1 Endogenous Jumps

In this exercise you will add endogenous jumps to the Lecture 5 model with log utility. The main advantage of log utility is that model solution \((q, \sigma_q, \chi, \iota, \kappa)\) remains unchanged, it is only the law of motion for \(\eta\) that is affected by runs. Use the model parameters from Problem Set 2, set \(\lambda = 0.05\) and introduce transfers \(\tau\), such that whenever there is a run and experts’ net wealth goes to zero, they receive a transfer (from households) in the size of \(\tau K_t\). The transfers are such that after the run \(\eta_t = \eta^* = 0.005\). Proceed in the following steps:

1. Obtain solution for the log utility model (no changes needed at this point).
2. Given that after the run \(\eta\) jumps to \(\eta^*\), write down expressions for \(j^q(\eta), j^\eta^*(\eta)\) and \(j^\eta^+\).\(^1\)
3. Characterize the vulnerability region in terms of \(j^q(\eta), \chi\) and \(\eta\). We still consider a jump to \(\eta^*\) to be self-fulfilling if it destroys the net worth of experts.\(^2\)
4. Find the value of \(\tau\) that ensures \(\eta_t = \eta^*\) after the run, given that \(N^\tau_e = \tau K_t\) after the run.
5. Express \(j^N(\eta), j^{N^\tau}(\eta)\) and \(j^{N^\tau_0}(\eta)\) in terms of \(\eta, \eta^*\) and \(j^\eta(\eta)\).
6. Derive \(j^{\tau^D}(\eta)\). For that, use the balance sheet of experts \((D^\tau_e = q_t K^\tau_e - N^\tau_e - OE^\tau_e)\) to compute the loss that is absorbed by debt when the jump occurs. Remember that the shock destroys experts’ net worth and part of the shock loads onto outside equity.

\(^1\)Hint: whenever there is a jump, we can ignore the drift and volatility terms. E.g. for \(\frac{dx_t}{\Delta t} = \mu x_t \Delta t + \sigma x_t \Delta Z_t + j^\eta x_t \Delta J_t\), denote by \(x^+_t\) the value right after the jump and by \(x^-_t\) right before the jump. When the jump occurs and \(dJ_t = 1\), \(dx_t = x^+_t - x^-_t = j^\eta x^-_t\).

\(^2\)Destruction of experts’ net worth is immediately followed by transfers, resulting in positive \(\eta^*\), but that does not affect the determination of vulnerability region.
7. Don’t forget to set all the \( j - s \) to zero outside of the vulnerability region! Compute the (geometric) drift of \( \eta \):

\[
\mu_t^\eta = (1 - \eta_t) \left( (\zeta_t^s - \sigma_t^a)(\sigma_t^a + \sigma_t^b) - (\zeta_t^b - \sigma_t^a) \right) - \frac{\eta_t}{1 - \eta_t} \left( N_t^\varepsilon - N_t^\nu \right) + \lambda \left( \nu_t \left( j_t^N - j_t^D \right) - \nu_t \left( j_t^N - j_t^N \right) \right) - \lambda j_t^s
\]

where \( \nu_t^N = \nu_t^N + \nu_t^D N_t^\nu \), \( \nu_t^N = \frac{j_t^N}{1 + \zeta_t^s \sigma_t^b} \), \( \zeta_t^s = \sigma_t^N \), and \( \sigma_t^N \) is as before. Plot the (arithmetic) drift \( \eta_t^\mu \), explain the differences with the baseline model (\( \lambda = 0 \)).

8. Given that now the law of motion for \( \eta \) is as follows:

\[
d\eta_t = \mu_t^\eta dt + \sigma_t^\eta dZ_t + j_t^\eta dJ_t
\]

compute the stationary distribution. In addition to the usual \( M \) matrix, you need to construct a \( \Lambda \) matrix that will reflect the jumps, so that the relevant matrix for the KFE will be \( M + \Lambda \).

2 Money Model with Stochastic Volatility

Consider the model of Lecture 10 with log utility and without government policy \((\mu = i = \sigma = q = \tau = 0)^3\). In this problem, we add stochastic volatility to the model. Suppose idiosyncratic risk \( \tilde{\sigma}_t \) evolves according to the exogenous stochastic process

\[
d\tilde{\sigma}_t = b(\tilde{\sigma}^{ss} - \tilde{\sigma}_t)dt + \nu \sqrt{\tilde{\sigma}_t} dZ_t,
\]

where \( \tilde{\sigma}^{ss} \), \( b \) and \( \nu \) are positive constants. There are no aggregate capital shocks, i.e. \( \sigma = 0 \).

1. Use goods market clearing and optimal investment to express \( q^K, q^B \) and \( \iota \) in terms of \( \theta := \frac{\tilde{\sigma}^{ss}}{\tilde{\sigma}^{ss} + \eta} \)

2. Derive the “money valuation equation” using martingale method and change of numeraire:

   (a) First, write down:

   - Returns on capital \((dr_t^{N,S})\) and bonds \((dr_t^B)\), laws of motion for individual wealth \(n_t^i\) and aggregate wealth \(N_t = \int n_t^i\) (simply integrate over all agents), and return on individual wealth

   \[
dr_t^{N,i} = \frac{dn_t^i}{n_t^i} + \frac{\tilde{\sigma}^i_t}{n_t^i} dt,
\]

   all in consumption numeraire.

   - Laws of motion for SDF in consumption numeraire \((\xi_t^s)\) and \(N_t\)-numeraire \((\xi_t^N)\), you don’t need to plug in for \( \mu_t^N \) and \( \sigma_t^N \) at this point.

   (b) Express the return on bonds in \(N_t\)-numeraire in terms of \( \mu_t^B \) and \( \sigma_t^B \). Note that \(dr_t^B = \frac{d(q^B_tK_t/B_t^o)}{q^B_tK_t/B_t^o} \), which in \(N_t\)-numeraire becomes

   \[
dr_t^{N,i} = \frac{d(q^B_tK_t/(B_tN_t))}{q^B_tK_t/(B_tN_t)}.
\]

   (c) Write down the return on individual wealth in \(N_t\)-numeraire \(dr_t^{N,i}\)

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\( ^3\)There can still be a (constant) supply of bonds \(B_t \neq 0\).
(d) Use the martingale pricing condition:

\[
\frac{\mathbb{E}[dr_t^{i,n}]}{dt} - \frac{\mathbb{E}[dr_t^B]}{dt} = \zeta_t(\sigma_t^{i,n} - \hat{\sigma}_t^{n}) + \hat{\zeta}_t(\hat{\sigma}_t^{i,n} - \hat{\sigma}_t^B)
\]

and capital market clearing condition to derive an expression of the form \( \mu_t^Q = f(\vartheta_t, \hat{\sigma}_t) \), where function \( f \) only depends on model parameters (the “money valuation equation”)

3. Solve the model numerically:

- Apply Ito’s lemma to \( \vartheta_t = \vartheta(\hat{\sigma}_t) \), and equate the drift term with \( \vartheta_t \mu_t^Q \). This gives you an HJB-looking equation for \( \vartheta(\hat{\sigma}) \).
- Set \( a = 0.2, \phi = 1, \delta = 0.05, \rho = 0.01, \hat{\sigma}^{ss} = 0.2, b = 0.05, \nu = 0.02 \).
- Before solving for \( \vartheta(\hat{\sigma}) \), suggest a grid for \( \hat{\sigma} \) (you can work with its law of motion). Motivate your choice of the upper and the lower boundaries.
- Solve the model using value function iteration.
- Plot \( \vartheta, q^B, q^K, r^f, \zeta, \hat{\zeta} \) as functions of \( \hat{\sigma} \).²

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²Where \( \zeta_t \) and \( \hat{\zeta}_t \) are the prices of aggregate and individual risk in \( N_t \)-numeraire, respectively.

²To compute \( r^f \) you would be using Ito’s formula and the martingale pricing formula for \( dr^{k,i} \) or \( dr^B \).