Jumps due to multiple equilibria

- Bank runs: Diamond Dybvig
- Liquidity spirals: Brunnermeier Pedersen
- Sudden stops: Calvo, Mendoza, ...
- Currency attacks: Obstfeld (2nd generation models), Morris Shin
- Twin crisis models: Kaminsky Reinhart (3rd generation models)
- Loss of safe asset status: (after introducing safe asset in world with idiosyncratic risk)
Endogenous Risk due to **Amplification**

Initial exogenous shock $\sigma dZ_t$ /trigger

$i$’s best response
Endogenous Risk due to **Multiple Equilibria Jumps**

- No exogenous shock, but sunspot process
- Higher strategic complementarities
Two Type/Sector Model with Outside Equity

- **Expert sector**

  ![Diagram of expert sector]

  - Capital: $\kappa^e_t q_t K_t$
  - Debt: $N^e_t$
  - Outside equity: $\geq \alpha \kappa^e_t$

- **Household sector**

  ![Diagram of household sector]

  - Loans: $\kappa^h_t q_t K_t$
  - Equity: $\kappa^h_t q_t K_t$
  - Net worth: $q_t K_t - N^e_t$

- Experts must hold fraction $\chi^e_t \geq \alpha \kappa^e_t$ (skin in the game constraint)

- Return on inside equity $N_t$ can differ from outside equity
  - Issue outside equity at required return from HH
  - In related model, He and Krishnamurthy 2013 impose that inside and outside equity have same return
Two Type Model Setup

**Expert sector**
- Output: \( y_t^e = \alpha^e k_t^e \)
- Consumption rate: \( c_t^e \)
- Investment rate: \( \iota_t^e \)

\[
\frac{dk_t^{i,e}}{k_t^{i,e}} = (\Phi(i_t^{i,e}) - \delta)dt + \sigma dZ_t
\]

\[
E_0\left[\int_0^\infty e^{-\rho_t^e (c_t^e)_{1-\gamma}}dt\right]
\]

Can only issue
- Risk-free debt
- Equity, but most hold \( \chi_t^e \geq \alpha \kappa_t \)

**Household sector**
- Output: \( y_t^h = \alpha^h k_t^h \)
- Consumption rate: \( c_t^h \)
- Investment rate: \( \iota_t^h \)

\[
\frac{dk_t^{i,h}}{k_t^{i,h}} = (\Phi(i_t^{i,h}) - \delta)dt + \sigma dZ_t
\]

\[
E_0\left[\int_0^\infty e^{-\rho_t^h (c_t^h)_{1-\gamma}}dt\right]
\]
Unanticipated Run on Experts

- Can unanticipated withdrawal of all experts’ funding be self-fulfilling?
- Unanticipated crash – jump to $\eta^e = 0$
  - Absent a run: solution as in earlier lecture, since unanticipated
  - When do jump capital losses wipe out experts’ net worth?

\[
(q(\eta^e_t) - q(0)) \left( \theta^e_t K + \theta^{e,OE}_t \frac{\eta^e_t}{\chi^e_t} \right) K_t \geq \eta^e_t q(\eta^e_t) K_t
\]

\[
q(\eta^e_t) \left( 1 - \frac{\eta^e_t}{\chi^e(\eta^e_t)} \right) \geq q(0) \quad \text{or} \quad q(\eta^e_t) \left( 1 - \frac{1}{\theta^e_t K + \theta^{e,OE}_t} \right) \geq q(0)
\]

- Vulnerability region:
  - High price (not very low $\eta^e$)
  - “high risk-leverage” (not very high $\eta^e$)

- After run: $\eta^0 = 0$ forever
2 Types of Runs and Modeling Challenges

- What type of run? What’s the trigger?
  - **Funding supply run**: Depositor/households run
    - Household withdraw funding to experts
  - **Funding demand run**: Other experts run
    - Each expert tries to pay back debt and fire-sells assets
    - Drop in $q$ is driver

- Model advantage: Always jump to the same point $q(\eta^e = 0)$!

- Modeling Challenges: (see Mendo (2020))
  1. Experts are whipped out forever.
     - **OLG structure**:
       - Death: all agents die with Poisson rate $\lambda^d$,
       - Birth: fraction $\psi$ of newborns are experts
  2. With anticipated run, expert fear “infinite marginal utility state” $\eta^e = 0$.
     - Transfer of $\tau K$ to bankrupt experts after run
     - Also fixes challenge 1.
     - To keep $\tau$ small also introduce relative performance penalty (then take limit $\tau \to 0$)
**Economic insights**

- **Volatility Paradox also in Jump risk**
  Reduction in exogenous risk $\sigma$
  $\Rightarrow$ higher leverage
  1. Increase in price risk $\sigma^q$ (Brownian)
     - $(\sigma + \sigma^q)$ stays roughly stable
  2. Increase in run risk $j^q(\eta)$ (Jump)
     - Total risk can be higher
     - Low risk environment is “risky”
       - E.g. through better Brownian risk insurance
       - Recall for very low $\eta$, $j^q(\eta) = 0$, there is no run risk since price $q$ is already low and can not drop much further

- **No runs in very bad crisis times**
  - vulnerability region doesn’t start at $\eta = 0$

- **Invariance of Relative Capital Demand**
  - If experts lever up more, risk is held by household due to default risk
From Ito to Levy and Cox Processes

- **Ito process:**
  \[ dX_t = \mu_t X_t dt + \sigma_t X_t dZ_t \] (geometric)

  the Brownian “shocks” \( dZ_t \) are i.i.d. and small s.t. continuous path
  - For non-normal shocks within \( dt \) one needs discontinuities

- **Levy process**
  \[ dL_t = adt + bdZ_t + dJ_t \] – most general class with i.i.d. increments
  \[ dX_t = \mu_t X_t dt + \sigma_t X_t dZ_t + j_t X_t dJ_t \]

- **Restrict attention to Poisson processes:**
  - Levy jump process can be written as integral w.r.t. Poisson random measures
  - Poisson process with arrival rate \( \lambda > 0 \):
    - \( J \) takes on values in \( \mathbb{N}_0 = \{0, 1, 2, \ldots\} \)
    - Increments \( J_{t+\Delta t} - J_t \) are Poisson distributed with Parameter \( \lambda \Delta t \)
    - Stochastic integral w.r.t. Poisson process simply sums up the values of the integrand
      - \( \int_0^T a_t dJ_t = \sum_{n=1}^{J_T} a_{r_n} \)
  - **Cox process:** \( \lambda_t \) can be time-varying

- **Compensated Jump process**
  \( J_t - \int_0^t \lambda_s ds \) is martingale
  - If \( \int_0^t a_s dJ_s \) and \( a_t \) uses info only up to right before \( t \) then \( J_t - \int_0^t a_s \lambda_s ds \) is martingale
Ito formulas

\[ df(X_t) = f'(X_t)(\mu^X_t X_t dt + \sigma^X_t X_t dZ_t) + \frac{1}{2} f''(X_t)(\sigma^X_t X_t)^2 dt + (f(X_t) - f(X_{t-}))dJ_t \]
\[ = \left(f'(X_t)\mu^X_t X_t + \frac{1}{2} f''(X_t)(\sigma^X_t X_t)^2\right)dt + f'(X_t)\sigma^X_t X_t dZ_t + \left(f\left((1 + j^X_t)x_{t-}\right) - f(x_{t-})\right)dJ_t \]

**Power rule:**

\[ \frac{dx^Y_t}{x^Y_t} = (\gamma \mu^X_t + \gamma (\gamma - 1)(\sigma^X_t)^2)dt + \gamma \sigma^X_t dZ_t + \left((1 + j^X_t)^\gamma - 1\right) dJ_t \]

**Product rule:**

\[ \frac{d(x_t y_t)}{x_t y_t} = (\mu^X_t + \mu^Y_t + \sigma^X_t \sigma^Y_t)dt + (\sigma^X_t + \sigma^Y_t)dZ_t + (j^X_t + j^Y_t + j^X_t j^Y_t)dJ_t \]

**Quotient rule:**

\[ \frac{d(x_t / y_t)}{x_t / y_t} = (\mu^X_t - \mu^Y_t + (\sigma^Y_t)^2 - \sigma^X_t \sigma^Y_t)dt + (\sigma^X_t - \sigma^Y_t)dZ_t + \frac{j^X_t - j^Y_t}{1 + j^Y_t}dJ_t \]

**Memorize simple rules:**

- \[ 1 + j^X_t = (1 + j^X_t)^\gamma \]
- \[ 1 + j^X_y = (1 + j^X_t)(1 + j^Y_t) \]
- \[ 1 + j^{X/Y}_t = \frac{1 + j^X_t}{1 + j^Y_t} \]
Solving MacroModels Step-by-Step

0. Postulate aggregates, price processes & obtain return processes

1. For given $C/N$-ratio and SDF processes for each $i$ finance block
   a. Real investment $\iota$ + Goods market clearing (static)
      - Toolbox 1: Martingale Approach, HJB vs. Stochastic Maximum Principle Approach
   b. Portfolio choice $\theta$ + Asset market clearing or
      Asset allocation $\kappa$ & risk allocation $\chi$
      - Toolbox 2: “price-taking social planner approach” – Fisher separation theorem
   c. “Money evaluation equation” $\vartheta$
      - Toolbox 3: Change in numeraire to total wealth (including SDF)

2. Evolution of state variable $\eta$ (and $K$)

3. Value functions
   a. Value fcn. as fcn. of individual investment opportunities $\omega$
      - Special cases: log-utility, constant investment opportunities
   b. Separating value fcn. $V^i(n^i; \eta, K)$ into $v^i(\eta) u(K)(n^i/n^i)^{1-\gamma}$
   c. Derive $C/N$-ratio and $\zeta$ price of risk

4. Numerical model solution
   a. Transform BSDE for separated value fcn. $v^i(\eta)$ into PDE
   b. Solve PDE via value function iteration

5. KFE: Stationary distribution, Fan charts
0. Postulate Aggregates and Processes

- **Individual capital evolution:**
  \[ \frac{d k_{t}^{i,i}}{k_{t}^{i,i}} = (\Phi(\lambda_{t}^{i,i}) - \delta)dt + \sigma dZ_t + d\Delta_{t}^{k,i,i} \]

  - Where \( \Delta_{t}^{k,i,i} \) is the individual cumulative capital purchase process

- **Capital aggregation:**
  - Within sector \( i \):
    \[ K_{t}^{i} \equiv \int k_{t}^{i,i} d\lambda \]
  - Across sectors:
    \[ K_{t} \equiv \sum_{i} K_{t}^{i} \]
  - Capital share:
    \[ \kappa_{t}^{i} \equiv \frac{K_{t}^{i}}{K_{t}} \]
    \[ \frac{dK_{t}}{K_{t}} = (\Phi(\lambda_{t}^{i}) - \delta)dt + \sigma dZ_t \]

- **Net worth aggregation:**
  - Within sector \( i \):
    \[ N_{t}^{i} \equiv \int n_{t}^{i,i} d\lambda \]
  - Across sectors:
    \[ N_{t} \equiv \sum_{i} N_{t}^{i} \]
  - Wealth share:
    \[ \eta_{t}^{i} \equiv \frac{N_{t}^{i}}{N_{t}} \]

- **Value of capital stock:**
  \[ q_{t}K_{t} \]
  Postulate
  \[ dq_{t}/q_{t} = \mu_{t}^{q}dt + \sigma_{t}^{q}dZ_t + j_{t}^{q}dJ_{t} \]
0. Postulate Aggregates and Processes

- Individual capital evolution:
  \[
  \frac{d k_t^{i,i}}{k_t^{i,i}} = \left( \Phi \left( i_t^{i,i} \right) - \delta \right) dt + \sigma d Z_t + d \Delta_t^{k,i,i}
  \]
  where \( \Delta_t^{k,i,i} \) is the individual cumulative capital purchase process.

- Capital aggregation:
  - Within sector \( i \):
    \[
    K_t^i \equiv \int k_t^{i,i} \, d i
    \]
  - Across sectors:
    \[
    K_t \equiv \sum_i K_t^i
    \]
  - Capital share:
    \[
    \kappa_t^i \equiv K_t^i / K_t
    \]

- Net worth aggregation:
  - Within sector \( i \):
    \[
    N_t^i \equiv \int n_t^{i,i} \, d i
    \]
  - Across sectors:
    \[
    N_t \equiv \sum_i N_t^i
    \]
  - Wealth share:
    \[
    \eta_t^i \equiv N_t^i / N_t
    \]

- Value of capital stock:
  \[
  q_t K_t
  \]
  Postulate
  \[
  \frac{dq_t}{q_t} = \mu_t^q \, dt + \sigma_t^q \, d Z_t + j_t^q \, d J_t
  \]

- Postulated SDF-process:
  \[
  \frac{d \xi_t}{\xi_t} = \mu_t^{\xi} \, dt + \sigma_t^{\xi} \, d Z_t + j_t^{\xi} \, (d J_t - \lambda_t \, dt)
  \]
  \( \mu_t^{\xi}, \sigma_t^{\xi}, j_t^{\xi}, \lambda_t \) are the SDF parameters.

- Sunspot arrival rate:
  Since only risky debt and not risk-free debt is traded.
0. Postulate Aggregates and Processes

- ... from price processes to return processes (using Ito)
  - Use Ito product rule to obtain capital gain rate (in absence of purchases/sales)
    - Define $\tilde{k}^i_t$: \[ \frac{d\tilde{k}^i_t}{k^i_t} = (\Phi(i^i_t) - \delta)dt + \sigma dZ_t + d\Delta^k_{t} \]
      without purchases/sales
    - Dividend yield
    - E[Capital gain rate]= \frac{d(qtk^i_t)}{(qtk^i_t)}
    - Price of risk
    - Return on defaultable debt
      \[ dr^D_t = r_t dt + j^D_t dJ_t \]
    - Postulate SDF-process: (Example: $\xi^i_t = e^{-\rho t}V'(n^i_t)$.)
      \[ \frac{d\xi^i_t}{\xi^i_t} = -r^F_t dt - \xi^i_t dZ_t - \nu^i_t (dJ_t - \lambda_t dt) \]
1a. Individual Agent Choice of $\iota$, $\theta$, $c/n$

- Choice of $\iota$ is static problem (and separable) for each $t$
  
  \[ \max_{\iota^i_t} \frac{d\rho^k_t(\iota^i_t)}{\iota^i_t} \]
  
  \[ = \max_{\iota^i_t} \left( \frac{a^i - \iota^i_t}{q_t} + \Phi(\iota^i_t) - \delta + \mu^q + \sigma \sigma^q \right) dt + (\sigma + \sigma^q_t) dZ_t + j^q_t dJ_t \]

- FOC: $\frac{1}{q_t} = \Phi'(\iota^i_t)$, Tobin’s $q$
  
  - All agents $\iota^i_t = \iota_t \Rightarrow \frac{dK_t}{K_t} = (\Phi(\iota_t) - \delta) dt + \sigma dZ_t$
  
  - Special functional form:
    
    \[ \Phi(i) = \frac{1}{\phi} \log(\phi i + 1) \Rightarrow \phi i = q - 1 \]

- Goods market clearing: $(A(\kappa) - \iota_t)K_t = \sum_i C^i_t$.
  
  \[ \kappa_t a^e K_t + (1 - \kappa_t) a^h K_t - \iota(q_t)K_t = \eta^e_t \frac{C^e_t}{N_t} q_t K_t + (1 - \eta^e_t) \frac{C^h_t}{N^h_t} q_t K_t \]
Solving MacroModels Step-by-Step

0. Postulate aggregates, price processes & obtain return processes

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   b. Portfolio choice $\theta$ + Asset market clearing or Asset allocation $\kappa$ & risk allocation $\chi$ 
      ▪ Toolbox 2: “price-taking social planner approach” – Fisher separation theorem 
   c. “Money evaluation equation” $\vartheta$ 
      ▪ Toolbox 3: Change in numeraire to total wealth (including SDF)

2. Evolution of state variable $\eta$ (and $K$)  
   \textit{forward equation}

3. Value functions  
   a. Value fcn. as fcn. of individual investment opportunities $\omega$ 
      ▪ Special cases: log-utility, constant investment opportunities 
   b. Separating value fcn. $V^i(n^i; \eta, K)$ into $v^i(\eta)u(K)(n^i/n^i)^{1-\gamma}$ 
   c. Derive $C/N$-ratio and $\zeta$ price of risk

4. Numerical model solution  
   a. Transform BSDE for separated value fcn. $v^i(\eta)$ into PDE 
   b. Solve PDE via value function iteration

5. KFE: Stationary distribution, Fan charts
1a. Individual Agent Choice of \( \iota, \theta, c/n \)

\[
\max_{\{\iota_t, \theta_t, c_t\}_{t=0}^{\infty}} E \left[ \int_0^\infty e^{-\rho t} u(c_t) dt \right]
\]

s.t. \( \frac{d n_t}{n_t} = -\frac{c_t}{n_t} dt + \sum_j \theta^j_t dr^j_t \) + labor income/endow/taxes

- Portfolio Choice: Martingale Approach
  - Let \( x_t^A \) be the value of a “self-financing trading strategy” (reinvest dividends)
  - Theorem: \( \xi_t x_t^A \) follows a Martingale, i.e. drift = 0.
    - Let \( \frac{dx_t^A}{x_t^A} = \mu_t^A dt + \sigma_t^A dZ_t + j_t^A dJ_t \),
    - Recall SDF \( \frac{d \xi_t^i}{\xi_t^i} = -r_t^{F,i} dt - \xi_t^i dZ_t - \nu_t^i (dJ_t - \lambda_t dt) \)
    - By Ito product rule
      \[
      \frac{d(\xi_t^i x_t^A)}{\xi_t^i x_t^A} = (-r_t^{F,i} + \mu_t^A - \xi_t^i \sigma_t^A + \nu_t^i \lambda_t) dt + (\sigma_t^A - \xi_t^i) dZ_t + (j_t^A - \nu_t^i - \nu_t^i j_t^A) dJ_t
      \]
      \[
      \frac{d(\xi_t^i x_t^A)}{\xi_t^i x_t^A} = (-r_t^{F,i} + \mu_t^A - \xi_t^i \sigma_t^A + \lambda_t j_t^A - \lambda_t \nu_t^i j_t^A) dt + (\sigma_t^A - \xi_t^i) dZ_t + (j_t^A - \nu_t^i - \nu_t^i j_t^A)(dJ_t - \lambda_t dt)
      \]
      martingale
  - Expected return: \( \mu_t^A + \lambda j_t^A = r_t^{F,i} + \xi_t^i \sigma_t^A + \lambda \nu_t^i j_t^A \)
1a. Individual Agent Choice of $\iota$, $\theta$, $c/n$

- Expected return: $\mu_t^A + \lambda j_t^A = r_t^{F,i} + \zeta_t^i \sigma_t^A + \nu_t^i \lambda j_t^A$
  - $r_t^{F,i}$ is the shadow risk-free rate (need not to be same across groups)
  - $\zeta_t^i$ is the price of Brownian risk of agents $i$,
  - $\zeta_t^i \sigma_t^A$ is the required Brownian risk premium of agents $i$
  - $\nu_t^i \lambda_t$ is the price of Poisson upside risk if $j_t^A > 0$
    For risk-neutral agents $\nu_t^i = 0$

- Remark:
  - $dr^{e,K}$ experts return on capital
  - $dr^{h,OE}$ households return on outside equity
  - $dr^{h,D}$ households’ return on debt is risky (due to bankruptcy)
1a. Individual Agent Choice of $i, \theta, c/n$

- Expected return:
  $$\mu^A_t + \lambda j^A_t = r^F,i_t + \zeta^i_t \sigma^A_t + \nu^i_t \lambda j^A_t$$
  - $r^F,i_t$ is the shadow risk-free rate (need not to be same across groups)
  - $\zeta^i_t$ is the price of Brownian risk of agents $i$
  - $\zeta^i_t \sigma^A_t$ is the required Brownian risk premium of agents $i$
  - $\nu^i_t \lambda_t$ is the price of Poisson upside risk if $j^A > 0$
    - For risk-neutral agents $\nu^i_t = 0$

- Remark:
  - For CRRA utility: SDF is
    $$\xi_t = e^{-\rho \omega_t^{1-\gamma} n_t^{-\gamma}}$$
    $$1 - \nu_t = (1 + j_t^{\omega})^{1-\gamma} (1 + j_t^n)^{-\gamma}$$
  - For log utility: $\nu_t = 1 - \frac{1}{1 + j_t^n} = \frac{j_t^n}{1 + j_t^n}$
  - For Epstein-Zin: part of $\omega_t$-process
1a. Individual Agent Choice of \( \lambda, \theta, c/n \)

- Of experts with outside equity issuance (after plugging in households’ outside equity choice)

\[
\frac{a^e - \lambda_t}{q_t} + \Phi(\lambda_t) - \delta + \mu_t^q + \sigma \sigma_t^q - \left[ \frac{\chi_t^e}{\kappa_t^e} r_t^{F,e} + \left(1 - \frac{\chi_t^e}{\kappa_t^e}\right) r_t^{F,h}\right] + \lambda_t j_t^q = \\
\left[ \zeta_t^e \frac{\chi_t^e}{\kappa_t^e} + \zeta_t^h \left(1 - \frac{\chi_t^e}{\kappa_t^e}\right) \right] (\sigma + \sigma^q) + \left[ \nu_t^e \frac{\chi_t^e}{\kappa_t^e} + \nu_t^h \left(1 - \frac{\chi_t^e}{\kappa_t^e}\right) \right] \lambda_t j_t^q
\]

- Of households’ capital choice

\[
\frac{a^h - \lambda_t}{q_t} + \Phi(\lambda_t) - \delta + \mu_t^q + \sigma \sigma_t^q - r_t^{F,h} + \lambda_t \left( j_t^q - j_t^{r,D} \right) \\
\leq \zeta_t^h (\sigma + \sigma^q) + \nu_t^h \lambda_t \left( j_t^q - j_t^{r,D} \right)
\]

with equality if \( \kappa_t^e < 1 \)

- Note: Later approach replaces this step with Fisher Separation Social Planners’ choice (see below)
Solving MacroModels Step-by-Step

0. Postulate aggregates, price processes & obtain return processes

1. For given $C/N$-ratio and SDF processes for each $i$ finance block
   a. Real investment $i +$ Goods market clearing (static)
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   b. Portfolio choice $\theta +$ Asset market clearing or Asset allocation $\kappa$ & risk allocation $\chi$
      ▪ Toolbox 2: “price-taking social planner approach” – Fisher separation theorem
   c. “Money evaluation equation” $\vartheta$
      ▪ Toolbox 3: Change in numeraire to total wealth (including SDF)

2. Evolution of state variable $\eta$ (and $K$) forward equation

3. Value functions forward equation backward equation
   a. Value fcn. as fcn. of individual investment opportunities $\omega$
      ▪ Special cases: log-utility, constant investment opportunities
   b. Separating value fcn. $V^i (n^i; \eta, K)$ into $v^i (\eta) u(K) (n^i/n^i)^{1-\gamma}$
   c. Derive $C/N$-ratio and $\varsigma$ price of risk

4. Numerical model solution
   a. Transform BSDE for separated value fcn. $v^i (\eta)$ into PDE
   b. Solve PDE via value function iteration

5. KFE: Stationary distribution, Fan charts
1b. Asset/Risk Allocation across $I$ Types

- **Price-Taking Planner’s Theorem:**
  A social planner that takes prices as given chooses an physical asset allocation, $\kappa_t$, and Brownian risk allocation, $\chi_t$, and a Jump risk allocation, $\zeta_t$, that coincides with the choices implied by all individuals’ portfolio choices.

  
  ![](https://via.placeholder.com/150)

  \[
  \begin{align*}
  \zeta_t &= (\zeta_t^1, ..., \zeta_t^I) \\
  \chi_t &= (\chi_t^1, ..., \chi_t^I) \\
  \zeta_t &= (\zeta_t^1, ..., \zeta_t^I) \\
  \sigma(\chi_t) &= (\chi_t^1 \sigma^N, ..., \chi_t^I \sigma^N) \\
  j(\zeta_t) &= (\zeta_t^1 j_t^N, ..., \zeta_t^I j_t^N)
  \end{align*}
  \]

  - Return on total wealth
    
    \[
    \text{Let } dN_t/N_t = \mu_t^N dt + \sigma_t^N dZ_t + j_t^N dJ_t
    \]

- **Planner’s problem**
  \[
  \max_{\{\kappa_t, \chi_t, \zeta_t\}} \frac{E_t[dr_t^N(\kappa_t)]}{dt} - \zeta_t \sigma(\chi_t) - \lambda v_j(\zeta_t)
  \]
  subject to friction: $F(\kappa_t, \chi_t, \zeta_t) \leq 0$

- **Example:**
  1. $\chi_t = \zeta_t = \kappa_t$ (can’t issue outside equity to offload Brownian or risky debt to offload Jump risk)
  2. $\chi_t \geq \alpha \kappa_t$ (skin in the game constraint, outside equity up to a limit)
1b. Allocation of Capital/Risk: 2 Types

- **Expert:** \( \boldsymbol{\theta}^e = (\theta_{t}^{e,K}, \theta_{t}^{e,OE}, \theta_{t}^{e,D}) \) for capital, outside equity, debt
  
  **Restrictions:**
  
  - \( \theta_{t}^{e,K} \geq 0, \)
  - \( \theta_{t}^{e,OE} \leq 0, \) only issue outside equity
  - \( \theta_{t}^{e,OE} \geq -(1 - \alpha)\theta_{t}^{e,K} \) skin in the game

  \[
  \begin{align*}
  &\text{maximize} \\
  &\theta_{t}^{e,K} E[dr_{t}^{e,K}] / dt + \theta_{t}^{e,OE} E[dr_{t}^{OE}] / dt + \theta_{t}^{e,D} E[dr_{t}^{D,e}] / dt - \varsigma_{t}^{e} (\theta_{t}^{e,K} + \theta_{t}^{e,OE}) \sigma r_{t}^{e,K} \\
  &- \lambda_{t} \nu_{t}^{e} ((\theta_{t}^{e,K} + \theta_{t}^{e,OE}) j_{t}^{eK} + \theta_{t}^{e,D} j_{t}^{eD}) \\ 
  \end{align*}
  \]

  Note \( j_{t}^{D} \) is just the jump due to the loss and not the change in D due to rebalancing.

- **Household:** \( \boldsymbol{\theta}^h = (\theta_{t}^{h,K}, \theta_{t}^{h,OE}, \theta_{t}^{h,D}) \)

  \[
  \begin{align*}
  &\text{maximize} \\
  &\theta_{t}^{h,K} E[dr_{t}^{h,K}] / dt + \theta_{t}^{h,OE} E[dr_{t}^{OE}] / dt + \theta_{t}^{h,D} E[dr_{t}^{D,h}] / dt - \varsigma_{t}^{h} (\theta_{t}^{h,K} + \theta_{t}^{h,OE}) \sigma r_{t}^{h,K} \\
  &- \lambda_{t} \nu_{t}^{h} ((\theta_{t}^{h,K} + \theta_{t}^{h,OE}) j_{t}^{hK} + \theta_{t}^{h,D} j_{t}^{hD}) \\ 
  \end{align*}
  \]

  \( \theta_{t}^{h,K} \geq 0 \\
  \theta_{t}^{h,OE} \geq 0 \)
1b. Allocation of Capital/Risk: 2 Types

- **Example 2:** 2 Type + with outside equity
  \[
  \max \left\{ \kappa^e_t a^e + (1 - \kappa^e_t) a^h - \nu_t \right\} + \Phi(\nu_t) - \delta + \left[ - (\chi^e_t \zeta^e_t + (1 - \chi^e_t) \zeta^h_t) (\sigma + \sigma^q_t) \right]
  \]

- **FOC** \( \chi \): Case 1: \( \zeta^e_t (\sigma + \sigma^q_t) + \cdots > \zeta^h_t (\sigma + \sigma^q_t) + \cdots \Rightarrow \chi^e_t = \alpha \kappa^e_t 

  Case 2: \( \chi^e_t > \alpha \kappa^e_t \)

- **Case 1:** plug \( \chi^e_t = \alpha \kappa^e_t \) in objective
  a. \( \text{FOC}_\kappa: \frac{a^e - a^h}{q_t} > \alpha (\zeta^e_t - \zeta^h_t) (\sigma + \sigma^q_t) + \cdots \Rightarrow \kappa^e_t = 1 \)
  b. \( = \Rightarrow \kappa^e_t < 1 \)

- **Case 2:**
  a. \( \text{FOC}_\kappa: \frac{a^e - a^h}{q_t} > 0 \Rightarrow \kappa^e_t = 1 \)
  b. \( = 0 \Rightarrow \kappa^e_t < 1 \) impossible

\( \chi^e_t = \alpha \kappa^e_t \)
### Invariance of Relative Capital Demand

- One of the insights of Mendo (2020) is that self-fulfilling jumps do not influence the relative demand for capital of experts relative to households. I.e. the excess market return that experts demand to hold capital is not affected.

- Subtract experts pricing condition from households

\[
\mu_t^{r,k,e} - \mu_t^{r,k,h} \geq \frac{\chi_t^e}{\kappa_t^e} (\zeta_t^e - \zeta_t^h)(\sigma + \sigma_t^q) - \frac{\chi_t^e}{\kappa_t^e} \lambda_t (1 - \nu_t^h) \left( \frac{\partial j_t^D}{\partial \theta_t^{e,K}} (\theta_t^{e,K} - 1) + j_t^q - j_t^{r,D} \right)
\]

- Losses are split between experts and households (via defaultable debt)

- Since experts’ losses are capped by their net worth due to limited liability, all additional losses from increasing capital holding, $\theta_t^{e,K}$, are born by households
Solving MacroModels Step-by-Step

0. Postulate aggregates, price processes & obtain return processes

1. For given $C/N$-ratio and SDF processes for each $i$ finance block
   a. Real investment $i$ + Goods market clearing (static)
      - Toolbox 1: Martingale Approach, HJB vs. Stochastic Maximum Principle Approach
   b. Portfolio choice $\theta$ + Asset market clearing or Asset allocation $\kappa$ & risk allocation $\chi$
      - Toolbox 2: “price-taking social planner approach” – Fisher separation theorem
   c. “Money evaluation equation” $\vartheta$
      - Toolbox 3: Change in numeraire to total wealth (including SDF)

2. Evolution of state variable $\eta$ (and $K$)

3. Value functions
   a. Value fcn. as fcn. of individual investment opportunities $\omega$
      - Special cases: log-utility, constant investment opportunities
   b. Separating value fcn. $V^i(n^i; \eta, K)$ into $v^i(\eta)u(K)(n^i/n^i)^{1-\gamma}$
   c. Derive $C/N$-ratio and $\zeta$ price of risk

4. Numerical model solution
   a. Transform BSDE for separated value fcn. $v^i(\eta)$ into PDE
   b. Solve PDE via value function iteration

5. KFE: Stationary distribution, Fan charts
**Toolbox 3: Change of Numeraire**

- $x_t^A$ is a value of a self-financing strategy/asset in $\}$
- $Y_t$ price of € in $\}$ (exchange rate)
  \[ \frac{dY_t}{Y_t} = \mu_t^Y dt + \sigma_t^Y dZ_t + j_t^Y dJ_t \]

- $x_t^A / Y_t$ value of the self-financing strategy/asset in €

  \[ e^{-\rho t} u'(c_t) Y_t \frac{x_t^A}{Y_t} = \hat{\xi}_t \]

  Recall \( \mu_t^A - \mu_t^B + \lambda_t (j_t^A - j_t^B) = \left( -\sigma_t^\xi \right) \left( \sigma_t^A - \sigma_t^B \right) + \nu_t \lambda_t (j_t^A - j_t^B) \)

- Frequency of sunspots, \( \lambda_t \), are not dependent on numeraire

- Price of Brownian risk $\xi^{€} = \xi^{\$} - \sigma^Y$
- Price of Jump risk $\nu_t^{€} = \nu_t^\$ - j_t^Y + \nu_t^\$ j_t^Y$
Change of Numeraire: SDF

- SDF in good numeraire is
  \[ d\xi_t^i/\xi_{t-}^i = -r_t^F,i dt - \zeta_t^i dZ_t - \nu_t^i (dJ_t - \lambda_t dt) \]

- SDF in total net worth numeraire is
  \[ d\hat{\xi}_t^i/\hat{\xi}_{t-}^i = \mu_t^i dt - (\zeta_t^i - \sigma_t^N) dZ_t - (\nu_t^i - j_t^N + \nu_t^i j_t^N) dJ_t \]
  \[ = \hat{r}_t^F,i dt - (\zeta_t^i - \sigma_t^N) dZ_t - (\nu_t^i - j_t^N + \nu_t^i j_t^N)(dJ_t - \lambda_t dt) \]
  \[ = \hat{\zeta}_t^i \]
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4. Numerical model solution  
   a. Transform BSDE for separated value fcn. $v^i(\eta)$ into PDE  
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5. KFE: Stationary distribution, Fan charts
2. GE: Markov States and Equilibria

- Equilibrium is a map

  Histories of shocks $\{Z_s, s \in [0, t]\}$

  $\rightarrow$ prices $q_t, \zeta_t^i, \iota_t^i, \theta_t^e$

  net worth distribution

  $\eta_t^e = \frac{N_t^e}{q_tK_t} \in (0,1)$

  net worth share

- All agents maximize utility
  - Choose: portfolio, consumption, technology

- All markets clear
  - Consumption, capital, money, outside equity
2. Law of Motion of Wealth Share $\eta_t$

- **Method 1**: Using Ito’s quotation rule $\eta_t^i = N_t^i / (q_t K_t)$
  - Recall $\frac{dN_t^i}{N_t^i} = \frac{C_t}{N_t^i} dt + r_{bm}^i dt + \zeta_t^i \left( \frac{\chi_t^i (\sigma + \sigma^q_t)}{\eta_t^i} - \sigma_{bm}^i \right) dt + \nu (j_t^N - j_{bm}^i) dt$  
  - Transfers in case $dJ_t = 1$ in vulnerability region: $\tau^i K_t$

- **Method 2**: Change of numeraire + Martingale Approach
  - New numeraire: Total wealth in the economy, $N_t$
  - Apply Martingale Approach for value of $i$’s portfolio
    - Simple algebra to obtain drift of $\eta_t^i$: $\mu_t^\eta_t^i$
    - Note that change of numeraire does not affect ratio $\eta_t^i$!
2. $\mu^\eta$ Drift of Wealth Share: Many Types

- **New Numeraire**
  - "Total net worth" in the economy, $N_t$ (without superscript)
  - Type $i$’s portfolio net worth = net worth share

- **Martingale Approach with new numeraire**
  - Asset $A = i$’s portfolio return in terms of total wealth,
    \[
    \left( \frac{C^i_t}{N^i_t} + \mu_t^i + \lambda_t j^i \right) dt + \sigma_t^i dZ_t + \tilde{\sigma}^i_t d\tilde{Z}_t
    \]

  - Dividend yield
  - E[capital gains] rate

  - Asset $B$ (benchmark asset that everyone can hold, e.g. risk-free asset or money (in terms of total economy wide wealth as numeraire))
    \[
    r^b_{t} dt + \sigma^b_{t} dZ_t
    \]

  - Apply our martingale asset pricing formula
    \[
    \mu_t^A - \mu_t^B + \lambda_t(j^A_t - j^B_t) = \hat{\sigma}^i_t(\sigma_t^A - \sigma_t^B) + \lambda_t \hat{\nu}_t(j^A_t - j^B_t)
    \]
2. $\mu^\eta$ Drift of Wealth Share: Many Types

- Asset pricing formula (relative to benchmark asset)

$$\mu_t^\eta + \frac{C_t^i}{N_t^i} - r_t^{bm} + \lambda_t \left( j_t^\eta^i - j_t^{bm} \right) = \left( \zeta_t^i - \sigma_t^N \right) \left( \sigma_t^\eta - \sigma_t^{bm} \right) + \lambda_t \hat{\nu}_t^i \left( j_t^\eta^i - j_t^{bm} \right)$$

- Add up across types (weighted), (capital letters without superscripts are aggregates for total economy)

$$\sum_{i'} \eta_{t'}^i \mu_t^\eta + \frac{C_t^i}{N_t^i} - r_t^{bm} + \lambda_t \sum_{i'} \eta_{t'}^i j_t^{\eta'^i} - \lambda_t j_t^{bm} = 0$$

$$\sum_{i'} \eta_{t'}^i \zeta_t^i \left( \sigma_t^{\eta'^i} - \sigma_t^{bm} \right) + \lambda_t \sum_{i'} \eta_{t'}^i \hat{\nu}_t^i \left( j_t^{\eta'^i} - j_t^{bm} \right) = 0$$

- Subtract from first equation

$$\mu_t^\eta + \lambda_t j_t^{\eta^i} = \frac{C_t^i}{N_t^i} - \frac{C_t^i}{N_t^i} + \hat{\zeta}_t^i \left( \sigma_t^\eta - \sigma_t^{bm} \right) - \sum_{i'} \eta_{t'}^i \hat{\zeta}_t^i \left( \sigma_t^{\eta'^i} - \sigma_t^{bm} \right)$$

$$+ \lambda_t \hat{\nu}_t^i \left( j_t^\eta^i - j_t^{bm} \right) - \lambda_t \sum_{i'} \eta_{t'}^i \hat{\nu}_t^i \left( j_t^{\eta'^i} - j_t^{bm} \right)$$

due to change in numeraire
2. \( \mu^\eta \) Drift of Wealth Share: Two Types \( i \in \{e, h\} \)

- Subtract from each other yield net worth share dynamics
  \[
  \mu^\eta_t + \lambda^\eta_t j^\eta_t = \frac{C_t}{N_t} - \frac{C^e_t}{N^e_t} + (1 - \eta^e_t) \hat{\zeta}_t^e \left( \sigma^\eta_t - \sigma^{bm}_t \right) - (1 - \eta^e_t) \hat{\zeta}_t^h \left( \sigma^\eta_t - \sigma^{bm}_t \right) \\
  + (1 - \eta^e_t) \lambda_t \hat{v}_t^e \left( j^\eta_t - j^{bm}_t \right) - (1 - \eta^e_t) \lambda_t \hat{v}_t^h \left( j^\eta_t - j^{bm}_t \right)
  \]

- In our model, benchmark asset is risky debt,
  - \( \sigma^{bm}_t = -\sigma^N_t \),
  - \( j^{bm}_t = \frac{j^{rD}_t - j^N_t}{1 + j^N} \) (since \( j^{rD}_t \) return on risky debt jump in c-numeraire, \( j^N_t \) wealth jump)
    - Apply quotient rule for jumps

- \( \mu^e_t + \lambda^e_t j^e_t \)
  \[
  \frac{C_t}{N_t} - \frac{C^e_t}{N^e_t} + (1 - \eta^e_t) \hat{\zeta}_t^e \left( \sigma^e_t + \sigma^N_t \right) - (1 - \eta^e_t) \hat{\zeta}_t^h \left( \sigma^h_t + \sigma^N_t \right) \\
  + (1 - \eta^e_t) \lambda_t \hat{v}_t^e \left( j^e_t - \frac{j^{rD}_t - j^N_t}{1 + j^N} \right) - (1 - \eta^e_t) \lambda_t \hat{v}_t^h \left( j^h_t - \frac{j^{rD}_t - j^N_t}{1 + j^N} \right)
  \]
2. $\sigma^\eta$ Volatility of Wealth Share

- Since $\eta^i_t = N^i_t/N_t$,
  \[
  \sigma^\eta_t = \sigma^N_t - \sigma^N = \sigma^N_t - \sum_{i'} \eta^i_{t'} \sigma^{N^i_t}_{t'}
  \]
  \[
  = (1 - \eta^i_t)\sigma^N_t - \sum_{i' \neq i} \eta^i_{t'} \sigma^{N^i_t}_{t'}
  \]

- Note for 2 types example
  \[
  j^{\eta^e}_t = \frac{j^{N^i}_t - j^{N}_t}{1 + j^{N^i}_t} = \frac{j^{N^i}_t - \sum_{i'} \eta^i_{t'} j^{N^i_{t'}}}{1 + \sum_{i'} \eta^i_{t'} j^{N^i_{t'}}} = \frac{(1 - \eta^e_t)j^{N^i}_t - \sum_{i' \neq i} \eta^i_{t'} j^{N^i_{t'}}}{1 + \sum_{i'} \eta^i_{t'} j^{N^i_{t'}}}
  \]
- Note for 2 types example
  \[
  j^{\eta^e}_t = \frac{(1 - \eta^e_t)(j^{N^e}_t - j^{N^h}_t)}{1 + \eta^e_t j^{N^e}_t + (1 - \eta^e_t)j^{N^h}_t}
  \]
Note:

- OLG structure and
- transfers $\tau K_t$

also affects net worth evolution and still has to be incorporated!
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2. Evolution of state variable $\eta$ (and $K$) forward equation

3. Value functions backward equation
   a. Value fcn. as fcn. of individual investment opportunities $\omega$
      ▪ Special cases: log-utility, constant investment opportunities
   b. Separating value fcn. $V^i(n^i; \eta, K)$ into $v^i(\eta)u(K)(n^i/n^i)^{1-\gamma}$
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5. KFE: Stationary distribution, Fan charts
Value Functions

- For log utility
  - Price of Brownian risk \( \zeta_t^i = \sigma_t^i \)
  - Price of Jump risk \( \nu_t = 1 - \frac{1}{1 + j_t^n} = \frac{j_t^n}{1 + j_t^n} \) (see earlier slide)

- For CRRA/EZ utility
  - Generalize earlier lecture
  - Add jump terms in value function BSDEs
Value Function Process for CRRA

\[
\frac{dV_t^i}{V_t^i} = \frac{d(v_t^iK_t^{1-\gamma})}{v_t^iK_t^{1-\gamma}}
\]

- By Ito’s product rule

\[
= \left( \mu_t^v + (1 - \gamma)(\Phi(t_t) - \delta) - \frac{1}{2} \gamma(1 - \gamma)(\sigma^2) + (1 - \gamma)\sigma_t^v \right)dt + j_t^v dJ_t
\]

+ volatility terms

- Recall by consumption optimality for CRRA utility

\[
\frac{dV_t^i}{V_t^i} - \rho^i dt + \frac{c_t^i}{n_t} dt \text{ follows a martingale}
\]

- Hence, drift above \( = \rho^i - \frac{c_t^i}{n_t} - \lambda_t j_t^v \) Still have to solve for \( \mu_t^v, \sigma_t^v \)

Poll 47: Why martingale?

a) Because we can “price” net worth with SDF
b) because \( \rho^i \) and \( c_t^i/n_t \) cancel out
Value Function Process for EZ

\[
\frac{dV_t^i}{V_t^i} = \frac{d(v_t^iK_t^{1-\gamma})}{v_t^iK_t^{1-\gamma}}
\]

- By Ito’s product rule

\[
= \left( \mu_t^i + (1 - \gamma)(\Phi(\nu_t) - \delta) - \frac{1}{2}\gamma(1 - \gamma)(\sigma^2) + (1 - \gamma)\sigma \sigma_t^v \right) dt + j_t^v dJ_t + \text{volatility terms}
\]

- Recall by consumption optimality for CRRA utility

\[
\frac{dV_t^i}{V_t^i} - \rho^i dt + \frac{c_t^i}{n_t^i} dt \text{ follows a martingale}
\]

- Hence, drift above

\[
= -\frac{\partial f}{\partial U}(c_S, v_t \frac{K_t^{1-\gamma}}{1-\gamma}) - \frac{c_t^i}{n_t^i} - \lambda_t j_t^v
\]

Still have to solve for \(\mu_t^v, \sigma_t^v\)

Poll 48: Why martingale?

a) Because we can “price” net worth with SDF

b) because \(\rho^i\) and \(\frac{c_t^i}{n_t^i}\) cancel out
If we relax the assumption $EIS = 1$, then the consumption-wealth ratio of agents will vary with investment opportunities (which do depend on the exact specification of (perceived) run risk even under log utility) and that will clearly affect $q$ through goods market clearing.

If we keep $EIS = 1$, but vary the risk aversion, then the $q$ function will only be affected if capital is allocated differently for the same value of $\eta$ (because the average consumption-wealth ratio in the economy does not change and then goods market clearing gives us a one-to-one mapping between $q$ and the capital allocation). So, we would have to check whether the invariance of capital demands result is only true because there are no hedging demands or whether that result generalizes even if there are hedging demands. I don’t have the equations in front of me right now, but my guess would be that also that result is not robust and thus the capital allocation and $q$ will be affected even if $EIS = 1$. 