{\vartheta}^{B,(k)}$, so that the limit $\bar{\vartheta}^B = \lim_{k\to\infty} \bar{\vartheta}^{B,(k)}$ exists.

Finally, if there was an equilibrium solution such that $\vartheta_t^B > \bar{\vartheta}^B$ (with positive probability) at some time t, this would lead to a contradiction: we can then find k so large that $\vartheta_t^B > \bar{\vartheta}^{B,(k)}$ (with positive probability) contradicting the previous conclusion that each $\bar{\vartheta}^{B,(k)}$ is an upper bound for equilibrium solutions. □

Proof of Lemma A.11. We first show that any equilibrium solution satisfies $\vartheta_t^B \leq \vartheta_t^{B*}$. This essentially follows from Lemma A.15. Let $\bar{\vartheta}^B$ as there. We show that $\bar{\vartheta}^B = \vartheta^{B*}$.

Note that $\bar{\vartheta}^B$ is constant conditional on the state $\tilde{\sigma}_t$ because all $\bar{\vartheta}^{B,(k)}$ are. Therefore, it is fully characterized by the two values $\bar{\vartheta}^{B,j}$ for $j \in \{l, h\}$. These values satisfy $\bar{\vartheta}^{B,j} = \lim_{k\to\infty} \bar{\vartheta}^{B,(k),j}$. Because the dependence of equation (46) on $\bar{\vartheta}^{B,(k),j}$ and $\bar{\vartheta}^{B,(k-1),-j}$is continuous, we can take the limit $k \to \infty$ and obtain that $(\bar{\vartheta}^{B,l}, \bar{\vartheta}^{B,j})$ must solve the equation system

$$
0 = f(\bar{\vartheta}^{B,j}, \tilde{\sigma}^j) + \breve{\mu}^{\mathcal{B},j} + \lambda^j - \lambda^j \frac{\bar{\vartheta}^{B,-j}}{\bar{\vartheta}^{B,j}}, \qquad j \in \{l, h\}
$$

But this is also solved by $(\vartheta^{B*,l}, \vartheta^{B*,h})$ and there is a unique solution. Hence, $(\bar{\vartheta}^{B,l}, \bar{\vartheta}^{B,j}) = (\vartheta^{B*,l}, \vartheta^{B*,h})$ which implies $\bar{\vartheta}^B = \vartheta^{B*}$.

For the second part of the lemma, suppose that $\vartheta_{t_0}^B = \vartheta_{t_0}^{B*}$ at some time t_0 (after some exogenous history). To see that then $\vartheta_t^B = \vartheta_t^{B*}$ and $\vartheta_t^C = 0$ for all $t \geq t_0$, we first restrict attention only to times before the next state transition. Before the state transition, if $\vartheta_t^B = \vartheta_t^{B*}$, then

$$
\begin{aligned}
\dot{\vartheta}_t^B &= f(\vartheta_t^{B*} + \vartheta_t^C, \tilde{\sigma}_t)\vartheta_t^{B*} + \breve{\mu}_t^{\mathcal{B}} \vartheta_t^{B*} - \lambda_t(\vartheta_t^{B,+} - \vartheta_t^{B*}) \\
&\geq f(\vartheta_t^{B*}, \tilde{\sigma}_t)\vartheta_t^{B*} + \breve{\mu}_t^{\mathcal{B}} \vartheta_t^{B*} - \lambda_t(\vartheta_t^{B,+} - \vartheta_t^{B*}) \\
&= \lambda_t \left(\vartheta_t^{B*,+} - \vartheta_t^{B,+} \right) \geq 0,
\end{aligned}
$$

whereas $\dot{\vartheta}_t^{B*} = 0$. Hence, if $\vartheta_{t_0}^B = \vartheta_{t_0}^{B*}$, then $\vartheta_t^B \geq \vartheta_t^{B*}$ for all $t \geq t_0$ before the next state transition. As any equilibrium solution also has to satisfy the opposite inequality, it must be that $\vartheta_t^B = \vartheta_t^{B*}$ for all $t \geq t_0$ before the next state transition. Next, note that the inequality from the first to the second line above is strict if $\vartheta_t^C > 0$. In this case, $\dot{\vartheta}_t^B > 0$ and hence $\vartheta_{t+\varepsilon}^B$ would have to exceed $\vartheta_{t+\varepsilon}^{B*}$ (conditional on no state transition), which is not a valid equilibrium solution. Therefore, it must be that $\vartheta_t^C = 0$ for all $t \geq t_0$ before the next state transition. With an identical argument, we can also conclude that $\vartheta_t^{B,+} = \vartheta_t^{B*,+}$ for all $t \geq t_0$ before the next state transition.

But this latter conclusion implies that also $\vartheta_{t_1}^B = \vartheta_{t_1}^{B*}$ at the next state transition time t_1. With what has been proven already, we can then conclude that $\vartheta_t^B = \vartheta_t^{B*}$, $\vartheta_t^C = 0$, and $\vartheta_t^{B,+} = \vartheta_t^{B*,+}$ for all $t \geq t_1$ before the next state transition after time t_1. Inductively, we can then conclude the three equations for all $t \geq t_0$ before the next n state transitions after time t_0 have taken place (for any n). This is sufficient to show that the equations hold for all $t \geq t_0$ regardless of the number of state transitions that have already happened since t_0.

□

A.7 Generalized Equilibrium and Private Ponzi Schemes

In this appendix we define the generalized notion of equilibrium with Ponzi schemes discussed in the main text and relate equilibria with Ponzi schemes to equilibria with cryptocoin bubbles. To keep matters simple, we assume that there are no cryptocoins in the setting with Ponzi schemes.

Because of the absence of cryptocoins, households are forced to choose $\theta_t^{C,i} = 0$ at all times, but otherwise household net worth still evolves according to equation (4). Consider the following variant of the household problem for agent i: taking the initial net worth n_0^i and the returns $dr_t^{\mathcal{B}}$ and $dr_t^{K,i}(\cdot)$ as given, choose the consumption-wealth ratio $\{\hat{c}_t^i\}_{t\geq 0}$, real investment $\{\iota_t^i\}_{t\geq 0}$, and the portfolio share $\{\theta_t^{B,i}\}_{t\geq 0}$ in bonds to maximize utility V_0^i subject to the net worth evolution (4) (with $\theta_t^{C,i} = 0$) and a generalized solvency constraint $n_t^i \geq \underline{n}_t^i$ for all $t \geq T$. Here, $T \in [0, \infty)$ is a given time from which on the constraint is imposed and $\{\underline{n}_t^i\}_{t\geq 0}$ is a given stochastic process such that $\underline{n}_t^i \leq 0$ for all t.

Definition A.5. We say the generalized solvency constraint $n_t^i \geq \underline{n}_t^i$ is *at most asymptotically binding* if the set of optimal solutions to the household problem is independent of the time T.

The previous definition formalizes the notation that the solvency constraint is supposed to not constrain the household at any finite time but only represent a "constraint at infinity". The idea of the generalized equilibrium concept is to treat the solvency constraints, like prices, as objects that are determined in equilibrium. This is consequent as solvency or no Ponzi constraints do not represent fundamental restrictions of the physical environment. Instead, they are meant to formalize the notion that other agents' optimizing behavior does not leave room for a given agent to roll over debt indefinitely. But as for the absence of bubbles, it is an economically more sound approach to derive such restrictions instead of imposing them from the outset.

In the following definition, we restrict attention to the special case that every agent i faces the same relaxed solvency constraint $n_t^i \geq \underline{n}_t$ with a, consequently deterministic, time path $\underline{n}_t$. This is consistent with our previous approach to restrict attention to symmetric equilibria throughout this paper (compare Definition A.1). However, it should be remarked that one could easily generalize the definition and allow for possibly asymmetrically relaxed solvency constraints across different agents.

Even in this special case, it is insufficient to define a symmetric equilibrium here because the optimal choices of $\hat{c}_t^i$ and $\theta_t^{B,i}$ may now depend on n_t^i. Nevertheless, to simplify matters slightly, we define here only equilibria under ex-ante symmetry, in which all agents start with the same asset holdings:

Definition A.6 (Generalized Competitive Equilibrium)**.** Given $K_0 > 0$, a (ex-ante symmetric) *generalized competitive equilibrium* (without cryptocoins) consists of absolutely continuous time paths

$$[0, \infty) \to \mathbb{R}^7, t \mapsto (\breve{\mu}_t^{\mathcal{B}}, \tau_t, q_t^B, q_t^K, \iota_t, K_t, \underline{n}_t)$$

for government policy $(\breve{\mu}_t^{\mathcal{B}}, \tau_t)$, asset prices (q_t^B, q_t^K), an investment rate ι_t, capital stock K_t, and a lower bound on net worth $(\underline{n}_t)$ as well as a collection of stochastic processes

$$\{\hat{c}_t^i, \theta_t^{B,i}, n_t^i\}_{t \geq 0, i \in [0,1]}$$

for household consumption and portfolio choices and (measured) net worth such that

(i) $\breve{\mu}_t^{\mathcal{B}}$ and τ_t satisfy the government budget constraint given prices (for all $t \geq 0$);

(ii) prices are nonnegative, $q_t^B, q_t^K \geq 0$ (for all $t \geq 0$);

(iii) the capital stock K_t satisfies the evolution equation

$$dK_t = (\Phi(\iota_t) - \delta)K_t dt;$$

(iv) for all agents i, $\{\hat{c}_t^i, \iota_t, \theta_t^{B,i}\}_{t \geq 0}$ solves the (generalized) household problem for initial $n_0^i = (q_0^B + q_0^K)K_0$ with solvency constraint $n_t^i \geq \underline{n}_t$ given the returns $dr_t^{\mathcal{B}}$ and $dr^{K,i}(\cdot)$ implied by prices and government policies and $\{n_t^i\}_{t \geq 0}$ is the associated net worth process under the optimal choice;

(v) $\underline{n}_t \leq 0$ for all t and, for each i, the constraints $n_t^i \geq \underline{n}_t$ in the household problem bind at most asymptotically;

(vi) all markets clear (for all $t \geq 0$):

$$\int \hat{c}_t^i n_t^i di + \mathfrak{g} K_t + \iota_t K_t = a K_t \qquad \text{goods market clearing}$$

$$\int \theta_t^{B,i} n_t^i di = q_t^B K_t \qquad \text{bond market clearing}$$

We say that a (generalized) equilibrium features private Ponzi schemes if the time path for $\underline{n}_t$ cannot be replaced with $\underline{n}_t \equiv 0$ without violating some equilibrium condition. As stated in the main text, whenever parameters are such that bubbles can exist in our baseline model, i.e. whenever $\tilde{\sigma}^2 \geq \rho$, then also equilibria exist that feature private Ponzi schemes.

The previous claim follows immediately from the following proposition. This proposition also provides a link between equilibria with cryptcoin bubbles and (certain) equilibria with Ponzi schemes. The idea is to turn measured wealth from cryptocoin holdings into implicit wealth arising from the ability to run Ponzi schemes.

Proposition A.5. *Let $(\breve{\mu}_t^{\mathcal{B}*}, \tau_t^*, q_t^{B*}, q_t^{C*}, q_t^{K*}, \hat{c}_t^*, \iota_t^*, \theta_t^{B*}, \theta_t^{C*})$ be an equilibrium in the baseline model (according to Definition A.1) with $\breve{\mu}^{\mathcal{C}} = 0$ ("$*$-equilibrium"). For this equilibrium, denote by r_t^{f*} the risk-free rate at time t.*

Then, for any initial capital stock $K_0 > 0$, there is an equilibrium according to Definition A.6 with private Ponzi schemes and

$$\underline{n}_t = -\exp\left(\int_0^t r_s^{f*}\right) q_0^{C*} K_0.$$

This equilibrium features the same capital and bond valuations and the same aggregate consumption and investment as the $*$-equilibrium, provided the latter is started at the same initial condition for K_0.

Furthermore, if the $*$-equilibrium is started at a symmetric initial wealth allocation, then both equilibria feature the same cross-sectional consumption allocation.

Proof. The proof is constructive. We use the $*$-equilibrium to construct a Ponzi scheme equilibrium according to Definition A.6.

We start with the deterministic portion of the equilibrium objects. Denote by K_t^* the aggregate capital stock in the $*$-equilibrium under the initial condition K_0, i.e. the solution to

$$dK_t^* = \left(\Phi(\iota_t^*) - \delta\right) K_t^* dt, \qquad K_0^* = K_0.$$

With this notation, define

$$(\breve{\mu}_t^{\mathcal{B}}, \tau_t, q_t^B, q_t^K, \iota_t, K_t, \underline{n}_t) = \left(\breve{\mu}_t^{\mathcal{B}*}, \tau_t^*, q_t^{B*}, q_t^{K*}, \iota_t^*, K_t^*, -\exp\left(\int_0^t r_s^{f*}\right) q_0^{C*} K_0\right).$$

We claim that this is the deterministic portion of a valid generalized equilibrium. The properties (i), (ii), and (iii) of Definition A.6 are clearly satisfied by construction. Also by construction, $\underline{n}$ satisfies the equation stated in the proposition.

For the remaining properties in Definition A.6, we also need to construct a suitable collection of stochastic processes $\{\hat{c}_t^i, \theta_t^{B,i}, n_t^i\}_{t\geq 0, i\in[0,1]}$. Let for each i,

$$\hat{c}_t^i := \frac{n_t^i + q_t^{C*}K_t^*}{n_t^i}\hat{c}_t^*, \qquad \theta_t^{B,i} := \theta_t^{B*} + \theta_t^{C*}\frac{n_t^i - q_t^{K*}K_t^*}{n_t^i}$$

and let $\{n_t^i\}_{t\geq 0}$ be the implied net worth process according to equation (4). We show next that these choices satisfy properties (iv)–(vi) in Definition A.6.

We start with the market clearing conditions, property (vi). For the goods market,

$$\begin{aligned}
\int \hat{c}_t^i n_t^i di &= \int \frac{n_t^i + q_t^{C*}K_t^*}{n_t^i} n_t^i di \cdot \hat{c}_t^* \\
&= \left((q_t^B + q_t^K)K_t + q_t^{C*}K_t^* \right) \hat{c}_t^* \\
&= \left(q_t^{B*} + q_t^{K*} + q_t^{C*} \right) \hat{c}_t^* K_t^* \\
&= (a - \iota_t^* - \mathfrak{g})\, K_t^* = (a - \iota_t - \mathfrak{g})\, K_t.
\end{aligned}$$

Here, the last line follows from the fact that the $*$-variables form an equilibrium according to Definition A.1. Similarly, for the bond market,

$$\begin{aligned}
\int \theta_t^{B,i} n_t^i di &= \int n_t^i di \cdot \theta_t^{B*} + \int \left(n_t^i - q^{K*}K_t^* \right) di \cdot \theta_t^{C*} \\
&= (q_t^{B*} + q_t^{K*})\theta_t^{B*}K_t^* + \left((q_t^{B*} + q_t^{K*})K_t^* - q_t^{K*}K_t^* \right) \theta_t^{C*} \\
&= \frac{(q_t^{B*} + q_t^{K*})q_t^{B*} + q_t^{B*}q_t^{C*}}{q_t^{B*} + q_t^{C*} + q_t^{K*}} K_t^* \\
&= q_t^{B*}K_t^* = q_t^B K_t,
\end{aligned}$$

where the third line follows because the $*$-variables satisfy the asset market clearing conditions in Definition A.1.

Next, for property (v), we simply show that $n_t^i > \underline{n}_t$ almost surely for all $t \geq 0$. This is sufficient because, if the conjectured choices are indeed optimal as shown below, then it is indeed true that the constraints are not binding along the optimal solution path at any finite time t. To show this, let $\hat{n}_t^i := n_t^i - \underline{n}_t$ and note that

$$d\hat{n}_t^i = dn_t^i - d\underline{n}_t$$

$$= -\hat{c}^i_t n^i_t dt + n^i_t dr^{K,i}_t(\iota_t) + \theta^{B,i}_t n^i_t \left(dr^{\mathcal{B}}_t - dr^{K,i}_t(\iota_t) \right) - r^{f*}_t \underline{n}_t dt$$
$$= -\left(n^i_t + q^{C*}_t K^*_t \right) \hat{c}^*_t dt + n^i_t dr^{K,i}_t(\iota^*_t) + \left(\left(\theta^{B*}_t + \theta^{C*}_t \right) n^i_t - \theta^{C*}_t q^{K*} K^*_t \right) \left(r^{f*}_t dt - dr^{K,i}_t(\iota^*_t) \right) - r^{f*}_t \underline{n}_t dt$$

Now use $\theta^{C*}_t q^{K*}_t K^*_t = \frac{q^{C*}_t q^{K*}_t}{q^{B*}_t + q^{C*}_t + q^{K*}_t} K^*_t = \left(1 - \theta^{B*}_t - \theta^{C*}_t\right) q^{C*}_t K^*_t$ and $q^{C*}_t K^*_t = -\underline{n}_t$. The latter equation follows from the definition of $\underline{n}_t$ and the fact that $r^{f*}_t = \mu^{q,C,*}_t - \mu^{K*}_t - \breve{\mu}^{\mathcal{C}} = \mu^{q,C,*}_t - \mu^{K*}_t$ in the $*$-equilibrium. Substituting both equations into the previous equation yields

$$d\hat{n}^i_t = -\left(n^i_t - \underline{n}_t \right) \hat{c}^*_t dt + n^i_t dr^{K,i}_t(\iota^*_t) + \left(\left(\theta^{B*}_t + \theta^{C*}_t \right) n^i_t - \left(\theta^{B*}_t + \theta^{C*}_t - 1 \right) \underline{n}_t \right) \left(r^{f*}_t dt - dr^{K,i}_t(\iota^*_t) \right) - r^{f*}_t \underline{n}_t dt$$
$$= -\left(n^i_t - \underline{n}_t \right) \hat{c}^*_t dt + \left(n^i_t - \underline{n}_t \right) dr^{K,i}_t(\iota^*_t) + \left(\theta^{B*}_t + \theta^{C*}_t \right) \left(n^i_t - \underline{n}_t \right) \left(r^{f*}_t dt - dr^{K,i}_t(\iota^*_t) \right)$$
$$= \hat{n}^i_t \left(-\hat{c}^*_t dt + dr^{K,i}_t(\iota_t) + \left(\theta^{B*}_t + \theta^{C*}_t \right) \left(r^{f*}_t dt - dr^{K,i}_t(\iota_t) \right) \right) \tag{47}$$

The previous equation implies $\hat{n}^i_t = n^{i*}_t$ provided we impose in the $*$-equilibrium the initial condition $n^{i*}_0 = \hat{n}^i_0$. Because $n^{i*}_t > 0$, we can conclude $\hat{n}^i_t > 0$.

For property (iv), we use the stochastic maximum principle in the finite-horizon problem with terminal condition $n^i_T \geq \underline{n}^i_T$ and take the limit $T \to \infty$. Because we have already shown that the constraints $n^i_t \geq \underline{n}_t$ are not binding at any finite time t along the conjectured optimal net worth path, this is sufficient to establish optimality in the actual problem with the additional constraints. Denote by λ^i_t the costate process for agent i and by $\tilde{\sigma}^i_{\lambda,t}$ its (arithmetic) volatility loading with respect to $d\tilde{Z}^i_t$. Maximization of the Hamiltonian over $[0, T]$ is associated with the first-order conditions with respect to $\hat{c}^i_t$, ι^i_t, and $\theta^{B,i}_t$, respectively (these are also sufficient due to concavity)

$$\lambda^i_t = e^{-\rho t} \frac{1}{\hat{c}^i_t n^i_t}$$
$$0 = \frac{d}{d\iota^i_t} \frac{\mathbb{E}_t[dr^{K,i}_t(\iota^i_t)]}{dt}$$
$$\lambda^i_t \left(\frac{\mathbb{E}_t[dr^{K,i}_t(\iota^i_t)]}{dt} - \frac{dr^{\mathcal{B}}_t}{dt} \right) = -\tilde{\sigma}_{\lambda,t} \tilde{\sigma}$$

In addition, $\lambda^i_t \geq 0$ has to satisfy the costate equation

$$\mathbb{E}_t[d\lambda^i_t] = -\left(\lambda^i_t \left((1 - \theta^B_t) \frac{\mathbb{E}_t[dr^{K,i}_t(\iota^i_t)]}{dt} + \theta^B_t \frac{dr^{\mathcal{B}}_t}{dt} \right) + \tilde{\sigma}_{\lambda,t}(1 - \theta^B_t) \tilde{\sigma} \right) dt \tag{48}$$

and the terminal condition

$$\mathbb{E}[\lambda^i_T (n^i_T - \underline{n}^i_T)] = 0.$$

In the limit $T \to \infty$, these conditions remain unchanged, except for the last which is replaced by the limit condition

$$\lim_{T\to\infty} \mathbb{E}[\lambda_T^i(n_T^i - \underline{n}_T^i)] = 0.$$

We show that all conditions are satisfied if we choose $\lambda_t^i := e^{-\rho t}\frac{1}{c_t^i} = e^{-\rho t}\frac{1}{\hat{c}_t^i n_t^i}$.

With this choice, the first first-order condition is satisfied by construction. The second first-order condition only depends on ι_t^i and q_t^K and is satisfied because $\iota_t^i = \iota_t^*$, $q_t^K = q_t^{K*}$, and an identical condition holds in the $*$-equilibrium. For the final first-order condition, note that

$$\begin{aligned}
\tilde{\sigma}_{\lambda,t} &= -\lambda_t^i \tilde{\sigma}_t^{c,i} = -\lambda_t^i \frac{n_t^i(1-\theta_t^{B,i})}{n_t^i + q_t^{C*}K_t^*}\tilde{\sigma} \\
&= -\lambda_t^i \frac{n_t^i - \theta_t^{B*}n_t^i - \theta_t^{C*}n_t^i + \theta_t^{C*}q^{K*}K_t^*}{n_t^i + q_t^{C*}K_t^*}\tilde{\sigma} \\
&= -\lambda_t^i \left(1-\theta_t^{B*}-\theta_t^{C*}\right)\frac{n_t^i + q^{C*}K_t^*}{n_t^i + q_t^{C*}K_t^*}\tilde{\sigma} = -\lambda_t^i \left(1-\theta_t^{B*}-\theta_t^{C*}\right)\tilde{\sigma}.
\end{aligned} \tag{49}$$

Hence, the third first-order condition is equivalent to

$$\frac{\mathbb{E}_t[dr_t^{K,i,*}(\iota_t^*)]}{dt} - \frac{dr_t^{\mathcal{B}*}}{dt} = \left(1-\theta_t^{B*}-\theta_t^{C*}\right)\tilde{\sigma}^2. \tag{50}$$

This equation is precisely the Merton portfolio choice condition in the $*$-equilibrium and therefore satisfied at all times.

Finally, consider the costate equation (48). Using $\lambda_t^i = e^{-\rho t}\frac{1}{c_t^i}$ and Ito's lemma, we obtain for the left-hand side of this equation

$$\mathbb{E}_t[d\lambda_t^i] = -\rho\lambda_t^i dt - \lambda_t^i \frac{\mathbb{E}_t[dc_t^i]}{c_t^i} + \lambda_t^i \left(\tilde{\sigma}_t^{c,i}\right)^2 dt.$$

We know $c_t^i = \rho \hat{n}_t^i$, where $\hat{n}_t^i = n_t^i - \underline{n}_t$ as defined previously. Hence, $\frac{\mathbb{E}_t[dc_t^i]}{c_t^i} = \frac{\mathbb{E}_t[d\hat{n}_t^i]}{\hat{n}_t^i}$. By equation (47), we can therefore write $\mathbb{E}_t[d\lambda_t^i]$ as follows:

$$\mathbb{E}_t[d\lambda_t^i] = -\lambda_t^i \left(\left(1-\theta_t^{B*}-\theta_t^{C*}\right)\frac{\mathbb{E}_t[dr_t^{K,i}(\iota_t^*)]}{dt} + \left(\theta_t^{B*}+\theta_t^{C*}\right) r_t^{f*} - \left(\tilde{\sigma}_t^{c,i}\right)^2\right) dt.$$

Now use equations (49) and (50) to conclude

$$\begin{aligned}
\mathbb{E}_t[d\lambda_t^i] &= -\lambda_t^i \left(\left(1-\theta_t^{B*}-\theta_t^{C*}\right)\left(r_t^{f*} + \left(1-\theta_t^{B*}-\theta_t^{C*}\right)\tilde{\sigma}^2\right) + \left(\theta_t^{B*}+\theta_t^{C*}\right) r_t^{f*} - \left(1-\theta_t^{B*}-\theta_t^{C*}\right)^2\tilde{\sigma}^2\right) dt \\
&= -\lambda_t^i r_t^{f*} dt.
\end{aligned}$$

To prove that the costate equation (48) holds, we show that also its right-hand side evaluates to $-\lambda_t^i r_t^{f*} dt$. This merely requires a few replacements using equilibrium relationships. First note that that $\frac{\mathbb{E}_t[dr_t^{K,i}(\iota_t^i)]}{dt} = \frac{\mathbb{E}_t[dr_t^{K,i}(\iota_t^*)]}{dt}$ and $\frac{dr_t^{\mathcal{B}}}{dt} = r_t^{f*} dt$ along the conjectured equilibrium path. Substituting these equations and equations (49) and (50) into the right-hand side of equation (48) yields

$$
\begin{aligned}
&-\left(\lambda_t^i\left((1-\theta_t^B)\frac{\mathbb{E}_t[dr_t^{K,i}(\iota_t^i)]}{dt}+\theta_t^B\frac{dr_t^{\mathcal{B}}}{dt}\right)+\tilde{\sigma}_{\lambda,t}(1-\theta_t^B)\tilde{\sigma}\right)dt \\
=&-\left(\lambda_t^i\left(r_t^{f*}+\left(1-\theta_t^B\right)\left(1-\theta_t^{B*}-\theta_t^{C*}\right)\tilde{\sigma}^2\right)-\lambda_t^i\left(1-\theta_t^{B*}-\theta_t^{C*}\right)\left(1-\theta_t^B\right)\tilde{\sigma}^2\right)dt \\
=&-\lambda_t^i r_t^{f*} dt.
\end{aligned}
$$

Hence, equation (48) holds and therefore the choices are indeed optimal.

This concludes the proof of the proposition. □