Modern Macro, Money, and International Finance

Eco529

Lecture 09: Macro-Finance with Jumps

Markus K. Brunnermeier

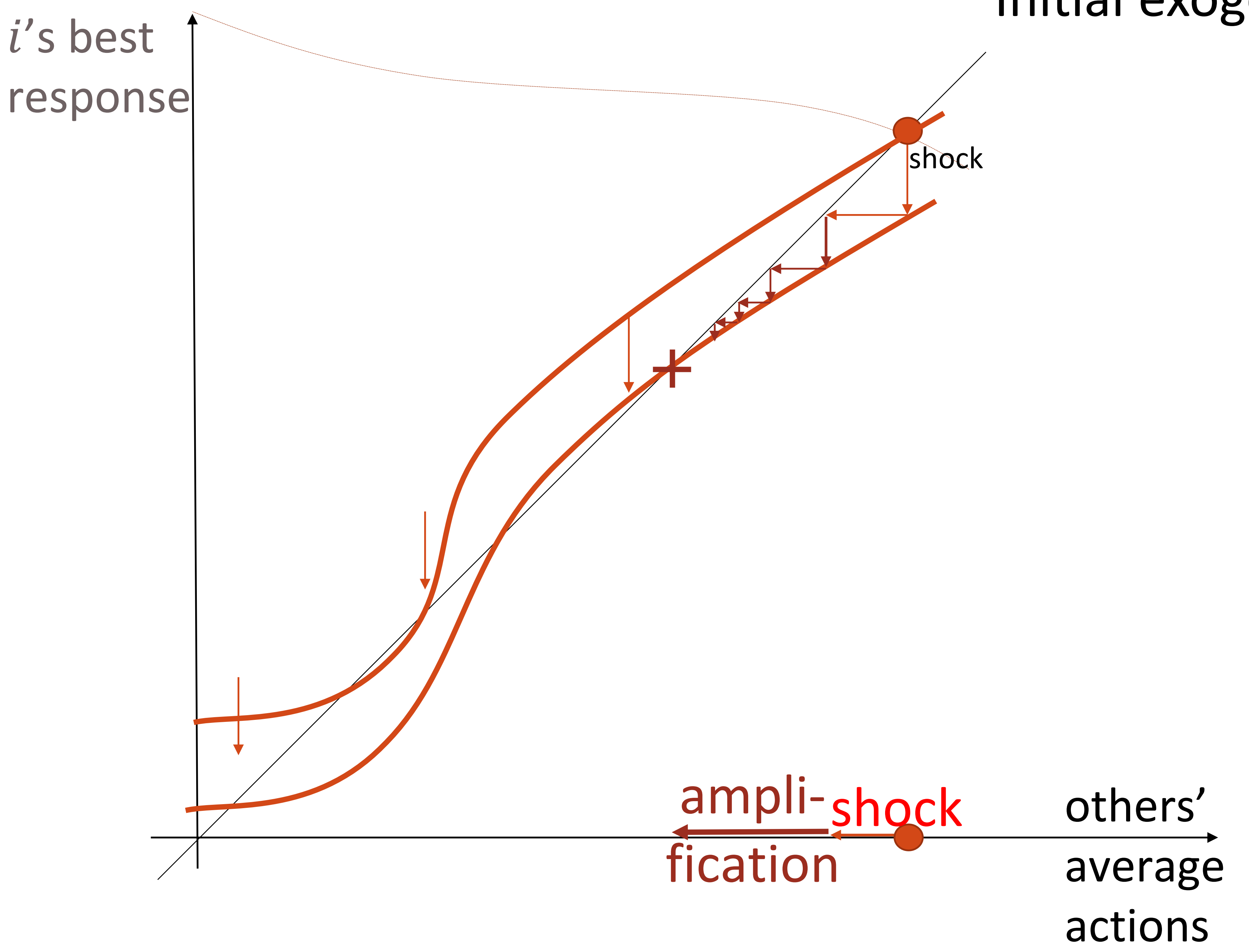
Princeton University

Jumps due to multiple equilibria

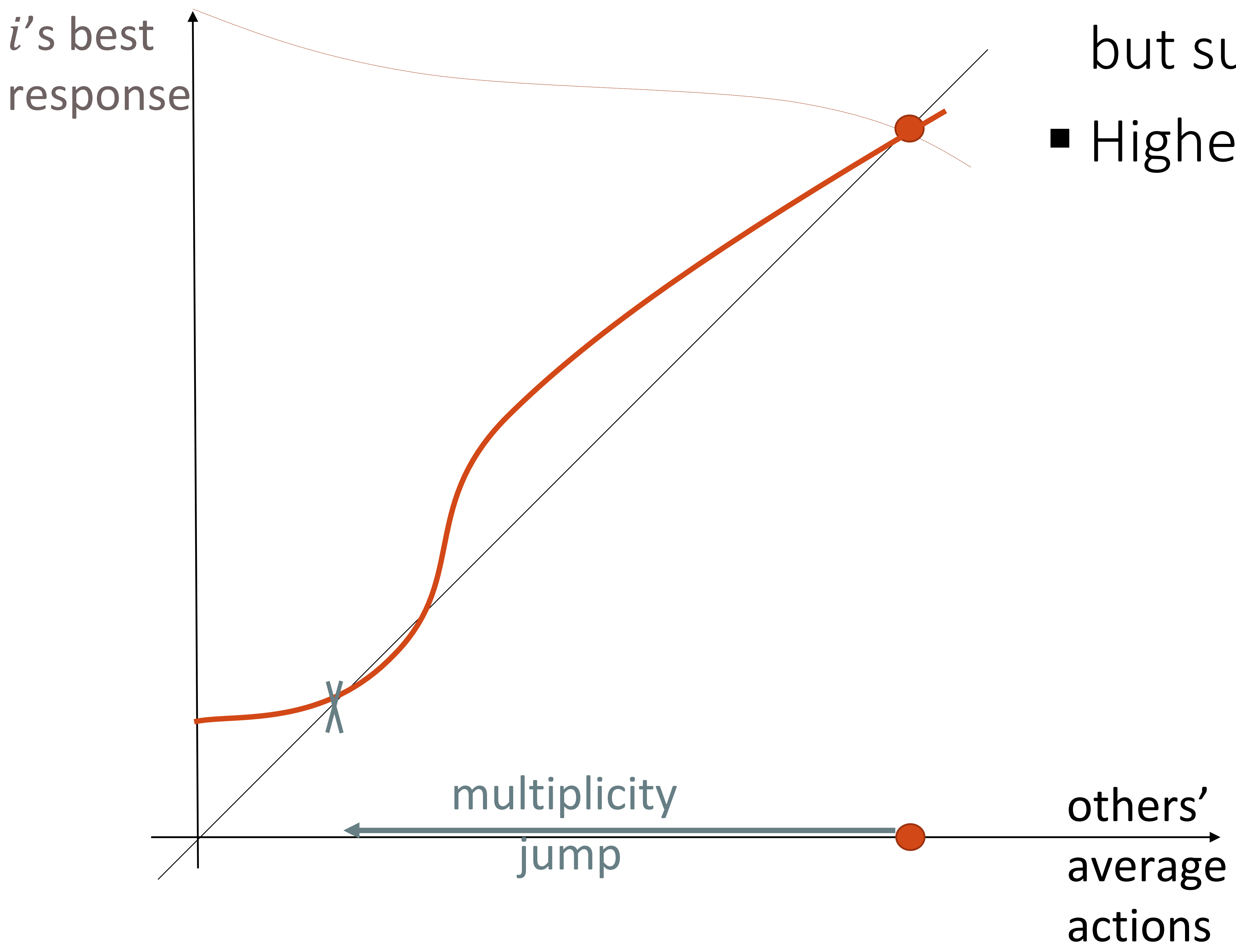
▪ Bank runs	Diamond Dybvig
▪ Liquidity spirals	Brunnermeier Pedersen
▪ Sudden stops	Calvo, Mendoza, …
▪ Currency attacks	Obstfeld (2nd generation models), Morris Shin
▪ Twin crisis models	Kaminsky Reinhart (3rd generation models)
▪ Loss of safe asset status	(after introducing safe asset in world with idiosyncratic risk)

Endogenous Risk due to Amplification

Initial exogenous shock σdZ_t /trigger



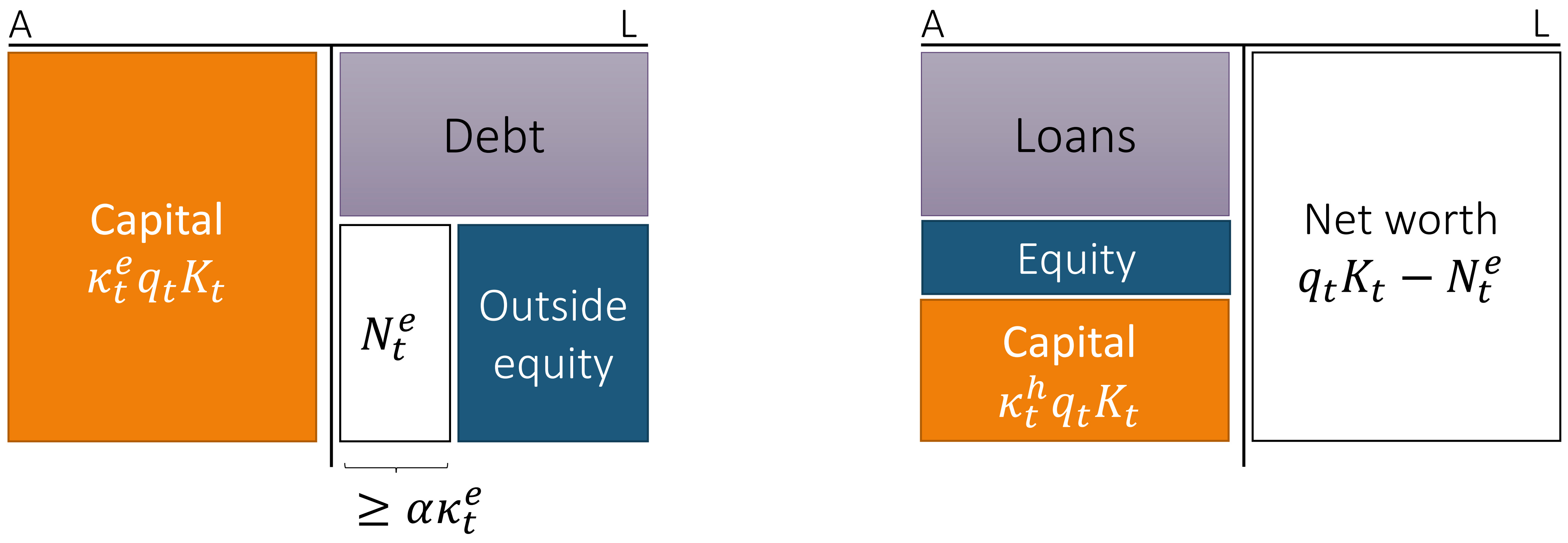
Endogenous Risk due to Multiple Equilibria Jumps



- No exogenous shock, but sunspot process
- Higher strategic complementarities

Two Type/Sector Model with Outside Equity

- Expert sector   Household sector



- Experts must hold fraction $\chi_t^e \geq \alpha\kappa_t^e$ (skin in the game constraint)
- Return on inside equity N_t can differ from outside equity
  - Issue outside equity at required return from HH
  - In related model, He and Krishnamurthy 2013 impose that inside and outside equity have same return

Two Type Model Setup

Expert sector

- Output: $y_t^e = a^e k_t^e$
- Consumption rate: c_t^e
- Investment rate: ι_t^e

$$\frac{dk_t^{\tilde{\imath},e}}{k_t^{\tilde{\imath},e}} = \left(\Phi\left(\iota_t^{\tilde{\imath},e}\right) - \delta\right)dt + \sigma dZ_t$$

- $E_0[\int_0^\infty e^{-\rho^e t}\frac{(c_t^e)^{1-\gamma}}{1-\gamma}dt]$

Can only issue

- Risk-free debt
- Equity, but most hold $\chi_t^e \geq \alpha\kappa_t$

Household sector

- Output: $y_t^h = a^h k_t^h$
- Consumption rate: c_t^h
- Investment rate: ι_t^h

$$\frac{dk_t^{\tilde{\imath},h}}{k_t^{\tilde{\imath},h}} = \left(\Phi\left(\iota_t^{\tilde{\imath},h}\right) - \delta\right)dt + \sigma dZ_t$$

- $E_0[\int_0^\infty e^{-\rho^h t}\frac{(c_t^h)^{1-\gamma}}{1-\gamma}dt]$

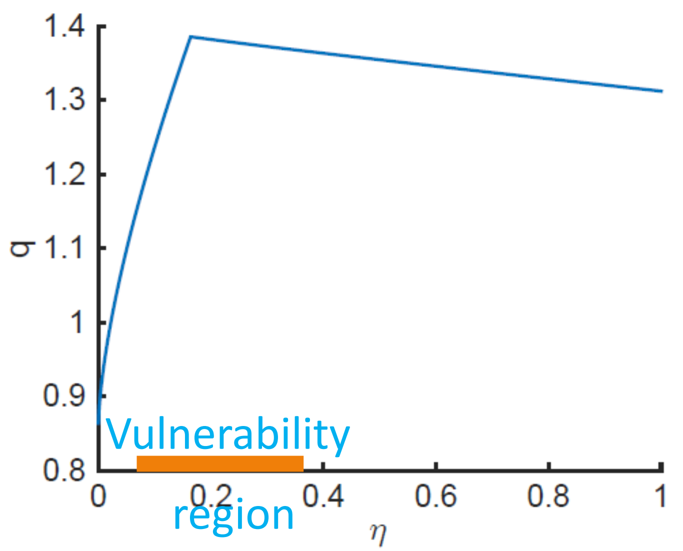
Unanticipated Run on Experts

- Can unanticipated withdrawal of all experts' funding be self-fulfilling?
- Unanticipated crash – jump to $\eta^e = 0$
  - Absent a run: solution as in earlier lecture, since unanticipated
  - When do jump capital losses wipe out experts' net worth?

$$\left(q(\eta_t^e) - q(0)\right) \underbrace{\left(\theta_t^{e,K} + \theta_t^{e,OE}\right)\eta_t^e}_{\chi_t^e} K_t \geq \eta_t^e q(\eta_t^e) K_t$$

$$q(\eta_t^e)\left(1 - \frac{\eta_t^e}{\chi^e(\eta_t^e)}\right) \geq q(0) \text{ or } q(\eta_t^e)\left(1 - \frac{1}{\theta_t^{e,K} + \theta_t^{e,OE}}\right) \geq q(0)$$

  - Vulnerability region:
    - High price (not very low η^e)
    - "high risk-leverage" (not very high η^e)



  - After run: $\eta^0 = 0$ forever

2 Types of Runs and Modeling Challenges

- What type of run? What's the trigger?
  - Funding supply run: Depositor/households run
    - Household withdraw funding to experts
  - Funding demand run: Other experts run
    - Each expert tries to pay back debt and fire-sells assets
    - Drop in q is driver

- Model advantage: Always jump to the same point $q(\eta^e = 0)$!

- Modeling Challenges: (see Mendo (2020))
  1. Experts are whipped out forever.
     - OLG structure:
       - Death: all agents die with Poisson rate λ^d,
       - Birth: fraction ψ of newborns are experts
  2. With anticipated run, expert fear "infinite marginal utility state" $\eta^e = 0$.
     - Transfer of τK to bankrupt experts after run
     - Also fixes challenge 1.
     - To keep τ small also introduce relative performance penalty (then take limit $\tau \to 0$)

Economic insights

- Volatility Paradox also in Jump risk
  Reduction in exogenous risk σ
  $\Rightarrow$ higher leverage
  1. Increase in price risk σ^q (Brownian)
     - $(\sigma + \sigma^q)$ stays roughly stable
  2. Increase in run risk (Jump)
     - Total risk can be higher
       - Low risk environment is "risky"
         - E.g. through better Brownian risk insurance
       - Recall for very low η, $j^q(\eta) = 0$, there is no run risk
         since price q is already low and can not drop much further
- No runs in very bad crisis times
  - vulnerability region doesn't start at $\eta = 0$
- Invariance of Relative Capital Demand
  - If experts lever up more, risk is held by household due to default risk

From Ito to Levy and Cox Processes

- Ito process: $dX_t = \mu_t^X X_t dt + \sigma_t^X X_t dZ_t$ (geometric)
  the Brownian "shocks" dZ_t are i.i.d. and small s.t. continuous path
  - For non-normal shocks within dt one needs discontinuities
- Levy process $dL_t = a dt + b dZ_t + dJ_t$ – most general class with i.i.d. increments
  $$dX_t = \mu_t^X X_t dt + \sigma_t^X X_t dZ_t + j_t^X X_{t-} dJ_t$$
  - Restrict attention to Poisson processes:
    - Levy jump process can be written as integral w.r.t. Poisson random measures
    - Poisson process with arrival rate $\lambda > 0$:
    - J takes on values in $\mathbb{N}_0 = \{0,1,2,\dots\}$
    - Increments $J_{t+\Delta t} - J_t$ are Poisson distributed with Parameter $\lambda \Delta t$
    - Stochastic integral w.r.t. Poisson process simply sums up the values of the integrand
      - $\int_0^T a_t dJ_t = \sum_{n=1}^{J_T} a_{\tau_n}$
  - Cox process: λ_t can be time-varying
  - Compensated Jump process $J_t - \int_0^t \lambda_s ds$ is martingale
    - If $\int_0^t a_s dJ_s$ and a_t uses info only up to right before t then $J_t - \int_0^t a_s \lambda_s ds$ is martingale

Ito formulas

- $df(X_t) = f'(X_t)(\mu_t^X X_t dt + \sigma_t^X X_t dZ_t) + \frac{1}{2} f''(X_t)(\sigma_t^X X_t)^2 dt + (\boldsymbol{f(X_t) - f(X_{t-})}) \boldsymbol{dJ_t}$

  $= \left(f'(X_t)\mu_t^X X_t + \frac{1}{2} f''(X_t)(\sigma_t^X X_t)^2\right) dt + f'(X_t)\sigma_t^X X_t dZ_t + \left(\boldsymbol{f\left((1 + j_t^X) X_{t-}\right) - f(X_{t-})}\right) \boldsymbol{dJ_t}$
- Power rule:
  - $\frac{dX_t^\gamma}{X_t^\gamma} = (\gamma \mu_t^X + \gamma(\gamma - 1)(\sigma_t^X)^2) dt + \gamma \sigma_t^X dZ_t + \left(\boldsymbol{(1 + j_t^X)^\gamma - 1}\right) \boldsymbol{dJ_t}$
- Product rule:
  - $\frac{d(X_t Y_t)}{X_t \ \ Y_t} = (\mu_t^X + \mu_t^Y + \sigma_t^X \sigma_t^Y) dt + (\sigma_t^X + \sigma_t^Y) dZ_t + (\boldsymbol{j_t^X + j_t^Y + j_t^X j_t^Y}) \boldsymbol{dJ_t}$
- Quotient rule:
  - $\frac{d(X_t / Y_t)}{X_t \ \ / Y_t} = (\mu_t^X - \mu_t^Y + (\sigma_t^Y)^2 - \sigma_t^X \sigma_t^Y) dt + (\sigma_t^X - \sigma_t^Y) dZ_t + \boldsymbol{\frac{j_t^X - j_t^Y}{1 + j_t^Y} dJ_t}$
- Memorize simple rules:
  - $\boldsymbol{1 + j_t^{X^\gamma} = (1 + j_t^X)^\gamma}$
  - $\boldsymbol{1 + j_t^{XY} = (1 + j_t^X)(1 + j_t^Y)}$
  - $\boldsymbol{1 + j_t^{X/Y} = \frac{1 + j_t^X}{1 + j_t^Y}}$

Solving MacroModels Step-by-Step

0. Postulate aggregates, price processes & obtain return processes
1. For given C/N-ratio and SDF processes for each i *finance block*
   a. Real investment ι + Goods market clearing *(static)*
   - *Toolbox 1:* Martingale Approach, HJB vs. Stochastic Maximum Principle Approach

   b. Portfolio choice θ + Asset market clearing or
   Asset allocation κ & risk allocation χ
   - *Toolbox 2:* "price-taking social planner approach" – Fisher separation theorem

   c. "Money evaluation equation" ϑ
   - *Toolbox 3:* Change in numeraire to total wealth (including SDF)
2. Evolution of state variable η (and K) *forward equation*
3. Value functions *backward equation*
   a. Value fcn. as fcn. of individual investment opportunities ω
   - *Special cases: log-utility, constant investment opportunities*

   b. Separating value fcn. $V^i(n^{\tilde{i}};\eta,K)$ into $v^i(\eta)u(K)(n^{\tilde{i}}/n^i)^{1-\gamma}$
   c. Derive C/N-ratio and ς price of risk
4. Numerical model solution
   a. Transform BSDE for separated value fcn. $v^i(\eta)$ into PDE
   b. Solve PDE via value function iteration
5. KFE: Stationary distribution, Fan charts

0. Postulate Aggregates and Processes

- Individual capital evolution:

$$\frac{dk_t^{\tilde{\imath},i}}{k_t^{\tilde{\imath},i}} = \left(\Phi\left(\iota^{\tilde{\imath},i}\right) - \delta\right)dt + \sigma dZ_t + d\Delta_t^{k,\tilde{\imath},i}$$

  - Where $\Delta_t^{k,\tilde{\imath},i}$ is the individual cumulative capital purchase process
- Capital aggregation:
  - Within sector i: $K_t^i \equiv \int k_t^{\tilde{\imath},i} d\tilde{\imath}$
  - Across sectors: $K_t \equiv \sum_i K_t^i$
  - Capital share: $\kappa_t^i \equiv K_t^i / K_t$

$$\frac{dK_t}{K_t} = \left(\Phi\left(\iota_t^i\right) - \delta\right)dt + \sigma dZ_t$$

- Net worth aggregation:
  - Within sector i: $N_t^i \equiv \int n_t^{\tilde{\imath},i} d\tilde{\imath}$
  - Across sectors: $N_t \equiv \sum_i N_t^i$
  - Wealth share: $\eta_t^i \equiv N_t^i / N_t$
- Value of capital stock: $q_t K_t$

  Postulate $dq_t/q_t = \mu_t^q dt + \sigma_t^q dZ_t + j_t^q dJ_t$

Same Brownian

-

0. Postulate Aggregates and Processes

- Individual capital evolution:

$$\frac{dk_t^{\tilde{\imath},i}}{k_t^{\tilde{\imath},i}} = \left(\Phi\left(\iota^{\tilde{\imath},i}\right) - \delta\right)dt + \sigma dZ_t + d\Delta_t^{k,\tilde{\imath},i}$$

  - Where $\Delta_t^{k,\tilde{\imath},i}$ is the individual cumulative capital purchase process
- Capital aggregation:
  - Within sector i: $K_t^i \equiv \int k_t^{\tilde{\imath},i} d\tilde{\imath}$
  - Across sectors: $K_t \equiv \sum_i K_t^i$
  - Capital share: $\kappa_t^i \equiv K_t^i / K_t$

$$\frac{dK_t}{K_t} = \left(\Phi\left(\iota_t^i\right) - \delta\right)dt + \sigma dZ_t$$

- Net worth aggregation:
  - Within sector i: $N_t^i \equiv \int n_t^{\tilde{\imath},i} d\tilde{\imath}$
  - Across sectors: $N_t \equiv \sum_i N_t^i$
  - Wealth share: $\eta_t^i \equiv N_t^i / N_t$
- Value of capital stock: $q_t K_t$

  Postulate $dq_t/q_t = \mu_t^q dt + \sigma_t^q dZ_t + j_t^q dJ_t$

Sunspot arrival rate

- Postulated SDF-process: $\frac{d\xi_t^i}{\xi_t^i} = \underbrace{\mu_t^{\xi i}}_{\equiv -r_t^{F,i}} dt + \underbrace{\sigma_t^{\xi i}}_{\equiv -\varsigma_t^i} dZ_t + \underbrace{j_t^{\xi i}}_{\equiv -\nu_t^i} (dJ_t - \lambda_t dt)$ (c is numeraire)

Since only risky debt and not risk-free debt is traded

0. Postulate Aggregates and Processes

- … from price processes to return processes (using Ito)
  - Use Ito product rule to obtain capital gain rate (in absence of purchases/sales)
    - Define $\check{k}_t^{\tilde{\imath}}$: $\frac{d\check{k}_t^{\tilde{\imath},i}}{\check{k}_t^{\tilde{\imath},i}} = \left(\Phi(\iota_t^{\tilde{\imath},i}) - \delta\right)dt + \sigma dZ_t + d\Delta_t^{k,\tilde{\imath},i}$ without purchases/sales

Dividend yield

E[Capital gain rate]= $\frac{d(q_t \check{k}^i{}_t)}{(q_t \check{k}_t^i)}$

$$dr_t^k(\iota_t^{\tilde{\imath},i}) = \left(\frac{a^i - \iota_t^i}{q} + \Phi(\iota_t^i) - \delta + \mu_t^q + \sigma\sigma_t^q\right)dt + \left(\sigma + \sigma_t^q\right)dZ_t + j_t^q dJ_t$$

For aggregate capital return, Replace a^i with $A(\kappa)$

- Return on defaultable debt

$$dr_t^D = r_t dt + j_t^{r^D} dJ_t$$

- Postulate SDF-process: (Example: $\xi_t^i = e^{-\rho t}V'(n_t^i)$.)

$$\frac{d\xi_t^i}{\xi_t^i} = -r_t^{F,i}dt - \varsigma_t^i dZ_t - \nu_t^i(dJ_t - \lambda_t dt)$$

Price of risk (ς_t^i) Price of jump/run risk (ν_t^i)

1a. Individual Agent Choice of ι, θ, c/n

- Choice of ι is static problem (and separable) for each t
- $\max_{\iota_t^i} dr_t^k(\iota_t^i)$

$$= \max_{\iota_t^i} \left(\frac{a^i - \iota_t^i}{q_t} + \Phi(\iota_t^i) - \delta + \mu^q + \sigma\sigma^q \right) dt + (\sigma + \sigma_t^q) dZ_t + j_t^q dJ_t$$

For aggregate capital return,
Replace a^i with $A(\kappa)$

- FOC: $\frac{1}{q_t} = \Phi'(\iota_t^i)$ Tobin's q
  - All agents $\iota_t^i = \iota_t \Rightarrow \frac{dK_t}{K_t} = (\Phi(\iota_t) - \delta)\, dt + \sigma dZ_t$
  - Special functional form:
    - $\Phi(\iota) = \frac{1}{\phi}\log(\phi\iota + 1) \Rightarrow \phi\iota = q - 1$
- Goods market clearing: $(A(\kappa) - \iota_t)K_t = \sum_i C_t^i$.

$$\kappa_t a^e K_t + (1 - \kappa_t) a^h K_t - \iota(q_t) K_t = \eta_t^e \frac{C_t^e}{N_t^e} q_t K_t + (1 - \eta_t^e) \frac{C_t^h}{N_t^h} q_t K_t$$

Solving MacroModels Step-by-Step

0. Postulate aggregates, price processes & obtain return processes
1. For given C/N-ratio and SDF processes for each i *finance block*
   a. Real investment ι + Goods market clearing *(static)*
   - *Toolbox 1:* Martingale Approach, HJB vs. Stochastic Maximum Principle Approach

   b. Portfolio choice θ + Asset market clearing or
   Asset allocation κ & risk allocation χ
   - *Toolbox 2:* "price-taking social planner approach" – Fisher separation theorem

   c. "Money evaluation equation" ϑ
   - *Toolbox 3:* Change in numeraire to total wealth (including SDF)
2. Evolution of state variable η (and K) *forward equation*
3. Value functions *backward equation*
   a. Value fcn. as fcn. of individual investment opportunities ω
   - *Special cases: log-utility, constant investment opportunities*

   b. Separating value fcn. $V^i(n^{\tilde{i}};\eta,K)$ into $v^i(\eta)u(K)\left(n^{\tilde{i}}/n^i\right)^{1-\gamma}$
   c. Derive C/N-ratio and ς price of risk
4. Numerical model solution
   a. Transform BSDE for separated value fcn. $v^i(\eta)$ into PDE
   b. Solve PDE via value function iteration
5. KFE: Stationary distribution, Fan charts

1a. Individual Agent Choice of ι, θ, c/n

$$\max_{\{\iota_t,\boldsymbol{\theta}_t,c_t\}_{t=0}^{\infty}} E\left[\int_0^{\infty} e^{-\rho t} u(c_t)dt\right]$$

s.t. $\frac{dn_t}{n_t} = -\frac{c_t}{n_t}dt + \sum_j \theta_t^j dr_t^j$ + labor income/endow/taxes

n_0 given

- Portfolio Choice: Martingale Approach
  - Let x_t^A be the value of a "self-financing trading strategy"(reinvest dividends)
- Theorem: $\xi_t x_t^A$ follows a Martingale, i.e. drift $= 0$.
  - Let $\frac{dx_t^A}{x_t^A} = \mu_t^A dt + \sigma_t^A dZ_t + j_t^A dJ_t,$
  - Recall SDF $\frac{d\xi_t^i}{\xi_t^i} = -r_t^{F,i}dt - \varsigma_t^i dZ_t - \nu_t^i(dJ_t - \lambda_t dt)$
  - By Ito product rule

$$\frac{d(\xi_t^i x_t^A)}{\xi_t^i x_t^A} = \left(-r_t^{F,i} + \mu_t^A - \varsigma_t^i \sigma_t^A + \nu_t^i \lambda_t\right)dt + \left(\sigma^A - \varsigma_t^i\right)dZ_t + \left(j_t^A - \nu_t^i - \nu_t^i j_t^A\right)dJ_t$$

$$\frac{d\left(\xi_t^i x_t^A\right)}{\xi_t^i x_t^A} = \left(-r_t^{F,i} + \mu_t^A - \varsigma_t^i \sigma_t^A + \lambda_t j_t^A - \lambda_t \nu_t^i j_t^A\right)dt + \underbrace{\left(\sigma^A - \varsigma_t^i\right)dZ_t + \left(j_t^A - \nu_t^i - \nu_t^i j_t^A\right)(dJ_t - \lambda_t dt)}_{martingale}$$

  - Expected return: $\mu_t^A + \lambda j_t^A = r_t^{F,i} + \varsigma_t^i \sigma_t^A + \lambda \nu_t^i j_t^A$

1a. Individual Agent Choice of ι, θ, c/n

- Expected return: $\mu_t^A + \lambda j_t^A = r_t^{F,i} + \varsigma_t^i \sigma_t^A + \nu_t^i \lambda j_t^A$
  - $r_t^{F,i}$ is the shadow risk-free rate (need not to be same across groups)
  - ς_t^i is the price of Brownian risk of agents i,
    $\varsigma_t^i \sigma_t^A$ is the required Brownian risk premium of agents i
  - $\nu_t^i \lambda_t$ is the price of Poisson upside risk if $j^A > 0$
    For risk-neutral agents $\nu_t^i = 0$

- Remark:
  - $dr^{e,K}$ experts return on capital
  - $dr^{h,OE}$ households return on outside equity
  - $dr^{h,D}$ households' return on debt is risky (due to bankruptcy)

1a. Individual Agent Choice of ι, θ, c/n

- Expected return: $$\mu_t^A + \lambda j_t^A = r_t^{F,i} + \varsigma_t^i \sigma_t^A + \nu_t^i \lambda j_t^A$$
  - $r_t^{F,i}$ is the shadow risk-free rate (need not to be same across groups)
  - ς_t^i is the price of Brownian risk of agents i,
    $\varsigma_t^i \sigma_t^A$ is the required Brownian risk premium of agents i
  - $\nu_t^i \lambda_t$ is the price of Poisson upside risk if $j^A > 0$
    For risk-neutral agents $\nu_t^i = 0$

- Remark:
  - For CRRA utility: SDF is $\xi_t = e^{-\rho} \omega_t^{1-\gamma} n_t^{-\gamma}$
    $$1 - \nu_t = (1 + j_t^\omega)^{1-\gamma} (1 + j_t^n)^{-\gamma}$$
  - For log utility: $$\nu_t = 1 - \frac{1}{1+j_t^n} = \frac{j_t^n}{1+j_t^n}$$
  - For Epstein-Zin: part of ω_t-process

1a. Individual Agent Choice of ι, θ, c/n

- Of experts with outside equity issuance (after plugging in households' outside equity choice)

$$\frac{a^e-\iota_t}{q_t}+\Phi(\iota_t)-\delta+\mu_t^q+\sigma\sigma_t^q-\left[\frac{\chi_t^e}{\kappa_t^e}r_t^{F,e}+\left(1-\frac{\chi_t^e}{\kappa_t^e}\right)r_t^{F,h}\right]+\lambda_t\, j_t^q=$$
$$\left[\varsigma_t^e\frac{\chi_t^e}{\kappa_t^e}+\varsigma_t^h\left(1-\frac{\chi_t^e}{\kappa_t^e}\right)\right](\sigma+\sigma^q)+\left[\nu_t^e\frac{\chi_t^e}{\kappa_t^e}+\nu_t^h\left(1-\frac{\chi_t^e}{\kappa_t^e}\right)\right]\lambda_t\, j_t^q$$

- Of households' capital choice

$$\frac{a^h-\iota_t}{q_t}+\Phi(\iota_t)-\delta+\mu_t^q+\sigma\sigma_t^q-r_t^{F,h}+\lambda_t\left(j_t^q-j_t^{r^D}\right)$$
$$\leq\varsigma_t^h(\sigma+\sigma^q)+\nu_t^h\lambda_t(j_t^q-j_t^{r^D})$$

with equality if $\kappa_t^e<1$

- Note: Later approach replaces this step with Fisher Separation Social Planners' choice (see below)

Solving MacroModels Step-by-Step

0. Postulate aggregates, price processes & obtain return processes
1. For given C/N-ratio and SDF processes for each i *finance block*
    a. Real investment ι + Goods market clearing *(static)*
    - *Toolbox 1:* Martingale Approach, HJB vs. Stochastic Maximum Principle Approach

    b. Portfolio choice θ + Asset market clearing or
       Asset allocation κ & risk allocation χ
    - *Toolbox 2:* "price-taking social planner approach" – Fisher separation theorem

    c. "Money evaluation equation" ϑ
    - *Toolbox 3:* Change in numeraire to total wealth (including SDF)
2. Evolution of state variable η (and K) *forward equation*
3. Value functions *backward equation*
    a. Value fcn. as fcn. of individual investment opportunities ω
    - *Special cases: log-utility, constant investment opportunities*

    b. Separating value fcn. $V^i(n^{\tilde{\imath}};\eta,K)$ into $v^i(\eta)u(K)\left(n^{\tilde{\imath}}/n^i\right)^{1-\gamma}$
    c. Derive C/N-ratio and ς price of risk
4. Numerical model solution
    a. Transform BSDE for separated value fcn. $v^i(\eta)$ into PDE
    b. Solve PDE via value function iteration
5. KFE: Stationary distribution, Fan charts

1b. Asset/Risk Allocation across I Types

Let $dN_t/N_t = \mu_t^N dt + \sigma_t^N dZ_t + j_t^N dJ_t$

- **Price-Taking Planner's Theorem:**
  A social planner that takes prices as given chooses an physical asset allocation, κ_t, and Brownian risk allocation, χ_t, and a Jump risk allocation, ζ_t, that coincides with the choices implied by all individuals' portfolio choices.

$$\boldsymbol{\varsigma}_t = (\varsigma_t^1, \dots, \varsigma_t^I)$$
$$\boldsymbol{\chi}_t = (\chi_t^1, \dots, \chi_t^I)$$
$$\boldsymbol{\zeta}_t = (\zeta_t^1, \dots, \zeta_t^I)$$
$$\boldsymbol{\sigma}(\boldsymbol{\chi}_t) = (\boldsymbol{\chi}_t^1 \sigma^N, \dots, \boldsymbol{\chi}_t^I \sigma^N)$$
$$\boldsymbol{j}(\boldsymbol{\zeta}_t) = (\zeta_t^1 j_t^N, \dots, \zeta_t^I j_t^N)$$

Return on total wealth

- Planner's problem

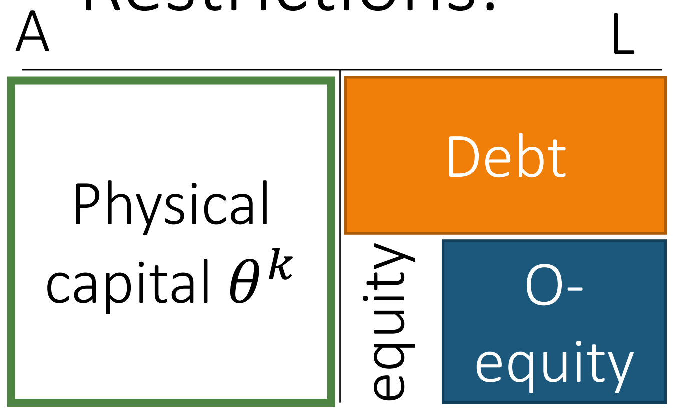
$$\max_{\{\boldsymbol{\kappa}_t, \boldsymbol{\chi}_t, \boldsymbol{\zeta}_t\}} \frac{E_t[dr_t^N(\kappa_t)]}{dt} - \boldsymbol{\varsigma}_t \sigma(\boldsymbol{\chi}_t) - \lambda \boldsymbol{\nu} j(\boldsymbol{\zeta}_t)$$

subject to friction: $F(\boldsymbol{\kappa}_t, \boldsymbol{\chi}_t, \zeta_t) \leq 0$

  - Example:
    1. $\chi_t = \zeta_t = \kappa_t$ (can't issue outside equity to offload Brownian or risky debt to offload Jump risk)
    2. $\chi_t \geq \alpha \kappa_t$ (skin in the game constraint, outside equity up to a limit)

1b. Allocation of Capital/Risk: 2 Types

- Expert: $\boldsymbol{\theta^e} = (\theta^{e,K}, \theta^{e,OE}, \theta^{e,D})$ for capital, outside equity, debt
  - Restrictions:



$\theta^{e,K} \geq 0,$

$\theta^{e,OE} \leq 0,$ only issue outside equity

$\theta^{e,OE} \geq -(1-\alpha)\theta^{e,K}$ skin in the game

maximize

$$\theta_t^{e,K} E[dr_t^{e,K}]/dt + \theta_t^{e,OE} E[dr_t^{OE}]/dt + \theta_t^{e,D} E[dr_t^{D,e}]/dt - \varsigma_t^e(\theta_t^{e,K} + \theta_t^{e,OE})\sigma^{r^{e,K}}$$
$$- \lambda_t \nu_t^e((\theta_t^{e,K} + \theta_t^{e,OE}) j_t^{r^{eK}} + \theta_t^{e,D} j_t^{r^D})$$

Note $j_t^{r^D}$ is just the jump due to the loss and not the change in D due to rebalancing.

- Household: $\boldsymbol{\theta^h} = (\theta^{h,K}, \theta^{h,OE}, \theta^{h,D})$

$\theta^{h,K} \geq 0$

$\theta^{h,OE} \geq 0$

maximize

$$\theta^{h,K} E[dr_t^{h,K}]/dt + \theta^{h,OE} E[dr_t^{OE}]/dt + \theta^{h,D} E[dr_t^{D,h}]/dt - \varsigma_t^h(\theta_t^{h,K} + \theta_t^{h,OE})\sigma^{r^{h,K}}$$
$$- \lambda_t \nu_t^h((\theta_t^{h,K} + \theta_t^{h,OE}) j_t^{r^{hK}} + \theta_t^{h,D} j_t^{r^D})$$

1b. Allocation of Capital/Risk: 2 Types

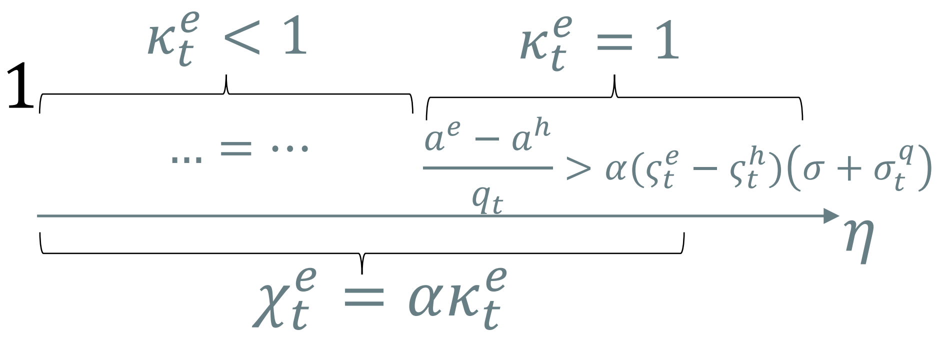
- Example 2: 2 Type + with outside equity

$$\max_{\{\kappa_t^e,\chi_t^e\}} \left[\frac{\kappa_t^e a^e + (1-\kappa_t^e)a^h - \iota_t}{q_t} + \Phi(\iota_t) - \delta + \right] - \left(\chi_t^e \varsigma_t^e + (1-\chi_t^e)\varsigma_t^h\right)(\sigma + \sigma_t^q)$$

- FOC_χ: Case 1: $\varsigma_t^e(\sigma + \sigma_t^q) + \cdots > \varsigma_t^h(\sigma + \sigma_t^q) + \cdots \Rightarrow \chi_t^e = \alpha\kappa_t^e$

  Case 2: $=$ $\chi_t^e > \alpha\kappa_t^e$

  - Case 1: plug $\chi_t^e = \alpha\kappa_t^e$ in objective
    a. $FOC_\kappa: \frac{a^e - a^h}{q_t} > \alpha(\varsigma_t^e - \varsigma_t^h)(\sigma + \sigma_t^q) + \cdots \Rightarrow \kappa_t^e = 1$
    b. $=$ $\Rightarrow \kappa_t^e < 1$
  - Case 2:
    a. $FOC_\kappa: \frac{a^e - a^h}{q_t} > 0$ $\Rightarrow \kappa_t^e = 1$
    b. $= 0 \Rightarrow \kappa_t^e < 1$ impossible



Invariance of Relative Capital Demand

- One of the insights of Mendo (2020) is that self-fulfilling jumps do not influence the relative demand for capital of experts relative to households.
  I.e. the excess market return that experts demand to hold capital is not affected.
- Subtract experts pricing condition from households
- $\mu_t^{r^{k,e}} - \mu_t^{r^{k,h}} \geq \frac{\chi_t^e}{\kappa_t^e}(\varsigma_t^e - \varsigma_t^h)(\sigma + \sigma_t^q) - \frac{\chi_t^e}{\kappa_t^e}\lambda_t(1-\nu_t^h)\underbrace{\left(\frac{\partial j_t^{r^D}}{\partial \theta_t^{e,K}}(\theta_t^{e,K} - 1) + j_t^q - j_t^{r^D}\right)}_{=0}$
- Losses are split between experts and households (via defaultable debt)
- Since experts' losses are capped by their net worth due to limited liability, all additional losses from increasing capital holding, $\theta_t^{e,K}$, are born by households

Solving MacroModels Step-by-Step

0. Postulate aggregates, price processes & obtain return processes
1. For given C/N-ratio and SDF processes for each i *finance block*
   a. Real investment ι + Goods market clearing *(static)*
   - *Toolbox 1:* Martingale Approach, HJB vs. Stochastic Maximum Principle Approach

   b. Portfolio choice θ + Asset market clearing or
   Asset allocation κ & risk allocation χ
   - *Toolbox 2:* "price-taking social planner approach" – Fisher separation theorem

   c. "Money evaluation equation" ϑ
   - *Toolbox 3:* Change in numeraire to total wealth (including SDF)
2. Evolution of state variable η (and K) *forward equation*
3. Value functions *backward equation*
   a. Value fcn. as fcn. of individual investment opportunities ω
   - *Special cases: log-utility, constant investment opportunities*

   b. Separating value fcn. $V^i(n^{\tilde{\imath}};\eta,K)$ into $v^i(\eta)u(K)\left(n^{\tilde{\imath}}/n^i\right)^{1-\gamma}$
   c. Derive C/N-ratio and ς price of risk
4. Numerical model solution
   a. Transform BSDE for separated value fcn. $v^i(\eta)$ into PDE
   b. Solve PDE via value function iteration
5. KFE: Stationary distribution, Fan charts

Toolbox 3: Change of Numeraire

- x_t^A is a value of a self-financing strategy/asset in \$
- Y_t price of € in \$ (exchange rate)

$$\frac{dY_t}{Y_t} = \mu_t^Y dt + \sigma_t^Y dZ_t + j_t^Y dJ_t$$

- x_t^A / Y_t value of the self-financing strategy/asset in €

$\underbrace{e^{-\rho t} u'(c_t)}_{=\xi_t} Y_t \frac{x_t^A}{Y_t}$ follows a martingale (+ SDF in new numeraire $\hat{\xi}_t = \xi_t Y_t$)

Frequency of sunspots, λ_t, are not dependent on numeraire

Recall $\mu_t^A - \mu_t^B + \lambda_t (j_t^A - j_t^B) = \underbrace{\left(-\sigma_t^\xi\right)}_{=\varsigma_t} \underbrace{(\sigma^A - \sigma_t^B)}_{risk} + \nu_t \lambda_t (j_t^A - j_t^B)$

$$\mu_t^{\frac{A}{Y}} - \mu_t^{\frac{B}{Y}} + \lambda_t \left(j_t^{\frac{A}{Y}} - j_t^{\frac{B}{Y}}\right) = \underbrace{\left(-\sigma_t^\xi - \sigma_t^Y\right)}_{price\ of\ risk} \underbrace{(\sigma^A - \sigma_t^B)}_{risk} + (\nu_t - j_t^Y + \nu_t j_t^Y)\, \lambda_t \frac{j_t^A - j_t^B}{1 + j_t^Y}$$

- Price of Brownian risk $\varsigma^€ = \varsigma^\$ - \sigma^Y$
- Price of Jump risk $\nu_t^€ = \nu_t^\$ - j_t^Y + \nu_t^\$ j_t^Y$

Change of Numeraire: SDF

- SDF in good numeraire is

$$d\xi_t^i/\xi_{t-}^i = -r_t^{F,i}dt - \varsigma_t^i dZ_t - \nu_t^i\,(dJ_t - \lambda_t dt)$$

- SDF in total net worth numeraire is

$$d\hat{\xi}_t^i/\hat{\xi}_{t-}^i = \mu_t^{\hat{\xi}^i}dt - \left(\varsigma_t^i - \sigma_t^N\right)dZ_t - \left(\nu_t^i - j_t^N + \nu_t^i j_t^N\right)dJ_t$$

$$= \hat{r}_t^{F,i}dt - \underbrace{\left(\varsigma_t^i - \sigma_t^N\right)}_{=\hat{\varsigma}_t^i} dZ_t - \left(\nu_t^i - j_t^N + \nu_t^i j_t^N\right)(dJ_t - \lambda_t dt)$$

Solving MacroModels Step-by-Step

0. Postulate aggregates, price processes & obtain return processes
1. For given C/N-ratio and SDF processes for each i *finance block*
   a. Real investment ι + Goods market clearing *(static)*
   - *Toolbox 1:* Martingale Approach, HJB vs. Stochastic Maximum Principle Approach

   b. Portfolio choice θ + Asset market clearing or
      Asset allocation κ & risk allocation χ
   - *Toolbox 2:* "price-taking social planner approach" – Fisher separation theorem

   c. "Money evaluation equation" ϑ
   - *Toolbox 3:* Change in numeraire to total wealth (including SDF)
2. Evolution of state variable η (and K) *forward equation*
3. Value functions *backward equation*
   a. Value fcn. as fcn. of individual investment opportunities ω
   - *Special cases: log-utility, constant investment opportunities*

   b. Separating value fcn. $V^i(n^{\tilde{i}};\eta,K)$ into $v^i(\eta)u(K)\left(n^{\tilde{i}}/n^i\right)^{1-\gamma}$
   c. Derive C/N-ratio and ς price of risk
4. Numerical model solution
   a. Transform BSDE for separated value fcn. $v^i(\eta)$ into PDE
   b. Solve PDE via value function iteration
5. KFE: Stationary distribution, Fan charts

2. GE: Markov States and Equilibria

- Equilibrium is a map

Histories of shocks ⇢ prices $q_t, \varsigma_t^i, \iota_t^i, \theta_t^e$

$\{\boldsymbol{Z}_s, s \in [0, t]\}$

net worth distribution

$$\eta_t^e = \frac{N_t^e}{q_t K_t} \in (0,1)$$

net worth share

- All agents maximize utility
  - Choose: portfolio, consumption, technology
- All markets clear
  - Consumption, capital, money, outside equity

2. Law of Motion of Wealth Share η_t

- Method 1: Using Ito's quotation rule $\eta_t^i = N_t^i/(q_t K_t)$
  - Recall

$$\frac{dN_t^i}{N_t^i} = -\frac{C_t^i}{N_t^i}dt + r_t^{bm}dt + \underbrace{\varsigma_t^i}_{price\ of\ risk} \underbrace{\left(\frac{\chi_t^i}{\eta_t^i}(\sigma+\sigma_t^q) - \sigma^{bm}\right)}_{excess\ risk} dt + \nu\left(j_t^{N^i} - j_t^{bm}\right)dt$$

$$+ \frac{\chi_t^i}{\eta_t^i}(\sigma + \sigma_t^q)dZ_t + j_t^{N^i}dJ_t$$

bm = benchmark asset
(tradable by everyone)

I ignored OLG terms for now

  - $\frac{d\eta_t^i}{\eta_t^i} = \ldots$ (lots of algebra)

Transfers in case $dJ_t = 1$ in vulnerability region: $\tau^i K_t$

- Method 2: Change of numeraire + Martingale Approach
  - New numeraire: Total wealth in the economy, N_t
  - Apply Martingale Approach for value of i's portfolio
    - Simple algebra to obtain drift of η_t^i: $\mu_t^{\eta^i}$
      Note that change of numeraire does not affect ratio η^i!

2. μ^{η} Drift of Wealth Share: Many Types

- New Numeraire
  - "Total net worth" in the economy, N_t (without superscript)
  - Type i's portfolio net worth = net worth share

- Martingale Approach with new numeraire
  - Asset A $= i$'s portfolio return in terms of total wealth,

$$\left(\underbrace{\frac{C_t^i}{N_t^i}}_{\text{Dividend yield}} + \underbrace{\mu_t^{\eta^i} + \lambda_t j^{\eta^i}}_{\text{E[capital gains] rate}}\right) dt + \sigma_t^{\eta^i} dZ_t + \tilde{\sigma}_t^{\eta^i} d\tilde{Z}_t$$

  - Asset B (benchmark asset that everyone can hold, e.g. risk-free asset or money (in terms of total economy wide wealth as numeraire))

$$r_t^{bm} dt + \sigma_t^{bm} dZ_t$$

Hat notation $\hat{}$ indicates total net worth numeraire

  - Apply our martingale asset pricing formula

$$\mu_t^A - \mu_t^B + \lambda_t (j_t^A - j_t^B) = \hat{\varsigma}_t^i (\sigma_t^A - \sigma_t^B) + \lambda_t \hat{\nu}_t (j_t^A - j_t^B)$$

2. μ^η Drift of Wealth Share: Many Types

- Asset pricing formula (relative to benchmark asset)

$$\mu_t^{\eta^i} + \frac{C_t^i}{N_t^i} - r_t^{bm} + \lambda_t \left(j_t^{\eta^i} - j_t^{bm} \right) = \left(\varsigma_t^i - \sigma_t^N \right) \left(\sigma_t^{\eta^i} - \sigma_t^{bm} \right) + \lambda_t \hat{\nu}_t^i \left(j_t^{\eta^i} - j_t^{bm} \right)$$

due to change in numeraire

- Add up across types (weighted),
  (capital letters without superscripts are aggregates for total economy)

$$\underbrace{\sum_{i'}^{I} \eta_t^{i'} \mu_t^{\eta^{i'}}}_{=0} + \frac{C_t}{N_t} - r_t^{bm} + \underbrace{\lambda_t \sum_{i'}^{I} \eta_t^{i'} j_t^{\eta^{i'}}}_{=0} - \lambda_t j_t^{bm} =$$

$$\sum_{i'} \eta_t^{i'} \hat{\varsigma}_t^{i'} \left(\sigma_t^{\eta^{i'}} - \sigma_t^{bm} \right) + \lambda_t \sum_{i'} \eta_t^{i'} \hat{\nu}_t^{i'} \left(j_t^{\eta^{i'}} - j_t^{bm} \right)$$

- Subtract from first equation

$$\mu_t^{\eta^i} + \lambda_t j_t^{\eta^{i'}} = \frac{C_t}{N_t} - \frac{C_t^i}{N_t^i} + \hat{\varsigma}_t^i \left(\sigma^{\eta^i} - \sigma^{bm} \right) - \sum_{i'} \eta_t^{i'} \hat{\varsigma}_t^{i'} \left(\sigma_t^{\eta^{i'}} - \sigma_t^{bm} \right)$$

$$+ \lambda_t \hat{\nu}_t^i \left(j_t^{\eta^i} - j_t^{bm} \right) - \lambda_t \textstyle\sum_{i'} \eta_t^{i'} \hat{\nu}_t^{i'} \left(j_t^{\eta^{i'}} - j_t^{bm} \right)$$

2. μ^η Drift of Wealth Share: Two Types $i \in \{e, h\}$

- Subtract from each other yield net worth share dynamics

$$\mu_t^{\eta^e} + \lambda_t j_t^{\eta^e}$$
$$= \frac{C_t}{N_t} - \frac{C_t^e}{N_t^e} + (1-\eta_t^e)\hat{\varsigma}_t^e\left(\sigma_t^{\eta^e} - \sigma_t^{bm}\right) - (1-\eta_t^e)\hat{\varsigma}_t^h\left(\sigma_t^{\eta^h} - \sigma_t^{bm}\right)$$
$$+(1-\eta_t^e)\lambda_t\hat{v}_t^e\left(j_t^{\eta^e} - j_t^{bm}\right) - (1-\eta_t^e)\lambda_t\hat{v}_t^h\left(j_t^{\eta^h} - j_t^{bm}\right)$$

- In in our model, benchmark asset is risky debt,
  - $\sigma_t^{bm} = -\sigma_t^N$,
  - $j_t^{bm} = \frac{j^{r^D} - j^N}{1+j^N}$ (since $j_t^{r^D}$ return on risky debt jump in c-numeraire, j_t^N wealth jump)
    - Apply quotient rule for jumps

- $\mu_t^{\eta^e} + \lambda_t j_t^{\eta^e}$

$$= \frac{C_t}{N_t} - \frac{C_t^e}{N_t^e} + (1-\eta_t^e)\hat{\varsigma}_t^e\left(\sigma_t^{\eta^e} + \sigma_t^N\right) - (1-\eta_t^e)\hat{\varsigma}_t^h\left(\sigma_t^{\eta^h} + \sigma_t^N\right)$$
$$+(1-\eta_t^e)\lambda_t\hat{v}_t^e\left(j_t^{\eta^e} - \frac{j^{r^D} - j^N}{1+j^N}\right) - (1-\eta_t^e)\lambda_t\hat{v}_t^h\left(j_t^{\eta^h} - \frac{j^{r^D} - j^N}{1+j^N}\right)$$

2. σ^η Volatility of Wealth Share

- Since $\eta_t^i = N_t^i / N_t$,

$$\sigma_t^{\eta^i} = \sigma_t^{N^i} - \sigma_t^N = \sigma_t^{N^i} - \sum_{i'} \eta_t^{i'} \sigma_t^{N^{i'}}$$

$$= \left(1 - \eta_t^i\right)\sigma_t^{N^i} - \sum_{i^- \neq i} \eta_t^{i^-} \sigma_t^{N^{i^-}}$$

$$j_t^{\eta^i} = \frac{j_t^{N^i} - j_t^N}{1 + j_t^N} = \frac{j_t^{N^i} - \sum_{i'} \eta_t^{i'} j_t^{N^{i'}}}{1 + \sum_{i'} \eta_t^{i'} j_t^{N^{i'}}} = \frac{\left(1 - \eta_t^i\right) j_t^{N^i} - \sum_{i^- \neq i} \eta_t^{i^-} j_t^{N^{i^-}}}{1 + \sum_{i'} \eta_t^{i'} j_t^{N^{i'}}}$$

- Note for 2 types example

$$j_t^{\eta^e} = \frac{(1 - \eta_t^e)(j_t^{N^e} - j_t^{N^h})}{1 + \eta_t^e j_t^{N^e} + (1 - \eta_t^e) j_t^{N^h}}$$

- …

- Note:
  - OLG structure and
  - transfers τK_t

  also affects net worth evolution and still has to be incorporated!

Solving MacroModels Step-by-Step

0. Postulate aggregates, price processes & obtain return processes
1. For given C/N-ratio and SDF processes for each i *finance block*
   a. Real investment ι + Goods market clearing *(static)*
   - *Toolbox 1:* Martingale Approach, HJB vs. Stochastic Maximum Principle Approach

   b. Portfolio choice θ + Asset market clearing or
   Asset allocation κ & risk allocation χ
   - *Toolbox 2:* "price-taking social planner approach" – Fisher separation theorem

   c. "Money evaluation equation" ϑ
   - *Toolbox 3:* Change in numeraire to total wealth (including SDF)
2. Evolution of state variable η (and K) *forward equation*
3. Value functions *backward equation*
   a. Value fcn. as fcn. of individual investment opportunities ω
   - *Special cases: log-utility, constant investment opportunities*

   b. Separating value fcn. $V^i(n^{\tilde{i}};\eta,K)$ into $v^i(\eta)u(K)\left(n^{\tilde{i}}/n^i\right)^{1-\gamma}$
   c. Derive C/N-ratio and ς price of risk
4. Numerical model solution
   a. Transform BSDE for separated value fcn. $v^i(\eta)$ into PDE
   b. Solve PDE via value function iteration
5. KFE: Stationary distribution, Fan charts

Value Functions

- For log utility
  - Price of Brownian risk $\varsigma_t^i = \sigma_t^{n^i}$
  - Price of Jump risk $\nu_t = 1 - \frac{1}{1+j_t^n} = \frac{j_t^n}{1+j_t^n}$ (see earlier slide)

- For CRRA/EZ utility
  - Generalize earlier lecture
    add jump terms in value function BSDEs

Value Function Process for CRRA

$$\frac{dV_t^i}{V_t^i} = \frac{d\left(v_t^i K_t^{1-\gamma}\right)}{v_t^i K_t^{1-\gamma}}$$

- By Ito's product rule

$$= \left(\mu_t^{v^i} + (1-\gamma)(\Phi(\iota_t) - \delta) - \frac{1}{2}\gamma(1-\gamma)(\sigma^2) + (1-\gamma)\sigma\sigma_t^{v^i}\right)dt + j_t^{v^i} dJ_t$$
$$+ volatility\ terms$$

- Recall by consumption optimality for CRRA utility
  $\frac{dV_t^i}{V_t^i} - \rho^i dt + \frac{c_t^i}{n_t^i} dt$ follows a martingale

Poll 47: Why martingale?
a) Because we can "price" net worth with SDF
b) because ρ^i and c_t^i/n_t^i cancel out

- Hence, drift above $= \rho^i - \frac{c_t^i}{n_t^i} - \lambda_t j_t^{v^i}$

Still have to solve for $\mu_t^{v^i}$, $\sigma_t^{v^i}$

Value Function Process for EZ

$$\frac{dV_t^i}{V_t^i} = \frac{d\left(v_t^i K_t^{1-\gamma}\right)}{v_t^i K_t^{1-\gamma}}$$

- By Ito's product rule

$$= \left(\mu_t^{v^i} + (1-\gamma)(\Phi(\iota_t) - \delta) - \frac{1}{2}\gamma(1-\gamma)(\sigma^2) + (1-\gamma)\sigma\sigma_t^{v^i}\right) dt + j_t^{v^i} dJ_t$$
$$+ \, volatility \; terms$$

- Recall by consumption optimality for CRRA utility
  $\frac{dV_t^i}{V_t^i} - \rho^i dt + \frac{c_t^i}{n_t^i} dt$ follows a martingale

Poll 48: Why martingale?
a) Because we can "price" net worth with SDF
b) because ρ^i and c_t^i/n_t^i cancel out

- Hence, drift above $= -\frac{\partial f}{\partial U}\left(c_s, v_t \frac{K_t^{1-\gamma}}{1-\gamma}\right) - \frac{c_t^i}{n_t^i} - \lambda_t j_t^{v^i}$

Still have to solve for $\mu_t^{v^i}$, $\sigma_t^{v^i}$

- If we relax the assumption EIS=1, then the consumption-wealth ratio of agents will vary with investment opportunities (which do depend on the exact specification of (perceived) run risk even under log utility) and that will clearly affect q through goods market clearing.
- If we keep $EIS = 1$, but vary the risk aversion, then the q function will only be affected if capital is allocated differently for the same value of η (because the average consumption-wealth ratio in the economy does not change and then goods market clearing gives us a one-to-one mapping between q and the capital allocation). So, we would have to check whether the invariance of capital demands result is only true because there are no hedging demands or whether that result generalizes even if there are hedging demands. I don't have the equations in front of me right now, but my guess would be that also that result is not robust and thus the capital allocation and q will be affected even if $EIS = 1$.