Modern Macro, Money, and International Finance

Eco529
Lecture 05: Endogenous Risk Dynamics in Real Macro Model with Heterogenous Agents

Markus K. Brunnermeier
Princeton University
Course Overview

Real Macro-Finance Models with Heterogeneous Agents
1. A Simple Real Macro-finance Model
2. Endogenous (Price of) Risk Dynamics
3. A Model with Jumps due to Sudden Stops/Runs

Money Models
1. A Simple Money Model
2. Cashless vs. Cash Economy and “The I Theory of Money”
3. Welfare Analysis & Optimal Policy
   1. Fiscal, Monetary, and Macroprudential Policy

International Macro-Finance Models
1. International Financial Architecture

Digital Money
Risk premia, price of risk

- Risk premia = price of risk * (endogenous + exogenous risk)

- Exogenous risk – shock from outside
- Endogenous risk
  - Amplification: adverse feedback loops
  - Multiple equilibria: Run, Sudden Stops

- Non-linearities are key for financial stability
  - Around vs. away from steady state
Desired Model Properties

- Normal regime: stable around steady state
  - Experts are adequately capitalized
  - Experts can absorb macro shock
- Endogenous risk and price of risk
  - Fire-sales, liquidity spirals, fat tails
  - Spillovers across assets and agents
  - Market and funding liquidity connection
  - SDF vs. cash-flow news
- Volatility paradox
- (Financial innovation less stable economy)
- (“Net worth trap” double-humped stationary distribution)
Persistence Leads to Dynamic Amplification

- **Static** amplification occurs because fire-sales of capital from productive sector to less productive sector depress asset prices
  - Importance of *market liquidity* of physical capital

- **Dynamic** amplification occurs because a temporary shock translates into a persistent decline in output and asset prices
  - Forward  grow net worth
  - Backward  asset pricing
“Single Shock Critique”

- Critique: After the shock all agents in the economy know that the economy will deterministically return to the steady state.
  - Length of slump is deterministic (and commonly known)
    - No safety cushion needed
  - In reality an adverse shock may be followed by additional adverse shocks
    - Build-up extra safety cushion for an additional shock in a crisis

- Impulse response vs. volatility dynamics
Endogenous Volatility & Volatility Paradox

- **Endogenous Risk/Volatility Dynamics in BruSan**
  - Beyond Impulse responses
  - Input: constant volatility
  - Output: endogenous risk time-varying volatility

⇒ Precautionary savings
  - Role for money/safe asset
    - Later: in Money lecture

⇒ Nonlinearities in crisis ⇒ endogenous fat tails, skewness

- **Volatility Paradox**
  - Low exogenous (measured) volatility leads to high build-up of (hidden) endogenous volatility (Minsky)
Toolboxes: Technical Innovations

- Occasionally binding equity issuance constraint (in addition to natural borrowing limit due to risk aversion)

- Price setting social planner to find capital and risk allocation

- Change of numeraire
  - Easily incorporate aggregate fluctuations
  - To use martingale methods more broadly

- Newton Method to solve log-utility numerical example
Two Type/Sector Model with Outside Equity

- Expert sector

Skin in the Game Constraint:
Experts must hold fraction $\chi_t^e = \frac{\sigma N_t^e}{\sigma qK_t} \geq \alpha \kappa_t^e$ of aggregate capital risk with $\alpha \in (0,1)$ ($\chi_t^e > \kappa_t^e$ never happens in equilibrium)

- Household sector

Return on inside equity $N_t$ can differ from outside equity
- Issue outside equity at required return from HH
- In related model, He and Krishnamurthy 2013 impose that inside and outside equity have same return
Financial Frictions and Distortions  UPDATE!

- Skin in the game constraint
  - Retain certain fraction of risk

- Incomplete markets
  - “natural” leverage constraint *(BruSan)*
  - Costly state verification *(BGG)*

- + Leverage constraints
  (no “liquidity creation”)
  - Exogenous limit *(Bewley/Ayagari)*

- Collateral constraints
  - Next period’s price *(KM)*
    \[ Rb_t \leq q_{t+1}k_t \]
  - Next periods volatility *(VaR, JG)*
  - Current price
Two Type Model Setup

Expert sector

- Output: \( y_t^e = a^e k_t^e \) \( a^e \geq a^h \)

Household sector

- Output: \( y_t^h = a^h k_t^h \)

\[ A(\kappa) = \kappa^e a^e + (1 - \kappa^e) a^h \]

Poll 11: Why is it important that households can hold capital?

a) to capture fire-sales
b) for households to speculate
c) to obtain stationary distribution
Two Type Model Setup

Expert sector

- Output: \( y_t^e = a^e k_t^e \)  \( a^e \geq a^h \)
- Consumption rate: \( c_t^e \)
- Investment rate: \( i_t^e \)

\[
\frac{d k_t^{i,e}}{k_t^{i,e}} = \left( \Phi \left( i_t^{i,e} \right) - \delta \right) dt + \sigma dZ_t + d\Delta_t^{k,e}
\]

Household sector

- Output: \( y_t^h = a^h k_t^h \)
- Consumption rate: \( c_t^h \)
- Investment rate: \( i_t^h \)

\[
\frac{d k_t^{i,h}}{k_t^{i,h}} = \left( \Phi \left( i_t^{i,h} \right) - \delta \right) dt + \sigma dZ_t + d\Delta_t^{k,h}
\]

Physical capital evolution absent market transactions/fire-sales
Two Type Model Setup

**Expert sector**
- Output: \( y_t^e = a^e k_t^e \quad a^e \geq a^h \)
- Consumption rate: \( c_t^e \)
- Investment rate: \( l_t^e \)

\[
\frac{dk_t^{i,e}}{k_t^{i,e}} = (\Phi (l_t^{i,e}) - \delta) dt + \sigma dZ_t + d\Delta_t^{k,e}
\]

**Household sector**
- Output: \( y_t^h = a^h k_t^h \)
- Consumption rate: \( c_t^h \)
- Investment rate: \( l_t^h \)

\[
\frac{dk_t^{i,h}}{k_t^{i,h}} = (\Phi (l_t^{i,h}) - \delta) dt + \sigma dZ_t + d\Delta_t^{k,h}
\]

Poll 13: What are the modeling tricks to obtain stationary distribution?
- a) switching types
- b) agents die, OLG/perpetual youth models (without bequest motive)
- c) different preference discount rates
Two Type Model Setup

Expert sector

- Output: \[ y_t^e = a^e k_t^e \quad a^e \geq a^h \]
- Consumption rate: \[ c_t^e \]
- Investment rate: \[ i_t^e \]

\[
\frac{dk_t^{i,e}}{k_t^{i,e}} = \left( \Phi \left( i_t^{i,e} \right) - \delta \right) dt + \sigma dZ_t + d\Delta_{k}^{k,e}
\]

\[
E_0 \left[ \int_0^\infty e^{-\rho^e t} \frac{c_t^{e}}{1-\gamma} dt \right] \quad \rho^e \geq \rho^h
\]

Household sector

- Output: \[ y_t^h = a^h k_t^h \]
- Consumption rate: \[ c_t^h \]
- Investment rate: \[ i_t^h \]

\[
\frac{dk_t^{i,h}}{k_t^{i,h}} = \left( \Phi \left( i_t^{i,h} \right) - \delta \right) dt + \sigma dZ_t + d\Delta_{k}^{k,h}
\]

\[
E_0 \left[ \int_0^\infty e^{-\rho^h t} \frac{c_t^{h}}{1-\gamma} dt \right]
\]
Two Type Model Setup

Expert sector

- Output: \( y_t^e = a^e k_t^e \) \( a^e \geq a^h \)
- Consumption rate: \( c_t^e \)
- Investment rate: \( \dot{i}_t^e \)

\[
\frac{dk_t^{i,e}}{k_t^{i,e}} = \left( \Phi \left( i_t^{i,e} \right) - \delta \right) dt + \sigma dZ_t + d\Delta_{t}^{k,e}
\]

\[
E_0 \left[ \int_0^\infty e^{-\rho_e t} \frac{c_t^e(1-\gamma)}{1-\gamma} dt \right] \rho^e \geq \rho^h
\]

Household sector

- Output: \( y_t^h = a^h k_t^h \)
- Consumption rate: \( c_t^h \)
- Investment rate: \( \dot{i}_t^h \)

\[
\frac{dk_t^{i,h}}{k_t^{i,h}} = \left( \Phi \left( i_t^{i,h} \right) - \delta \right) dt + \sigma dZ_t + d\Delta_{t}^{k,h}
\]

\[
E_0 \left[ \int_0^\infty e^{-\rho_h t} \frac{c_t^h(1-\gamma)}{1-\gamma} dt \right]
\]

Friction: Can only issue

- Risk-free debt
- Equity, but must hold \( \chi_t^e \geq \alpha \kappa_t \), i.e. \( \theta_t^{e,K} + \theta_t^{e,OE} \geq \alpha \theta_t^{e,K} \)
Recall Previous Lecture: HH can’t hold capital or equity

\[ a = 0.11, \rho = 5\%, \sigma = 0.1, \Phi(i) = \frac{\log(\phi i + 1)}{\phi}, \phi = 10 \]

Basak-Cuco
Preview of new, extended model

- Price of capital

- Amplification

Parameters: \( \rho^e = 0.06, \rho^h = 0.05, a^e = 0.11, a^h = 0.03, \delta = 0.05, \sigma = 0.1, \alpha = 0.50, \gamma = 2, \phi = 10 \)
Drift and Volatility of $\eta^e$

"Steady state" $\eta^*$,

$\eta^e = \alpha \kappa^e$
Solving MacroModels Step-by-Step

0. Postulate aggregates, price processes & obtain return processes

1. For given $C/N$-ratio and SDF processes for each $i$ finance block
   a. Real investment $\iota$ + Goods market clearing (static)
      - Toolbox 1: Martingale Approach, HJB vs. Stochastic Maximum Principle Approach
   b. Portfolio choice $\theta$ + Asset market clearing or Asset allocation $\kappa$ & risk allocation $\chi$
      - Toolbox 2: “price-taking social planner approach” – Fisher separation theorem
   c. “Money evaluation equation” $\varpi$
      - Toolbox 3: Change in numeraire to total wealth (including SDF)

2. Evolution of state variable $\eta$ (and $K$) forward equation

3. Value functions backward equation
   a. Value fcn. as fcn. of individual investment opportunities $\omega$
      - Special cases: log-utility, constant investment opportunities
   b. Separating value fcn. $V^i(n^i; \eta, K)$ into $v^i(\eta)u(K)(n^i/n^i)^{1-\gamma}$
   c. Derive $C/N$-ratio and $\zeta$ price of risk

4. Numerical model solution
   a. Transform BSDE for separated value fcn. $v^i(\eta)$ into PDE
   b. Solve PDE via value function iteration

5. KFE: Stationary distribution, Fan charts
0. Postulate Aggregates and Processes

- Individual capital evolution:

\[
\frac{dk_{t}^{i,i}}{k_{t}^{i,i}} = (\Phi(t^{i,i}) - \delta)dt + \sigma dZ_{t} + d\Delta_{t}^{k,i,i}
\]

- Where \( \Delta_{t}^{k,i,i} \) is the individual cumulative capital purchase process

\((c\ is\ numeraire)\)
0. Postulate Aggregates and Processes

- Individual capital evolution:
  \[
  \frac{dk_{t,i}^i}{k_{t,i}^i} = (\Phi(t_{t,i}^i) - \delta)dt + \sigma dZ_t + d\Delta_{t,k,i}^i
  \]
  - Where \( \Delta_{t,k,i}^i \) is the individual cumulative capital purchase process

- Capital aggregation:
  - Within sector \( i \):
    \[ K_t^i \equiv \int k_{t,i}^i d\bar{i} \]
  - Across sectors:
    \[ K_t \equiv \sum_i K_t^i \]
  - Capital share:
    \[ k_t^i \equiv K_t^i / K_t \]

- Net worth aggregation:
  - Within sector \( i \):
    \[ N_t^i \equiv \int n_{t,i}^i d\bar{i} \]
  - Across sectors:
    \[ N_t \equiv \sum_i N_t^i \]
  - Wealth share:
    \[ \eta_t^i \equiv N_t^i / N_t \]

- Value of capital stock:
  \[ q_t K_t \]

- Postulate
  \[
  \frac{dq_t}{q_t} = \mu_t q_{t-d} + \sigma_t q_{t-Z_t}
  \]

- Postulated SDF-process:
  \[
  \frac{d\kappa_t}{\kappa_t} = \mu_{t,i} \kappa_t \equiv -(\delta_{t,i} + \kappa_t)
  \]
  \[ d\kappa_t = \kappa_t d\bar{Z}_t \] (\( c \) is numeraire)
0. Postulate Aggregates and Processes

- Individual capital evolution:
  \[ \frac{dk_{i,i}}{k_{i,i}} = (\Phi(i_{i,i}) - \delta)dt + \sigma dZ_t + d\Delta_{t}^{k,i,i} \]
  - Where \( \Delta_{t}^{k,i,i} \) is the individual cumulative capital purchase process

- Capital aggregation:
  - Within sector \( i \):
    \[ K_t^i \equiv \int k_{t,i}^i d\bar{i} \]
  - Across sectors:
    \[ K_t \equiv \sum_i K_t^i \]
  - Capital share:
    \[ \kappa_t^i \equiv \frac{K_t^i}{K_t} \]
    \[ \frac{dK_t}{K_t} = (\Phi(i_t^i) - \delta)dt + \sigma dZ_t \]

- Net worth aggregation:
  - Within sector \( i \):
    \[ N_t^i \equiv \int n_{t,i}^i d\bar{i} \]
  - Across sectors:
    \[ N_t \equiv \sum_i N_t^i \]
  - Wealth share:
    \[ \eta_t^i \equiv \frac{N_t^i}{N_t} \]

(c is numeraire)
0. Postulate Aggregates and Processes

- Individual capital evolution:
  \[
  \frac{dk_{t,i}^i}{k_{t,i}^i} = \left( \Phi(t_i^i) - \delta \right) dt + \sigma dZ_t + d\Delta_t^{k_{t,i}^i, i}
  \]
  Where \(\Delta_t^{k_{t,i}^i, i}\) is the individual cumulative capital purchase process

- Capital aggregation:
  - Within sector \(i\):
    \[K_t^i \equiv \int k_{t,i}^i d\bar{i}\]
  - Across sectors:
    \[K_t \equiv \sum_i K_t^i\]
  - Capital share:
    \[\kappa_t^i \equiv K_t^i / K_t\]
    \[
    \frac{dK_t}{K_t} = \left( \Phi(t_t^i) - \delta \right) dt + \sigma dZ_t
    \]
  - Net worth aggregation:
  - Within sector \(i\):
    \[N_t^i \equiv \int n_{t,i}^i d\bar{i}\]
  - Across sectors:
    \[N_t \equiv \sum_i N_t^i\]
  - Wealth share:
    \[\eta_t^i \equiv N_t^i / N_t\]

- Value of capital stock:
  \[q_t K_t\]
  Postulate \[dq_t/q_t = \mu_t^q dt + \sigma_t^q dZ_t\]
  \((c \text{ is numeraire})\)
0. Postulate Aggregates and Processes

- Individual capital evolution:
  \[
  \frac{dk_{t}^{i,i}}{k_{t}^{i,i}} = \left(\Phi(\bar{\lambda}_{t}^{i,i}) - \delta\right)dt + \sigma dZ_{t} + d\Delta_{t}^{k,i,i}
  \]
  - Where \(\Delta_{t}^{k,i,i}\) is the individual cumulative capital purchase process

- Capital aggregation:
  - Within sector \(i\):
    \[K_{t}^{i} \equiv \int k_{t}^{i,i} d\bar{\lambda}\]
  - Across sectors:
    \[K_{t} \equiv \sum_{i} K_{t}^{i}\]
  - Capital share:
    \[\kappa_{t}^{i} \equiv \frac{K_{t}^{i}}{K_{t}}\]

- Net worth aggregation:
  - Within sector \(i\):
    \[N_{t}^{i} \equiv \int n_{t}^{i,i} d\bar{\lambda}\]
  - Across sectors:
    \[N_{t} \equiv \sum_{i} N_{t}^{i}\]
  - Wealth share:
    \[\eta_{t}^{i} \equiv \frac{N_{t}^{i}}{N_{t}}\]

- Value of capital stock:
  \[q_{t}K_{t}\]
  - Postulate
  \[dq_{t}/q_{t} = \mu_{t}^{q} dt + \sigma_{t}^{q} dZ_{t}\]
  (\(c\) is numeraire)
0. Postulate Aggregates and Processes

- Individual capital evolution:
  \[
  \frac{dk_t^{i,i}}{k_t^{i,i}} = (\Phi(\lambda_t^i) - \delta)dt + \sigma dZ_t + d\Delta_t^{k,i,i}
  \]
  - Where \( \Delta_t^{k,i,i} \) is the individual cumulative capital purchase process

- Capital aggregation:
  - Within sector \( i \):
    \[ K_t^i \equiv \int k_t^{i,i} d\lambda_t^i \]
  - Across sectors:
    \[ K_t \equiv \sum_i K_t^i \]
  - Capital share:
    \[ k_t^i \equiv K_t^i / K_t \]

\[
\frac{dK_t}{K_t} = (\Phi(\lambda_t^i) - \delta)dt + \sigma dZ_t
\]

- Net worth aggregation:
  - Within sector \( i \):
    \[ N_t^i \equiv \int n_t^{i,i} d\lambda_t^i \]
  - Across sectors:
    \[ N_t \equiv \sum_i N_t^i \]
  - Wealth share:
    \[ \eta_t^i \equiv N_t^i / N_t \]

- Value of capital stock:
  \[ q_t K_t \]

\[ dq_t / q_t = \mu_t^q dt + \sigma_t^q dZ_t \]

- Postulated SDF-process:
  \[
  \frac{d\xi_t^i}{\xi_t^i} = \mu_t^\xi dt + \sigma_t^\xi dZ_t
  \]
  \[ \equiv -r_t \]
  \[ \equiv -\xi_t^i \] (c is numeraire)
0. Postulate Aggregates and Processes

- ... from price processes to return processes (using Ito)
  - Use Ito product rule to obtain (in absence of purchases/sales)
    - Define $\bar{k}_t^i$: $\frac{d\bar{k}_t^i}{\bar{k}_t^i} = \left( \Phi \left( l_t^i \right) - \delta \right) dt + \sigma dZ_t + \frac{d\Delta_t}{t}$ without purchases/sales

  
  \[
  dr_t^k \left( l_t^i \right) = \left( \frac{a_t^i - l_t^i}{q} + \Phi \left( l_t^i \right) - \delta + \mu_t^q + \sigma \sigma_t^q \right) dt + (\sigma + \sigma_t^q) dZ_t
  \]

- Postulate SDF-process: (Example: $\xi_t^i = e^{-\rho t} V'(n_t)$.)

  \[
  \frac{d\xi_t^i}{\xi_t^i} = -r_t dt - \xi_t^i dZ_t
  \]

  Poll 26: Why does drift of SDF equal risk-free rate
  - a) no idio risk
  - b) $e^{-r_F} = E[SDF]$ *1
  - c) no jump in consumption

Recall discrete time $e^{-r_F} = E[SDF]$. 

For aggregate capital return, replace $a_t^i$ with $A(\kappa)$.
The Big Picture

- Output $A(\kappa)$
- Consumption + investment
- Physical assets
- Capital growth $\Phi(\ell) - \delta$
- Net worth distribution $\eta$
- Value function
- Precautionary saving
- Drift
- Volatility
- Debt accumulation
- Outside equity
- Allocation of physical assets
- Risk amplification
- Price of risk $\zeta$
- Backward equation with expectations
- Forward equation with expectations

Mathematical formulas:

\[ \chi \geq \chi \]

\[ A(A(\kappa)) \]

\[ \Phi(\ell) - \delta \]

\[ \eta \]

\[ \text{Drift} \]

\[ \text{Volatility} \]
Solving MacroModels Step-by-Step

0. Postulate aggregates, price processes & obtain return processes

1. For given $C/N$-ratio and SDF processes for each $i$, **finance block**
   a. Real investment $\iota$ + Goods market clearing *(static)*
      - *Toolbox 1*: Martingale Approach, HJB vs. Stochastic Maximum Principle Approach (previous lecture)
   b. Portfolio choice $\theta$ + Asset market clearing or Asset allocation $\kappa$ & risk allocation $\chi$
      - *Toolbox 2*: “price-taking social planner approach” – Fisher separation theorem
   c. “Money evaluation equation” $\vartheta$
      - *Toolbox 3*: Change in numeraire to total wealth (including SDF)

2. Evolution of state variable $\eta$ (and $K$) **forward equation**

3. Value functions **backward equation**
   a. Value fcn. as fcn. of individual investment opportunities $\omega$
      - Special cases: log-utility, constant investment opportunities
   b. Separating value fcn. $V^i(n^i; \eta, K)$ into $v^i(\eta) w(K) (n^i/n^i)^{1-\gamma}$
   c. Derive $C/N$-ratio and $\zeta$ price of risk

4. Numerical model solution
   a. Transform BSDE for separated value fcn. $v^i(\eta)$ into PDE
   b. Solve PDE via value function iteration

5. KFE: Stationary distribution, Fan charts
1a. Individual Agent Choice of $\ell$

- Choice of $\ell$ is static problem (and separable) for each $t$
- \[
\max_{\ell_t^i} dr_t^k(\ell_t^i) = \max_{\ell_t^i} \left( \frac{a^i - \ell_t^i}{q_t} + \Phi(\ell_t^i) - \delta + \mu^q + \sigma \sigma^q \right)
\]
  For aggregate capital return,
  Replace $a^i$ with $A(\kappa)$

- FOC: \[\frac{1}{q_t} = \Phi'(\ell_t^i)\] Tobin’s $q$
  - All agents $\ell_t^i = \ell_t \Rightarrow \frac{dK_t}{K_t} = (\Phi(\ell_t) - \delta) \, dt + \sigma dZ_t$
  - Special functional form:
    - $\Phi(\ell) = \frac{1}{\phi} \log(\phi \ell + 1) \Rightarrow \phi \ell = q - 1$
  - Goods market clearing: \[(A(\kappa) - \ell_t)K_t = \sum_i C_t^i.\]
Solving MacroModels Step-by-Step

0. Postulate aggregates, price processes & obtain return processes

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      - Toolbox 1: Martingale Approach, HJB vs. Stochastic Maximum Principle Approach (previous lecture)
   b. Portfolio choice $\theta$ + Asset market clearing or Asset allocation $\kappa$ & risk allocation $\chi$
      - Toolbox 2: “price-taking social planner approach” – Fisher separation theorem
   c. “Money evaluation equation” $\psi$
      - Toolbox 3: Change in numeraire to total wealth (including SDF)

2. Evolution of state variable $\eta$ (and $K$) **forward equation**

3. Value functions **backward equation**
   a. Value fcn. as fcn. of individual investment opportunities $\omega$
      - Special cases: log-utility, constant investment opportunities
   b. Separating value fcn. $V^i(n^i; \eta, K)$ into $v_i(\eta)u(K)(n^i/n^i)^{1-\gamma}$
   c. Derive $C/N$-ratio and $\zeta$ price of risk

4. Numerical model solution
   a. Transform BSDE for separated value fcn. $v^i(\eta)$ into PDE
   b. Solve PDE via value function iteration

5. KFE: Stationary distribution, Fan charts
1b. Individual Agent Choice of $\theta \Rightarrow$ asset/risk allocation

- **Approach 1**: Portfolio optimization
  - Step 1: Optimization e.g. via Martingale Approach – recall: $\mu_t^A = r_t^i + \zeta_t^i \sigma_t^A$
    - Of experts with outside equity issuance (after plugging in households’ outside equity choice)
    $$\frac{a^e_t - r_t}{q_t^e} + \Phi(u_t) - \delta + \mu_t^q + \sigma_t^q = r_t + [\zeta_t^e \chi_t^e / \kappa_t^e + \zeta_t^h (1 - \chi_t^e / \kappa_t^e)] (\sigma + \sigma^q)$$
    - Of households’ capital choice
    $$\frac{a^h_t - r_t}{q_t^h} + \Phi(u_t) - \delta + \mu_t^q + \sigma_t^q \leq r_t + \zeta_t^h (\sigma + \sigma^q)$$
    with equality if $\kappa_t^e < 1$
  - Step 2: Capital market clearing to obtain asset/risk allocation $\kappa_t^e, \chi_t^e$ from portfolio weights $\theta$'s

- **Approach 2**: Price-taking Social Planner Approach
1b. **Toolbox: Price Taking Social Planner ⇒ Asset/Risk Allocation**

- **Price-Taking Planner’s Theorem:**
  A social planner that takes prices as given chooses a physical asset allocation, $\kappa_t$, and risk allocation, $\chi_t$, that coincides with the choices implied by all individuals’ portfolio choices.

- **Planner’s problem**

  \[
  \max_{\{\kappa_t, \chi_t\}} E_t \left[ \frac{d r_t}{dt} (\kappa_t) \right] / dt - \varsigma_t \sigma(\chi_t) \]

  subject to friction: $F(\kappa_t, \chi_t) \leq 0$

- **Example:**
  1. $\chi_t = \kappa_t$ (if one holds capital, one has to hold risk)
  2. $\chi_t \geq \alpha \kappa_t$ (skin in the game constraint, outside equity up to a limit)
**1b. Toolbox: Price Taking Social Planner ⇒ Asset/Risk Allocation**

- **Sketch of Proof of Theorem**

1. **Fisher Separation Theorem:** (delegated portfolio choice by firm)
   - FOC yield the martingale approach solution
   - Each individual agent \((i, \tilde{i})\) portfolio maximization is equivalent to the maximization problem of a firm
   \[
   \max_{\{\theta^{j,i}\}} E_t \left[ dr^{n(i,\tilde{i})} \right] / dt - \zeta \sigma^n
   \]
   \[
   dr^{n(i,\tilde{i})} = \sum_j \theta^{j,i} dr^j = \sum_j \theta^{j,i} E[dr^j] + \sum_j \theta^{j,i} \sigma^j dZ_t
   \]
   is linear in \(\theta\)s
   - Either bang-bang solution for \(\theta\)s s.t. portfolio constraints bind
   - Or prices/returns/risk premia are s.t. that firm is indifferent

2. **Aggregate**
   - Taking \(\eta\)-weighted sum to obtain return on aggregate wealth

3. **Use market clearing to relate \(\theta\)s to \(\kappa\)s & \(\chi\)s (incl. \(\theta\)-constraint)**
1b. **Toolbox**: Price Taking Social Planner $\Rightarrow$ Asset/Risk Allocation

- **Expert**: $\theta^e = (\theta^{e,K}, \theta^{e,OE}, \theta^{e,D})$ for capital, outside equity, debt

- **Restrictions**:
  - $\theta^{e,K} \geq 0$
  - $\theta^{e,OE} \leq 0$
  - $\theta^{e,OE} \geq -(1 - \alpha)\theta^{e,K}$

  only issue outside equity

- **Household**: $\theta^h = (\theta^{h,K}, \theta^{h,OE}, \theta^{h,D})$

  maximize

  \[
  \theta^{e,K}_t E[dr^{e,K}_t]/dt + \theta^{e,OE}_t E[dr^{OE}_t]/dt + \theta^{e,D}_t r_t - \varsigma^e_t (\theta^{e,K}_t + \theta^{e,OE}_t)\sigma^{r,e,K}_t
  \]

  \[
  \theta^{h,K}_t \geq 0
  \]

  $\theta^{h,OE} \geq 0$

  $\theta^{h,OE} \geq 0$

  maximize

  \[
  \theta^{h,K}_t E[dr^{h,K}_t]/dt + \theta^{h,OE}_t E[dr^{OE}_t]/dt + \theta^{h,D}_t r_t - \varsigma^e_t (\theta^{h,K}_t + \theta^{h,OE}_t)\sigma^{r,h,K}_t
  \]
### 1b. **Toolbox:** Price Taking Social Planner ⇒ Asset/Risk Allocation

- **Aggregate** $\eta$-weighted sum of expert + HH max problem
  \[ \eta^e \{...\} + \eta^h \{...\} \]

- \[ \kappa_t^e := \left( \eta^e_t \theta_t^{e,K} + \eta^h_t \theta_t^{h,K,OE} \right) E \left[ dr_t^{e,K} \right] / dt + \eta^h_t \theta_t^{h,K} E \left[ dr_t^{h,K} \right] / dt + \]

  \[ \kappa_t^h := \left( \eta^e_t \theta_t^{e,OE} + \eta^h_t \theta_t^{h,OE} \right) E \left[ dr_t^{O} \right] / dt + \left( \eta^e_t \theta_t^{e,D} + \eta^h_t \theta_t^{h,D} \right) r_t \]

- \[ -\zeta_t^e \eta_t^e \left( \theta_t^{e,K} + \theta_t^{e,OE} \right) \sigma_t^{r,K} - \zeta_t^h \eta_t^h \left( \theta_t^{h,K} + \theta_t^{h,OE} \right) \sigma_t^{r,K} \]

- \[ =: \chi_t^e \]

- \[ =: \chi_t^h \]
1b. **Toolbox: Price Taking Social Planner ⇒ Asset/Risk Allocation**

- Aggregate $\eta$-weighted sum of expert + HH max problem
  \[ \eta^e \{ \ldots \} + \eta^h \{ \ldots \} \]

\[ \eta^e_t \theta^e_t \kappa^e_t := \eta^e \{ \ldots \} + \eta^h \{ \ldots \} \]

\[ \eta^h_t \theta^h_t \kappa^h_t := \left( \eta^e \theta^e + \eta^h \theta^h \right) \]

\[ E \left[ dr_t^e \right] / dt + \eta^h_t \theta^h \kappa^h_t \left[ dr_t^h \right] / dt + \]

\[ -\zeta^e_t \eta^e_t \left( \theta^e_t + \theta^e_k \right) \sigma_t^K + \zeta^h_t \eta^h_t \left( \theta^h_t + \theta^h_k \right) \sigma_t^K \]

\[ \chi^e_t = 0 \]

\[ \chi^h_t = 0 \]

Poll 36: Why = 0?

a) because marginal benefits = marginal costs at optimum
b) due to martingale behavior
c) because outside equity and debt are in zero net supply
1b. **Toolbox:** Price Taking Social Planner ⇒ Asset/Risk Allocation

- Translate constraints:
  1. \( \chi_t^e \leq \kappa_t^e \) experts cannot buy outside equity of others  
     
     - only important for the case with idio risk
  2. \( \chi_t^e = \eta_t^e \theta_{t,K}^e + \eta_t^e \theta_{t,OE}^e \geq \alpha \kappa_t^e \)

- Price-taking social planers problem

\[
\max_{\{\kappa_t^e, \kappa_t^h = 1-\kappa_t^e, \chi_t^e \in [\alpha \kappa_t^e, \kappa_t^e], \chi_t^h = 1-\chi_t^e\}} \left[ \frac{\kappa_t^e a_t^e + \kappa_t^h a_t^h - \mu_t}{q_t} + \Phi(\mu_t) - \delta \right] - (\zeta_t^e \chi_t^e + \zeta_t^h \chi_t^h) \sigma_t^{r,K}
\]

End of Proof. Q.E.D.

- Linear objective (if frictions take form of constraints)
  1. Price of risk adjust such that objective becomes flat or
  2. Bang-bang solution hitting constraints
1b. **Toolbox:** Price Taking Social Planner ⇒ Asset/Risk Allocation

- **Example 1:** 2 Types + **no** outside equity ($\alpha = 1$)

$$\max_{\{\kappa^e_t, \chi^e_t\}} \left[ \frac{\kappa^e_t a^e + (1 - \kappa^e_t) a^h - \iota_t}{q_t} + \Phi(\iota_t) - \delta \right] - (\chi^e_t \zeta^e_t + (1 - \chi^e_t) \zeta^h_t)(\sigma + \sigma^q_t)$$

s.t. friction $\chi^e_t = \kappa^e_t$ if no outside equity can be issued

- **FOC** $\chi$: $\frac{a^e - a^h}{q_t} = (\zeta^e_t - \zeta^h_t)(\sigma + \sigma^q_t)$

- May hold only with inequality ($\geq$), if at constraint $\kappa^e_t = 1$
1b. Price Taking Social Planner ⇒ Asset/Risk Allocation

**Example 2:** 2 Types + with outside equity

\[
\max_{\{\kappa^e_t, \chi^e_t\}} \left[ \frac{\kappa^e_t a^e + (1 - \kappa^e_t) a^h - l_t}{q_t} \right] + \Phi(\iota_t) - \delta - (\chi^e_t \xi^e_t + (1 - \chi^e_t) \xi^h_t)(\sigma + \sigma^q_t) - \chi^e_t \xi^e_t + (1 - \chi^e_t) \xi^h_t \xi^e_t + (1 - \chi^e_t) \xi^h_t \xi^h_t + (1 - \chi^e_t) \xi^h_t \xi^q_t
\]

**FOC** $\chi$: Case 1: $\zeta^e_t (\sigma + \sigma^q_t) > \zeta^h_t (\sigma + \sigma^q_t) \Rightarrow \chi^e_t = \alpha \kappa^e_t$

Case 2: $\chi^e_t > \alpha \kappa^e_t$

**Case 1:** plug $\chi^e_t = \alpha \kappa^e_t$ in objective

\[ a. \quad \text{FOC}_{\kappa}: \frac{a^e - a^h}{q_t} > \alpha (\zeta^e_t - \zeta^h_t)(\sigma + \sigma^q_t) \Rightarrow \kappa^e_t < 1 \]

\[ b. \quad \Rightarrow \kappa^e_t = 1 \]

**Case 2:**

\[ a. \quad \text{FOC}_{\kappa}: \frac{a^e - a^h}{q_t} > 0 \Rightarrow \kappa^e_t = 1 \]

\[ b. \quad = 0 \Rightarrow \kappa^e_t < 1 \text{ impossible} \]
### 1b. Price Taking Social Planner ⇒ Asset/Risk Allocation

- Summarizing previous slide (2 types with outside equity)

<table>
<thead>
<tr>
<th>Cases</th>
<th>$\chi_t^e \geq \alpha \kappa_t^e$</th>
<th>$\kappa_t^e \leq 1$</th>
<th>$\frac{(a^e - a^h)}{q_t} \geq \alpha (\zeta_t^e - \zeta_t^h)(\sigma + \sigma_t^q)$</th>
<th>$(\zeta_t^e - \zeta_t^h)(\sigma + \sigma_t^q) \geq 0$</th>
</tr>
</thead>
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<td>$&lt;$</td>
<td>$=$</td>
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<tr>
<td>1b</td>
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<td>$=$</td>
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<td>$&gt;$</td>
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<tr>
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<td>$=$</td>
<td>$&gt;$</td>
<td>$=$</td>
</tr>
<tr>
<td></td>
<td>impossible</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**2 Types**

- **HHs’ short-sale constraint of capital binds, $\kappa_t^e = 1$**
- **Experts’ skin in the game constraint binds, $\chi_t^e = \alpha \kappa_t^e$**
- **Occasionally binding constraint (skin in the game constraint)**

**Experts’ skin in the game**

$\eta$

**HHs’ short-sale constraint**

$\eta$
Solving MacroModels Step-by-Step

0. Postulate aggregates, price processes & obtain return processes

1. For given $C/N$-ratio and SDF processes for each $i$ finance block
   a. Real investment $\iota$ + Goods market clearing (static)
      - Toolbox 1: Martingale Approach, HJB vs. Stochastic Maximum Principle Approach (previous lecture)
   b. Portfolio choice $\theta$ + Asset market clearing or Asset allocation $\kappa$ & risk allocation $\chi$
      - Toolbox 2: “price-taking social planner approach” – Fisher separation theorem
   c. “Money evaluation equation” $\varpi$
      - Toolbox 3: Change in numeraire to total wealth (including SDF)

2. Evolution of state variable $\eta$ (and $K$)

3. Value functions
   a. Value fcn. as fcn. of individual investment opportunities $\omega$
      - Special cases: log-utility, constant investment opportunities
   b. Separating value fcn. $V^i(n^i; \eta, K)$ into $v^i(\eta)u(K)(n^i/n^i)^{1-\gamma}$
   c. Derive $C/N$-ratio and $\zeta$ price of risk

4. Numerical model solution
   a. Transform BSDE for separated value fcn. $v^i(\eta)$ into PDE
   b. Solve PDE via value function iteration

5. KFE: Stationary distribution, Fan charts
Toolbox 3: Change of Numeraire

- $x_t^A$ is a value of a self-financing strategy/asset in $\$. 
- $Y_t$ price of € in $\$$(exchange rate) 
\[
\frac{dY_t}{Y_t} = \mu^Y_t dt + \sigma^Y_t dZ_t
\]

- $x_t^A/Y_t$ value of the self-financing strategy/asset in € 
\[
e^{-\rho_t u'(c_t)} Y_t \frac{x_t^A}{Y_t} = \xi_t
\]

Recall
\[
\mu^A_t - \mu^B_t = (-\sigma^\xi_t)(\sigma^A - \sigma^B_t) = \zeta_t
\]

\[
\mu^{A/Y}_t - \mu^{B/Y}_t = (-\sigma^\xi_t - \sigma^Y_t)(\sigma^A - \sigma^B_t + \sigma^Y_t)
\]

- Price of risk $\zeta^\xi € = \zeta^\$ − \sigma^Y$
**Toolbox 3: Change of Numeraire**

- $x^A_t$ is a value of a self-financing strategy/asset in $\$
- $Y_t$ price of € in $\$$(exchange rate)

\[
dY_t = \mu^Y_t dt + \sigma^Y_t dZ_t
\]

- $x^A_t/Y_t$ value of the self-financing strategy/asset in €

\[
e^{-\rho t} u'(c_t) \frac{x^A_t}{Y_t} \text{ follows a martingale}
\]

Recall $\mu^A_t - \mu^B_t = (-\sigma^A_t) (\sigma^A - \sigma^B_t)$

\[
\mu^A/Y_t - \mu^B/Y_t = (-\sigma^X_t - \sigma^Y_t) (\sigma^A - \sigma^Y_t - \sigma^B_t + \sigma^Y_t)
\]

- Price of risk $\xi^\mathcal{E} = \xi^\mathcal{\$} - \sigma^Y$

Poll 44: Why does the price of risk change, though real risk remains the same

a) because risk-free rate might not stay risk-free
b) because covariance structure changes
Solving MacroModels Step-by-Step

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4. Numerical model solution
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   b. Solve PDE via value function iteration

5. KFE: Stationary distribution, Fan charts
2. GE: Markov States and Equilibria

- Equilibrium is a map
  
  Histories of shocks \( \{Z_s, s \in [0, t]\} \) → prices \( q_t, \xi^i_t, \iota^i_t, \theta^i_t \)
  
  net worth distribution
  \[
  \eta^e_t = \frac{N^e_t}{q_tK_t} \in (0,1)
  \]
  
  net worth share

- All agents maximize utility
  - Choose: portfolio, consumption, technology

- All markets clear
  - Consumption, capital, money, outside equity
2. Law of Motion of Wealth Share $\eta_t$

**Method 1:** Using Ito’s quotation rule $\eta^i_t = N^i_t / (q_t K_t)$

- Recall
  \[
  \frac{dN^i_t}{N^i_t} = r_t dt + \frac{\chi^i_t}{\eta^i_t} \sigma^i_t dt + \frac{\chi^i_t}{\eta^i_t} (\sigma + \sigma^q_t) dZ_t - \frac{C^i_t}{N^i_t} dt
  \]

- \[
  \frac{d\eta^i_t}{\eta^i_t} = \ldots \text{(lots of algebra)}
  \]

**Method 2:** Change of numeraire + Martingale Approach

- New numeraire: Total wealth in the economy, $N_t$
- Apply Martingale Approach for value of $i$’s portfolio
  - Simple algebra to obtain drift of $\eta^i_t$: $\mu^\eta^i_t$
    - Note that change of numeraire does not affect ratio $\eta^i$!
2. $\mu^\eta$ Drift of Wealth Share: Many Types

- **New Numeraire**
  - “Total net worth” in the economy, $N_t$ (without superscript)
  - Type $i$’s portfolio net worth = net worth share

- **Martingale Approach with new numeraire**
  - Asset $A = i$’s portfolio return in terms of total wealth,
    \[
    \left( \frac{C_t^i}{N_t^i} + \mu_t^\eta_i \right) dt + \sigma_t^\eta_i dZ_t + \tilde{\sigma}_t^\eta_i d\tilde{Z}_t
    \]
    Dividend yield E[capital gains] rate
  - Asset $B$ (benchmark asset that everyone can hold, e.g. risk-free asset or money (in terms of total economy wide wealth as numeraire))
    \[
    r_t^m dt + \sigma_t^m dZ_t
    \]
  - Poll 48: Is risk-free asset, risk free in the new numeraire?
    - a) Yes
    - b) No

- Apply our martingale asset pricing formula
  \[
  \mu_t^A - \mu_t^B = \zeta_t^i (\sigma_t^A - \sigma_t^B)
  \]
2. $\mu^\eta$ Drift of Wealth Share: Many Types

- Asset pricing formula (relative to benchmark asset)
  \[ \mu^\eta_t + \frac{C^i_t}{N^i_t} - r_t^m = (\zeta^i_t - \sigma^N_t) \left( \sigma^\eta_t - \sigma^m_t \right) \]

- Add up across types (weighted),
  (capital letters without superscripts are aggregates for total economy)
  \[ \sum_{i'} \eta^i' \mu^i_t + \frac{C_t}{N_t} - r_t^m = \sum_{i'} \eta^i' \left( \zeta^i' - \sigma^N_t \right) \left( \sigma^\eta_t - \sigma^m_t \right) \]

- Poll 49: Why $= 0$?
  a) Because we have stationary distribution
  b) Because $\eta$s sum up to 1
  c) Because $\eta$s follow martingale

Benchmark asset everyone can trade
$\sigma^m_t = -\sigma^N_t$
2. $\mu^\eta$ Drift of Wealth Share: 2 Types

- Asset pricing formula (relative to benchmark asset)
  \[ \mu_t^\eta i + \frac{C_t^i}{N_t^i} - r_t^m = (\zeta_t^i - \sigma_t^N) \left( \sigma_t^\eta i - \sigma_t^m \right) \]

- Add up across types (weighted),
  \[(\sigma_t^\eta e - \sigma_t^N) \left( \sigma_t^\eta e - \sigma_t^m \right) + \eta_t^h \left( \zeta_t^h - \sigma_t^N \right) \left( \sigma_t^\eta h - \sigma_t^m \right)\]

- Subtract from each other yield net worth share dynamics
  \[ \mu_t^\eta e = (1 - \eta_t^e) \left( \zeta_t^e - \sigma_t^N \right) \left( \sigma_t^\eta e - \sigma_t^m \right) - (1 - \eta_t^e) \left( \zeta_t^h - \sigma_t^N \right) \left( \sigma_t^\eta h - \sigma_t^m \right) \]

For benchmark asset: risk-free debt
\[ \sigma_t^m = -\sigma_t^N \]
2. $\sigma^\eta$ Volatility of Wealth Share

- Recall Ito ratio rule (only volatility term)

- Since $\eta_t^e = N_t^e / N_t$,

  $$\sigma_t^{\eta^e} = \sigma_t^{N^e} - \sigma_t^N = \sigma_t^{N^i} - \sum_{i'} \eta_{t}^{i'} \sigma_t^{N^{i'}} = (1 - \eta_{t}^{i})\sigma_t^{N^i} - \sum_{i' \neq i} \eta_{t}^{i-} \sigma_t^{N^{i-}}$$

- Note for

  $$\sigma_t^{\eta^e} = (1 - \eta_{t}^{e})(\sigma_t^{n^e} - \sigma_t^{n^h})$$

  $$\sigma_t^{n^e} = \frac{\chi_t^e/\eta_t^e}{\theta^e, K + \theta^e, O_E} (\sigma + \sigma_t^q)$$

  $$\sigma_t^{n^h} = \frac{\chi_t^h}{\eta_t^h} (\sigma + \sigma_t^q) = \frac{1 - \chi_t^e}{1 - \eta_t^e} (\sigma + \sigma_t^q)$$

  Hence,

  $$\sigma_t^{\eta^e} = \frac{\chi_t^e - \eta_t^e}{\eta_t^e} (\sigma + \sigma_t^q)$$

- Note also, $\eta_t^e \sigma_t^{\eta^e} + \eta_t^h \sigma_t^{\eta^h} = 0 \Rightarrow \sigma_t^{\eta^h} = -\frac{\eta_t^e}{\eta_t^h} \sigma_t^{\eta^e} = -\frac{\eta_t^e}{1 - \eta_t^e} \sigma_t^{\eta^e}$

Change in notation in 2 type setting
Type-net worth is $n^i = N_t^i$
2. Amplification Formula: Loss Spiral

- Recall
  \[ \sigma_t \eta^e = \frac{\chi_t^e - \eta_t^e}{\eta_t^e} \left( \sigma + \sigma^q_t \right) \]

- By Ito’s Lemma on \( q(\eta^e) \)
  \[ \sigma^q_t = \frac{q'(\eta_t^e)}{q(\eta_t^e)} \eta_t^e \sigma_t \eta^e \]

- Total volatility
  \[ \sigma + \sigma^q_t = \frac{\sigma}{1 - \frac{q'(\eta_t^e)\chi_t^e - \eta_t^e}{q/\eta_t^e \eta_t^e}} \]

- Loss spiral
  - Market illiquidity (price impact elasticity)
2. Amplification Formula: Loss Spiral

- Recall
  \[ \sigma_t \eta^e = \frac{x_t^e - \eta_t^e}{\eta_t^e} \left( \sigma + \sigma_t^q \right) \]
  (leverage)

- By Ito’s Lemma on \( q(\eta^e) \)
  \[ \sigma_t^q = \frac{q'(\eta_t^e)}{q(\eta_t^e)} \eta_t^e \sigma_t^e \]

- Total volatility
  \[ \sigma + \sigma_t^q = \frac{\sigma}{1 - \frac{q'(\eta_t^e)}{q/\eta_t^e} \frac{x_t^e - \eta_t^e}{\eta_t^e}} \]

Poll 53: Where is the spiral?
- a) Sum of infinite geometric series (denominator)
- b) in \( q' \), since with constant price, no spiral

- Loss spiral
  - Market illiquidity (price impact elasticity)
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5. KFE: Stationary distribution, Fan charts
The Big Picture

allocation of physical assets

output $A(\kappa)$

consumption + investment

value function

precautionary

capital growth $K$ 

price of risk $\zeta$

net worth distribution $\eta$

volatility

drift

Backward equation with expectations

Forward equation

$\kappa$ 

accumulation Debt

Outside equity

$\chi \geq \chi$

$\Phi(\iota) - \delta$

$\delta$

$\kappa$

$\eta$

$\Phi$

$\sigma$

$\kappa$

$\delta$

$\Phi$

$\sigma$

$\kappa$

$\delta$

$\Phi$

$\sigma$

$\kappa$

$\delta$

$\Phi$

$\sigma$

$\kappa$

$\delta$

$\Phi$

$\sigma$

$\kappa$

$\delta$

$\Phi$

$\sigma$
3a. CRRA Value Function  

- Martingale Approach: works best in endowment economy
- Here: mix Martingale approach with value function (envelop condition)

\[ V^i(n^i_t; \eta_t, K_t) \] for individuals \( i \)

- For CRRA/power utility \( u(c^i_t) = \frac{(c^i_t)^{1-\gamma} - 1}{1-\gamma} \)

\( \Rightarrow \) increase net worth by factor, optimal \( c^i \) for all future states increases 
by this factor \( \Rightarrow \left( \frac{c^i_t}{n^i_t} \right) \)-ratio is invariant in \( n^i_t \)

\( \Rightarrow \) value function can be written as 
\[ V^i(n^i_t; \eta_t, K_t) = \frac{u(\omega^i(\eta_t,K_t)n^i_t)}{\rho^i} \]

- \( \omega_t^i \) Investment opportunity/ “net worth multiplier”
  - \( \omega^i(\eta_t,K_t) \)-function turns out to be independent of \( K_t \)
  - Change notation from \( \omega^i(\eta_t,K_t) \)-function to \( \omega_t^i \)-process
3a. CRRA Value Function: relate to $\omega$

- Value function can be written as $\frac{u(\omega_t n_t^i)}{\rho}$, that is

$$\frac{1}{\rho^i} \frac{(\omega_t n_t^i)^{1-\gamma}}{1-\gamma} - 1 = \frac{1}{\rho^i} \frac{(\omega_t)^{1-\gamma} (n_t^i)^{1-\gamma}}{1-\gamma}$$

- $\frac{\partial V}{\partial n_t^i} = u'(c_t^i)$ by optimal consumption (if no corner solution)

$$\frac{(\omega_t)^{1-\gamma} (n_t^i)^{-\gamma}}{\rho^i} = (c_t^i)^{-\gamma} \iff \frac{c_t^i}{n_t^i} = (\rho^i)^{1/\gamma} (\omega_t)^{1-1/\gamma}$$

- For log utility $\gamma = 1$
  - Consumption choice: $c_t^i = \rho^i n_t^i$
    - $\omega_t$ does not matter $\Rightarrow$ income and substitution effect cancel out

- Portfolio choice: myopic (no Mertonian hedging demand)
  - Volatility of investment of opportunity/net worth multiplier does not matter $\Rightarrow$ Myopic price of risk $\zeta_t^i = \sigma_t^i = \sigma_{c_t}^i$
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   ▪ forward equation

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   b. Separating value fcn. $V^i(n^i; \eta, K)$ into $v^i(\eta)u(K)(n^i/n^i)^{1-\gamma}$
   c. Derive $C/N$-ratio and $\zeta$ price of risk

4. Numerical model solution
   a. Transform BSDE for separated value fcn. $v^i(\eta)$ into PDE
   b. Solve PDE via value function iteration

5. KFE: Stationary distribution, Fan charts
4a. Replacing $l_t$

- Recall from optimal re-investment $\Phi'(l_t) = 1/q_t$
- For $\Phi(i) = \frac{1}{\phi} \log(\phi i + 1) \Rightarrow \phi i = q - 1$
4a. Replacing $\chi$, obtain $\kappa$ for good mkt clearing

- Recall from planner’s problem (Step 1b)

| Cases | $\chi^e_t \geq \alpha \kappa^e_t$ | $\kappa^e_t \leq 1$ | \(\frac{(a^e - a^h)}{q_t} \geq \alpha (\zeta^e_t - \zeta^h_t)(\sigma + \sigma^q_t)\) | \((\zeta^e_t - \zeta^h_t)(\sigma + \sigma^q_t) \geq 0\) |
|-------|-------------------------------|------------------------|--------------------------------|
| 1a    | 1                             | <                      | =                                | >                                  |
| 1b    | 1                             | =                      | >                                | >                                  |
| 2a    | >                             | =                      | >                                | =                                  |

impossible

- HHs’ short-sale constraint of capital binds, $\kappa^e_t = 1$
- Experts’ skin in the game constraint binds, $\chi^e_t = \alpha \kappa^e_t$
- Occasionally binding constraint (skin in the game constraint)
4a. Replacing $\chi$, obtain $\kappa$ for good mkt clearing

- **Determination of $\kappa_t$**
  - Based on difference in risk premia $(\zeta_t^e - \zeta_t^h)(\sigma + \sigma_t^q)$
  - For log utility:
    \[
    (\sigma_t^{n^e} - \sigma_t^{n^h}) (\sigma + \sigma_t^q) = \frac{\chi_t^e - \eta_t^e}{(1 - \eta_t^e) \eta_t^e} (\sigma + \sigma_t^q)
    \]
    - $= \text{since } \sigma_t^{n^e} = \frac{\chi_t^e - \eta_t^e}{\eta_t^e} (\sigma + \sigma_t^q)$, $\sigma_t^{n^h} = -\frac{\eta_t^e}{1 - \eta_t^e} \sigma_t^{n^e}$ and $\sigma_t^{n^e} - \sigma_t^{n^h} = \sigma_t^{n^e} - \sigma_t^{n^h}$
  - Hence,
    \[
    (a^e - a^h)/q_t \geq \alpha \frac{\chi_t^e - \eta_t^e}{(1 - \eta_t^e) \eta_t^e} (\sigma + \sigma_t^q)^2
    \]
    with equality if $\kappa_t^e < 1$

- **Determination of $\chi_t^e$**
  \[
  \chi_t^e = \max\{\alpha \kappa_t^e, \eta_t^e\}
  \]
4a. Replacing \( \chi \), obtain \( \kappa \) for good mkt clearing

- Need to determine diff in risk premia \( (\zeta_t^e - \zeta_t^h)(\sigma + \sigma_t^q) \):

- Recall

  \[
  \sigma_t^n = \frac{\chi_t^e - \eta_t^e}{\eta_t^e} (\sigma + \sigma_t^q) = \sigma_t^e - \sigma_t^h
  \]

  \[
  \sigma_t^h = -\frac{\eta_t^e}{1 - \eta_t^e} \sigma_t^e
  \]

  Hence,

  \[
  \sigma_t^n - \sigma_t^h = \frac{1}{1 - \eta_t^e} \sigma_t^e
  \]

  \[
  \Rightarrow (\zeta_t^e - \zeta_t^h)(\sigma + \sigma_t^q) = \left( -\frac{\partial \eta v_t^e}{v_t^e} + \frac{\partial \eta v_t^h}{v_t^h} + \frac{1}{(1 - \eta_t^e)\eta_t^e} \right) \eta_t^e \sigma_t^e (\sigma + \sigma_t^q)
  \]

  \[
  = \left( -\frac{\partial \eta v_t^e}{v_t^e} + \frac{\partial \eta v_t^h}{v_t^h} + \frac{1}{(1 - \eta_t^e)\eta_t^e} \right) (\chi_t^e - \eta_t^e)(\sigma + \sigma_t^q)^2
  \]

  Note, since \(-\frac{\partial \eta v_t^e}{v_t^e} + \frac{\partial \eta v_t^h}{v_t^h} + \frac{1}{(1 - \eta_t^e)\eta_t^e} > 0\),

  \[
  \chi_t^e > \eta_t^e \iff \alpha \kappa_t^e > \eta_t^e
  \]
4a. Market Clearing

- Output good market

\[(\kappa_t^e a^e + (1 - \kappa_t^e)a^h - \iota_t)K_t = C_t\]

... jointly restricts \(\kappa_t\) and \(q_t\)

\[
\kappa_t a^e + (1 - \kappa_t)a^h - \iota(q_t) = q_t[\eta_t^e \rho^e + (1 - \eta_t)\rho^h]
\]

\[
= \left(\frac{\eta_t^e q_t}{v_t^e}\right)^{1/\gamma} + \left(\frac{(1 - \eta_t^e)q_t}{v_t^h}\right)^{1/\gamma}
\]

\[
= \left(\frac{\eta_t^e q_t}{c_t^e/K_t}\right)^{1/\gamma} + \left(\frac{(1 - \eta_t^e)q_t}{c_t^h/K_t}\right)^{1/\gamma}
\]
4a. Market Clearing

- **Output good market**
  \[
  (\kappa_t^e a^e + (1 - \kappa_t^e)a^h - \iota_t)K_t = C_t, \\
  \kappa_t a^e + (1 - \kappa_t)a^h - \iota(q_t) = q_t [\eta_t \rho^e + (1 - \eta_t)\rho^h]
  \]

  ... jointly restricts \( \kappa_t \) and \( q_t \)

- **Capital market** is taken care off by price taking social planner approach

  \[ \theta_{t}^{e,K} = \frac{\kappa_t^e q_t K_t}{\eta_t^e q_t K_t} \]

- **Risk-free debt market** also taken care off by price taking social planner approach

  (would be cleared by Walras Law anyways)
4a. $\sigma^q(q, q')$

- Recall from “amplification slide” – Step 2

\[
\sigma + \sigma^q_t = \frac{\sigma}{1 - \frac{q'(\eta^e_t) \chi^e_t - \eta^e_t}{q/\eta^e_t \eta^e_t}}
\]

\[
\sigma^q = \frac{q'(\eta^e_t)}{q(\eta^e_t)} (\chi^e_t - \eta^e_t) (\sigma + \sigma^q_t)
\]

- Now all red terms are replaced, and we can solve ...
4b. Algorithm – Static Step

- Suppose we know functions \( v^e(\eta^e) \), \( v^h(\eta) \), have five static conditions:

1. \( \phi_t = q_t - 1 \)
2. Planner condition for \( \kappa^e_t: \frac{(a^e - a^h)/q_t}{\alpha^e} \geq \frac{\chi^e_t - \eta^e_t}{(1-\eta^e_t)\eta^e_t} (\sigma + \sigma^q_t)^2 \) \( \Rightarrow \) Get \( q(\eta^e), \kappa^e(\eta^e) \), \( \sigma^q(\eta^e) \)
3. Planner condition for \( \chi^e_t = \max\{\alpha \kappa^e_t, \eta^e_t\} \)
4. \( \kappa^e_t a^e + (1 - \kappa^e_t)a^h - \iota(q_t) = q_t [\eta_t \rho^e + (1 - \eta_t)\rho^h] \)
5. \( \sigma^q = \frac{q'(\eta^e_t)}{q(\eta^e)} (\chi^e_t - \eta^e_t)(\sigma + \sigma^q_t) \)

- Start at \( q(0) \), solve to the right, use different procedure for two \( \eta \) regions depending on \( \kappa^e \):

1. While \( \kappa^e < 1 \), solve ODE for \( q(\eta^e) \):
   - For given \( q(\eta) \), plug optimal investment (1) into (4)
   - Plug planner condition (3) into (2) and (5)
   - Solve ODE using three equilibrium condition (2),(4) and (5) via Newton’s method (see next slide)
2. When \( \kappa = 1 \), (2) is no longer informative, since \( \kappa^e = 1 \), solve (1) and (4) for \( q(\eta) \)
4b. Aside: Newton’s Method

- Find the root of equation system \( F(z_n) = 0 \) via iterative method
  \[ z_{n+1} = z_n - J_n^{-1}F(z_n) \]

  Where \( J_n \) is the Jacobian matrix, i.e., \( J_{ij} = \frac{\partial f_i(z)}{\partial z_j} \).

- Newton’s method does not guarantee global convergence.
- Commonly takes several-step iteration.
4b. Aside: Newton’s Method

\[ z_n = \begin{bmatrix} q_t \\ \kappa_t^e \\ \sigma + \sigma_t^q \end{bmatrix}, \]

\[ F(z_n) = \begin{bmatrix} \kappa_t^e a^e + (1 - \kappa_t^e)a^h - \iota(q_t) - q_t[\eta_t\rho^e + (1 - \eta_t)\rho^h] \\ q'(\eta_t^e)(\chi_t^e - \eta_t^e)(\sigma + \sigma_t^q) - \sigma^q q(\eta_t^e) \\ (\alpha^e - \alpha^h) - \alpha q_t \frac{\chi_t^e - \eta_t^e}{(1 - \eta_t^e)\eta_t^e} (\sigma + \sigma_t^q)^2 \end{bmatrix} \]

Plug in blue terms from optimal investment and Planner condition for \( \chi_t^e \)
Solution

- Price of capital

- Amplification

Parameters: $\rho^e = .06, \rho^h = .05, a^e = .11, a^h = .03, \delta = .05, \sigma = .1, \alpha = .50, \gamma = 2, \phi = 10$
Volatility Paradox

- Comparative Static w.r.t. $\sigma = .01, .05, .1$
Risk Sharing via Outside Equity

- Comparative Static w.r.t. Risk sharing $\alpha = 0.1, 0.2, 0.5$ (skin the game constraint)
Market Liquidity

- Comparative static w.r.t. $a^h = .03, -.03, -.09$
Solving MacroModels Step-by-Step

0. Postulate aggregates, price processes & obtain return processes

1. For given \( C/N \)-ratio and SDF processes for each \( i \) finance block
   a. Real investment \( \iota \) + Goods market clearing (static)
      ▪ Toolbox 1: Martingale Approach, HJB vs. Stochastic Maximum Principle Approach
   b. Portfolio choice \( \theta \) + Asset market clearing or Asset allocation \( \kappa \) & risk allocation \( \chi \)
      ▪ Toolbox 2: “price-taking social planner approach” – Fisher separation theorem
   c. “Money evaluation equation” \( \vartheta \)
      ▪ Toolbox 3: Change in numeraire to total wealth (including SDF)

2. Evolution of state variable \( \eta \) (and \( K \))

3. Value functions
   a. Value fcn. as fcn. of individual investment opportunities \( \omega \)
      ▪ Special cases: log-utility, constant investment opportunities
   b. Separating value fcn. \( V^i(n^i; \eta, K) \) into \( v^i(\eta)u(K)(n^i/n^i)^{1-\gamma} \)
   c. Derive \( C/N \)-ratio and \( \zeta \) price of risk

4. Numerical model solution
   a. Transform BSDE for separated value fcn. \( v^i(\eta) \) into PDE
   b. Solve PDE via value function iteration

5. KFE: Stationary distribution, Fan charts
From $\mu^\eta (\eta^e)$ & $\sigma^\eta (\eta^e)$ to Stationary Distribution

- Drift and Volatility of $\eta^e$

\[ \eta^e = \alpha \kappa^e \]

"Steady state" $\eta^*$, HHs' short-sale constraint of capital binds, $\kappa^e = 1$

Experts' skin in the game constraint binds, $\chi_t^e = \alpha \kappa^e$
5. Kolmogorov Forward Equation

- Given an initial distribution $f(\eta, 0) = f_0(\eta)$, the density diffusion follows PDE
  \[
  \frac{\partial f(\eta, t)}{\partial t} = -\frac{\partial [f(\eta, t)\mu(\eta)]}{\partial \eta} + \frac{1}{2} \frac{\partial^2 [f(\eta, t)\sigma^2(\eta)]}{\partial \eta^2}
  \]

- “Kolmogorov Forward Equation” is in physics referred to as “Fokker-Planck Equation”

- Corollary: if stationary distribution $f(\eta)$ exists, it satisfies the ODE
  \[
  0 = -\frac{\partial [f(\eta, t)\mu(\eta)]}{\partial \eta} + \frac{1}{2} \frac{\partial^2 [f(\eta, t)\sigma^2(\eta)]}{\partial \eta^2}
  \]
Solving MacroModels Step-by-Step

0. Postulate aggregates, price processes & obtain return processes

1. For given $C/N$-ratio and SDF processes for each $i$ finance block
   a. Real investment $\iota$ + Goods market clearing \textit{(static)}
      • \textit{Toolbox 1}: Martingale Approach, HJB vs. Stochastic Maximum Principle Approach
   b. Portfolio choice $\theta$ + Asset market clearing or Asset allocation $\kappa$ & risk allocation $\chi$
      • \textit{Toolbox 2}: “price-taking social planner approach” – Fisher separation theorem
   c. “Money evaluation equation” $\varphi$
      • \textit{Toolbox 3}: Change in numeraire to total wealth (including SDF)

2. Evolution of state variable $\eta$ (and $K$) \textit{forward equation}

3. Value functions \textit{backward equation}
   a. Value fcn. as fcn. of individual investment opportunities $\omega$
      • \textit{Special cases: log-utility, constant investment opportunities}
   b. Separating value fcn. $V^i(n^i; \eta, K)$ into $v^i(\eta)u(K)(n^i/n^i)^{1-\gamma}$
   c. Derive $C/N$-ratio and $\zeta$ price of risk

4. Numerical model solution
   a. Transform BSDE for separated value fcn. $v^i(\eta)$ into PDE
   b. Solve PDE via value function iteration

5. \textit{KFE}: Stationary distribution, Fan charts
5. Stationary Distribution

- Stationary distribution of $\eta^e$

\[ \eta^e = \alpha \kappa^e \]

Poll 78: Is the constraint always (not just occasionally) binding

a) yes

b) no, only for some parameters $\rho^e > \rho^h$
5. Stationary Distribution

- Stationary distribution of $\eta^e$

\[ \eta^e = \alpha \kappa^e \]

Experts' skin in the game constraint binds $\chi^e_t = \alpha \kappa^e_t$

Perfect risk-sharing region (infeasible)

Poll 79: What happens for $\rho^e = \rho^h$

a) experts take over the economy, $\eta \rightarrow 1$
b) there is a steady state at $\eta = \alpha$
5. Fan chart and distributional impulse response

- ... the theory to Bank of England’s empirical fan charts
- Starts at $\eta_0$, the median of stationary distribution
- Simulate a shock at 1% quantile of original Brownian shock ($dZ_t = -2.32 \, dt$) for a period of $\Delta t = 1$.
- Converges back to stationary distribution
5. Fan chart and distributional impulse response

- Starts at stationary distribution
- Simulate a shock at 1% quantile of original Brownian shock ($dZ_t = -2.32 \, dt$) for a period of $\Delta t = 1$.
- Converges back to stationary distribution
5. Density Diffusion

- Starts at stationary distribution
- Simulate a shock at 1% quantile of original Brownian shock \(dZ_t = -2.32 \, dt\) for a period of \(\Delta t = 1\).
- Converges back to stationary distribution
5. Density Diffusion Movies
5. Distributional Impulse Response

- Difference between path with and without shock
- Difference converges to zero in the long-run

\[ \sigma = 0.15 \]