ECO529: Modern Macro, Money and (International) Finance
Problem Set 4

Andrey Alexandrov

October 10, 2022
1. Endogenous Jumps

• Setup:
  ▶ Sunspots arrive with exogenous intensity $\lambda$
  ▶ Jump: $q(\eta) \rightarrow q(0)$ if self-fulfilling (i.e. if wipes experts out and $\eta \rightarrow 0$)

• Issue: $\eta = 0$ is absorbing
• Fix: introduce transfer $\tau K_t$ in case of a jump:
  ▶ $q(\eta) \rightarrow q(\eta^*)$
  ▶ Net $\rightarrow 0$, experts default on debt, receive $\tau K_t$
  ▶ $\eta \rightarrow \eta^*$ (self-fulfilling)
1. Endogenous Jumps

• Setup:
  ▶ Sunspots arrive with exogenous intensity $\lambda$
  ▶ Jump: $q(\eta) \rightarrow q(0)$ if self-fulfilling (i.e. if wipes experts out and $\eta \rightarrow 0$)

• Issue: $\eta = 0$ is absorbing
1. Endogenous Jumps

- Setup:
  - Sunspots arrive with exogenous intensity $\lambda$
  - Jump: $q(\eta) \to q(0)$ if self-fulfilling (i.e. if wipes experts out and $\eta \to 0$)

- Issue: $\eta = 0$ is absorbing

- Fix: introduce transfer $\tau K_t$ in case of a jump:
  - $q(\eta) \to q(\eta^*)$
  - $N_t^e \to 0$, experts default on debt, receive $\tau K_t$
  - $\eta \to \eta^*$ (self-fulfilling)
1. Endogenous Jumps

- Laws of motion:

\[
d\eta_t = \mu_{\eta,t} dt + \sigma_{\eta,t} dZ_t + j_\eta^- \eta_t^- dJ_t \\
dq_t = \mu_{q,t} t dt + \sigma_{q,t} dZ_t + j_\eta^- q_t^- dJ_t
\]

- Jump at \( t \): \( \eta_t^- \rightarrow \eta_t^+ \), \( d\eta_t = \eta_t^+ - \eta_t^- \) and \( dJ_t = 1 \)

\[
d\eta_t = \eta_t^+ - \eta_t^- = j_\eta^- \eta_t^- , \quad j_\eta^- = \frac{\eta^* - \eta_t}{\eta_t} \\
dq_t = q_t^+ - q_t^- = j_\eta^- q_t^- , \quad j_\eta^- = \frac{q^* - q_t}{q_t}
\]
1. Endogenous Jumps

- Vulnerability region:

\[
\chi_t (q_t - q^*) K_t \geq \eta_t q_t K_t \iff -j^q_t \chi_t \geq \eta_t
\]

Loss in \( N_t^e \) \( \iff \) Loss in \( D_t^e \)

- Once \( N_t^e \) reaches 0, the rest is absorbed by defaulting debt \( D_t^e \):

\[
\chi_t (q_t - q^*) K_t - \eta_t q_t K_t = -j^D_t D_t^e, - = -j^D_t (\chi_t - \eta_t) q_t K_t
\]

Loss in \( D_t^e \)

\[
j^D_t = \frac{\chi_t j^q_t + \eta_t}{\chi_t - \eta_t}
\]
1. Endogenous Jumps

- Transfer $\tau$ ensures $\eta_t \to \eta^*$:

$$N_t^{e,+} = \tau K_t \iff \tau = q^* \frac{N_t^{e,+}}{q^* K_t} = q^* \eta^*$$

- Computing $j^N$:

$$dN_t^e = j_t^N N_t^{e,-} = N_t^{e,+} - N_t^{e,-} = \tau K_t - \eta_t q_t K_t$$

$$j_t^N = \frac{q^* \eta^* K_t - q_t \eta_t K_t}{q_t \eta_t K_t} = \frac{q^* \eta^* - q_t \eta_t}{q_t \eta_t}$$
1. Endogenous Jumps

- Log utility $\implies$ add jumps ex-post
- Crucial assumption: jumps always wipe experts out

\[
\frac{d\eta_t}{\eta_t} = \mu_t^\eta dt + \sigma_t^\eta dZ_t + j_t^\eta dJ_t
\]

\[
\mu_t^\eta = (1 - \eta_t) \left[ \hat{\zeta}_t^e (\sigma_t^\eta - \sigma - \sigma_t^q) - \hat{\zeta}_t^h (\sigma_t^h - \sigma - \sigma_t^q) - \left( \frac{C_t^e}{N_t^e} - \frac{C_t^h}{N_t^h} \right) \right]
+ \lambda_t \left( \hat{\nu}_t^e \left( j_t^\eta - \frac{j_t^D - j_t^N}{1 + j_t^N} \right) - \hat{\nu}_t^h \left( j_t^h - \frac{j_t^D - j_t^N}{1 + j_t^N} \right) \right) - \lambda_t j_t^\eta
\]

\[
\sigma_t^\eta = \frac{\chi_t^i - \eta_t^i}{\eta_t^i} (\sigma + \sigma_t^q)
\]
1. Endogenous Jumps

- Stationary distribution: $\tilde{M}'g = 0$, $\tilde{M} = M + \Lambda$
- The usual matrix $M$ from drift and volatility
- An additional matrix $\Lambda$ that reflects jumps:
  - Denote vulnerability region by $V_r = \{\eta \mid -j^q(\eta)\chi(\eta) \geq \eta\}$
  - $\Lambda_{i,i} = -\lambda$ if $\eta_i \in V_r$
  - $\Lambda_{i,m} = \lambda$ if $\eta_i \in V_r$ and $\eta_m = \eta^*$
1. Endogenous Jumps
1. Endogenous Jumps

\[ \mu_\eta \]

\( \lambda = 0.0 \)
\( \lambda = 0.05 \)
\( \lambda = 0.1 \)

Stationary distribution

\[ 0 \quad 0.2 \quad 0.4 \quad 0.6 \quad 0.8 \quad 1 \]
\[ 0 \quad 1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6 \quad 7 \quad 8 \]
1. Endogenous Jumps

\[ \mu_\eta \]

\( \eta^* = 0.003 \)
\( \eta^* = 0.005 \)
\( \eta^* = 0.007 \)

Stationary distribution
2. Money Model with Stochastic Volatility

- One sector + time-varying idiosyncratic risk

\[
\frac{dk^i_t}{k^i_t} = \left( \Phi(\tilde{\ell}^i_t) - \delta \right) dt + \tilde{\sigma}_t d\tilde{Z}^i_t
\]

\[
d\tilde{\sigma}_t = b(\tilde{\sigma}^{ss} - \tilde{\sigma}_t) dt + \nu \sqrt{\tilde{\sigma}_t} dZ_t
\]

- No policy ($\mu^B = i = \sigma^B = g = \tau = 0$)
2. Money Model with Stochastic Volatility

- Optimal investment: $q^K = 1 + \phi \iota$

- Goods market clearing: $\rho (q^K + q^B) = a - \iota$

- Bond share in wealth: $\vartheta = \frac{q^B}{q^K + q^B}$

\[
q^B_t = q^B(\vartheta_t) = \vartheta_t \frac{1 + \phi a}{(1 - \vartheta_t) + \phi \rho}
\]

\[
q^K_t = q^K(\vartheta_t) = (1 - \vartheta_t) \frac{1 + \phi a}{(1 - \vartheta_t) + \phi \rho}
\]

\[
\iota_t = \iota(\vartheta_t) = \frac{(1 - \vartheta_t) a - \rho}{(1 - \vartheta_t) + \phi \rho}
\]
2. Money Model with Stochastic Volatility

- Change of numeraire + martingale method:
  - Asset A: capital
  - Asset B: bonds

1. Express returns on A and B in $N_t$-numeraire
2. Apply martingale pricing formula
2. Money Model with Stochastic Volatility

• Returns on capital and bonds:

\[ dr_t^{k,i} = \frac{a - \iota_t}{q_t^K} dt + \frac{d \left( q_t^K k_t^{i} \right)}{q_t^K k_t^{i}} \]

\[ dr_t^B = i_t + \frac{d(1/P_t)}{1/P_t} \]

• SDF:

\[ \frac{d\xi_t^i}{\xi_t^i} = -r_t^f dt - \varsigma_t dZ_t - \tilde{\varsigma}_t d\tilde{Z}_t \]
2. Money Model with Stochastic Volatility

• Recall $\vartheta_t = \frac{q_t^B K_t}{(q_t^B + q_t^B) K_t} = \frac{q_t^B K_t}{N_t}$. In $N_t$-numeraire:

\[
d\hat{r}_t^{k,i} = \frac{a - \iota_t}{q_t^K} dt + \frac{d \left( q_t^K k_t^i / N_t \right)}{q_t^K k_t^i / N_t} = \frac{a - \iota_t}{q_t^K} dt + \frac{d \left( (1 - \vartheta_t) k_t^i / K_t \right)}{(1 - \vartheta_t) k_t^i / K_t} \\
= \left( \frac{a - \iota_t}{q_t^K} + \mu_t^{1-\vartheta} \right) dt + \tilde{\sigma}_t d\tilde{Z}_t = \left( \frac{\rho}{1 - \vartheta_t} - \mu_t^{\vartheta} \frac{\vartheta_t}{1 - \vartheta_t} \right) dt + \tilde{\sigma}_t d\tilde{Z}_t
\]
2. Money Model with Stochastic Volatility

• Recall \( \vartheta_t = \frac{q_t^B K_t}{(q_t^b + q_t^B) K_t} = \frac{q_t^B K_t}{N_t} \). In \( N_t \)-numeraire:

\[
\begin{align*}
\frac{d\hat{\rho}_t^B}{1/(P_t N_t)} &= \frac{d(\vartheta_t/B_t)}{\vartheta_t/B_t} = \mu_t^q dt + \sigma_t^q dZ_t \\
\frac{d\hat{\xi}_t}{\xi_t N_t} &= - (r^f_t - \mu_t^N) dt - (\varsigma_t - \sigma_t^N) dZ_t - \zeta_t d\tilde{Z}_t
\end{align*}
\]
2. Money Model with Stochastic Volatility

- Martingale formula:

$$\mathbb{E}[d\hat{r}_t^{k,i}] - \mathbb{E}[d\hat{r}_t^B] = \zeta_t (\sigma_t^{k,i} - \sigma_t^B) + \tilde{\zeta}_t (\tilde{\sigma}_t^{k,i} - \tilde{\sigma}_t^B)$$

$$\frac{\rho}{1 - \psi_t} - \mu_t \psi_t - \mu \Psi_t = (\tilde{\zeta}_t - \sigma_t^N)(-\sigma^\Psi_t) + \tilde{\zeta}_t \tilde{\sigma}_t$$

$$\mu_t = \rho - \tilde{\zeta}_t (1 - \psi_t) \tilde{\sigma}_t$$

$$\mu^\Psi_t = \rho - (1 - \psi_t)^2 \tilde{\sigma}_t^2$$
2. Money Model with Stochastic Volatility

- \( \vartheta_t = \vartheta(\tilde{\sigma}_t) \), Ito’s formula:

\[
d\vartheta_t = \left( \mu_{\tilde{\sigma},t} \vartheta'(\tilde{\sigma}) + \frac{\sigma^2_{\tilde{\sigma},t}}{2} \vartheta''(\tilde{\sigma}) \right) dt + \sigma_{\tilde{\sigma},t} \vartheta'(\tilde{\sigma}) dZ_t
\]

\[
\rho \vartheta(\tilde{\sigma}) = (1 - \vartheta(\tilde{\sigma}))^2 \tilde{\sigma}^2 \vartheta(\tilde{\sigma}) + b(\tilde{\sigma}^{ss} - \tilde{\sigma}) \vartheta'(\tilde{\sigma}) + \frac{\nu^2 \tilde{\sigma}}{2} \vartheta''(\tilde{\sigma})
\]

\[
\rho \vartheta_t(\tilde{\sigma}) = \partial_t \vartheta_t(\tilde{\sigma}) + (1 - \vartheta_t(\tilde{\sigma}))^2 \tilde{\sigma}^2 \vartheta_t(\tilde{\sigma}) + b(\tilde{\sigma}^{ss} - \tilde{\sigma}) \vartheta'_t(\tilde{\sigma}) + \frac{\nu^2 \tilde{\sigma}}{2} \vartheta''_t(\tilde{\sigma})
\]
2. Money Model with Stochastic Volatility

\[ d\tilde{\sigma}_t = b(\tilde{\sigma}^{ss} - \tilde{\sigma}_t)dt + \nu \sqrt{\tilde{\sigma}_t} dZ_t \]

- ‘Natural’ lower boundary: \( \tilde{\sigma} = 0 \)
- Upper boundary: ‘sufficiently’ large \( \tilde{\sigma} \)
2. Money Model with Stochastic Volatility

- LOM of $\tilde{\sigma}$ does not depend on model’s solution!
- Compute its distribution:
2. Money Model with Stochastic Volatility

\[ \theta \]

\[ q^K \]

\[ q^B \]

\[ r^f \]

\[ \zeta \]

\[ \bar{\zeta} \]
Stationary Distribution of Wealth (Shares)

- Consider the one-sector money model with constant idiosyncratic volatility \( \tilde{\sigma} \)
- Distribution of wealth share \( \eta_t = \frac{n_t}{N_t} \) is non-stationary
Stationary Distribution of Wealth (Shares)

- Consider the one-sector money model with constant idiosyncratic volatility $\tilde{\sigma}$
- Distribution of wealth share $\tilde{\eta}_t = \frac{n_t}{N_t}$ is non-stationary
Stationary Distribution of Wealth (Shares)

- Consider the one-sector money model with constant idiosyncratic volatility $\tilde{\sigma}$
- Distribution of wealth share $\eta_t = \frac{n_t}{N_t}$ is non-stationary
Stationary Distribution of Wealth (Shares)

- Consider the one-sector money model with constant idiosyncratic volatility $\tilde{\sigma}$
- Distribution of wealth share $\eta^i_t = \frac{n^i_t}{N_t}$ is non-stationary
Stationary Distribution of Wealth (Shares)

• Consider the one-sector money model with constant idiosyncratic volatility $\tilde{\sigma}$

• Distribution of wealth share $\eta_t = \frac{n_t^i}{N_t}$ is non-stationary

• Can make it stationary with an ad hoc fix:
  
  ▶ Idiosyncratic Poisson wealth shocks
  
  ▶ With intensity $\lambda$ wealth share $\eta^i$ is set to $\eta^*$
  
  ▶ Log-utility, same returns $\implies$ no effect on equilibrium

$$\frac{dn_t^i}{n_t^i} = \left( -\rho \underbrace{c_t/n_t^i}_{\left\{ \right.} + \underbrace{g}_{r_t^B=\Phi(\nu)-\delta} + (1 - \theta_t) \underbrace{\frac{a-\nu}{q_K}}_{E[r_t^K,\tilde{i}]-r_t^B} \right) dt + (1 - \theta_t)\tilde{\sigma} dZ_t^i + j_t^{n,i} dJ_t^i$$
Stationary Distribution of Wealth (Shares)

\[
\frac{dn_t^i}{n_t^i} = \left(-\rho + g + (1 - \vartheta_t) \frac{a - \ell}{q^K}\right) dt + \left(1 - \theta_t\right) \tilde{\sigma} dZ_t^i + j_{t}^{n,i} dJ_t^i
\]

\[
\frac{dN_t}{N_t} = g dt
\]

\[
\frac{d\eta_t^i}{\eta_t^i} = \left(-\rho + (1 - \vartheta_t) \frac{a - \ell}{q^K}\right) dt + \left(1 - \vartheta_t\right) \tilde{\sigma} dZ_t^i + j_{t}^{n,i} dJ_t^i
\]

\[= 0\]
Stationary Distribution of Wealth (Shares)

\[ \frac{d\eta_t^{\tilde{i}}}{\eta_t^{\tilde{i}}} = (1 - \vartheta)\sigma dZ_t^{\tilde{i}} + j_t^{\tilde{n},\tilde{i}} dJ_t^{\tilde{i}} \]

- Set \( j_t^{\tilde{n},\tilde{i}} = \frac{\eta^* - \eta_t^{\tilde{i}}}{\eta_t^{\tilde{i}}} \)
- KFE (for all \( \eta \neq \eta^* \)) is given by:
  \[ 0 = \frac{(1 - \vartheta)^2 \tilde{\sigma}^2}{2} \frac{\partial^2 g(\eta)}{\partial \eta^2} - \lambda g(\eta) \]
- There is a kink at \( \eta^* \)
Stationary Distribution of Wealth (Shares)

- KFE (for all $\eta \neq \eta^*$) is given by:

$$0 = g''(\eta)\eta^2 + 4g'(\eta)\eta + \left(2 - \frac{2\lambda}{(1 - \vartheta)^2\tilde{\sigma}^2}\right)g(\eta)$$

- Euler’s equation – has closed-form solutions

$$g(\eta) = C_1\eta^{\alpha_1} + C_2\eta^{\alpha_2} \text{ for } \eta < \eta^*$$

$$g(\eta) = C_3\eta^{\alpha_1} + C_4\eta^{\alpha_2} \text{ for } \eta \geq \eta^*$$

$$\int_0^\infty g(\eta)\,d\eta = 1, \quad \lim_{\eta \to 0} g(\eta) = \lim_{\eta \to \infty} g(\eta) = 0$$

- Continuity at $\eta^*$, $\alpha_1 = \frac{\alpha-3}{2}$, $\alpha_2 = -\frac{\alpha+3}{2}$, $\alpha = \sqrt{\frac{8\lambda}{(1 - \vartheta)^2\tilde{\sigma}^2} + 1}$
Stationary Distribution of Wealth (Shares)

- Solution under $\eta^* = 1$: $C_1 = C_4 = \frac{2\lambda}{(1-\vartheta)^2 \sigma^2 \alpha}$, $C_2 = C_3 = 0$
A (General) Model

\[
\max_{c, \theta} \mathbb{E} \left[ \int_0^\infty e^{-\rho t} u(c_t) dt \right]
\]

\[
dn_t = \left( -c_t + \gamma_t \right) dt + n_t \left( rdt + (1 - \gamma_t)(dr_t^k - rdt) \right)
\]

\[
dr_t^k = r^k dt + \tilde{\sigma}^k dZ_t^k
\]

\[
dy_t = -\nu y_t dt + \tilde{\sigma}^y dZ_t^y
\]
A (General) Model

\[
\max_{c,\theta} \mathbb{E} \left[ \int_0^\infty e^{-\rho t} u(c_t) \, dt \right]
\]

\[
dn_t = (-c_t + y_t) \, dt + n_t \left( r dt + (1 - \theta_t)(dr_t - rdt) \right)
\]

\[
dr_t = r^k dt + \tilde{\sigma}^k dZ_t
\]

\[
dy_t = -\nu y_t \, dt + \tilde{\sigma}^y dZ_t
\]

1. This class: \(\tilde{\sigma}^y = 0\)

2. Bewley-Huggett-Aiyagari model: \(\tilde{\sigma}^k = 0\)
Bewley-Huggett-Aiyagari Model

- Key difference: idiosyncratic risk is in endowment, not returns

\[
\max_c \mathbb{E} \left[ \int_0^\infty e^{-\rho t} u(c_t) dt \right]
\]

\[
dn_t = (-c_t + r\bar{n}_t + y_t) dt
\]

\[
dy_t = -\nu y_t dt + \tilde{\sigma} dZ_t
\]

- Risk does not scale with wealth \(\implies C/N\) (and portfolio) depends on wealth
Bewley-Huggett-Aiyagari Model

- Go to HJB directly

\[ \rho v(y, n) = \max_c \left[ u(c) + (-c + rn + y) \partial_n v(y, n) - vy \partial_y v(y, n) + \frac{\tilde{\sigma}^2}{2} \partial_{yy} v(y, n) \right] \]
Bewley-Huggett-Aiyagari Model

- Go to HJB directly

\[ \rho v(y, n) = \max_c \left[ u(c) + (-c + rn + y) \partial_n v(y, n) - \nu y \partial_y v(y, n) + \frac{\tilde{\sigma}^2}{2} \partial_{yy} v(y, n) \right] \]

- FOC: \( \partial_c u(c) = \partial_n v(y, n) \Rightarrow c(v, y, n) = (u')^{-1}(\partial_n v(y, n)) \)
Bewley-Huggett-Aiyagari Model

- Go to HJB directly

\[
\rho v(y, n) = \max_c \left[ u(c) + (-c + rn + y) \partial_n v(y, n) - \nu y \partial_y v(y, n) + \frac{\tilde{\sigma}^2}{2} \partial_{yy} v(y, n) \right]
\]

- FOC: \( \partial_c u(c) = \partial_n v(y, n) \implies c(v, y, n) = (u')^{-1}(\partial_n v(y, n)) \)

\[
\rho v(y, n) = u(v, y, n) + (-c(v, y, n) + rn + y) \partial_n v(y, n) - \nu y \partial_y v(y, n) + \frac{\tilde{\sigma}^2}{2} \partial_{yy} v(y, n)
\]

\[
\rho v = u(v) + M(v)v
\]