Modern Macro, Money, and International Finance
Eco 529
Lecture 10: One Sector Monetary Model with Heterogenous Agents

Markus K. Brunnermeier
Princeton University
Roadmap

- Intuition for different “Monetary Theories”

- Monetary Model with one sector with constant idiosyncratic risk
  - Safe Asset and Service Flows
  - Bubble (mining) or not
  - 2 Different Asset Pricing Perspectives/SDFs

- Monetary model with time-varying idiosyncratic risk
  - Safe asset has negative $\beta$
  - Calibration:
    - Debt valuation puzzle, Debt Laffer Curve, Flight-to-Safety and Equity excess volatility

- Medium of Exchange Role
The 3 Roles of Money

- **Unit of account**
  - Intratemporal: Numeraire bounded rationality/price stickiness
  - Intertemporal: Debt contract incomplete markets

- **Store of value**
  - “I Theory of Money without I”
    Less risky than other “capital” – no idiosyncratic risk
  - Fiscal theory of the price level

- **Medium of exchange**
  - Overcome double-coincidence of wants problem

- **Record keeping device – money is memory**
  - Virtual ledger
Money versus Credit

- Credit: zero net supply

- Money: positive net supply

- “Medium of Exchange Money” (double-coincidence of wants)
  - Credit renders money useless

- “Store of value Money” (safe asset)
  - Money/gov. debt still useful if bubble since it “partially completes markets”
    - Incomplete markets
    - OLG
Money, Reserves, and Government Debt

- **Cash**: extra convenience yield and zero interest $\Rightarrow$ lower return by $\Delta i$
  - Erodes due to Fintech revolution

- **Reserves**: Interest bearing $M \ (\Delta i)$
  - Special form of
    - Infinite maturity more like equity (no rollover risk)
    - Zero duration more like overnight debt
    - Banking system can’t offload it
      - Financial Repression

- Is QE simply swapping one debt for another one (reserves)?
(Narrow) Money, Gov. Debt, Safe Assets

- Service flows $\supseteq$ convenience yield
  1. **Safe asset:** [good friend analogy]
     - When one needs funds, one can sell at stable price ... since others buy
     - Partial insurance through re-trading - market liquidity
  2. **Collateral:** relax constraints (Lagrange multiplier)
  3. **Money (narrow):** relax double-coincidence of wants

- Higher Asset Price = lower expected return - **exorbitant privilege**
  
  \[
  \rho + \gamma \mathbb{E}[g_c] - \frac{1}{2} \gamma (\gamma + 1) \left\{ \text{Var}_t[g_c] + \text{Var}_t[\tilde{g}_c] \right\} - \left\{ \lambda \text{(Collateral Constr)} \right\} - \Delta i + \text{risk permium}
  \]

  Preference rate  
  Ramsey term  
  Precautionary savings/self-insurance  
  Aggregated Risk  
  Idiosyncratic Risk  
  Inflation risk + loss of safe asset status

- Problem: safe asset + money status might burst like a bubble
  - Multiple equilibria: [safe asset tautology]
Inflation Theories

- **Fiscal Theory of the Price Level (FTPL)**
  - **store of value**
  - Price level is determined by
    \[ \frac{B_t}{\varphi_t} = E_t [PV(\text{primary surpluses})] + \ldots \]
  - Fiscal vs. Monetary Dominance + Financial Dominance

- **Monetarism** – assumes Monetary Dominance
  - Fiscal implications of monetary policy
    induces government to change lump-sum taxes
  - so that per-period budget constraint holds (is unaltered)
  - Money \( M_t / \varphi_t \) serves as **medium of exchange**
    - Cash-in-advance constraint, transaction cost, shopping time model, ...
    - \( \Rightarrow \Delta i \) (convenience yield)
  - Price level is determined by \( M_t \nu = \varphi_t Y_t \) (if velocity, \( \nu \) is constant)

- **(New) Keynesianism** – assumes Monetary Dominance
  - Cashless limit
  - Interest rate policy
    - All price/inflation paths are explosive except for one
      - (Cochrane: What’s wrong with explosive nominal paths?)
Price Stickiness and Phillips Curve

- Flexible prices: Prices adjust immediately

- Sticky prices:
  - Since prices adjust sluggishly, output has to adjust
    - Inflation pressure: prices too low during transition period, output (demand) overshoots natural (= flexible price) level
    - Deflation pressure: prices too high during transition period, output (demand) undershoots natural level

- Sticky price models smooth out adjustment dynamics relative to equivalent flex price models
Inflation & Gov. Debt

CBO projection, May 2021
FTPL: US: Inflation – Fiscal Link

- Inflation Rate (%)
- Nominal Interest Rate (%)
- US Budget Surplus (% of GDP)
FTPL: UK: inflation-fiscal link + wars

UK Budget Surpluses, Nominal Interest Rate and Inflation
1680-2018

Source: ukpublicrevenues.co.uk, MeasuringWorth.com, Young (1925), Maddison (2010), Schmelzing (2020)
FTPL: Valuating Government Debt: Japan

- Think of a representative agent holding all gov. debt
  - His cash flow is primary surplus
  - $B_t = E_t[PV_r(\text{primary surpluses})] + \text{Bubble}$
  - ... link to inflation
  - Can surpluses be negative forever? Yes, if $r < g$ (e.g. due to safe asset nature)
FTPL: Primary surplus, \( r \) and \( g \) for United States

- Primary surplus/GDP
- Negative surplus in recession

- \( g \) GDP growth
- \( r \)
- \( r - g \)
FTPL Equation: Negative primary surplus forever?

- without creating inflation (devaluing debt)?
- Yes, if $r < g$

\[
\frac{B_t}{\varphi_t} = E_t[PV_r(\text{primary surpluses})] + \lim_{T \to \infty} PV_r \frac{B_T}{\varphi_T}
\]

- discount at $r$ (agents' SDF)
- grows at $g$ with constant deficit/GDP
- $\to -\infty$
- Bubble $\to +\infty$

To determine real value of gov. debt and price level
FTPL equation is not enough
(goods market clearing and wealth effect)
FTPL Equation: Negative primary surplus forever?

- without creating inflation (devaluing debt)?
- Yes, if $r < g$

\[
\frac{B_t}{\sigma_t} = E_t[PV_r(\text{primary surpluses})] \quad \text{grows at } g \text{ with constant deficit/GDP} \quad \lim_{T \to \infty} PV_r \frac{B_T}{\sigma_T} \quad \text{Bubble} \quad \rightarrow -\infty \quad \rightarrow +\infty
\]

- Discount at a different rate $r^{**} > g$ instead, so that

\[
\frac{B_t}{\sigma_t} = E_t[PV_{r^{**}}(\text{primary surpluses})] \quad + \quad E_t[PV_{r^{**}}(\text{service flow})] \quad > -\infty \quad < +\infty
\]

- Both terms meaningful
- Discount rate $r^{**} =$ representative agents’ risk-free rate $\neq m$ (Reis)
FTPL: Monetary vs. Fiscal Dominance

- **Monetary dominance**
  - Monetary tightening leads fiscal authority to reduce fiscal deficit

- **Fiscal dominance**
  - Interest rate increase does not reduce primary fiscal deficit
  - ... only lead to higher inflation

See YouTube video 4, minute 4:15
FTPL: Monetary vs. Fiscal Dominance

- **Monetary dominance**
  - Monetary tightening leads fiscal authority to reduce fiscal deficit

- **Fiscal dominance**
  - Interest rate increase does not reduce primary fiscal deficit
  - ... only lead to higher inflation

- **Financial dominance**
  - Monetary tightening causes havoc in financial markets
Roadmap

- Intuition for different “Monetary Theories”

- Monetary Model with one sector with constant idiosyncratic risk
  - Safe Asset and Service Flows
  - Bubble (mining) or not
  - 2 Different Asset Pricing Perspectives/SDFs

- Monetary model with time-varying idiosyncratic risk
  - Safe asset has negative CAPM-$\beta$
  - Calibration:
    - Debt valuation puzzle, Debt Laffer Curve, Flight-to-Safety and Equity excess volatility

- Medium of Exchange Role
Simplify to One Sector Model

Expert sector
- Output: \( y_t^e = a^e k_t^e \)
- Consumption rate: \( c_t^e \)
- Investment rate: \( \frac{dk_t^{i,e}}{k_t^{t,e}} = \left( \Phi \left( i_t^{i,e} \right) - \delta \right) dt + d\Delta_t^{k,i,e} + \sigma dZ_t + \bar{\sigma} d\bar{Z}_t \)
- \( E_0 \left[ \int_0^\infty e^{-\rho^e t \left( c_t^e \right)^{1-\gamma} \frac{1}{1-\gamma}} dt \right] \)
  - Can only issue
- Risk-free debt
- Equity, but most hold \( \chi_t^e \geq \alpha \kappa_t \)

Household sector
- Output: \( y_t^h = a^h k_t^h \)
- Consumption rate: \( c_t^h \)
- Investment rate: \( \frac{dk_t^{i,h}}{k_t^{i,h}} = \left( \Phi \left( i_t^{i,h} \right) - \delta \right) dt + d\Delta_t^{k,i,h} + \sigma dZ_t + \bar{\sigma} d\bar{Z}_t \)
- \( E_0 \left[ \int_0^\infty e^{-\rho^h t \left( c_t^h \right)^{1-\gamma} \frac{1}{1-\gamma}} dt \right] \)
Model Overview

- Continuous time, infinite horizon, one consumption good
- Continuum of agents
  - Operate capital with time-varying idiosyncratic risk, AK production technology
  - Can trade capital and government bond, Extension: add diversified equity claims
- Government
  - Exogenous spending
  - Taxes output
  - Issues (nominal) bonds
- Financial Frictions: incomplete markets
  - Agents cannot trade idiosyncratic risk
  - Extension with equity: must retain skin in the game
- Aggregate risk: fluctuations in volatility of idio risk (& capital productivity)
Model with Capital + Safe Asset

- Each heterogenous citizen $\tilde{i} \in [0,1]$

$$\mathbb{E}_t \left[ \int_t^\infty e^{-\rho s} \left( \frac{\epsilon_s^{1-\gamma}}{1-\gamma} + f(gK_s) \right) ds \right]$$

where $K_t := \int k_t^i d\tilde{i}$

s.t.

$$\frac{dn_t^i}{n_t^i} = -\frac{c_t^i}{n_t^i} dt + dr_t^B + (1 - \theta_t^i) \left( dr_t^{K,i}(\tilde{i}) - dr_t^B \right)$$

- Each citizen operates physical capital $k_t^i$
  - Output (net investment)
    $$y_t^i = a_t^i k_t^i - \tilde{i}_t^i k_t^i dt$$
  - Output tax
    $$\tau a_t^i k_t^i \tilde{i}_t^i dt$$

$$\frac{dk_t^i}{k_t^i} = \left( \Phi(\tilde{i}_t^i) - \delta \right) dt + \tilde{\sigma}_t d\tilde{Z}_t^i + d\Delta_{k_t^i}$$

- $d\tilde{Z}_t^i$ idiosyncratic Brownian

- Aggregate risk $dZ_t$ : Heston model (time-varying idiosyncratic risk)
  - $d\tilde{\sigma}_t^2 = -\psi \left( \tilde{\sigma}_t^2 - (\tilde{\sigma}_0)^2 \right) dt - \sigma \tilde{\sigma}_t dZ_t$
  - $a_t = a(\tilde{\sigma}_t)$

- Financial Friction: Incomplete markets: no $d\tilde{Z}_t^i$ claims
Government: Taxes, Bond/Money Supply, Gov. Budget

- **Policy Instruments**
  - Government spending $g_t K_t$ (with exogenous $g_t$)
  - Proportional output tax $\tau a K_t$
  - Nominal value of total government debt supply $d B_t = \mu_t B_t dt$
  - Floating nominal interest rate $i_t$ on outstanding bonds

- **Government budget constraint (BC)**

\[
\begin{align*}
\left( \mu_t^B - i_t \right) B_t + \phi_t K_t \left( \tau a - g_t \right) &= 0 \\
\tilde{\mu}_t^B := & \tilde{\mu}_t \\
\bar{s} := &\end{align*}
\]

Primary surplus (per $K_t$)
What’s a Safe Asset? What is its Service Flow?

\[ \frac{B_t}{\varphi_t} = E_t[PV\xi^*(primary\ surpluses)] + E_t[PV\xi^*(service\ flow)] \] (FTPL*)
What’s a Safe Asset? What is its Service Flow?

\[ \frac{B_t}{\mathcal{S}_t} = \mathbb{E}_t[PV_{\xi^{**}}(\text{primary surpluses})] + \mathbb{E}_t[PV_{\xi^{**}}(\text{service flow})] \]  

Example: = 0

Portfolio of Safe asset
Cash flow asset

A
B

0
0

CF
CF

CF
CF

0
0

CF
CF

0
0

CF
CF

...
What’s a Safe Asset? What is its Service Flow?

\[ \frac{B_t}{\sigma_t} = \mathbb{E}_t[PV_{\xi^*}(\text{primary surpluses})] + \mathbb{E}_t[PV_{\xi^*}(\text{service flow})] \quad \text{(FTPL**)} \]

- Value come from re-trading
- Insures by partially completing markets
- Re-duces \( \mathbb{V}ar_t[\tilde{\sigma}_c] \)
- Can be "bubbly" = fragile

...
What’s a Safe Asset?

- In incomplete markets setting (Bewley, Aiyagari, BruSan, ...)
- Good friend analogy (Brunnermeier Haddad, 2012)
  - When one needs funds, one can sell at stable price... since others buy
    - Idiosyncratic shock: Partial insurance through re-trading - low bid-ask spread
    - Aggregate (volatility) shock: Appreciate in value in times of crises

Safe asset definition

- Tradeable: no asymmetric info – info insensitive
  - Service flow is derived from “dynamic re-trading”

Individual \( \beta_t^i \) = \(-\frac{\text{Cov}_t[\frac{d\xi_t^i}{\xi_t^i}, dr_t]}{\text{Var}_t[\frac{d\xi_t^i}{\xi_t^i}]} \) \leq 0

where \( \xi_t^i \) is SDF of agent \( i \)

Note: \(-\text{Cov}_t[\frac{d\xi_t^i}{\xi_t^i}, dr_t] = \zeta_t^i \sigma_t^r + \xi_t^i \sigma_t^{r,i} \)
where \( d\xi_t^i/\xi_t^i = -r_t^f dt - \zeta_t^i dZ_t - \xi_t^i d\tilde{Z}_t \)
Solving MacroModels Step-by-Step

0. Postulate aggregates, price processes & obtain return processes

1. For given $C/N$-ratio and SDF processes for each $i$ finance block
   a. Real investment $\iota$ + Goods market clearing (static)
      - Toolbox 1: Martingale Approach, HJB vs. Stochastic Maximum Principle Approach
   b. Portfolio choice $\theta$ + Asset market clearing or Asset allocation $\kappa$ & risk allocation $\chi$
      - Toolbox 2: “price-taking social planner approach” – Fisher separation theorem
      - Toolbox 3: Change in numeraire to total wealth (including SDF)
        - “Money evaluation/FTPL equation” $\varphi$

2. Evolution of state variable $\eta$ (and $K$)

3. Value functions
   a. Value fcn. as fcn. of individual investment opportunities $\omega$
      - Special cases: log-utility, constant investment opportunities
   b. Separating value fcn. $V^i(n^i; \eta, K)$ into $v^i(\eta)(\bar{\eta}^i)^{1-\gamma} u(K)(n^i/n^i)^{1-\gamma}$
   c. Derive $\bar{\rho} = C/N$-ratio and $\zeta$ price of risk

4. Numerical model solution
   a. Transform BSDE for separated value fcn. $v^i(\eta)$ into PDE
   b. Solve PDE via value function iteration

5. KFE: Stationary distribution, Fan charts
Assets, Aggregate Resource Constraint, and Markets

- **Assets: capital and bonds**
  - $q_t^K$ Capital price
  - $q_t^B := \frac{B_t}{\varphi_t}$ value of the bonds per unit of capital
  - $q_t := \frac{q_t^K \varphi_t}{q_t^K + q_t^B}$ Share of bond wealth
- **Postulate Ito price processes**
  - $dq_t^K/q_t^K = \mu_t^{q,K} dt + \sigma_t^{q,K} dZ_t$, $dq_t^B/q_t^B = \mu_t^{q,B} dt + \sigma_t^{q,B} dZ_t$
  - $d\varphi_t/\varphi_t = \mu_t^\varphi dt + \sigma_t^\varphi dZ_t$
  - SDF for each $\tilde{i}$ agent $d\xi_t^{\tilde{i}}/\xi_t^{\tilde{i}} = -r_t^{\tilde{i}} dt - \zeta_t^{\tilde{i}} dZ_t - \zeta_t^{\tilde{i}} d\tilde{Z}_t$

- **Aggregate resource constraints:**
  - **Output:** $C_t + \iota_t K_t + g K_t = a K_t$
  - **Capital:** $\int \left( k_t^{\tilde{i}} d\Delta_t^{k,\tilde{i}} \right) d\tilde{i} = 0$

- **Markets:** Walrasian goods, bonds, and capital markets

---

Poll 34: Why do risk-free rate and price of risk not depend on individual $\tilde{i}$?

- a) risk-free bond can be traded
- b) aggregate risk can be traded
- c) CRRA utility for all agents with same $\gamma$
0. Return on Gov. Bond/Money

- Number of Bonds/coins follows:

\[
\frac{dB_t}{B_t} = (\mu_t^B + i_t) dt + \sigma_t^B dZ_t
\]

- Where \( i_t \) is interest paid on government bonds/outside money (reserves)

- Return on Gov. Bond/Money: in output numeraire

\[
dr_t^B = i_t dt + \frac{d(q_t^B K_t/B_t)}{q_t^B K_t/B_t} - \text{inflation}
\]

\[
= \frac{d(q_t^B K_t)}{q_t^B K_t} - \mu_t^B dt - \sigma_t^B dZ_t + \sigma_t^B (\sigma_t^B - \sigma - \sigma_{t,q,B}^B) dt
\]

- Seigniorage (excluding interest paid to money holders)
0. Distribution of “Seigniorage”

1. Proportionally to bond/money holdings
   - No real effects, only nominal

2. Proportionally to capital holdings
   - Bond/Money return decreases with $dB_t$ (change in debt level/money supply)
   - Capital return increases
   - Pushes citizens to hold more capital

3. Proportionally to net worth
   - Fraction of seigniorage goes to capital - same as 2.
   - Rest of seigniorage goes to money holders - same as 1.

4. Per capita
   - No real effects: people simply borrow against the transfers they expect to receive
0. Return on Capital (with seigniorage rebate terms)

\[ dr_t^{K,\bar{i}} = \frac{a(1-\tau)-\bar{i}_t}{q_t^K} dt + \frac{d(q_t^K k_t^{\bar{i}})}{q_t^K k_t^{\bar{i}}} = \left( \frac{a(1-\tau)-\bar{i}_t}{q_t^K} + \Phi(i_t) - \delta + \mu_t q^K \right) dt + \sigma_t q^K dZ_t + \tilde{\sigma} d\tilde{Z}_t \]

- Use government budget constraint to substitute out \( \tau \) (and \( B_t/\bar{o}_t = q_t^B K_t \))

\[ \frac{(\mu_t^B - i_t) q_t^B}{\tilde{\mu}_t^{B,t}} + (\tau a - g) = 0 \]

\[ dr_t^{K,\bar{i}} = \left( \frac{\bar{a}}{\bar{a}-\bar{g}} - i_t^{\bar{a}} + \Phi(i_t) - \delta + \mu_t^{q^K} + \frac{q_t^B}{q_t^K} \tilde{\mu}_t^{B,t} \right) dt + \sigma_t q^K dZ_t + \tilde{\sigma} d\tilde{Z}_t \]

Simplified case:

\[ \sigma^B = 0 \]
1. Optimal Choices

- Optimal investment rate
  \[ \phi_{it} = q_t^K - 1 \]

- Consumption
  \[ \frac{c_t}{n_t} =: \tilde{\rho}_t \Rightarrow C_t = \tilde{\rho}_t(q_t^B + q_t^K)K_t \]

- Looking ahead to Step 3:
  When is \( \frac{c}{n} \) constant? Recall \( \frac{c}{n} = \rho^{1/\gamma} \omega^{1-1/\gamma} \)
  - Log utility, \( \gamma = 1 \): \( \tilde{\rho} = \rho \)
  - In steady state:
    \( \omega \) investment opportunity/net worth multiplier is constant
1. Optimal Choices & Market Clearing

- Optimal investment rate
  \[ \phi_t = q_t^K - 1 \]

- Consumption
  \[ \frac{c_t}{n_t} = \tilde{\rho}_t \Rightarrow C_t = \tilde{\rho}_t (q_t^B + q_t^K)K_t = (a - \iota_t - \varpi_t)K_t \]

- Portfolio
  Solve for \( \theta_t \) later

  \[ 1 - \theta_t = 1 - \vartheta_t \]

Goods market

Capital market

Bond market

clears by Walras law
### Equilibrium (before solving for portfolio choice)

<table>
<thead>
<tr>
<th></th>
<th>Equilibrium</th>
</tr>
</thead>
<tbody>
<tr>
<td>$q^B_t$</td>
<td>$\vartheta_t \cdot \frac{1 + \phi \tilde{a}}{(1 - \vartheta_t) + \phi \tilde{p}_t}$</td>
</tr>
<tr>
<td>$q^K_t$</td>
<td>$(1 - \vartheta_t) \cdot \frac{1 + \phi \tilde{a}}{(1 - \vartheta_t) + \phi \tilde{p}_t}$</td>
</tr>
<tr>
<td>$\iota_t$</td>
<td>$\frac{(1 - \vartheta_t)\tilde{a} - \tilde{p}_t}{(1 - \vartheta_t) + \phi \tilde{p}_t}$</td>
</tr>
</tbody>
</table>

- Moneyless equilibrium with $q^B_t = 0 \Rightarrow \vartheta_t = 0$
- Next, determine portfolio choice.
Solving MacroModels Step-by-Step

0. Postulate aggregates, price processes & obtain return processes

1. For given $C/N$-ratio and SDF processes for each $i$
   a. Real investment $\iota$ + Goods market clearing (static)
      - Toolbox 1: Martingale Approach, HJB vs. Stochastic Maximum Principle Approach
   b. Portfolio choice $\theta$ + Asset market clearing or Asset allocation $\kappa$ & risk allocation $\chi$
      - Toolbox 2: “price-taking social planner approach” – Fisher separation theorem
      - Toolbox 3: Change in numeraire to total wealth (including SDF)
        - “Money evaluation/FTPL equation” $\vartheta$

2. Evolution of state variable $\eta$ (and $K$) \textbf{forward equation}

3. Value functions \textbf{backward equation}
   a. Value fcn. as fcn. of individual investment opportunities $\omega$
      - Special cases: log-utility, constant investment opportunities
   b. Separating value fcn. $V^i(n^i; \eta, K)$ into $v^i(\eta)(\bar{\eta}^{i})^{1-\gamma} u(K)(n^i/n^i)^{1-\gamma}$
   c. Derive $\bar{\rho} = C/N$-ratio and $\zeta$ price of risk

4. Numerical model solution
   a. Transform BSDE for separated value fcn. $v^i(\eta)$ into PDE
   b. Solve PDE via value function iteration

5. KFE: Stationary distribution, Fan charts
1.b Portfolio choice $\theta$: Bond/Money Evaluation/FTPL Equation

- Recall: Expected return: $\mu_t^A = r_t^i + \zeta_t^i \sigma_t^A$
  - Excess expected return to risky asset B: $\mu_t^A - \mu_t^B = \zeta_t^i (\sigma_t^A - \sigma_t^B)$

- Alternative derivations:
  - In consumption numeraire
    i. Expected excess return of capital w.r.t. bond return
    ii. Expected excess return of net worth (portfolio) w.r.t. bond return
  - In total net worth numeraire
    iii. Expected excess return of capital w.r.t. bond return
    iv. Expected excess return of individual net worth (=net worth share) w.r.t. bond return (per bond)
1.1 Portfolio choice $\theta$: Bond/Money Evaluation/FTPL Equation

- For stationary setting with $\sigma = \sigma_t^B = 0$

- Asset pricing equation (martingale method):
  \[
  \mu_t^A - \mu_t^B = \zeta_t^i (\sigma_t^A - \sigma_t^B)
  \]

\[
\begin{align*}
\mathbb{E}_t \left[ d\ln K_t \right] &= \frac{\ddot{a} - \iota_t}{q_t^K} + \frac{q_t^K}{q_t^K} \dddot{\mu} + \Phi(\iota_t) - \delta + \mu_t^q + \sigma \sigma_t^q = r_t^f + \zeta_t^i \sigma_t^q + \tilde{c}_t \tilde{\sigma} \\
\mathbb{E}_t \left[ d\ln B_t \right] &= \left[ \tilde{\mu} - \Phi(\iota_t) - \delta + \mu_t^q + \sigma \sigma_t^q \right] = r_t^f + \zeta_t^i \sigma_t^q
\end{align*}
\]

- Goods market clearing: $\tilde{\rho}(q_t^K + q_t^K) = (\ddot{a} - \iota_t)K_t \Rightarrow \frac{\ddot{a} - \iota_t}{q_t^K} = \frac{\tilde{\rho}}{1 - \theta_t}$

- Price of idiosyncratic risk: $\tilde{\zeta}_t = \gamma \tilde{\sigma}_t^n = (1 - \theta_t)\gamma \tilde{\sigma}$

- Capital market clearing: $1 - \theta_t = 1 - \theta_t$

Poll 43: Why is the price of idiosyncratic risk simply $\gamma \tilde{\sigma}_t^n$ for CRRA?

a) because idiosyncratic risk for $\sigma = 0$
there is no idiosyncratic investment opportunity
b) idiosyncratic risk is always myopic (like for log-u)
1.b Portfolio choice $\theta$: Bond/Money Evaluation/FTPL Equation

- For stationary setting with $\sigma = \sigma^B_t = 0$

- Asset pricing equation (martingale method):
  \[ \mu^A_t - \mu^B_t = \zeta_t^i (\sigma^A_t - \sigma^B_t) \]

- Goods market clearing:
  \[ \ddot{\rho}(q^K_t + q^B_t)K_t = (\ddot{a} - \iota_t)K_t \Rightarrow \frac{\ddot{a} - \iota_t}{q^K_t} = \frac{\ddot{\rho}}{1 - \theta_t} \]

- Price of idiosyncratic risk:
  \[ \zeta_t = \gamma \tilde{\sigma}^n_t = (1 - \theta_t)\gamma \tilde{\sigma} \]

- Capital market clearing:
  \[ 1 - \theta_t = 1 - \theta_t \]

- Money valuation Equation:
  \[ \mu_t^\theta = \rho + \tilde{\mu}^B_t - (1 - \theta_t)^2 \gamma \tilde{\sigma}^2 \]

  - In steady state $\mu^\theta_t = 0$: 
    \[ (1 - \theta) = \sqrt{\ddot{\rho} + \tilde{\mu}^B_t} / (\sqrt{\gamma} \tilde{\sigma}) \]
1.b Deriving FTPL - traditional

- Money valuation equation for log utility $\gamma = 1$
  \[
  \vartheta_t \mu_t^\vartheta = \vartheta_t (\rho + \widehat{\gamma} - (1 - \vartheta_t)^2 \hat{\sigma}^2 - g + \ddot{\mu}_t^B)
  \]
  
- Integrate forward
  \[
  \vartheta_0 = \mathbb{E} \int_0^\infty e^{-rf_t} e^{gt} (-\ddot{\mu}_t^B) \vartheta_t dt \quad \text{recall gov. budget constraint } \ddot{\mu}_t^B = -s/q_t^B
  \]
  
  \[
  = \mathbb{E} \int_0^\infty e^{-rf_t} e^{gt} \frac{s}{q_t^B} \vartheta_t dt
  \]
  
  \[
  = \mathbb{E} \int_0^\infty e^{-(rf-g)t} \frac{sK_t}{N_t} dt
  \]

- Multiply by $N_0$: $\vartheta_0 N_0 = \mathbb{E} \left[ \int_0^\infty e^{-(rf-g)t} \frac{N_0}{N_t} sK_t \right] dt = \frac{B_0}{\varphi_0} = q^B_0 K_0 = \vartheta_0 N_0$

- FTPL equation: $\frac{B_0}{\varphi_0} = \mathbb{E} \left[ \int_0^\infty e^{-rf_t} sK_t dt \right]$ if $g < r^f$ since $K_t = e^{gt}K_0$
1.b Deriving FTPL – separating service flow with SDF $\xi_t^{**}$

- Money valuation equation for log utility $\gamma = 1$
  $$\vartheta_t \mu_t^\vartheta = \vartheta_t (\rho - (1 - \vartheta_t)^2 \tilde{\sigma}^2 + \ddot{\mu}_t^B)$$

- Integrate forward
  $$\vartheta_0 = \mathbb{E} \int_0^\infty e^{-\rho t} (-\ddot{\mu}_t^B + (1 - \vartheta_t)^2 \tilde{\sigma}^2) \vartheta_t dt$$
  $$= \mathbb{E} \int_0^\infty e^{-\rho t} \frac{s}{q_B^t} \vartheta_t dt + \mathbb{E} \int_0^\infty e^{-\rho t} (1 - \vartheta_t)^2 \tilde{\sigma}^2 \vartheta_t dt$$
  $$= \mathbb{E} \int_0^\infty e^{-\rho t} \frac{sK_t}{N_t} dt + \mathbb{E} \int_0^\infty e^{-\rho t} (1 - \vartheta_t)^2 \tilde{\sigma}^2 \frac{B_t}{\vartheta_t N_t} dt$$

- Multiply by $N_0$
  $$\vartheta_0 N_0 = \frac{B_0}{\vartheta_0} = \mathbb{E} \left[ \int_0^\infty e^{-\rho t} \frac{N_0}{N_t} sK_t dt \right] + \mathbb{E} \left[ \int_0^\infty e^{-\rho t} \frac{N_0}{N_t} (1 - \vartheta_t)^2 \tilde{\sigma}^2 \frac{B_t}{\vartheta_t N_t} dt \right]$$

- Service flow

1. Multiply by $N_0$
What is Quasi-SDF $\xi_t^{**} = \int \xi_t^i \eta_t^i \, di$?

- $\xi_t^{**} = \int \xi_t^i \eta_t^i \, d\tilde{t}$
  
  $$= \int e^{-\rho t} \frac{u'(c_t^i)}{u'(c_0^i)} \eta_t^i \, d\tilde{t} = \int e^{-\rho t} \left(\frac{c_t^i}{c_0^i}\right)^{-\gamma} \eta_t^i \, d\tilde{t} = \int e^{-\rho t} \left(\frac{\bar{p} n_t^i}{\bar{p} n_0^i}\right)^{-\gamma} \eta_t^i \, d\tilde{t}$$

- For log utility $\gamma = 1$: $\xi_t^{**} = \int e^{-\rho t} \left(\frac{n_0^i}{n_t^i}\right) \eta_t^i \, d\tilde{t} = e^{-\rho t} \frac{N_0}{N_t}$

- Total net worth (incl. bubble wealth) $= N_t = \mathbb{E}_t \left[ \int_t^{\infty} \frac{\int \xi_s^i \eta_s^i \, di}{\int \xi_t^i \eta_t^i \, di} C_s \, ds \right]$

- Net worth share weighted SDF
- “Representative agent SDF”
- Complete markets: $\xi_t^{**} = \xi_t$
Eliciting the service flow term - 2 Asset pricing perspectives

- **Buy and Hold Perspective:**
  - market cap = \( P_0 = \lim_{T \to \infty} \left( \mathbb{E} \left[ \int_0^T \xi_t^i \text{ AssetCashflow}_t \ dt \right] + \mathbb{E}[\xi_T^i P_T] \right) \)
    - If all agents \( i \) are marginal investors of aggregate risk asset

Agent \( i \)'s SDF, \( \xi_t^i: d\xi_t^i / \xi_t^i = -r_t^f dt - \zeta dZ_t - \xi_t^i d\tilde{Z}_t \)

First aggregate and then iterate (over time)
Eliciting the service flow term - 2 Asset pricing perspectives

Buy and Hold Perspective:
- Market cap = \( P_0 = \lim_{T \to \infty} \left( \mathbb{E} \left[ \int_0^T \xi_t^i \text{AssetCashflow}_t \, dt \right] + \mathbb{E} \left[ \xi_T^i P_T \right] \right) \)
  - If all agents \( i \) are marginal investors of aggregate risk asset

Dynamic Trading Perspective:
- Dynamic trading strategy leads to cashflows conditional on idiosyncratic risks
- Denote \( \eta^i \) the share of asset held by agent \( i \)
- \( = \lim_{T \to \infty} \left( \int \mathbb{E} \left[ \int_0^\infty \xi_t^i \eta_t^i \text{AssetCashflow}_t + \eta_t^i \text{TradingCashflow}_t \right] dt \right) di + \cdots \)
- \( = \mathbb{E} \left[ \int_0^\infty \int \frac{\xi^i \eta^i}{\xi^*_t} \text{AssetCashflow}_t dt \right] + \mathbb{E} \left[ \int_0^\infty \int \frac{\xi^i \eta^i}{\xi^*_t} \text{TradingCashflow}_t dt \right] \)
  - Discount rate \( E[dr^\eta]/dt = r^f + \tilde{\sigma} \)
  - \( \xi^i \) and \( \eta^i \) are negatively correlated \( \Rightarrow \) depresses weighted “Quasi-SDF” (higher discount rate)

\[ \frac{d\xi_t^i}{\xi_t^i} = -r_t^f \, dt - \zeta dZ_t - \xi_t^i \tilde{\sigma} d\tilde{Z}_t \]

First aggregate and then iterate (over time)

First iterate (over time) then aggregate
Buy and Hold Perspective:

Expected bond return

\[ \text{Ramsey term} \quad \text{Precautionary savings/self-insurance} \]

\[ = \rho + \gamma \mu^c - \frac{1}{2} \gamma (\gamma + 1) \left\{ (\sigma^c)^2 + (\bar{\sigma}^c)^2 \right\} + \text{risk premium} - \text{convience yield} \]

Risk-free rate \( r_f \) =

Dynamic Trading Perspective:

Expected bond return

\[ = \rho + \gamma \mu^c - \frac{1}{2} \gamma (\gamma + 1) (\sigma^c)^2 + \text{risk premium} - \left\{ \frac{1}{2} \gamma (\gamma + 1) (\bar{\sigma}^c)^2 + \text{convience yield} \right\} \]

Risk-free rate \( r_f^{**} \) =

\( r_f^{**} \) = “representative agent’s” risk-free rate
Recall: Equilibrium (before solving for portfolio choice)

- \( \mu^g_t = \rho + \mu^B_t - (1 - \vartheta_t)^2 \gamma \bar{\sigma}^2 \)
- In steady state \( \mu^g_t = 0 \): \( (1 - \vartheta) = \sqrt{\tilde{\rho} + \mu^B_t} / (\sqrt{\gamma} \bar{\sigma}) \)

<table>
<thead>
<tr>
<th>( q^B_t = )</th>
<th>( \vartheta_t )</th>
<th>( \frac{1 + \phi \tilde{a}}{(1 - \vartheta_t) + \phi \tilde{\rho}_t} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( q^K_t = (1 - \vartheta_t) )</td>
<td>( \frac{1 + \phi \tilde{a}}{(1 - \vartheta_t) + \phi \tilde{\rho}_t} )</td>
<td></td>
</tr>
<tr>
<td>( \iota_t = )</td>
<td>( \frac{(1 - \vartheta_t) \tilde{a} - \tilde{\rho}_t}{(1 - \vartheta_t) + \phi \tilde{\rho}_t} )</td>
<td></td>
</tr>
</tbody>
</table>
Two Stationary Equilibria

<table>
<thead>
<tr>
<th>Non-Monetary</th>
<th>Monetary</th>
</tr>
</thead>
<tbody>
<tr>
<td>$q^B_0 = 0$</td>
<td>$q^B = \frac{\left(\sqrt{\gamma} \bar{\sigma} - \sqrt{\bar{\rho} + \bar{\mu}^B}\right) (1 + \phi \tilde{a})}{\sqrt{\bar{\rho} + \bar{\mu}^B} + \phi \sqrt{\bar{\gamma} \bar{\sigma} \bar{\rho}}} = \frac{\left(\sqrt{\gamma} \bar{\sigma} - \sqrt{\bar{\rho} - s/q^B}\right) (1 + \phi \tilde{a})}{\sqrt{\bar{\rho} - s/q^B} + \phi \sqrt{\bar{\gamma} \bar{\sigma} \bar{\rho}}}$</td>
</tr>
<tr>
<td>$q^K_0 = \frac{1 + \phi \tilde{a}}{1 + \phi \bar{\rho}_0}$</td>
<td>$q^K = \frac{\sqrt{\bar{\rho} + \bar{\mu}^B} (1 + \phi \tilde{a})}{\sqrt{\bar{\rho} + \bar{\mu}^B} + \phi \sqrt{\bar{\gamma} \bar{\sigma} \bar{\rho}}}$</td>
</tr>
<tr>
<td>$t = \frac{\tilde{a} - \bar{\rho}_0}{1 + \phi \bar{\rho}_0}$</td>
<td>$t = \frac{\tilde{a} \sqrt{\bar{\rho} + \bar{\mu}^B} - \sqrt{\bar{\gamma} \bar{\sigma} \bar{\rho}}}{\sqrt{\bar{\rho} + \bar{\mu}^B} + \phi \sqrt{\bar{\gamma} \bar{\sigma} \bar{\rho}}}$</td>
</tr>
</tbody>
</table>

- For log utility
  - $\bar{\rho} = \bar{\rho}_0 = \rho$
  - $\gamma = 1$

$\rho$ time preference rate
$\phi$ adjustment cost for investment rate
$\tilde{\mu}_t^B = \mu_t^B - i_t$ bond issuance rate beyond interest rate
$\tilde{a} = a - \varrho$ part of TFP not spend on gov.
Remarks

- **Real risk-free rate**
  
  \[ r^f = \Phi(\mu(B)) - \delta - \mu^B = g \]

- \[ \mu^B = 0 \Rightarrow s = 0 \] no primary surplus (no cash payoff for bond)
  
  - \[ q^B K = \frac{B}{P} > 0 \] bond trades at a *bubble* due to service flow

- \[ \mu^B > 0 \Rightarrow s < 0 \] primary deficit (constant fraction of GDP)
  
  - As long as \[ q^B > 0 \] “mine the bubble”

- \[ \mu^B < 0 \Rightarrow s > 0 \] and \( r > g \) primary surplus (constant fraction of GDP)
  
  - \[ q^B K_t = E_t[PV_{r f}(sK_t)] \] no bubble, but service flow
Service Flow Term, Convenience Yield, Ponzi Scheme

**Service flow**
- Convenience yield: relaxes collateral constraint or CIA constraint (money)
  - Traditional measure: BAA-US Treasury spread

- Here: Partially completing markets through retrading
  - Low interest rate (cash flow) asset can be issued by everyone
  - Hence, corporate-Treasury spread = 0

**Ponzi scheme** is not feasibly for everyone

**No Ponzi constraint** may be binding
- Who can run a Ponzi scheme?
  - ... assigned by equilibrium selection
  - Likely to government, private entities are subject to solvency constraint
  - ... still there is a Debt Laffer Curve

exorbitant privilege
Flight to Safety: Comparative static w.r.t. $\tilde{\sigma}$

- **Flight to safety** into bubbly gov. debt
  - $q^B$ rises (disinflation)
  - $q^K$ falls and so does $\iota$ and $g$

- Similar with stochastic idiosyncratic volatility
Recall: Expected return: $\mu_t^A = r_t^i + \zeta_t^i \sigma_t^A$

- Excess expected return to risky asset B: $\mu_t^A - \mu_t^B = \zeta_t^i(\sigma_t^A - \sigma_t^B)$

Alternative derivations:

- In consumption numeraire
  i. Expected excess return of capital w.r.t. bond return
  ii. Expected excess return of net worth (portfolio) w.r.t. bond return (homework!)

- In total net worth numeraire
  iii. Expected excess return of capital w.r.t. bond return (homework!)
  iv. Expected excess return of individual net worth (=net worth share) w.r.t. bond return (per bond)
Extra: 1.b Recall: Change to total net worth numeraire $N_t$

- **SDF in consumption numeraire**
  \[
  \frac{d\xi_t^i}{\xi_t^i} = -r_t^f dt - \zeta_t dZ_t - \tilde{\zeta}_t d\tilde{Z}_t
  \]

- **SDF in $N_t$-numeraire**
  \[
  \frac{d\tilde{\xi}_t^i}{\tilde{\xi}_t^i} = \frac{d(\xi_t^i N_t)}{(\xi_t^i N_t)} = -(r_t^f - \mu_t^N + \zeta_t \sigma_t^N) dt - (\zeta_t - \sigma_t^N) dZ_t - \tilde{\zeta}_t d\tilde{Z}_t
  \]

- **Return in consumption numeraire:**
  \[
  dr_t^j = \mu_t^{r_j} dt + \sigma_t^{r_j} dZ_t - \tilde{\sigma}_t^{r_j} d\tilde{Z}_t
  \]

- **Return in $N_t$-numeraire**
  \[
  dr_{t,N}^j = \left(\mu_t^{r_j} - \mu_t^N - \sigma_t^N \left(\sigma_t^{r_j} - \sigma_t^N\right)\right) dt + \left(\sigma_t^{r_j} - \sigma_t^N\right) dZ_t - \tilde{\sigma}_t^{r_j} d\tilde{Z}_t
  \]

- **Value of self-financing strategy investing in asset in the consumption numeraire, e.g. $x_t^j$ satisfies**
  \[
  dx_t^j / x_t^j = dr_t^j.
  \]
  The same holds in the $N_t$-numeraire, but now the value is $x_t^j / N_t$. 

59
**Extra: 1.b Alternative iii: Portfolio choice $\theta$ ($N_t$-numeraire)**

Total net worth $N_t$ relative to a single bond/coin of money

- Asset pricing equation (martingale method)

$$
\mathbb{E}\left[\frac{dr_{\theta/B}}{dt}\right] = \tilde{\rho}_t = \left( r_f^t - (\Phi(\mu_t) - \delta) - \mu^q_t d_t + q^{K+q_B} - \sigma^q_t d_t + q^{K+q_B} + \zeta_t \sigma^q_t d_t + q^{K+q_B} \right) + (\sigma^N - \sigma^q_t)0 + \zeta_t (1 - \theta_t) \tilde{\sigma}
$$

$$
\mathbb{E}\left[\frac{dr_{\theta/B}}{dt}\right] = i_t + \mu_t^{\theta/B} = \left( r_f^t - (\Phi(\mu_t) - \delta) - \mu^q_t d_t + q^{K+q_B} - \sigma^q_t d_t + q^{K+q_B} + \zeta_t \sigma^q_t d_t + q^{K+q_B} \right) + \frac{(\sigma^N - \sigma^q_t)\sigma_t^{\theta/B}}{\text{price of risk in } N_t\text{-numeraire}}
$$

$$
\tilde{\rho}_t - i_t - \mu_t^{\theta/B} = -(\sigma^N - \sigma^q_t)\sigma_t^{\theta/B} + \zeta_t (1 - \theta_t) \tilde{\sigma}
$$

- Remark: $\theta/B = \text{wealth share per bond}$

- Value of a single bond/coin in $N_t$-numeraire

$$
\frac{d(\theta_t/B_t)}{\theta_t/B_t} = \mu_t^{\theta} dt + \sigma_t^{\theta} dZ_t - \mu_t^B dt - \sigma_t^B dZ_t + \sigma_t^B (\sigma_t^B - \sigma_t^{\theta}) dt
$$

$$
= \mu_t^{\theta/B} dt + \sigma_t^{\theta/B} dZ_t \quad \text{(defining return-drift and volatility)}
$$

- Terms are shifted into risk-free rate in $N_t$-numeraire, which drop out when differencing
Solving MacroModels Step-by-Step

0. Postulate aggregates, price processes & obtain return processes

1. For given $C/N$-ratio and SDF processes for each $i$ finance block
   a. Real investment $\iota$ + Goods market clearing (static)
      - Toolbox 1: Martingale Approach, HJB vs. Stochastic Maximum Principle Approach
   b. Portfolio choice $\theta$ + Asset market clearing or Asset allocation $\kappa$ & risk allocation $\chi$
      - Toolbox 2: “price-taking social planner approach” – Fisher separation theorem
      - Toolbox 3: Change in numeraire to total wealth (including SDF)
        - “Money evaluation/FTPL equation” $\vartheta$

2. Evolution of state variable $\eta$ (and $K$) forward equation

3. Value functions backward equation
   a. Value fcn. as fcn. of individual investment opportunities $\omega$
      - Special cases: log-utility, constant investment opportunities
   b. Separating value fcn. $V^i(n^i; \eta, K)$ into $v^i(\eta)(\tilde{\eta}^i)^{1-\gamma} u(K)(n^i/n)^{1-\gamma}$
   c. Derive $\tilde{\rho} = C/N$-ratio and $\zeta$ price of risk

4. Numerical model solution
   a. Transform BSDE for separated value fcn. $v^i(\eta)$ into PDE
   b. Solve PDE via value function iteration

5. KFE: Stationary distribution, Fan charts
3a.+b. + Isolating Idio. Risk

- Rephrase the conjecture value function as
  \[ V_{t}^{\tilde{i}} = \frac{\left(\omega_{t}^{i} n_{t}^{\tilde{i}}\right)^{1-\gamma}}{1-\gamma} = \left(\frac{\omega_{t} N_{t}}{K_{t}}\right)^{1-\gamma} \left(\frac{n_{t}^{\tilde{i}}}{N_{t}}\right)^{1-\gamma} K_{t}^{1-\gamma} \]

- \( v_{t}^{i} \) depend only on aggregate state \( \eta_{t} \)

- Ito’s quotation rule
  \[ \frac{d\tilde{n}_{t}^{i}}{\tilde{n}_{t}^{i}} = \frac{d(n_{t}^{i}/N_{t})}{n_{t}^{i}/N_{t}} = (\mu_{t}^{n_{t}^{i}} - \mu_{t}^{N} + (\sigma_{t}^{N})^{2} - \sigma^{N} \sigma^{n_{t}^{i}}) dt + (\sigma_{t}^{n_{t}^{i}} - \sigma_{t}^{N}) dZ_{t} + \tilde{\sigma}_{n}^{i} d\tilde{Z}_{t} = \tilde{\sigma}_{n}^{i} d\tilde{Z}_{t} \]

- Ito’s Lemma
  \[ \frac{d(\tilde{n}_{t}^{i})^{1-\gamma}}{(\tilde{n}_{t}^{i})^{1-\gamma}} = -\frac{1}{2} \gamma (1-\gamma) \left(\tilde{\sigma}_{n}^{i}\right)^{2} dt + (1-\gamma) \tilde{\sigma}_{n}^{i} d\tilde{Z}_{t} \]
3b. BSDE for $\nu^i_t$

$$\frac{dV^i_t}{V^i_t} = \frac{d \left( \nu_t \left( \bar{\eta}^i_t \right)^{1-\gamma} (K_t)^{1-\gamma} \right)}{\nu_t \left( \bar{\eta}^i_t \right)^{1-\gamma} (K_t)^{1-\gamma}}$$

- By Ito's product rule
  $$= \left( \mu^\nu_t + (1 - \gamma)(\Phi(\iota_t) - \delta) - \frac{1}{2} \gamma(1 - \gamma) \left( \sigma^2 + (\tilde{\sigma}n^i)^2 \right) + (1 - \gamma)\sigma\sigma^\nu_t \right) dt + \text{volatility terms}$$

- Recall by consumption optimality
  $$\frac{dV^i_t}{V^i_t} - \rho dt + \frac{c^i_t}{n^i_t} dt$$ follows a martingale
  - Hence, drift above = $\rho - \frac{c^i_t}{n^i_t}$

- BSDE:
  $$\mu^\nu_t + (1 - \gamma)(\Phi(\iota_t) - \delta) - \frac{1}{2} \gamma(1 - \gamma) \left( \sigma^2 + (\tilde{\sigma}n^i)^2 \right) + (1 - \gamma)\sigma\sigma^\nu_t = \rho - \frac{c^i_t}{n^i_t}$$
3. Deriving \( C/N \)-ratio \( \tilde{\rho} \) in stationary setting

- In stationary equilibrium
\[
\mu_t^v + (1 - \gamma)(\Phi(\mu_t) - \delta) - \frac{1}{2} \gamma (1 - \gamma) \left( \sigma^2 + (\tilde{\sigma}^n_t)^2 \right) + (1 - \gamma)\tilde{\sigma}_t^v = \rho - \frac{c_t^i}{n_t^v} = \rho
\]

- Recall and plug in
  - \( \tilde{\sigma}^n_t = (1 - \theta)\tilde{\sigma} = \sqrt{\rho + \mu_B}/\sqrt{\nu} \) using \( (1 - \theta) = \sqrt{\rho + \mu_B}/(\sqrt{\nu}\tilde{\sigma}) \)
  - \( \nu = \frac{a\sqrt{\rho + \mu_B} - \sqrt{\nu}\tilde{\sigma}\rho}{\sqrt{\rho + \mu_B} + \kappa\sqrt{\nu}\tilde{\sigma}\rho} \)

yields an equation for \( \tilde{\rho} \)
\[
(1 - \gamma) \left( \frac{1}{\kappa} \log \frac{\sqrt{\rho + \mu_B}}{\sqrt{\rho + \mu_B} + \phi\sqrt{\nu}\tilde{\sigma}\rho} - \delta \right) - \frac{1}{2} \gamma (1 - \gamma) \left( \sigma^2 + \frac{\tilde{\rho} + \mu_B}{\gamma^2} \right) = \rho - \tilde{\rho}
\]

- For \( \gamma = 1 \): \( \tilde{\rho} = \rho \)
4. Numerical Solution

- Simpler than previous lectures since there is no state variable
- Generalization with $\tilde{\sigma}_t$ as a state variable.
Cross-sectional net worth distribution

- $\tilde{\eta}_t = \frac{\tilde{n}_t}{N_t}$ is non-stationary ... and log-normally distributed

- Next: Extend model with net worth reset jumps to $\eta^*$
Aside 1: Model with idiosyncratic net worth reset jumps

- With Poisson intensity $\lambda$ net worth $\tilde{\eta}_t$ jumps from $\tilde{\eta}_0$ to $\tilde{\eta}_1$
- Log-utility $\Rightarrow$ same returns, no impact on equilibrium

\[
\frac{dn_t^i}{n_t^i} = \left( -\rho + g + \frac{a - \ell}{q^K} \right) dt + \left( 1 - \theta_t \right) \tilde{\sigma} dZ_t^i + j_t^{n,i} dJ_t^i
\]

\[
\frac{dn_t^i}{\eta_t^i} = \left( -\rho + g + \left( 1 - \vartheta_t \right) \frac{a - \ell}{q^K} \right) dt + \left( 1 - \theta_t \right) \tilde{\sigma} dZ_t^i + j_t^{n,i} dJ_t^i
\]

- Set $j_t^{n,i} = \frac{\eta_t^i - \overline{\eta}_t}{\eta_t^i}$
- KFE (for all $\eta \neq \eta^*$) is given by:

\[
0 = \frac{\left( 1 - \vartheta \right)^2 \tilde{\sigma}^2}{2} \frac{\partial^2(\eta^* g(\eta))}{\partial \eta^2} - \lambda g(\eta)
\]

- There is a kink at $\eta^*$
Aside 1: Model Extension with idiosyncratic reset jumps

\[ \frac{d\eta_t}{\eta_t} = (1 - \vartheta)\bar{\sigma}dZ_t + j_t^{\eta_t}dJ_t \]

- Set \( j_t^{\eta_t} = \frac{\eta_t^* - \eta_t}{\eta_t} \)
- KFE (for all \( \eta \neq \eta^* \)) is given by:

\[
0 = \frac{(1 - \vartheta)^2 \bar{\sigma}^2}{2} \frac{\partial(\eta^2 g(\eta))}{\partial \eta} - \lambda g(\eta)
\]

- There is a kink at \( \eta^* \)

- Solution under \( \eta^* = 1 \):

\[
C_1 = C_4 = \frac{2\lambda}{(1 - \zeta)^2 \bar{\sigma}^2 \alpha}, \quad C_2 = C_3 = 0
\]

- KFE (for all \( \eta \neq \eta^* \)) is given by:

\[
0 = g''(\eta)\eta^2 + 4g'(\eta)\eta + \left(2 - \frac{2\lambda}{(1 - \vartheta)^2 \bar{\sigma}^2}\right)g(\eta)
\]

- Euler’s equation – has closed-form solutions

\[
g(\eta) = C_1\eta^{\alpha_1} + C_2\eta^{\alpha_2} \text{ for } \eta < \eta^*
\]

\[
g(\eta) = C_3\eta^{\alpha_3} + C_4\eta^{\alpha_4} \text{ for } \eta \geq \eta^*
\]

\[
\int_0^\infty g(\eta)d\eta = 1, \quad \lim_{\eta \to 0} g(\eta) = \lim_{\eta \to \infty} g(\eta) = 0
\]

- Continuity at \( \eta^* \), \( \alpha_1 = \frac{\alpha - 3}{2}, \alpha_2 = -\frac{\alpha + 3}{2}, \alpha = \frac{\beta \lambda}{(1 - \vartheta)^2 \bar{\sigma}^2 + 1} \)
## Models on Money as Store of Value

<table>
<thead>
<tr>
<th>Friction</th>
<th>OLG</th>
<th>Incomplete Markets + idiosyncratic risk</th>
</tr>
</thead>
<tbody>
<tr>
<td>Risk</td>
<td>deterministic</td>
<td>endowment risk</td>
</tr>
<tr>
<td></td>
<td></td>
<td>borrowing constraint</td>
</tr>
<tr>
<td></td>
<td></td>
<td>return risk</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Risk tied up with</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Individual capital</td>
</tr>
<tr>
<td>Only money</td>
<td>Samuelson</td>
<td>Bewley</td>
</tr>
<tr>
<td>With capital</td>
<td>Diamond</td>
<td>Aiyagari</td>
</tr>
<tr>
<td></td>
<td></td>
<td>“I Theory without I”</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Brunnermeier-Sannikov</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(AER PP 2016)</td>
</tr>
</tbody>
</table>
Aside 2: BruSan meets Bewley-Huggett-Aiyagari

\[ \max_{c,\theta} \mathbb{E} \int_{0}^{\infty} e^{-\rho t} u(c_t) dt \]

\[ dn_t = (-c_t^i + y_t^i) dt + n_t^i \left( r dt + (1 - \theta) (d r_t^k - r dt) \right) \]

\[ dr_t^k = r^k dt + \bar{\sigma}^k d Z_t^k \]

\[ dy_t^k = -\nu y_t^i dt + \bar{\sigma}^y d Z_t^y \]

Partial insurance via retrading

**BruSan:** \[ \bar{\sigma}^y = 0 \] ... retrade capital and safe asset + smooth consumption

**Bewley-Huggett-Aiyagari:** \[ \bar{\sigma}^k = 0 \] ... smooth consumption

- Risk does not scale with net worth \( \Rightarrow c/n \) and portfolio \( \theta \) depends on net worth
Roadmap

- Intuition for different “Monetary Theories”

- Monetary Model with one sector with constant idiosyncratic risk
  - Safe Asset and Service Flows
  - Bubble (mining) or not
  - 2 Different Asset Pricing Perspectives/SDFs

- Monetary model with time-varying idiosyncratic risk
  - Safe asset has negative $\beta$
  - Calibration:
    - Debt valuation puzzle, Debt Laffer Curve, Flight-to-Safety and Equity excess volatility

- Medium of Exchange Role
Time-varying Idio Risk + Equity Markets + Epstein-Zin

- **Equity Market**
  - Each citizen $\bar{\iota}$ can sell off a fraction $(1 - \bar{\chi})$ of capital risk to outside equity holders
  - Return $dr_t^{E,\bar{\iota}}$
    - Same risk as $dr_t^{K,\bar{\iota}}$
    - But $\mathbb{E}_t[dr_t^{E,\bar{\iota}}] < \mathbb{E}_t[dr_t^{K,\bar{\iota}}]$... due to insider premium
  - Prop.: Model equations as before but replace $\bar{\sigma}$ with $\bar{\chi}\bar{\sigma}$
- **Aggregate risk $dZ_t$: Heston model (time-varying idiosyncratic risk)**
  - $d\bar{\sigma}_t^2 = -\psi \left( \bar{\sigma}_t^2 - (\bar{\sigma}^0)^2 \right) dt - \sigma \bar{\sigma}_t dZ_t, \quad a_t = a(\bar{\sigma}_t)$
- **Monetary/bond issuing policy:** $dB_t/B_t = \mu_t^B dt + \sigma_t^B dZ_t$
- **Epstein-Zin preferences for calibration (EIS=1)**

$$V_t^i = E_t \left[ \int_t^{\infty} (1 - \gamma) \rho V_s^i \left( \log(c_s^i) - \frac{1}{1-\gamma} \log \left( (1 - \gamma) V_s^i \right) \right) ds \right]$$
What’s a Safe Asset? What is its Service Flow?

\[ \frac{B_t}{\varphi_t} = \mathbb{E}_t[PV\xi^*(primary\surpluses)] + \mathbb{E}_t[PV\xi^*(service\ flow)] \]

- Value come from re-trading
- Insures by partially completing markets
- Can be “bubbly” = fragile

In recessions:
- Risk is higher
- Service flow is more valuable
- Cash flows are lower
  (depends on fiscal policy)
Government: Taxes, Bond/Money Supply, Gov. Budget

- $\sigma_B^t \neq 0$ leads to stochastic “seigniorage revenue” (state contingent)
- Relabel tax revenue process to $\frac{d\tau_t}{\tau_t} = \mu_t^\tau dt + \sigma_t^\tau dZ_t$
  - Or should we label $s$ (primary surplus) as a process
- Government budget constraint (BC) [REDEFINE]
  $$dB_t - i_t B_t dt + \varnothing_t K_t \left( \frac{d\tau_t}{\tau_t} a_t - g dt \right) = 0$$
  
  Primary surplus (per $K_t$)

- Return on Gov. Bond/Money: in output numeraire

$$dr_t^B = dt + \frac{d\left( q_t^B K_t / B_t \right)}{q_t^B K_t / B_t} = \frac{d\left( q_t^B K_t \right)}{q_t^B K_t} - \mu_t^B dt - \sigma_t^B dZ_t + \sigma_t^B \left( \sigma_t^B - \sigma - \sigma_t^{q,B} \right) dt$$
2.b+c Alternative iii: Portfolio choice $\theta$ ($N_t$-numeraire)

Total net worth $N_t$ relative to single bond/coin

- Asset pricing equation (martingale method)

\[
\mathbb{E}\left[ \frac{dr_{t}^{\tilde{\eta}^i}}{dt} \right] = \tilde{\rho}_t = \left( r_t^f - (\Phi(\iota_t) - \delta) - \mu_t^{q^K+q^B} - \sigma_t q_t^{q^K+q^B} + \zeta_t \sigma_t^{q^K+q^B} \right) + (\zeta_t - \sigma_t^N) 0 + \tilde{\zeta}_t (1 - \theta_t) \tilde{\sigma} \\
\mathbb{E}\left[ \frac{dr_{t}^{\vartheta/B}}{dt} \right] = \mu_t^{\vartheta/B} = \left( r_t^f - (\Phi(\iota_t) - \delta) - \mu_t^{q^K+q^B} - \sigma_t^{q^K+q^B} + \zeta_t \sigma_t^{q^K+q^B} \right) + \frac{(\zeta_t - \sigma_t^N) \sigma_t^{\vartheta/B}}{\text{price of risk in } N_t-\text{numeraire}}
\]
2. b+c Alternative iii: Portfolio choice $\theta$ ($N_t$-numeraire)

Total net worth $N_t$ relative to a single bond/coin of money

- Asset pricing equation (martingale method)

\[
\mathbb{E} \left[ \frac{dr_t^{\tilde{N}}}{dt} \right] = \tilde{\rho}_t = \left( r_t^{f} - (\Phi(t) - \delta) - \mu_t^{qK+qB} - \sigma_t^{qK+qB} + \zeta_t \sigma_t^{qK+qB} \right) + (\zeta_t - \sigma_t^{N}) 0 + \tilde{\zeta}_t (1 - \theta_t) \tilde{\sigma}
\]

\[
\mathbb{E} \left[ \frac{dr_t^{\vartheta/B}}{dt} \right] = \mu_t^{\vartheta/B} = \left( r_t^{f} - (\Phi(t) - \delta) - \mu_t^{qK+qB} - \sigma_t^{qK+qB} + \zeta_t \sigma_t^{qK+qB} \right) + \frac{(\zeta_t - \sigma_t^{N}) \sigma_t^{\vartheta/B}}{\text{price of risk in } N_t\text{-numeraire}}
\]

- Remark:

\[
\tilde{\rho}_t - \mu_t^{\vartheta/B} = -(\zeta_t - \sigma_t^{N}) \sigma_t^{\vartheta/B} + \tilde{\zeta}_t (1 - \theta_t) \tilde{\sigma}
\]

- Value of a single bond/coin in $N_t$-numeraire

\[
\frac{d(\vartheta_t/B_t)}{\vartheta_t/B_t} = \mu_t^{\vartheta} + \sigma_t^{\vartheta} dZ_t - \mu_t^{B} dt - \sigma_t^{B} dZ_t + \sigma_t^{B} (\sigma_t^{B} - \sigma_t^{\vartheta}) dt
\]

\[
= \mu_t^{\vartheta/B} dt + \sigma_t^{\vartheta/B} dZ_t \quad \text{(defining return-drift and volatility)}
\]

- Terms are shifted into risk-free rate in $N_t$-numeraire, which drop out when differencing
2.b+c Alternative iii: Portfolio choice $\theta$ ($N_t$-numeraire)

Total net worth $N_t$ relative to single bond/coin of money

- Asset pricing equation (martingale method)

$$
\mathbb{E}\left[ \frac{dr_t^\eta}{dt} \right] = \tilde{\rho}_t = \left( r_t^f - (\Phi(t) - \delta) - \mu_t^q + qB - \sigma_t^q + qB + \zeta_t^q + qB \right) + (\zeta_t - \sigma_t^N)0 + \tilde{\zeta}_t (1 - \theta_t) \tilde{\sigma}
$$

$$
\mathbb{E}\left[ \frac{dr_t^{\vartheta/B}}{dt} \right] = \mu_t^{\vartheta/B} = \left( r_t^f - (\Phi(t) - \delta) - \mu_t^q + qB - \sigma_t^q + qB + \zeta_t^q + qB \right) + \frac{(\zeta_t - \sigma_t^N)\sigma_t^{\vartheta/B}}{\text{price of risk in } N_t\text{-numeraire}}
$$

$$
\tilde{\rho}_t - \mu_t^{\vartheta/B} = -(\zeta_t - \sigma_t^N)\sigma_t^{\vartheta/B} + \tilde{\zeta}_t (1 - \theta_t) \tilde{\sigma}
$$

- Price of Risk: $\zeta_t = -\sigma_t^p + \sigma_t^{p+q} + \gamma \sigma$, $\tilde{\zeta}_t = \gamma \tilde{\sigma}_t^N = \gamma (1 - \theta_t) \tilde{\sigma}$

$$
\tilde{\rho}_t - \mu_t^{\vartheta/B} = (\sigma_t^p - (\gamma - 1)\sigma)\sigma_t^{\vartheta/B} + \gamma (1 - \theta_t)^2 \tilde{\sigma}^2
$$

- Capital market clearing: $1 - \theta = 1 - \vartheta$

Recall

$$
\mu_t^{\vartheta/B} = \mu_t^\vartheta - \mu_t^\vartheta B + \sigma_t^B (\sigma_t^B - \sigma_t^\vartheta)
$$

$$
\sigma_t^{\vartheta/B} = \sigma_t^\vartheta - \sigma_t^B
$$
FTPL Equation with Bubble: 2 Perspectives

- Agent $\tilde{i}$’s SDF, $\xi_t^{\tilde{i}}$: $d\xi_t^{\tilde{i}}/\xi_t^{\tilde{i}} = -r_t^f dt - \zeta_t^e dZ_t - \tilde{\zeta}_t d\tilde{Z}_t$

- Buy and Hold Perspective:

  - $\frac{B_0}{P_0} = \lim_{T \to \infty} \left( \mathbb{E} \left[ \int_0^T \xi_t^{\tilde{i}} s_t K_t dt \right] + \mathbb{E} \left[ \xi_T^{\tilde{i}} B_T/P_T \right] \right)$

  - Bubble is possible: $\lim_{T \to \infty} \mathbb{E}[\xi_t^{\tilde{i}} B_T/P_T] > 0$ if $r_t^f + \zeta_t^e \sigma_t^B \leq g_t$ (on average) $g - \bar{\mu}^B = $ discount rate

- Dynamic Trading Perspective:

  - Value cash flow from individual bond portfolios, including trading cash flows
  - Integrate over citizens weighted by net worth share $\eta_t^i$
  - Bond as part of a dynamic trading strategy

  - $\frac{B_0}{P_0} = \mathbb{E} \left[ \int_0^{\xi_t^{\tilde{i}} *} \left( \int \xi_t^{\tilde{i}} \eta_t^i di \right) s_t K_t dt \right] + \mathbb{E} \left[ \int_0^{\xi_t^{\tilde{i}} *} \left( \int \xi_t^{\tilde{i}} \eta_t^i di \right) (\tilde{\sigma}_t^e)^2 B_t/P_t dt \right]$

  - Discount rate $E[dr^\eta]/dt = r^f + \tilde{\zeta} \tilde{\sigma}$
  - $\xi_t^i$ and $\eta_t^i$ are negatively correlated $\Rightarrow$ depresses weighted “Quasi-SDF” (higher discount rate)
Numerical Steps

\[ \vartheta_t = \vartheta(\tilde{\sigma}_t), \text{Ito's formula:} \]

\[ d\vartheta_t = \left( \mu_{\tilde{\sigma}, t} \vartheta'(\tilde{\sigma}) + \frac{\sigma_{\tilde{\sigma}, t}^2}{2} \vartheta''(\tilde{\sigma}) \right) dt + \sigma_{\tilde{\sigma}, t} \vartheta'(\tilde{\sigma}) dZ_t \]

\[ \rho \vartheta(\tilde{\sigma}) = (1 - \vartheta(\tilde{\sigma}))^2 \tilde{\sigma}^2 \vartheta(\tilde{\sigma}) + b(\tilde{\sigma}_{ss} - \tilde{\sigma}) \vartheta'(\tilde{\sigma}) + \frac{\nu^2 \tilde{\sigma}}{2} \vartheta''(\tilde{\sigma}) \]

\[ \rho \vartheta_t(\tilde{\sigma}) = \partial_t \vartheta_t(\tilde{\sigma}) + (1 - \vartheta_t(\tilde{\sigma}))^2 \tilde{\sigma}^2 \vartheta_t(\tilde{\sigma}) + b(\tilde{\sigma}_{ss} - \tilde{\sigma}) \vartheta'_t(\tilde{\sigma}) + \frac{\nu^2 \tilde{\sigma}}{2} \vartheta''_t(\tilde{\sigma}) \]

- For \( \tilde{\sigma} \)-grid: from 0 to \( \tilde{\sigma} \) large enough (look at stationary distribution of \( \tilde{\sigma} \))
Bond and Capital Value for time-varying idiosyncratic risk $\tilde{\sigma}_t$

- Gov. debt value rises in recessions
- Capital price
Dynamic Trading Perspective Decomposition

\[
\frac{\mathcal{B}_0}{\mathcal{P}_0} = \mathbb{E} \left[ \int_0^\infty \xi_t^{**} s_t K_t \, dt \right] + \mathbb{E} \left[ \int_0^\infty \xi_t^{**} \gamma (\tilde{\sigma}_t)^2 \frac{\mathcal{B}_t}{\mathcal{P}_t} \, dt \right]
\]
Calibration

- Exogenous processes:
  - **Recessions** feature high idiosyncratic risk and low consumption
    - $\tilde{\sigma}_t$: Heston (1993) model of stochastic volatility
      \[ d\tilde{\sigma}_t^2 = -\psi \left( \tilde{\sigma}_t^2 - (\tilde{\sigma}^0)^2 \right) dt - \sigma \tilde{\sigma}_t dZ_t \]
    - $a_t$: $a_t = a(\tilde{\sigma}_t)$
      \[ a_t(\tilde{\sigma}_t) = a^0 - \alpha (\tilde{\sigma}_t - \tilde{\sigma}^0) \]
    - $\rho_t = 0$

- Government (bubble-mining policy)
  \[ \tilde{\mu}_t^B = \tilde{\mu}_t^{B,0} + \alpha^B (\tilde{\sigma}_t - \tilde{\sigma}^0) \]

- Calibration to US data (1970-2019, period length is one year)
## Parameters

<table>
<thead>
<tr>
<th>parameter</th>
<th>description</th>
<th>value</th>
<th>target</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\tilde{\sigma}^0$</td>
<td>$\tilde{\sigma}_t$ stoch. steady state</td>
<td>0.54</td>
<td>external calibration</td>
</tr>
<tr>
<td>$\psi$</td>
<td>$\tilde{\sigma}^2_t$ mean reversion</td>
<td>0.67</td>
<td>MLE targeting common idiosyncratic volatility (Herskovic et al. 2018)</td>
</tr>
<tr>
<td>$\sigma$</td>
<td>$\tilde{\sigma}^2_t$ volatility</td>
<td>0.4</td>
<td>Heaton, Lucas (1996, 2000, 2001), Angeletos (2007) (range [0.2, 0.6])</td>
</tr>
<tr>
<td>$\bar{\chi}$</td>
<td>undiversifiable idio. risk</td>
<td>0.3</td>
<td>calibration to match model moments</td>
</tr>
<tr>
<td>$\gamma$</td>
<td>risk aversion</td>
<td>6</td>
<td>chosen jointly to match (approximately)</td>
</tr>
<tr>
<td>$\rho$</td>
<td>time preference</td>
<td>0.138</td>
<td>- volatility of $Y, C, I, S/Y$</td>
</tr>
<tr>
<td>$a^0$</td>
<td>$a_t$ stoch. steady state</td>
<td>0.63</td>
<td>- average $C/Y, G/Y, S/Y, I/K, q^K K/Y, q^B K/Y$</td>
</tr>
<tr>
<td>$g$</td>
<td>gov. expenditures</td>
<td>0.138</td>
<td>- mean equity premium</td>
</tr>
<tr>
<td>$\mu^B,0$</td>
<td>$\tilde{\mu}_t^B$ stoch. steady state</td>
<td>0.0023</td>
<td>- equity Sharpe ratio</td>
</tr>
<tr>
<td>$\alpha^a$</td>
<td>$a_t$ slope</td>
<td>0.071</td>
<td></td>
</tr>
<tr>
<td>$\alpha^B$</td>
<td>$\tilde{\mu}_t^B$ slope</td>
<td>0.12</td>
<td></td>
</tr>
<tr>
<td>$\phi$</td>
<td>capital adjustment cost</td>
<td>8.5</td>
<td></td>
</tr>
<tr>
<td>$\delta$</td>
<td>depreciation rate</td>
<td>0.055</td>
<td>economic growth rate (ultimately irrelevant for all results)</td>
</tr>
</tbody>
</table>
## Quantitative Model Fit

<table>
<thead>
<tr>
<th>moment symbol</th>
<th>description</th>
<th>model</th>
<th>data</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\sigma(Y)$</td>
<td>output volatility</td>
<td>1.3%</td>
<td>1.3%</td>
</tr>
<tr>
<td>$\sigma(C)/\sigma(Y)$</td>
<td>relative consumption volatility</td>
<td>0.63</td>
<td>0.64</td>
</tr>
<tr>
<td>$\sigma(S/Y)$</td>
<td>surplus volatility</td>
<td>1.1%</td>
<td>1.1%</td>
</tr>
<tr>
<td>$\mathbb{E}[S/Y]$</td>
<td>average surplus-output ratio</td>
<td>-0.0004</td>
<td>-0.0005</td>
</tr>
<tr>
<td>$\mathbb{E}[q^K K/Y]$</td>
<td>average capital-output ratio</td>
<td>3.48</td>
<td>3.73</td>
</tr>
<tr>
<td>$\mathbb{E}[q^B K/Y]$</td>
<td>average debt-output ratio</td>
<td>0.74</td>
<td>0.71</td>
</tr>
<tr>
<td>$\mathbb{E}[d\tilde{r}^E - d\tilde{r}^B]$</td>
<td>average (unlevered) equity premium</td>
<td>3.62%</td>
<td>3.40%</td>
</tr>
<tr>
<td>$\frac{\mathbb{E}(d\tilde{r}^E - d\tilde{r}^B)}{\sigma(d\tilde{r}^E - d\tilde{r}^B)}$</td>
<td>equity sharpe ratio</td>
<td>0.31</td>
<td>0.31</td>
</tr>
</tbody>
</table>
Two Debt Valuation Puzzles

- Properties of US primary surpluses
  - Average surplus $\approx 0$
  - Procyclical surplus ($> 0$ in booms, $< 0$ in recessions)

- Two valuation puzzles from standard perspective: (Jiang, Lustig, van Nieuwerburgh, Xiaolan, 2019, 2020)
  1. “Public Debt Valuation Puzzle”
     - Empirical: $E[\text{PV(surpluses)}] < 0$, yet $\frac{B}{\phi} > 0$
     - Our model: bubble/service flow component overturns results
  2. “Gov. Debt Risk Premium Puzzle”
     - Debt should be positive $\beta$ asset, but market don’t price it this way
     - Our model: can be rationalized with countercyclical bubble/service flow
Debt Laffer Curve ≠ MMT  Debt Sustainability Analysis 1

- Issue bonds at a faster rate $\tilde{\mu}^B$ (esp. in recessions)
  - $\Rightarrow$ tax precautionary self insurance $\Rightarrow$ tax rate
  - $\Rightarrow$ real value of bonds, $\frac{B}{\phi}$, $\Rightarrow$ “tax base”
- Less so in recession due to flight-to-safety

Sizeable revenue only if Gov. debt has negative $\beta$
Service Flow Term, Convenience Yield, Ponzi Scheme

- **Ponzi scheme** is not feasibly for everyone
  - **No Ponzi constraint** may be binding
  - Who can run a Ponzi scheme? exorbitant privilege
    - ... assigned by equilibrium selection
    - Likely to government, private entities are subject to solvency constraint
      - ... still there is a Debt Laffer Curve

- **Service flow**
  - Convenience yield: relaxes collateral constraint or CIA constraint (money)
    - Traditional measure: BAA-US Treasury spread
  
  - Here: Partially completing markets through retrading
    - Low interest rate (cash flow) asset can be issued by everyone
      - Hence, corporate-Treasury spread = 0
Why Does Safe Asset Survive in Presence of ETFs?

- Diversified stock portfolio is free of idiosyncratic risk
  - Trading in stocks (ETF) can also self-insure idiosyncratic risk
    - Good friend in idiosyncratically bad times

- But: poor hedge against aggregate risk, losses value in recessions
  - Positive $\beta$
    - Bad friend in aggregate bad times

- Why positive $\beta$? (after all $r^f$ goes down in recessions, lowers discount rate)
  - Equity are claims to capital, but marginal capital holder is insider
  - Insider bears idiosyncratic risk, must be compensated
  - $\tilde{\sigma}_t \uparrow \Rightarrow$ insider premium $E_t[dr_t^K] - E_t[dr_t^E] \uparrow \Rightarrow$ payouts to stockholders fall
Stock Market Volatility due to Flight to Safety

- "Aggregate Intertemporal Budget Constraint
  \[ q^K_t K_t + q^B_t K_t = \mathbb{E}_t [ \int_t^{\infty} \int \xi^S_i \eta^i_{S_i} d_i \int \xi^T_i \eta^i_{T_i} d_i C_s ds ] \] (*)

- Lucas-type models: \( q^B = 0 \) (also \( C_t = Y_t \), no idiosyncratic risk)
  - Value of equity (Lucas tree) = PV of consumption claim
  - Volatility equity values require volatile RHS of (*)

- This model: even for constant RHS of (*), \( q^K_t K_t \) can be volatile due to flight to safety:
  - Increase in \( \tilde{\sigma}_t \) \( \Rightarrow \) Portfolio reallocation from capital to bonds, \( q^K_t K_t \downarrow, B_t/\varnothing_t \uparrow \)

- Quantitatively relevant? Yes
  - Excess return volatility
    - 2.9% in equivalent bondless model \( (s = 0 \) and no bubble)
    - 12.9% in our model
Loss of Safe Asset Status – Equilibrium selection

- When government debt has a (stationary) bubble, other equilibria possible
  - Stationary no bubble equilibrium
  - Nonstationary equilibria that converge to the no bubble equilibrium

- Implies fragility: bubbles may pop, loss of safe asset status

- Are there policies to prevent a loss of safe asset status?
  1. Create a “fundamentally safe asset”
     - Raise (positive) surpluses to generate safe cash flow component $q_{t}^{B,CF}$
     - If surpluses always exceed a (positive) fraction of total output, no bubble
     - But: gives up revenues from bubble mining
  2. Off-equilibrium tax backing
     - Sufficient to (credibly) promise policy 1 off equilibrium
     - See “FTPL with a Bubble”
Roadmap

- Intuition for different “Monetary Theories”

- Monetary Model with one sector with constant idiosyncratic risk
  - Safe Asset and Service Flows
  - Bubble (mining) or not
  - 2 Different Asset Pricing Perspectives/SDFs

- Monetary model with time-varying idiosyncratic risk
  - Safe asset has negative CAPM-\( \beta \)
  - Calibration:
    - Debt valuation puzzle, Debt Laffer Curve, Flight-to-Safety and Equity Excess Volatility

- Medium of Exchange Role
ADD “Medium of Exchange” to Store of Value

- Fiscal Theory of the Price Level (FTPL)  
  - SDF is time-varying + Bubble term

- ADD
  - Cash-in-advance constraint, transaction cost, shopping time model, ...
  - \( \Rightarrow \Delta i \) (convenience yield)
  - Price level is determined by \( M_t v(\cdot) = \phi_t Y_t \)
The 4 Roles of Money

- **Unit of account**
  - Intratemporal: Numeraire bounded rationality/price stickiness
  - Intertemporal: Debt contract incomplete markets

- **Store of value**
  - “I Theory of Money without I”
  - Fiscal theory of the price level

- **Medium of exchange**
  - Overcome double-coincidence
Medium of Exchange – Transaction Role

- Overcome double-coincidence of wants

Quantity equation: \( \phi_t T_t = \nu M_t \)

- \( \nu \) (nu) is velocity  (Monetarism: \( \nu \) exogenous, constant)
- \( T \) transactions
  - Consumption
  - New investment production
  - Transaction of physical capital
  - Transaction of financial claims

\[ \begin{align*}
C & \quad Y \\
\iota K & \\
d \Delta^k &
\end{align*} \]

produce own machines

infinite velocity

infinite velocity
Models of Medium of Exchange

- Reduced form models
  - Cash in advance
    \[ T_t = \nu \frac{M_t}{\phi_t} \]
  - Shopping time models
  - Money in the utility function
    - New Keynesian Models
    - No satiation point
    - New Monetary Economics
    \[ c_t \leq \sum_{j \in M} \nu^j \theta^j n_t \]
    \[ c = (c^c, l) \]
    consume money
    CES
  - Only assets \( j \in M \) with money-like features

For generic setting encompassing all models:
see Brunnermeier-Niepelt 2018
Cash in Advance

- Liquidity/cash in advance constraint
  - \( c_t \leq \sum_{j \in M} v^j \theta^j n_t \)  
  - Lagrange multiplier \( \hat{\lambda}_t \)
  - Asset \( j \in M \) which relaxes liquidity/CIA constraint

- Price of liquid/money asset

\[
q_{t}^{j \in M} = E_t \left[ \frac{\xi_{t+\Delta}}{\xi_t} \left( x_{t+\Delta} + q_{t+\Delta}^{j \in M} \right) \right] - \hat{\lambda}_t v^j q_{t}^{j \in M} \]

\[
q_{t}^{j \in M} = E_t \left[ \frac{\xi_{t+\Delta}}{\xi_t} \frac{1}{1 + \hat{\lambda}_t v^j} \left( x_{t+\Delta} + q_{t+\Delta}^j \right) \right]
\]

\[
q_{t}^{j \in M} = \lim_{T \to \infty} E_t \left[ \sum_{\tau=1}^{(T-t)/\Delta} \frac{\xi_{t+\tau\Delta}}{\xi_t} \frac{\Lambda^j_{t+\tau\Delta}}{\Lambda^j_t} x_{t+\tau\Delta} \right] + \lim_{T \to \infty} E_t \left[ \frac{\xi_T}{\xi_t} \frac{\Lambda^j_T}{\Lambda^j_t} q_T \right]
\]

As if SDF is multiplied by “liquidity multiplier” (Brunnermeier-Niepelt)
Cash in Advance

- Liquidity/cash in advance constraint
  - $c_t = \sum_{j \in M} \nu^j \theta^j n_t$  
    Lagrange multiplier $\hat{\lambda}_t$
  - Asset $j \in M$ which relaxes liquidity/CIA constraint

$$q^{j \in M}_t = \lim_{T \to \infty} E_t \left[ \int_t^T \frac{\xi_\tau \Lambda^j_\tau}{\xi_t \Lambda^j_t} x_\tau d\tau \right] + \lim_{T \to \infty} E_t \left[ \frac{\xi_T \Lambda^j_T}{\xi_t \Lambda^j_t} q_T \right]$$

- “Money bubble” easier to obtain due to liquidity service
  - Condition absent aggregate risk: $r^M < g$ easier to obtain since $r^M < r^f$

- Stochastic Maximum Principle approach (with constraints)

$$\mu^r,j_t = r^f_t + \zeta_t \sigma^r,j_t + \tilde{\zeta}_t \tilde{\sigma}^r,j_t - \lambda_t \nu_t$$

where $\lambda_t = \hat{\lambda}_t / V'(n_t)$

(Shadow) risk-free rate of illiquid asset
Add Cash in Advance to BruSan Model

- Return on money (no bonds)
  - Store of value – as before
  - Liquidity service (medium of exchange)
    \[
    E\left[ d\tau_t^M \right] = \Phi(\mu_t) - \delta + \mu_t^p + \sigma \sigma_t^p - \mu^M = r_t^f + \zeta_t (\sigma + \sigma_t^p) - \lambda_t v^M
    \]
- In steady state
  \[
  \Phi(\mu) - \delta - \left( \frac{\mu^M - \lambda v^M}{\mu^M} \right) = r^f + \zeta \sigma
  \]

- Solving the model as before ...
  - By simply replace \( \mu^M \) with \( \mu^M - \lambda v_t^M \)
  - Special case: \( \dot{\mu}^M = 0 \), i.e. \( \mu^M = \lambda v^M \), \( \gamma = 1 \) \( \Rightarrow \) explicit solution as fcn of \( \bar{\rho} \)
    - Same \( q^K \) and \( q^B \) as a function of \( \zeta \),
    - But \( \bar{\rho} \neq \rho \) if CIA constraint binds in steady state, otherwise \( \bar{\rho} = \rho \)
      1. Assume it binds, i.e. \( \zeta = \nu \vartheta \)
      2. Recall from slide 21 for \( \dot{\mu}^M = 0 \) and \( \gamma = 1 \), \( \vartheta = \frac{\sigma - \sqrt{\zeta}}{\sigma} \)
      3. Equate 1. and 2. to obtain quadratic solution for \( \bar{\rho} \)
        1. If \( \vartheta < \rho \), then solution equals \( \bar{\rho} \)
        2. If \( \vartheta > \rho \), then \( \bar{\rho} = \rho \) and hence CIA doesn’t bind, \( \lambda = 0 \), above solution
  - “Occasionally” binding CIA constraint (outside of steady state)
  - for sufficiently high \( \tilde{\sigma} \), store of value (insurance motive) \( \Rightarrow \lambda_t = 0 \)
Add Money in Utility to BruSan Model

- Money in utility function $u(c, M/\varphi) = u(c, \theta n)$

- Can be expressed as equality constraint
  - Difference to CIA inequality: No satiation point

- DiTella add MiU to BruSan 2016 AER PP
The 4 Roles of Money

- **Unit of account**
  - Intratemporal: Numeraire bounded rationality/price stickiness
  - Intertemporal: Debt contract incomplete markets

- **Store of value**
  - “I Theory of Money without I”
    Less risky than other “capital” – no idiosyncratic risk
  - Fiscal theory of the price level

- **Medium of exchange**
  - Overcome double-coincidence of wants problem

- **Record keeping device – money is memory**
  - Virtual ledger
Extra Slides
Related Literature

- **Safe Asset:**
  - Gorton-Pennachi (1990), Dang et al (2017), ...
  - Brunnermeier et al. (2017), ESBies,

- **Equity premium**
  - Constantinides-Duffie (1996) – imposes “aggregate” transversality condition

- **Public Debt Evaluation Puzzles:**
  - Jiang et al. (2020,2021)

- **Fiscal debt sustainability $r$ vs. $g$:**
  - Bassetto-Cui (2018), Reis (2020)

$m$ instead of $\xi^{**}/r^{**}$
Deriving FTPL equation (in cts time)

- Nominal government budget constraint
  \[
  \left( \mu_t^B B_t + \mu_t^M M_t + \varphi_t T_t \right) dt = (i_t B_t + i_t^m M_t + \varphi_t G_t) dt
  \]

- Multiply by nominal SDF $\xi_t/\varphi_t$, rearrange
  \[
  \left[ (\mu_t^B - i_t) \frac{\xi_t}{\varphi_t} B_t + (\mu_t^M - i_t) \frac{\xi_t}{\varphi_t} M_t \right] dt = -\xi_t (T_t - G_t - \frac{(i_t - i_t^m) M_t}{\Delta i_t}) dt
  \]

- Suppose $\xi_t/\varphi_t$ prices the nominal bond
  - Then $E_t \left[ \frac{d(\xi_t/\varphi_t)}{(\xi_t/\varphi_t)} \right] = i_t dt$
  - Substitute into above, use product rule, take expectations
    \[
    E_t \left[ d \left( \frac{\xi_t}{\varphi_t} (B_t + M_t) \right) \right] = -E_t \left[ \xi_t \left( T_t - G_t - \Delta i_t \frac{M_t}{\varphi_t} \right) dt \right]
    \]

- In integral form
  \[
  \frac{B_t + M_t}{\varphi_t} = E_t \int_t \frac{\xi_s}{\xi_t} (T_s - G_s) ds + E_t \int_t \frac{\xi_s}{\xi_t} \Delta i_s \frac{M_s}{\varphi_s} ds + \frac{\xi_T}{\xi_t} \frac{B_T + M_T}{\varphi_T}
  \]
Deriving FTPL equation (in cts time)

- Take limit $T \to \infty$

$$\frac{B_t + M_t}{\xi_t} = E_t \int_t^\infty \frac{\xi_s}{\xi_t} (T_s - G_s)ds + E_t \int_t^\infty \frac{\xi_s}{\xi_t} \Delta i_s \frac{M_s}{\phi_s} ds + \lim_{T \to \infty} E_t \frac{\xi_T B_T + M_T}{\phi_T}$$

- Remark 1:
  - Literature focuses on settings, in which private-sector transversality eliminates the bubble term
  - Here: fiscal theory in setting, in which where transversality does not rule out bubbles

- Remark 2:
  - The sum of the three limits on the right may not be well-defined mathematically, because they can be infinite with opposite signs
  - The limit of the sum may nevertheless exist and be finite
    - This is what matters economically (cannot separately trade the bubble and fundamental components)
3 Forms of Seigniorage

\[
\frac{B_t + M_t}{\phi_t} = E_t \int_t^\infty \frac{\xi_s}{\xi_t} (T_s - G_s)ds + E_t \int_t^\infty \frac{\xi_s}{\xi_t} \Delta i_s \frac{M_s}{\phi_s} ds + \lim_{T \to \infty} E_t \frac{\xi_T}{\xi_t} \frac{B_T + M_T}{\phi_T}
\]

1. **Surprise devaluation**
   - Irrational expectations
   - Small (Hilscher, Raviv, Reis 2014)
     - Inflation options imply likelihood of exceeding 5% of GDP is less than 1%

2. **Exploiting liquidity benefits of “narrow” cash**
   - Only for “narrow” cash that provides medium-of-exchange services
   - \(\Delta i = i - i^M\)
   - 0.36 % of GDP, NPV = 20% (at most 30%) of GDP, (Reis 2019)

3. **“Money bubble mining”**