Modern Macro, Money, and International Finance

Eco529

Lecture 06: Real Macro Model with Heterogenous Agents
CRRA and Epstein-Zin Preferences

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Course Overview

Real Macro-Finance Models with Heterogeneous Agents

1. A Simple Real Macro-finance Model
2. Endogenous (Price of) Risk Dynamics
3. A Model with Jumps due to Sudden Stops/Runs

Money Models

1. A Simple Money Model
2. Cashless vs. Cash Economy and "The I Theory of Money"
3. Welfare Analysis & Optimal Policy
    1. Fiscal, Monetary, and Macroprudential Policy

International Macro-Finance Models

1. International Financial Architecture

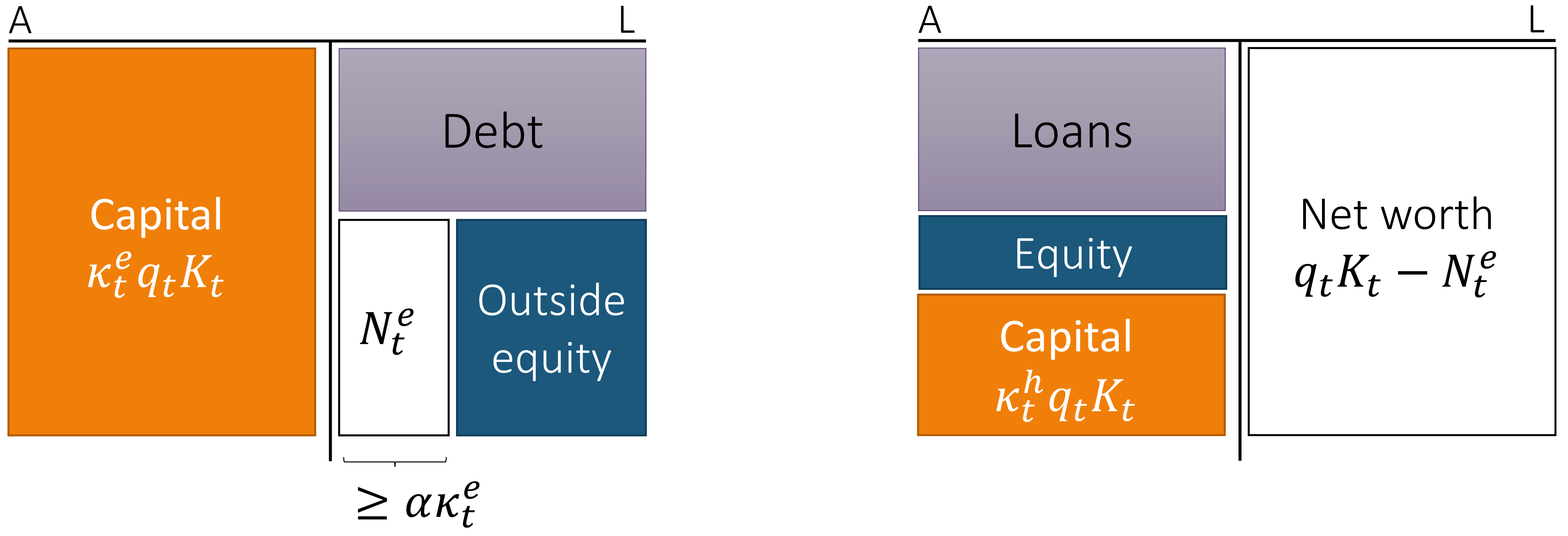
Digital Money

Two Type/Sector Model with Outside Equity

BruSan 2017:
Handbook of Macroeconomics,
Lecture Notes, Chatper 3

Expert sector

Household sector



- Skin in the Game Constraint:
  Experts must hold fraction $\chi_t^e \geq \alpha \kappa_t^e$ of aggregate capital risk with $\alpha \in (0,1)$ ($\chi_t^e > \kappa_t^e$ never happens in equilibrium)
- Return on inside equity N_t can differ from outside equity
  - Issue outside equity at required return from HH
  - In related model, He and Krishnamurthy 2013 impose that inside and outside equity have same return

Two Type Model Setup

Expert sector

- Output: $y_t^e = a^e k_t^e$ $\quad a^e \geq a^h$
- Consumption rate: c_t^e
- Investment rate: ι_t^e

$$\frac{dk_t^{\tilde{\imath},e}}{k_t^{\tilde{\imath},e}} = \left(\Phi\left(\iota_t^{\tilde{\imath},e}\right) - \delta\right)dt + \sigma dZ_t + d\Delta_t^{k,e}$$

- $E_0[\int_0^\infty e^{-\rho^e t} \frac{(c_t^e)^{1-\gamma}}{1-\gamma} dt]$ $\quad \rho^e \geq \rho^h$

Household sector

- Output: $y_t^h = a^h k_t^h$
- Consumption rate: c_t^h
- Investment rate: ι_t^h

$$\frac{dk_t^{\tilde{\imath},h}}{k_t^{\tilde{\imath},h}} = \left(\Phi\left(\iota_t^{\tilde{\imath},h}\right) - \delta\right)dt + \sigma dZ_t + d\Delta_t^{k,h}$$

- $E_0[\int_0^\infty e^{-\rho^h t} \frac{(c_t^h)^{1-\gamma}}{1-\gamma} dt]$

Friction: Can only issue

- Risk-free debt
- Equity, but must hold $\chi_t^e \geq \alpha \kappa_t$

Recursive Epstein-Zin Utility: Separating EIS from Risk Aversion

- For all $t \geq 0$

$$U_t = \mathbb{E}_t\left[\int_t^{\infty} f(c_s, U_s)ds\right]$$

  - With the aggregator

$$f(c,U) = \frac{1-\gamma}{1-\psi^{-1}}\rho U\left(\left(\frac{c}{((1-\gamma)\rho U)^{1/(1-\gamma)}}\right)^{1-\psi^{-1}} - 1\right)$$

  - Special Case: EIS $\psi \to 1$

$$f(c,U) = (1-\gamma)\rho U\left(\log c - \frac{1}{1-\gamma}\log((1-\gamma)\rho U)\right)$$

  - Special Case: EIS $\psi = \gamma^{-1} \neq 1$

$$f(c,U) = \frac{c^{1-\gamma}}{1-\gamma} - \rho U \quad \ldots U_t = \mathbb{E}_t[\int_t^{\infty} e^{-\rho^e t} \quad \frac{(c_s^e)^{1-\gamma}}{1-\gamma} ds]$$

Solving MacroModels Step-by-Step

0. Postulate aggregates, price processes & obtain return processes
1. For given C/N-ratio and SDF processes for each i *finance block*
    a. Real investment ι + Goods market clearing *(static)*
    - *Toolbox 1:* Martingale Approach, HJB vs. Stochastic Maximum Principle Approach

    b. Portfolio choice θ + Asset market clearing or
    Asset allocation κ & risk allocation χ
    - *Toolbox 2:* "price-taking social planner approach" – Fisher separation theorem

    c. "Money evaluation equation" ϑ
    - *Toolbox 3:* Change in numeraire to total wealth (including SDF)
2. Evolution of state variable η (and K) *forward equation*
3. Value functions *backward equation*
    a. Value fcn. as fcn. of individual investment opportunities ω
    - *Special cases: log-utility, constant investment opportunities*

    b. Separating value fcn. $V^i(n^{\tilde{i}};\eta,K)$ into $v^i(\eta)u(K)\left(n^{\tilde{i}}/n^i\right)^{1-\gamma}$
    c. Derive C/N-ratio and ς price of risk
4. Numerical model solution
    a. Transform BSDE for separated value fcn. $v^i(\eta)$ into PDE
    b. Solve PDE via value function iteration
5. KFE: Stationary distribution, Fan charts

3a. CRRA/EZ Value Function

Applies separately for each type of agent

- Martingale Approach: works best in endowment economy
- Here: mix Martingale approach with value function (envelop condition)

- $V^i(n_t^i; \boldsymbol{\eta_t}, K_t)$ for individuals $\boldsymbol{i}$
- For CRRA/power utility $u(c_t^i) = \frac{(c_t^i)^{1-\gamma}}{1-\gamma}$ and for EZ

⇒ increase net worth by factor, optimal c^i for all future states increases by this factor ⇒ $\left(\frac{c_t^i}{n_t^i}\right)$-ratio is invariant in n_t^i

- ⇒ value function can be written as $V^i(n_t^i; \boldsymbol{\eta_t}, K_t) = \frac{1}{\rho^i} \frac{\left(\omega^i(\boldsymbol{\eta_t}, K_t) n_t^i\right)^{1-\gamma}}{1-\gamma}$
- ω_t^i Investment opportunity/ "net worth multiplier"
  - $\omega^i(\boldsymbol{\eta_t}, K_t)$-function turns out to be independent of K_t
  - Change notation from $\omega^i(\boldsymbol{\eta_t}, K_t)$-function to ω_t^i-process

3b. Value Process $V_t^i = V^i(n_t^i; \eta_t, K_t)$ for CRRA

- Recall Martingale approach: if x_t is the value of a portfolio with return $\frac{dn_t^i}{n_t^i} + \frac{c_t^i}{n_t^i} dt$, then $\xi_t^i x_t^i$must be a martingale

$$\frac{d(\xi_t^i n_t^i)}{\xi_t^i n_t^i} = -\frac{c_t^i}{n_t^i} dt + martingale$$

- For CRRA, the SDF is given by $\xi_t^i = e^{-\rho^i t} \frac{\partial V_t^i}{\partial n^i}$
- For our CRRA value function guess $V_t^i = \frac{1}{\rho^i} \frac{(\omega_t^i n_t^i)^{1-\gamma}}{1-\gamma}$, $\frac{\partial V^i}{\partial n^i} = \frac{1}{\rho^i} (\omega^i)^{1-\gamma} (n_t^i)^{-\gamma}$
- $\xi_t^i n_t^i = e^{-\rho^i t} \frac{1}{\rho^i} (\omega^i)^{1-\gamma} (n_t^i)^{1-\gamma} = (1-\gamma) e^{-\rho^i t} \underbrace{\frac{1}{\rho^i} \frac{(\omega_t^i n_t^i)^{1-\gamma}}{1-\gamma}}_{V_t^i}$
- Apply Ito to obtain $\frac{d(\xi_t^i n_t^i)}{\xi_t^i n_t^i} = -\rho^i dt + \frac{dV_t^i}{V_t^i}$
- Hence,

$$\frac{dV_t^i}{V_t^i} = \frac{d(e^{\rho^i t} \xi_t^i n_t^i)}{\xi_t^i n_t^i} = \left(\rho^i - \frac{c_t^i}{n_t^i}\right) dt + martingale$$

- Next, let's compute the drift of $\frac{dV_t^i}{V_t^i}$

3b. Value Process $V_t^i = V^i(n_t^i; \eta_t, K_t)$ for EZ

- For our EZ value function SDF is

$$\xi_t^i = e^{\left(\int_0^t \frac{\partial f}{\partial U}(c_s, V_s) ds\right)} \frac{\partial V_t^i}{\partial n^i}$$

- $\xi_t^i n_t^i = e^{\left(\int_0^t \frac{\partial f}{\partial U}(c_s, V_s) ds\right)} \frac{1}{\rho^i} (\omega^i)^{1-\gamma} (n_t^i)^{1-\gamma} = (1-\gamma) e^{\left(\int_0^t \frac{\partial f}{\partial U}(c_s, V_s) ds\right)} \underbrace{\frac{1}{\rho^i} \frac{(\omega_t^i n_t^i)^{1-\gamma}}{1-\gamma}}_{V_t^i}$

- Hence,

$$\frac{dV_t^i}{V_t^i} = \frac{d\left(e^{-\int_0^t \frac{\partial f}{\partial U}(c_s, V_s) ds} \xi_t^i n_t^i\right)}{\xi_t^i n_t^i} = \left(-\frac{\partial f}{\partial U}(c_s, V_s) - \frac{c_t^i}{n_t^i}\right) dt + martingale$$

- Note $\frac{\partial f}{\partial U} = \rho^i \frac{\psi^{-1} - \gamma}{1 - \psi^{-1}} \left(\frac{c^{1-\psi^{-1}}}{\left((1-\gamma)\rho^i U\right)^{\frac{1-\psi^{-1}}{1-\gamma}}} \right) - \rho^i$

- for $\psi \to 1$: $\frac{\partial f}{\partial U} = \rho^i \left((1 - \gamma^i) \log c^i - \log\left((1 - \gamma^i) \rho^i U\right) \right) - \rho^i$

3b. CRRA Value Fcn: De-scale by K_t

- Drift of $\frac{dV_t^i}{V_t^i}$, we could use Ito on $V_t^i = \frac{1}{\rho^i}\frac{\left(\omega_t^i n_t^i\right)^{1-\gamma}}{1-\gamma}$, but
  - *Poll 10: What could be the problem?*
    - *a. Net worth n_t is unbounded*
    - *b. Net worth $n_t(\eta_t)$ and N-multiplier $\omega_t(\eta_t)$ are not differentiable (if $q(\eta_t)$, $q^B(\eta_t)$ have a kink).*
    - *c. N-multiplier is not scale invariant*
    - *d. Avoids need to compute second derivative q'' anywhere in solution code*
  - Answer: b,d

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  - *Poll 11: What could be the problem?*
    - *a. Net worth n_t is unbounded*
    - *b. Net worth $n_t(\eta_t)$ and N-multiplier $\omega_t(\eta_t)$ are not differentiable (if $q(\eta_t)$, $q^B(\eta_t)$ have a kink).*
    - *c. N-multiplier is not scale invariant*
    - *d. Avoids need to compute second derivative q'' anywhere in solution code*
  - Answer: b,d
- Change of variable: use v_t^i which is $V_t^i = v_t^i \frac{K_t^{1-\gamma}}{1-\gamma}$
  - Let's de-scale the problem w.r.t. K_t

$$V_t^i = \frac{1}{\rho^i}\frac{(\omega_t^i n_t^i)^{1-\gamma}}{1-\gamma} = \underbrace{\frac{(\omega_t^i N_t^i / K_t)^{1-\gamma}}{\rho^i}}_{v_t^i :=} \underbrace{\frac{K_t^{1-\gamma}}{1-\gamma}}_{u(K):=}$$

  - define v_t^i (which is twice differentiable in η_t)
  - Note that in equilibrium $n^i = N^i$ (all experts/HH are the same)
- State variable K_t is easy to handle due to scale invariance

3b. Value Function Process for CRRA

$$\frac{dV_t^i}{V_t^i} = \frac{d\left(v_t^i K_t^{1-\gamma}\right)}{v_t^i K_t^{1-\gamma}}$$

- By Ito's product rule

$$= \left(\mu_t^{v^i} + (1-\gamma)(\Phi(\iota_t) - \delta) - \frac{1}{2}\gamma(1-\gamma)(\sigma^2) + (1-\gamma)\sigma\sigma_t^{v^i}\right)dt$$
$$+ volatility\ terms$$

- Recall by consumption optimality for CRRA utility
  $\frac{dV_t^i}{V_t^i} - \rho^i dt + \frac{c_t^i}{n_t^i} dt$ follows a martingale

Poll 12: Why martingale?
a) Because we can "price" net worth with SDF
b) because ρ^i and c_t^i/n_t^i cancel out

- Hence, drift above $= \rho^i - \frac{c_t^i}{n_t^i}$

Still have to solve for $\mu_t^{v^i}$, $\sigma_t^{v^i}$

3b. Value Function Process for EZ

$$\frac{dV_t^i}{V_t^i} = \frac{d\left(v_t^i K_t^{1-\gamma}\right)}{v_t^i K_t^{1-\gamma}}$$

- By Ito's product rule

$$= \left(\mu_t^{v^i} + (1-\gamma)(\Phi(\iota_t) - \delta) - \frac{1}{2}\gamma(1-\gamma)(\sigma^2) + (1-\gamma)\sigma\sigma_t^{v^i}\right) dt$$
$$+\ volatility\ terms$$

- Recall by consumption optimality for CRRA utility

$\frac{dV_t^i}{V_t^i} - \rho^i dt + \frac{c_t^i}{n_t^i} dt$ follows a martingale

Poll 13: Why martingale?
a) Because we can "price" net worth with SDF
b) because ρ^i and c_t^i/n_t^i cancel out

- Hence, drift above $= -\frac{\partial f}{\partial U}\left(c_s, v_t \frac{K_t^{1-\gamma}}{1-\gamma}\right) - \frac{c_t^i}{n_t^i}$ Still have to solve for $\mu_t^{v^i}$, $\sigma_t^{v^i}$

3b. Value Function Process for EZ

$$\frac{dV_t^i}{V_t^i} = \frac{d\left(v_t^i K_t^{1-\gamma}\right)}{v_t^i K_t^{1-\gamma}}$$

- By Ito's product rule

$$= \left(\mu_t^{v^i} + (1-\gamma)(\Phi(\iota_t) - \delta) - \frac{1}{2}\gamma(1-\gamma)(\sigma^2) + (1-\gamma)\sigma\sigma_t^{v^i}\right) dt$$
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- Recall by consumption optimality for CRRA utility

$\frac{dV_t^i}{V_t^i} - \rho^i dt + \frac{c_t^i}{n_t^i} dt$ follows a martingale

- Hence, drift above $= \rho^i - \rho^i \frac{\psi^{-1}-\gamma}{1-\psi^{-1}}\left(\frac{(c_t/K_t)^{1-\psi^{-1}}}{(\rho^i v_t)^{\frac{1-\psi^{-1}}{1-\gamma}}} - 1\right) - \frac{c_t^i}{n_t^i}$

Still have to solve for $\mu_t^{v^i}, \sigma_t^{v^i}$

3b. CRRA Value Fcn BSDE

- Only conceptual interim solution
  - We will transform it into a PDE in Step 4 below
- From last slide

$$\underbrace{\mu_t^{v^i} + (1-\gamma)(\Phi(\iota_t) - \delta) - \frac{1}{2}\gamma(1-\gamma)\sigma^2 + (1-\gamma)\sigma\sigma_t^{v^i}}_{=:\mu_t^{V^i}} = \rho - \frac{c_t^i}{n_t^i}$$

- Can solve for $\mu_t^{v^i}$, then v_t^i must follow

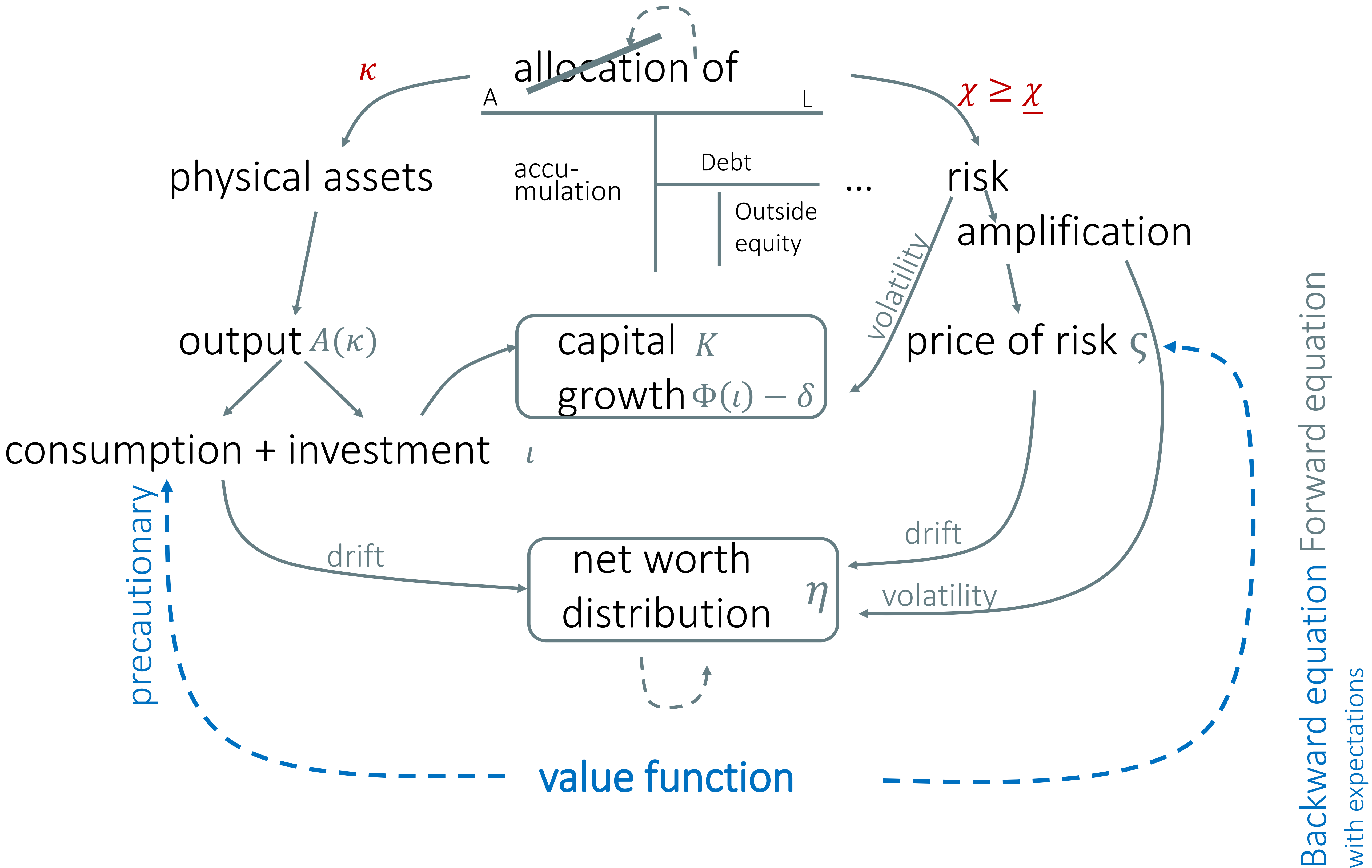
$$\frac{dv_t^i}{v_t^i} = f\left(\eta_t^i, v_t^i, \sigma_t^{v^i}\right) dt + \sigma_t^{v^i} dZ_t$$

with

$$f\left(\eta_t^i, v_t^i, \sigma_t^{v^i}\right) = \rho^i - \frac{c_t^i}{n_t^i} - (1-\gamma)(\Phi(\iota_t) - \delta) + \frac{1}{2}\gamma(1-\gamma)(\sigma^2) - (1-\gamma)\sigma\sigma_t^{v^i}$$

- Together with terminal condition v_T^i (possibly a constant for 1000 periods ahead), this is a backward stochastic differential equation (BSDE)
- A solution consists of processes v^i and σ^{v^i}
- Can use numerical BSDE solution methods (as random objects, so only get simulated paths)
- To solve this via a PDE we also need to get state evolution

The Big Picture



3c. Get ςs from Value Function Envelop

- Recall $V^i(n_t^i;\boldsymbol{\eta_t},K_t) = \frac{u(\omega^i(\boldsymbol{\eta}_t,K_t)n_t^i)}{\rho^i}$
- For envelop condition $\frac{\partial V_t}{\partial n_t} = \frac{\partial u(c_t)}{\partial c_t}$
  - To obtain $\frac{\partial V^i(n_t^i;\boldsymbol{\eta}_t,K_t)}{\partial n_t^i} = \frac{(\omega^i(\boldsymbol{\eta}_t,K_t))^{1-\gamma}}{\rho^i}\left(n_t^i\right)^{-\gamma}$

$$= \underbrace{\frac{\left(\omega_t^i n_t^i/K_t\right)^{1-\gamma}}{\rho^i}}_{v_t^i :=}\left(\frac{K_t}{n_t^i}\right)^{1-\gamma}\left(n_t^i\right)^{-\gamma},$$

  - $\Rightarrow \frac{\partial V_t}{\partial n_t^i} = v_t^i\left(\frac{K_t}{n_t^i}\right)^{1-\gamma}\left(n_t^i\right)^{-\gamma} = (c_t^i)^{-\gamma} = \frac{\partial u(c_t^i)}{\partial c_t^i}$
- In equilibrium $N_t^i = n_t^i$ and $C_t^i = c_t^i$ & using $N_t^i = \eta_t^i q_t K_t$

$$\frac{v_t^i}{\eta_t^i q_t} K_t^{-\gamma} = (C_t^i)^{-\gamma}$$

- Ito's quotient rule $\sigma_t^{v^i} - \sigma_t^{\eta^i} - \sigma_t^q - \gamma\sigma = -\gamma\sigma_t^{c^i} = -\varsigma_t^i$

3c. Get $\frac{C_t^i}{N_t^i}$ from Value Function Envelop

- CRRA Envelop condition $v_t^i \left(\frac{K_t}{N_t^i}\right)^{1-\gamma} \left(n_t^i\right)^{-\gamma} = (c_t^i)^{-\gamma}$
- using $K_t/N_t^i = 1/\eta_t^i q_t$

$$\frac{C_t^i}{N_t^i} = \frac{c_t^i}{n_t^i} = \frac{(\eta_t^i q_t)^{1/\gamma-1}}{(v_t^i)^{1/\gamma}}$$

- Aggregate level (two agents case)

$$\frac{C_t}{N_t} = \frac{C_t^e + C_t^h}{N_t^e + N_t^h} = \eta_t^e \frac{C_t^e}{N_t^e} + \eta_t^h \frac{C_t^h}{N_t^h} = \frac{1}{q_t}\left[\left(\frac{\eta_t^e q_t}{v_t^e}\right)^{1/\gamma} + \left(\frac{\eta_t^h q_t}{v_t^h}\right)^{1/\gamma}\right]$$

- EZ with $\frac{\partial V_t^i}{\partial n^i}$ =xxx , envelop condition becomes

$$\Rightarrow c_t^i = \left(\rho^i\right)^{\psi} \omega_t^{1-\psi} n_t$$

Solving MacroModels Step-by-Step

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   a. Real investment ι + Goods market clearing *(static)*
   - *Toolbox 1:* Martingale Approach, HJB vs. Stochastic Maximum Principle Approach

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Recall: 2 Ways of Solving ODE

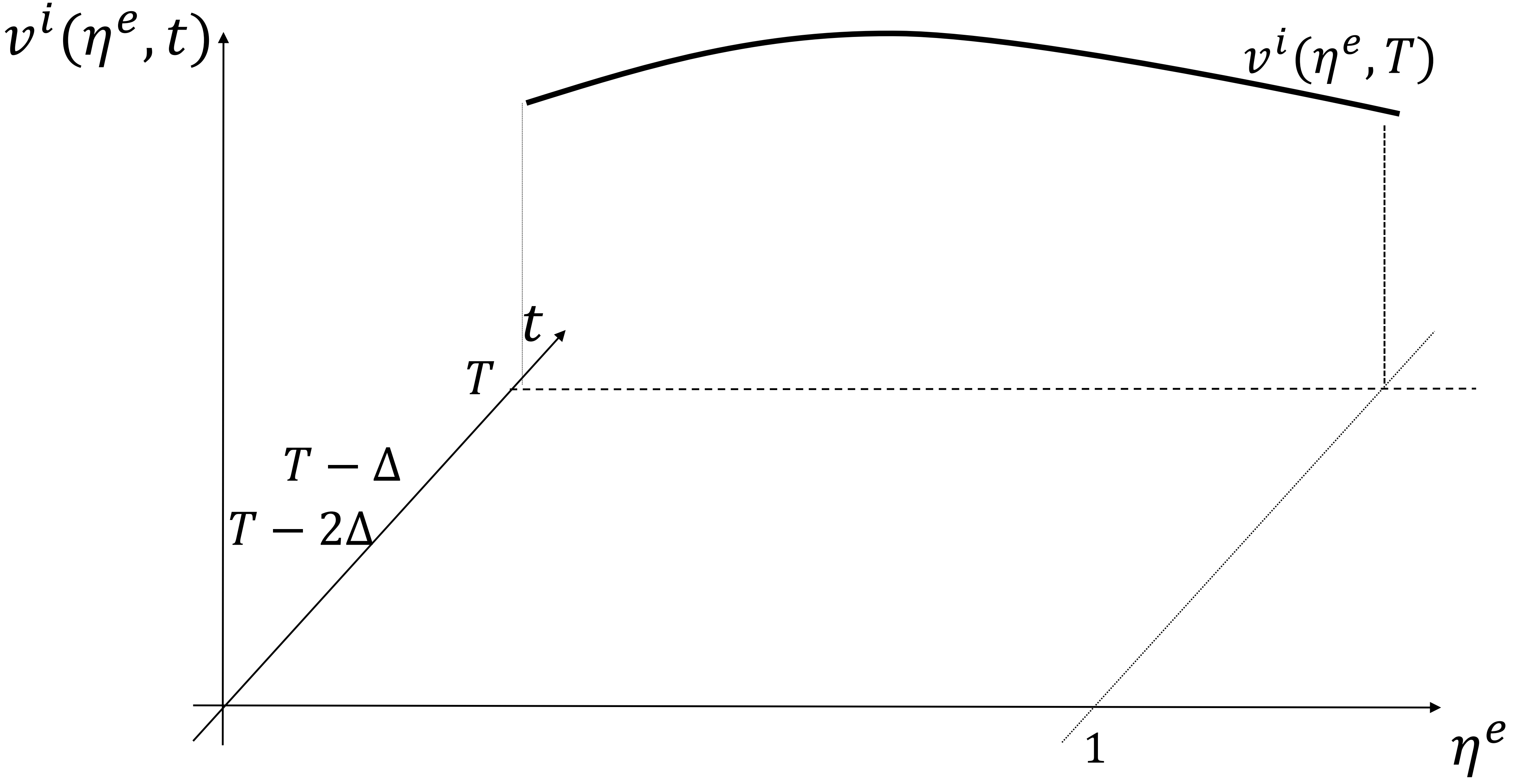
- Propose function and iterate $q(\eta)$

- Start from boundary condition and solve step-by-step $v^e(\eta), v^h(\eta)$ (Newton Method)

4. Value function Iteration - Big picture

- Add time, t, as an additional state variable $v^e(\eta^e, t)$, $v^h(\eta^e, t)$
- Convert BSDE into PDE using Ito's Lemma
  - $\mu_t^{v^e} v_t^e = \partial_t v_t^e + \eta_t^e \mu_t^{\eta^e} \partial_\eta v_t^e + \frac{1}{2}\left(\eta_t^e \sigma_t^{\eta^e}\right)^2 \partial_{\eta\eta} v_t^e$
  - $\mu_t^{v^h} v_t^h = \partial_t v_t^h + \eta_t^e \mu_t^{\eta^e} \partial_\eta v_t^h + \frac{1}{2}\left(\eta_t^e \sigma_t^{\eta^e}\right)^2 \partial_{\eta\eta} v_t^h$
- Guess terminal value functions $v^e(\eta^e, T)$ and $v^h(\eta^e, T)$ (far in the future $t = T$)
- … and iterate back to $t = 0$
  - In each step use
    - From Step 2: $\mu_t^{v^e} v_t^e$, $\mu_t^{v^h} v_t^h$
    - From Step 3: $\eta_t^e \mu_t^{\eta^e}$ and $\eta_t^e \sigma_t^{\eta^e}$ (η-evolution)
    - Portfolio choice, planners' problem, (static conditions)
    - Market clearing
  - To calculate all terms in these $\mu_{t-\Delta}^{v^i} v_{t-\Delta}^i$, $\eta_{t-\Delta}^e \mu_{t-\Delta}^{\eta^e}$ and $\eta_{t-\Delta}^e \sigma_{t-\Delta}^{\eta^e}$

4. Value Function Iteration – Big Picture



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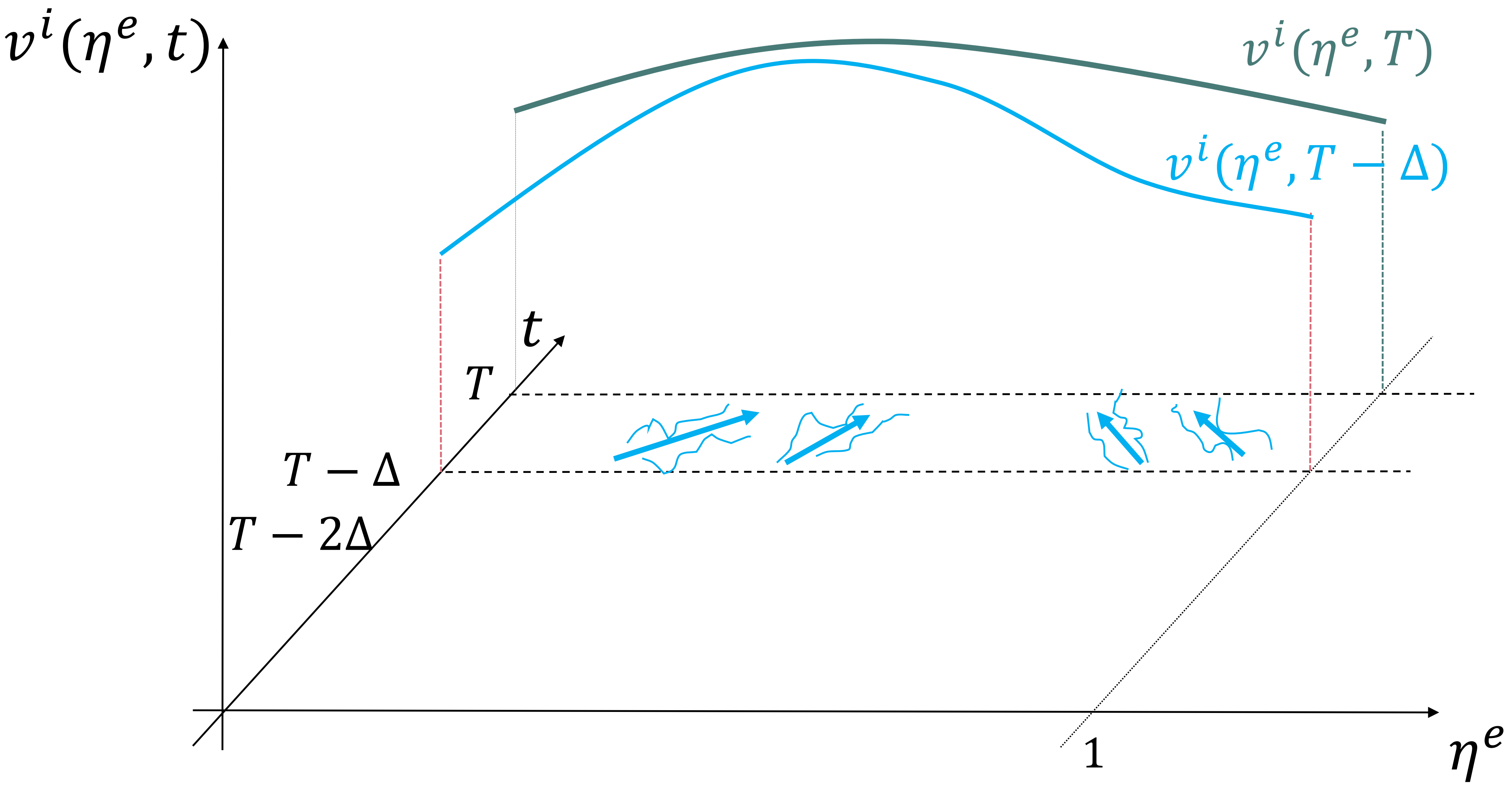


- Obtain descaled value function $v^i(\eta^e, T-\Delta)$
- Repeat previous steps

4. Value function Iteration - Big picture

- Add time, t, as an additional state variable $v^e(\eta^e, t)$, $v^h(\eta^e, t)$
- Convert BSDE into PDE using Ito's Lemma
  - $\mu_t^{v^e} v_t^e = \partial_t v_t^e + \eta_t^e \mu_t^{\eta^e} \partial_\eta v_t^e + \frac{1}{2}\left(\eta_t^e \sigma_t^{\eta^e} \right)^2 \partial_{\eta\eta} v_t^e$
  - $\mu_t^{v^h} v_t^h = \partial_t v_t^h + \eta_t^e \mu_t^{\eta^e} \partial_\eta v_t^h + \frac{1}{2}\left(\eta_t^e \sigma_t^{\eta^e} \right)^2 \partial_{\eta\eta} v_t^h$
- Guess terminal value functions $v^e(\eta^e, T)$ and $v^h(\eta^e, T)$ (far in the future $t = T$)
- … and iterate back to $t = 0$
  - In each step use
    - From Step 3: $\mu_t^{v^e} v_t^e$, $\mu_t^{v^h} v_t^h$
    - From Step 2: $\eta_t^e \mu_t^{\eta^e}$ and $\eta_t^e \sigma_t^{\eta^e}$ (η-evolution)
    - Portfolio choice, planners' problem, (static conditions)
    - Market clearing
  - To calculate all terms in these $\mu_{t-\Delta}^{v^i} v_{t-\Delta}^i$, $\eta_{t-\Delta}^e \mu_{t-\Delta}^{\eta^e}$ and $\eta_{t-\Delta}^e \sigma_{t-\Delta}^{\eta^e}$

Short-hand notation:
$\partial_x f$ for $\partial f / \partial x$

4a. PDE Value Function Iteration

Short-hand notation:
$\partial_x f$ for $\partial f / \partial x$

- Postulate $v_t^i = v^i(\eta_t^e, t)$
- By Ito's Lemma
- $\frac{dv_t^i}{v_t^i} = \frac{\partial_t v_t^i + (\eta^e \mu_t^{\eta^e})\partial_\eta v_t^i + \frac{1}{2}\left(\eta_t^e \sigma_t^{\eta^e}\right)^2 \partial_{\eta\eta} v_t^i}{v_t^i} dt + \frac{(\eta^e \sigma_t^{\eta^e})\partial_\eta v_t^i}{v_t^i} dZ_t$
- That is,
- $\mu_t^{v^i} v_t^i = \partial_t v_t^i + (\eta^e \mu_t^{\eta^e})\partial_\eta v_t^i + \frac{1}{2}\left(\eta_t^e \sigma_t^{\eta^e}\right)^2 \partial_{\eta\eta} v_t^i$
- $\sigma_t^{v^i} v_t^i = \left(\eta^e \sigma_t^{\eta^e}\right) \partial_\eta v_t^i$
- Equating with Step 3 (plug in $\mu_t^{v^i}$)⇒

$$\partial_t v_t^i + \left(\eta^e \mu_t^{\eta^e} + (1-\gamma)\sigma \eta_t^e \sigma_t^{\eta^e}\right) \partial_\eta v_t^i + \frac{1}{2}\left(\eta_t^e \sigma_t^{\eta^e}\right)^2 \partial_{\eta\eta} v_t^i$$
$$= \left(\rho^i - (1-\gamma)(\Phi(\iota_t) - \delta) + \frac{1}{2}\gamma(1-\gamma)\sigma^2\right) v_t^i - \frac{c_t^i}{n_t^i} v_t^i$$

4a. PDE Value Fcn: Replacing Terms

$$\partial_t v_t^i + \left(\eta^e \mu_t^{\eta^e} + (1-\gamma)\sigma\eta_t^e \sigma_t^{\eta^e}\right)\partial_\eta v_t^i + \frac{1}{2}\left(\eta_t^e \sigma_t^{\eta^e}\right)^2 \partial_{\eta\eta} v_t^i$$
$$= \left(\rho^i - (1-\gamma)(\Phi(\iota_t) - \delta) + \frac{1}{2}\gamma(1-\gamma)\sigma^2\right) v_t^i - \frac{c_t^i}{n_t^i} v_t^i$$

1. Replace "blue terms" using results from Step 2.

$$\mu_t^{\eta^e} = (1-\eta_t^e)(\varsigma_t^e - \sigma_t^q - \sigma)\left(\sigma_t^{\eta^e} - \underbrace{\sigma_t^M}_{=0}\right) - (1-\eta_t^e)\left(\varsigma_t^h - \sigma_t^q - \sigma\right)\left(\sigma_t^{\eta^h} - \underbrace{\sigma_t^M}_{=0}\right) - \left(\frac{C_t^e}{N_t^e} - \frac{C_t}{N_t}\right)$$

$$\sigma_t^{\eta^e} = \frac{\chi_t^e - \eta_t^e}{\eta_t^e}(\sigma + \sigma_t^q) \qquad \sigma_t^{\eta^h} = -\frac{\eta_t^e}{1-\eta_t^e}\sigma_t^{\eta^e}$$

2.

.

$$\varsigma_t^e = -\sigma_t^{v^e} + \sigma_t^{\eta^e} + \sigma_t^q + \gamma\sigma, \qquad \varsigma_t^h = -\sigma_t^{v^h} + \sigma_t^{\eta^h} + \sigma_t^q + \gamma\sigma$$

$$\frac{C_t^i}{N_t^i} = \frac{(\eta_t^i q_t)^{1/\gamma - 1}}{(v_t^i)^{1/\gamma}}$$

Recall from Ito's Lemma

$$\sigma_t^{v^i} v_t^i = (\eta^e \sigma_t^{\eta^e})\partial_\eta v_t^i$$

$$\frac{C_t}{N_t} = \frac{1}{q_t}\left[\left(\frac{\eta_t^e q_t}{v_t^e}\right)^{1/\gamma} + \left(\frac{(1-\eta_t^e) q_t}{v_t^h}\right)^{1/\gamma}\right]$$

3. Replace "red terms" $\iota_t, \sigma_t^q, \chi_t^e$ (see below)

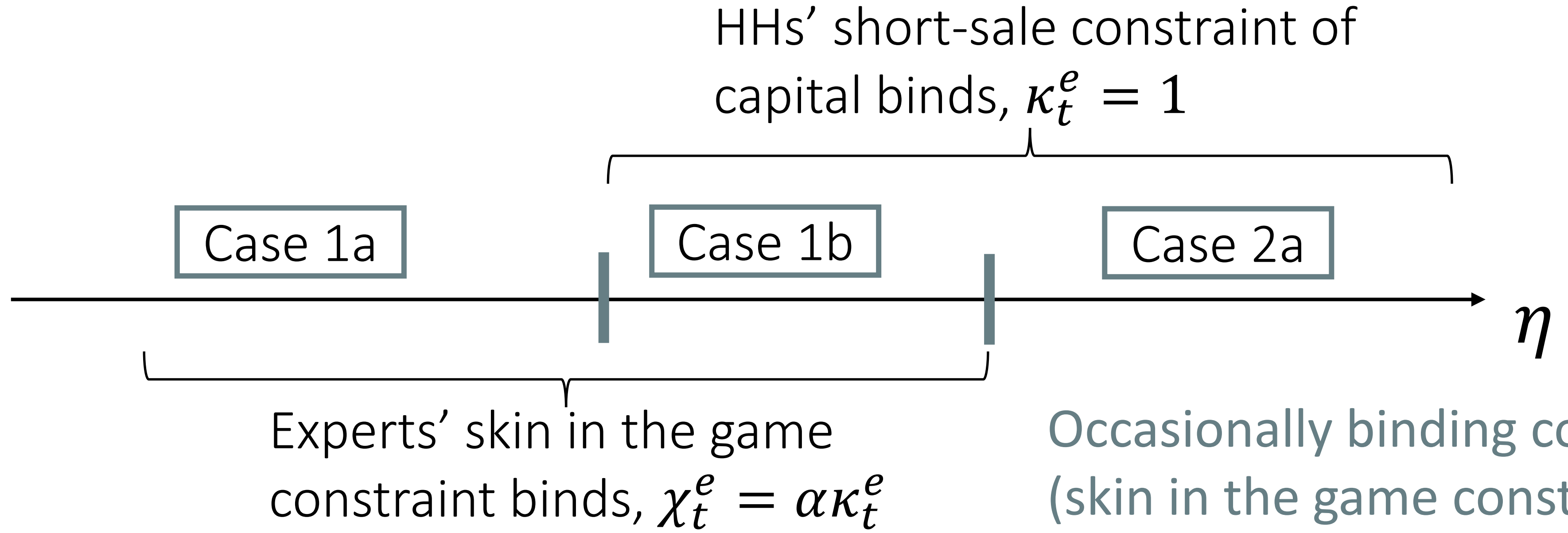
4a. Replacing ι_t

- Recall from optimal re-investment $\Phi'(\iota_t) = 1/q_t$
  - For $\Phi(\iota) = \frac{1}{\phi}\log(\phi\iota + 1) \Rightarrow \boxed{\phi\iota = q - 1}$

4a. Replacing χ, obtain κ for good mkt clearing

- Recall from planner's problem (Step 1b)

Cases	$\chi_t^e \geq \alpha\kappa_t^e$	$\kappa_t^e \leq 1$	$\frac{(a^e - a^h)}{q_t} \geq \alpha(\varsigma_t^e - \varsigma_t^h)(\sigma + \sigma_t^q)$ Shift a capital unit to expert Benefit: LHS Cost: RHS	$(\varsigma_t^e - \varsigma_t^h)(\sigma + \sigma_t^q) \geq 0$ Required risk premium of experts vs. HH
1a	=	<	=	>
1b	=	=	>	>
2a	>	=	>	=
impossible				



Occasionally binding constraint
(skin in the game constraint)

4a. Replacing χ, obtain κ for good mkt clearing

- Need to determine diff in risk premia $(\varsigma_t^e - \varsigma_t^h)(\sigma + \sigma_t^q)$:
- Recall
  - diff in price of risk: $\varsigma_t^e - \varsigma_t^h = -\sigma_t^{v^e} + \sigma_t^{v^h} + \frac{\sigma_t^{\eta^e}}{1-\eta_t^e}$
  - By Ito's lemma $\sigma_t^{v^e} = \frac{\partial_\eta v_t^e}{v_t^e} \eta_t^e \sigma_t^{\eta^e}$ and $\sigma_t^{v^h} = \frac{\partial_\eta v_t^h}{v_t^h} \eta_t^e \sigma_t^{\eta^e}$

$$\Rightarrow (\varsigma_t^e - \varsigma_t^h)(\sigma + \sigma_t^q) = \left(-\frac{\partial_\eta v_t^e}{v_t^e} + \frac{\partial_\eta v_t^h}{v_t^h} + \frac{1}{(1-\eta_t^e)\eta_t^e} \right) \eta_t^e \sigma_t^{\eta^e} (\sigma + \sigma_t^q)$$

$$= \left(-\frac{\partial_\eta v_t^e}{v_t^e} + \frac{\partial_\eta v_t^h}{v_t^h} + \frac{1}{(1-\eta_t^e)\eta_t^e} \right) (\chi_t^e - \eta_t^e)(\sigma + \sigma_t^q)^2$$

- Note, since $-\frac{\partial_\eta v_t^e}{v_t^e} + \frac{\partial_\eta v_t^h}{v_t^h} + \frac{1}{(1-\eta_t^e)\eta_t^e} > 0$,

4a. Replacing χ, obtain κ for good mkt clearing

- Determination of κ_t

$$(a^e - a^h)/q_t \geq \alpha\left(-\frac{\partial_\eta v_t^e}{v_t^e} + \frac{\partial_\eta v_t^h}{v_t^h} + \frac{1}{(1-\eta_t^e)\eta_t^e}\right)(\chi_t^e - \eta_t^e)(\sigma + \sigma_t^q)^2$$

with equality if $\kappa_t^e < 1$

- Determination of χ_t^e

$$\chi_t^e = \max\{\alpha\kappa_t^e, \eta_t^e\}$$

4a. Market Clearing

- Output good market

$$\left(\kappa_t^e a^e + (1-\kappa_t^e)a^h - \iota_t\right)K_t = C_t$$

... jointly restricts κ_t and q_t

$$\kappa_t a^e + (1-\kappa_t)a^h - \iota(q_t) = \underbrace{\left(\frac{\eta_t^e q_t}{v_t^e}\right)^{1/\gamma}}_{C_t^e/K_t} + \underbrace{\left(\frac{(1-\eta_t^e)q_t}{v_t^h}\right)^{1/\gamma}}_{C_t^h/K_t}$$

4a. Market Clearing

- Output good market
$$\left(\kappa_t^e a^e + (1-\kappa_t^e)a^h - \iota_t\right)K_t = C_t,$$

… jointly restricts κ_t and q_t

- Capital market is taken care off by price taking social planner approach
$$\theta_t^e = \frac{\kappa_t^e q_t K_t}{\eta_t^e q_t K_t}$$

- Risk-free debt market also taken care off by price taking social planner approach
(would be cleared by Walras Law anyways)

4a. $\sigma^q(q, q')$

- Recall from "amplification slide" – Step 2

$$\sigma + \sigma_t^q = \frac{\sigma}{1 - \frac{q'(\eta_t^e)}{q/\eta_t^e} \frac{\chi_t^e - \eta_t^e}{\eta_t^e}}$$

$$\sigma^q = \frac{q'(\eta_t^e)}{q(\eta_t^e)} (\chi_t^e - \eta_t^e)(\sigma + \sigma_t^q)$$

- Now all red terms are replaced and we can solve …

4b. Algorithm – Static Step

- Suppose we know functions $v^e(\eta^e), v^h(\eta)$, have five static conditions:

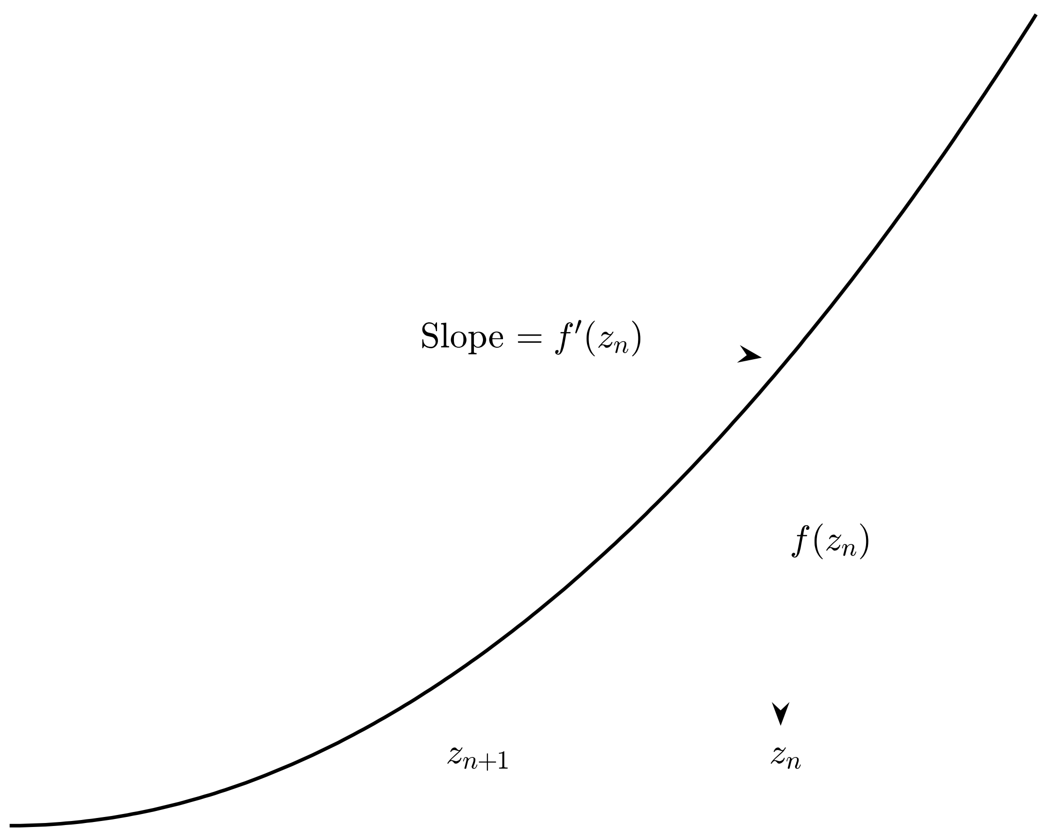
1. $\phi\iota_t = q_t - 1$
2. Planner condition for κ_t^e
3. Planner condition for $\chi_t^e = \max\{\alpha\kappa_t^e, \eta_t^e\}$
4. $\kappa_t^e a^e + (1-\kappa_t^e)a^h - \iota(q_t) = \left(\frac{\eta_t q_t}{v_t^e}\right)^{1/\gamma} + \left(\frac{(1-\eta_t)q_t}{v_t^h}\right)^{1/\gamma}$
5. $\sigma^q = \frac{q'(\eta_t^e)}{q(\eta_t^e)}(\chi_t^e - \eta_t^e)(\sigma + \sigma_t^q)$

$\Rightarrow$ Get $q(\eta^e)$, $\kappa^e(\eta^e)$, $\sigma^q(\eta^e)$

- Start at $q(0)$, solve to the right, use different procedure for two η-regions depending on κ:

1. While $\kappa^e < 1$, solve ODE for $q(\eta^e)$:
   - For given $q(\eta)$, plug optimal investment (1) into (4)
   - Plug planner condition (3) into (2) and (5)
   - Solve ODE using three equilibrium condition (2),(4) and (5) via Newton's method (see next slide)
2. When $\kappa = 1$, (2) is no longer informative, since $\kappa^e = 1$, solve (1) and (4) for $q(\eta)$

4b. *Aside:* Newton's Method



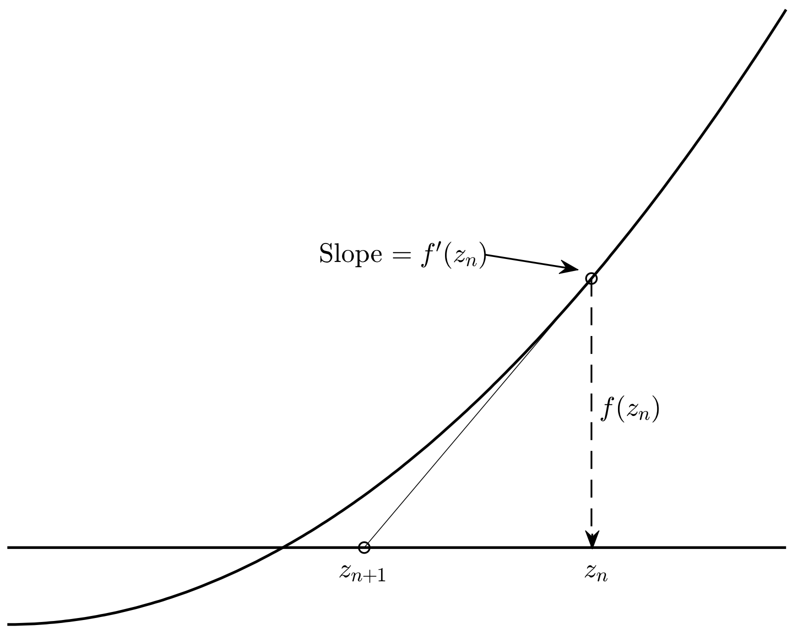
- Find the root of equation system $F(\boldsymbol{z}_n) = 0$ via iterative method

$$z_{n+1} = \mathrm{z_n} - J_n^{-1} F(\boldsymbol{z}_n)$$

Where J_n is the Jacobian matrix, i.e., $J_{ij} = \partial f_i(\boldsymbol{z}) / \partial z_j$.

- Newton's method does not guarantee global convergence.
- commonly take several-step iteration.

4b. *Aside:* Newton's Method



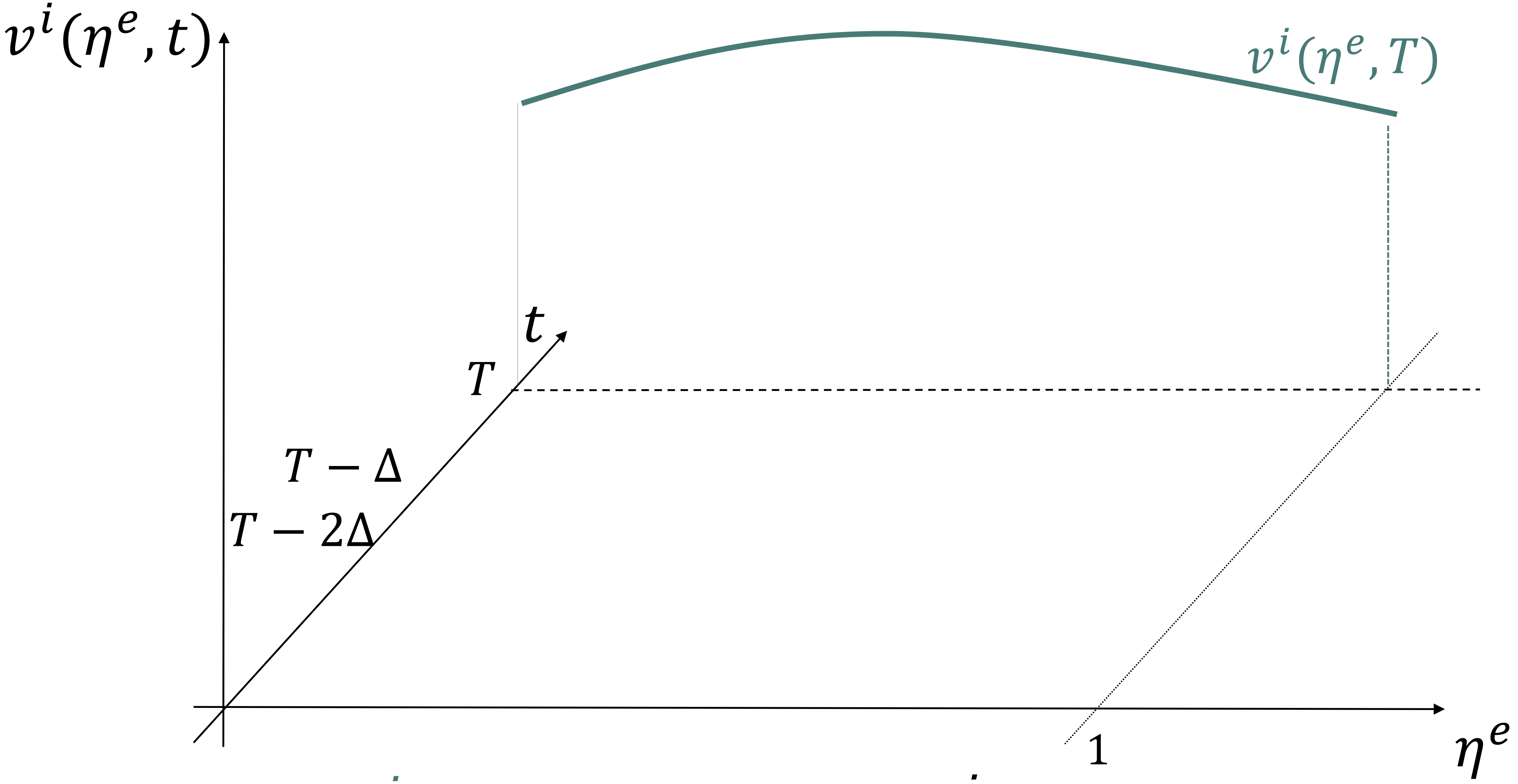
$$\mathbf{z}_n = \begin{bmatrix} q_t \\ \kappa_t^e \\ \sigma + \sigma_t^q \end{bmatrix},$$

$$\begin{bmatrix} \text{market clearing condtion} \\ \text{amplification condition} \\ \text{Planner condition for } \kappa_t^e \end{bmatrix}$$

$$F(\mathbf{z}_n) = \begin{bmatrix} \kappa_t^e a^e + (1-\kappa_t^e)a^h - \iota(q_t) - \left(\frac{\eta_t^e q_t}{v_t^e}\right)^{\frac{1}{\gamma}} - \left(\frac{(1-\eta_t^e) q_t}{v_t^h}\right)^{\frac{1}{\gamma}} \\ q'(\eta_t^e)(\chi_t^e - \eta_t^e)(\sigma + \sigma_t^q) - \sigma^q q(\eta_t^e) \\ (a^e - a^h) - \alpha q_t \left(-\frac{\partial_\eta v_t^e}{v_t^e} + \frac{\partial_\eta v_t^h}{v_t^h} + \frac{1}{(1-\eta_t^e)\eta_t^e}\right)(\chi_t^e - \eta_t^e)(\sigma + \sigma_t^q)^2 \end{bmatrix}$$

Plug in blue terms from optimal investment and Planner condition for χ_t^e

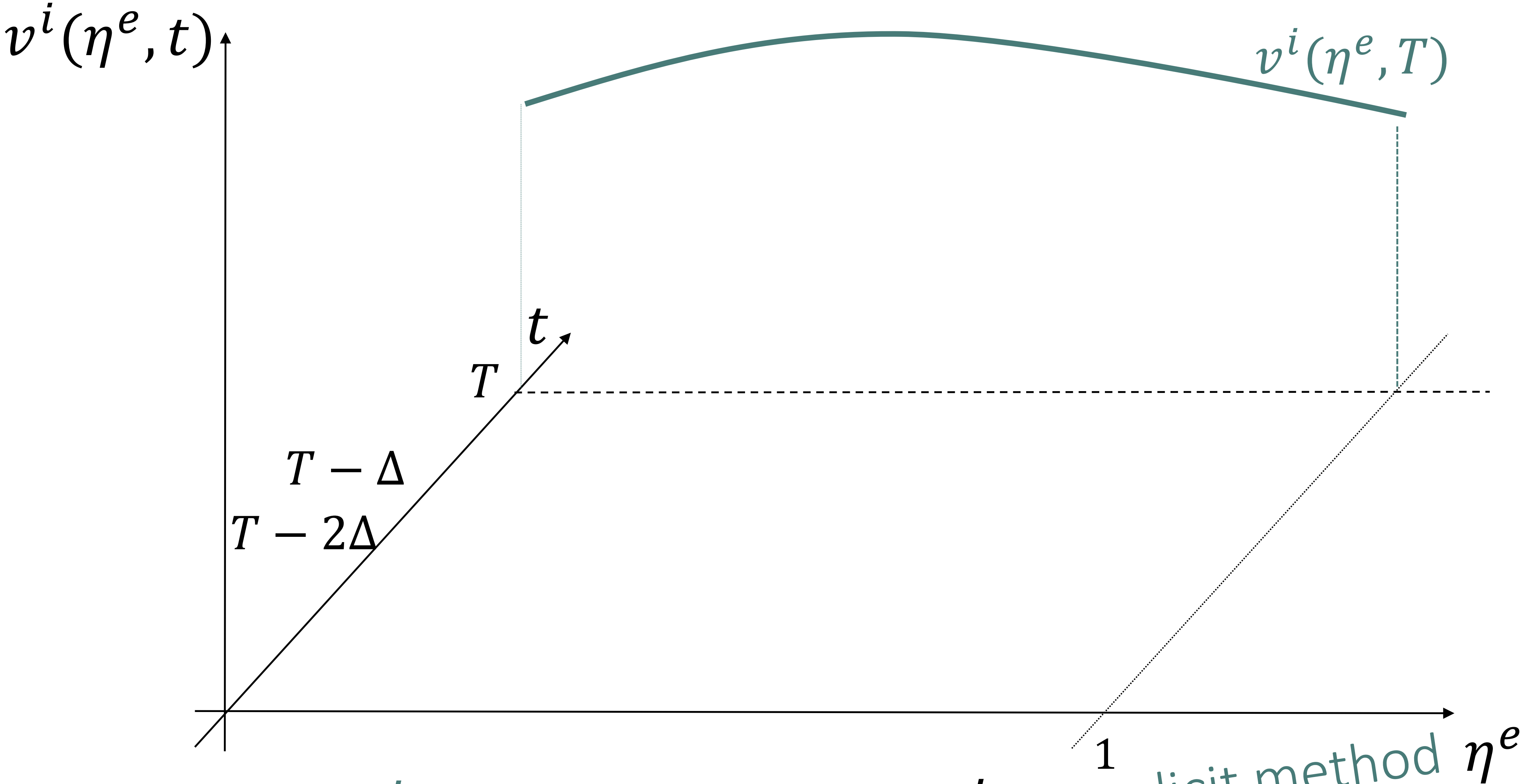
4. Value Function Iteration – Big Picture



- For given $v^i(\eta^e, T)$, derive SDF ξ_T^i
- Optimal investment, portfolio, consumption, at T as fcn. of η^e

4. Market clearing at T obtain PDE coefficient at T
   (pretend they are constant between T & $T - \Delta$)

4. Value Function Iteration – Big Picture



- For given $v^i(\eta^e, T)$, derive SDF ξ_T^i

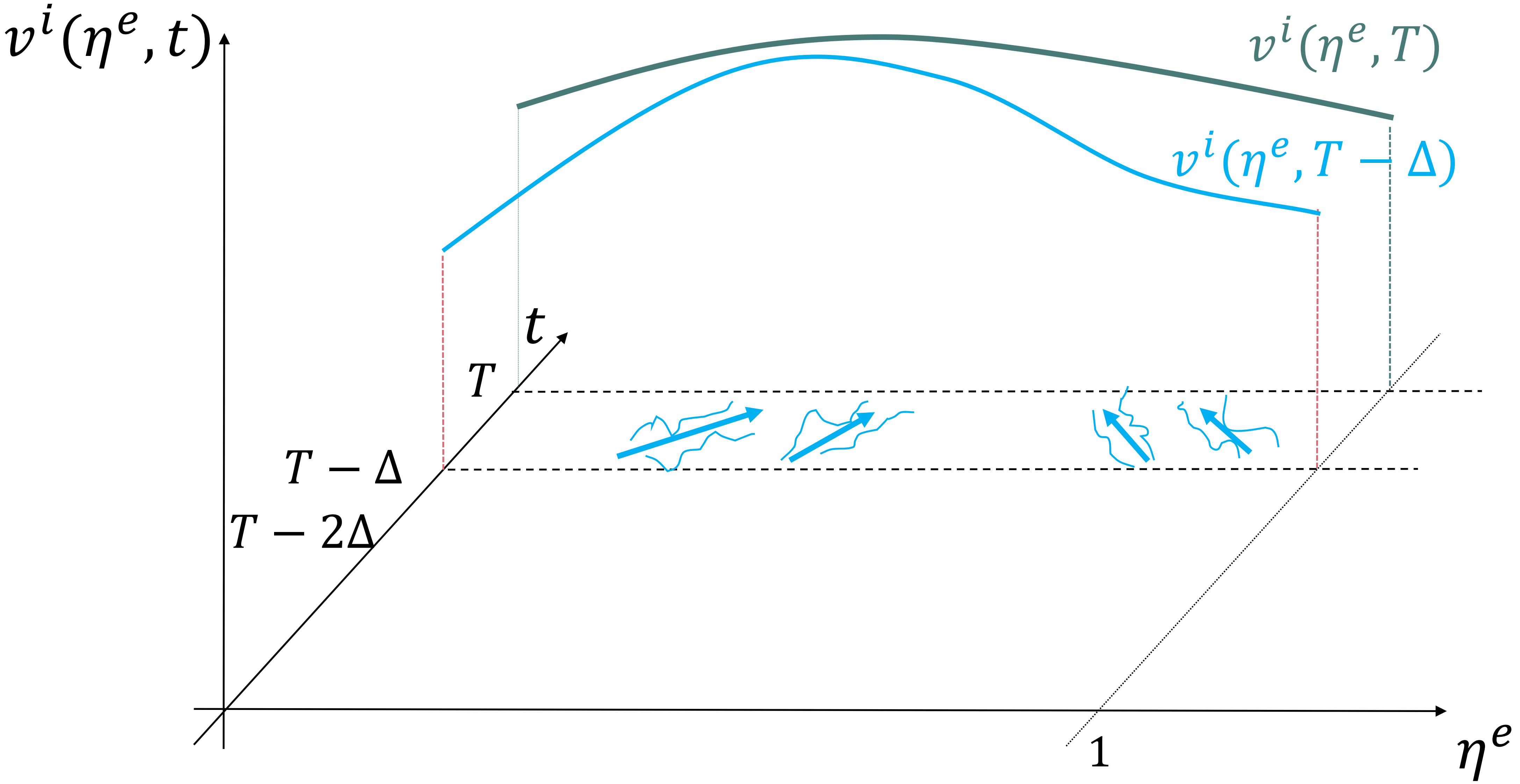
Explicit method

Implicit method uses $T - \Delta$

- Optimal investment, portfolio, consumption, at T as fcn. of η^e

4. Market clearing at T obtain PDE coefficient at T
(pretend they are constant between T & $T - \Delta$)

4. Value Function Iteration – Big Picture



- Obtain descaled value function $v^i(\eta^e, T-\Delta)$
- Repeat previous steps

4b. Pseudocode

1. Initialize two terminal functions $v^e(\boldsymbol{\eta^e}, T), v^h(\boldsymbol{\eta^e}, T)$ over $\boldsymbol{\eta^e}$-grid $(\eta_1^e, \eta_2^e, \cdots \eta_n^e)$
2. For $t \in \{T, T - \Delta t, T - 2\Delta t, \cdots 0\}$
   a. Compute $\partial_\eta v_t^i$ by first-order difference
   b. Start at $\eta_1^e \approx 0$ smallest grid point > 0 (autarky economy), find $q(0,t), \kappa^e(0,t), \sigma^q(0,t)$.
   c. For $\eta_i^e \in \{\eta_2^e, \eta_3^e, \cdots, \eta_n^e\}$
      i. If $\kappa^e(\eta_i^e, t) < 1$, solve ODE for $q(\eta_i^e, t), \kappa^e(\eta_i^e, t), \sigma^q(\eta_i^e, t)$ using Newton's method.
      ii. If $\kappa^e(\eta_i^e, t) = 1$, solve ODE for $q(\eta_i^e, t)$ from market clearing equation via Newton's method. Then find $\sigma^q(\eta_i^e, t)$ using amplification function
   d. Find $\mu^{\eta^e}(\boldsymbol{\eta^e}, t), \sigma^{\eta^e}(\boldsymbol{\eta^e}, t), \mu^{v^i}(\boldsymbol{\eta^e}, t)$.
   e. Update: obtain $v^e(\boldsymbol{\eta^e}, t - \Delta t)$ from $v^e(\boldsymbol{\eta^e}, t)$ via finite difference method (do $\mu_t^v v_t = \partial_t v_t^i + \mu_t^{\eta^e} \eta_t^e (\partial_\eta v_t^i) + \frac{1}{2}\left(\sigma_t^{\eta^e} \eta_t^e\right)^2 (\partial_{\eta\eta} v_t^i)$ for one time-step)

Upwind scheme: $$\partial_\eta f(n,t) = \begin{cases} \dfrac{f(\eta+1,t) - f(\eta,t)}{\Delta\eta} & \text{for} \quad \mu^\eta \eta > 0 \\ \dfrac{f(\eta,t) - f(\eta-1,t)}{\Delta\eta} & \text{for} \quad \mu^\eta \eta < 0 \end{cases}$$

Implicit scheme: $\partial_t f(\eta, t) = \frac{f(\eta,t+1) - f(\eta,t)}{\Delta t}$

2-order difference: $\partial_n f(\eta, t) = \frac{f(\eta+1) - 2f(\eta) + f(\eta-1)}{(\Delta\eta)^2}$

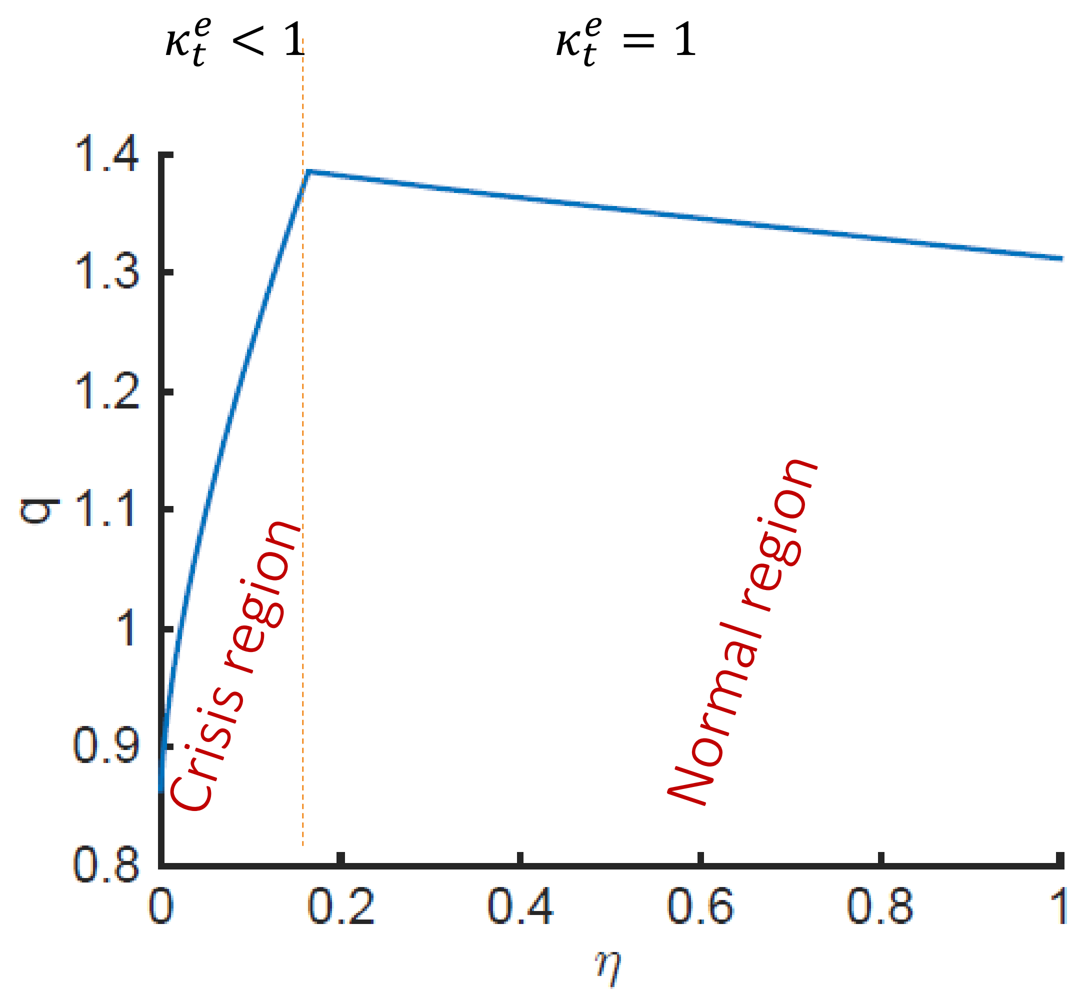
4b. Pseudocode – further questions

- Is there a boundary condition for $v^i(0)$?
  - Take smallest grid point – slightly above 0
  - Boundary condition is only needed if volatility > 0
    - For $v^h(0)$ this is ok, i.e. not an issue
    - For $v^e(0)$ one has to be careful and strictly speaking one has to look at the v^e of a single expert (assuming all other experts have zero net worth)
      - Numerical short-cut, which seems to work: simply set volatility equal to zero and it seems to work

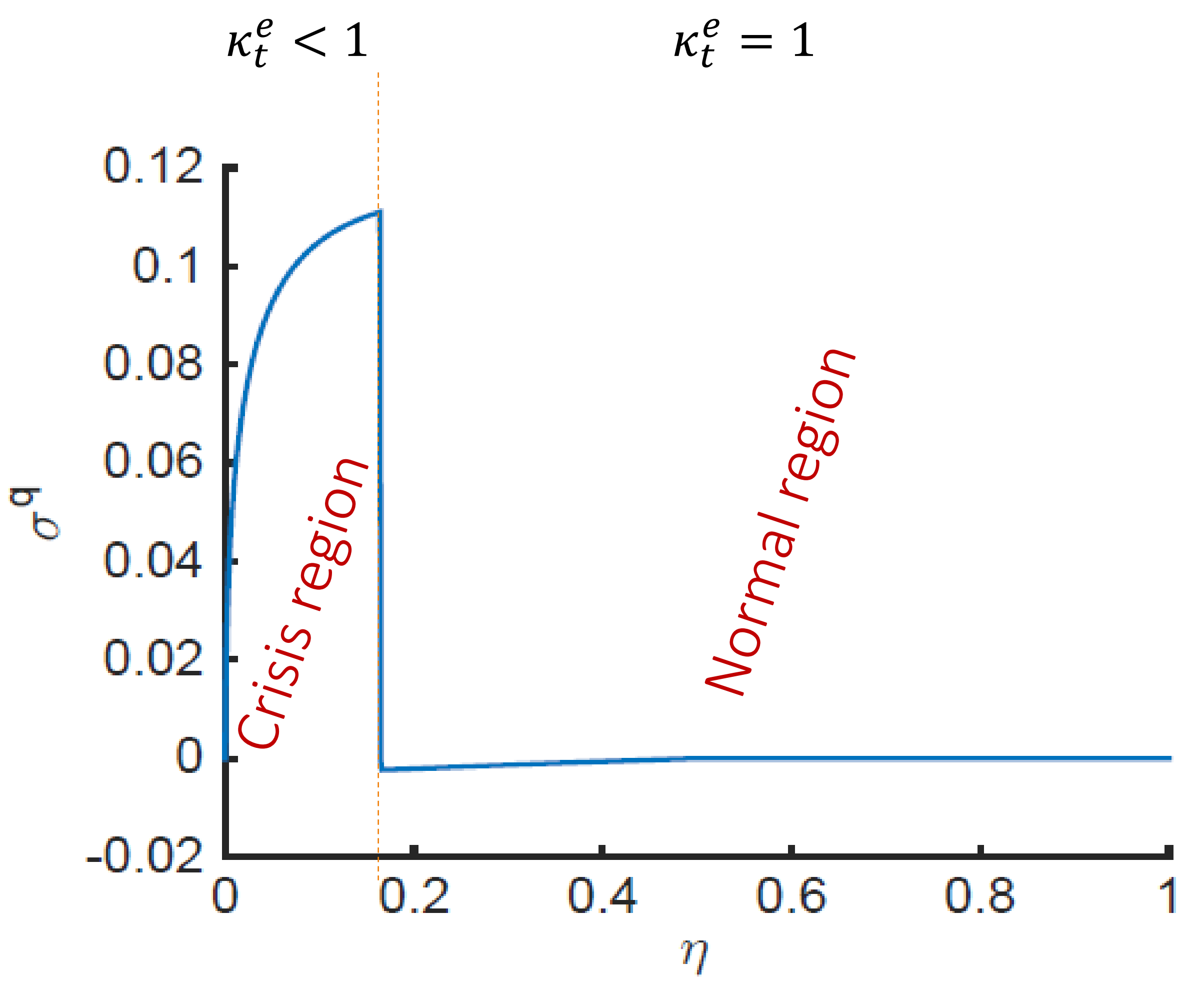
- Do we need specific conditions to ensure that the value function iteration converges?
  - Theoretically we only know these conditions for specific economies (representative agent, complete markets)
  - Here we need it in theory, but if code converges, we are fine.
    - Of course, we did not ensure uniqueness of the equilibrium

Solution

- Price of capital



Amplification



Parameters: $\rho^e = .06, \rho^h = .05, a^e = .11, a^h = .03,$
$\delta = .05, \sigma = .01, \alpha = .50, \gamma = 2, \phi = 10$