Course Overview

Real Macro-Finance Models with Heterogeneous Agents
1. A Simple Real Macro-finance Model
2. Endogenous (Price of) Risk Dynamics
3. A Model with Jumps due to Sudden Stops/Runs

Money Models
1. A Simple Money Model
2. Cashless vs. Cash Economy and “The I Theory of Money”
3. Welfare Analysis & Optimal Policy
   1. Fiscal, Monetary, and Macroprudential Policy

International Macro-Finance Models
1. International Financial Architecture

Digital Money
Simple Two Sector Model: Basak Cuoco (1998)

- Expert sector

- Household sector

See Lecture Notes, Chapter 2 or Handbook of Macroeconomics 2017, Chapter 18
Financial Frictions and Distortions

- Belief distortions
  - Match “belief surveys”

- Incomplete markets
  - “natural” leverage constraint \((\text{BruSan})\)
  - Costly state verification \((\text{BGG})\)

- + Leverage constraints
  - (no “liquidity creation”)
  - Exogenous limit

- Collateral constraints
  - Next period’s price \((\text{KM})\)
    \[
    R_b t \leq q_{t+1} k_t
    \]
  - Next periods volatility
  - Current price \((\text{VaR, JG})\)

- Search Friction \((\text{DGP})\)
Two Sector Model Setup

Expert sector

▪ Output: $y_t^e = a k_t^e$
▪ Consumption rate: $c_t^e$
▪ Investment rate: $\ell_t^e$

\[
\frac{dk_t^{e,\tilde{i}}}{k_t^{e,\tilde{i}}} = (\Phi(\ell_t^{e,\tilde{i}}) - \delta)dt + \sigma dZ_t + d\Delta_t^{k,\tilde{i},e}
\]

agent $\tilde{i}$ of type $i$ (expert, HH)

Household sector

▪ Consumption rate: $c_t^h$
Two Sector Model Setup

Expert sector

- Output: \( y_t^e = a k_t^e \)
- Consumption rate: \( c_t^e \)
- Investment rate: \( i_t^e \)
  \[
  \frac{dk_t^{e,i}}{k_t^{e,i}} = \left( \Phi(i_t^{e,i}) - \delta \right) dt + \sigma dZ_t + d\Delta t^{k,i,e}_t
  \]

- \( E_0 \left[ \int_0^\infty e^{-\rho t (c_t^e)^{1-\gamma}} dt \right] \)

Household sector

- Consumption rate: \( c_t^h \)

- \( E_0 \left[ \int_0^\infty e^{-\rho t (c_t^h)^{1-\gamma}} dt \right] \)

Log-utility in Basak Cuoco 1998
Two Sector Model Setup

Expert sector
- Output: \( y^e_t = ak^e_t \)
- Consumption rate: \( c^e_t \)
- Investment rate: \( \iota^e_t \)

\[
\frac{dk^e_{t,i}}{k^e_{t,i}} = (\Phi(\iota^e_{t,i}) - \delta) dt + \sigma dZ_t + d\Delta_{t,k^e_{t,i}}
\]

\[
E_0[\int_0^\infty e^{-\rho t}(c^e_t)^{1-\gamma} dt]
\]

Friction: Can only issue
- Risk-free debt

Household sector
- Consumption rate: \( c^h_t \)

\[
E_0[\int_0^\infty e^{-\rho t}(c^h_t)^{1-\gamma} dt]
\]
The Big Picture

Contents:
- Allocation of physical assets
- Risk amplification
- Capital output
- Net worth distribution
- Debt accumulation
- Value function

Equations:
- $\kappa = 1$
- $\chi = 1$
- $A(\kappa)$
- $K$ growth
- $\Phi(i) - \delta$
- $\eta$
- $\zeta$

Arrows and labels:
- Consumption + investment
- Drift
- Volatility
- Precautionary
- Backward equation
- Forward equation

With expectations
Solving MacroModels Step-by-Step

0. Postulate aggregates, price processes & obtain return processes

1. For given $C/N$-ratio and SDF processes for each $i$  
   a. Real investment $\iota$ + Goods market clearing (static)  
      Toolbox 1: Martingale Approach, HJB vs. Stochastic Maximum Principle Approach
   b. Portfolio choice $\theta$ + Asset market clearing  
      Asset allocation $\kappa$ & risk allocation $\chi$  
      Toolbox 2: “price-taking social planner approach” Fisher separation theorem
   c. “Money evaluation equation” $\theta$
      Toolbox 3: Change in numeraire to total wealth (including SDF)

2. Evolution of state variable $\eta$ (and $K$)

3. Value functions
   a. Value fcn. as fcn. of individual investment opportunities $\omega$
      Special cases: log-utility investment opportunities
   b. Separating value fcn. $V^i(n^i; \eta, K)$ into $v^i(\eta)u(K)(n^i/n^i)^{1-\gamma}$
   c. Derive $\check{\rho} = C/N$-ratio and $\zeta, \xi$ prices of risks

4. Numerical model solution
   a. Transform BSDE for separated value fcn. $v^i(\eta)$ into PDE
   b. Solve PDE via value function iteration

5. KFE: Stationary distribution, Fan charts
0. Postulate Aggregates and Processes

- **Individual capital evolution:**
  \[ \frac{dk_{i,i}^t}{k_{i,i}^t} = (\Phi(l_{i,i}) - \delta)dt + \sigma dZ_t + d\Delta_{i,i}^t \]
  - Where \( \Delta_{i,i}^t \) is the individual cumulative capital purchase process

- **Capital aggregation:**
  - Within sector \( i \):
    \[ K_t^i \equiv \int k_{i,i}^t d\tilde{i} \]
  - Across sectors:
    \[ K_t \equiv \sum_i K_t^i \]
  - Capital share:
    \[ k_t^i \equiv \frac{K_t^i}{K_t} \]
    \[ \frac{dK_t}{K_t} = (\Phi(l_t^i) - \delta)dt + \sigma dZ_t \]

- **Net worth aggregation:**
  - Within sector \( i \):
    \[ N_t^i \equiv \int n_{i,i}^t d\tilde{i} \]
  - Across sectors:
    \[ N_t \equiv \sum_i N_t^i \]
  - Wealth share:
    \[ \eta_t^i \equiv \frac{N_t^i}{N_t} \]

- **Value of capital stock:**
  \[ q_t K_t \]
  - Postulate
    \[ dq_t/q_t = \mu_t^q dt + \sigma_t^q dZ_t \]
  - Postulated SDF-process:
    \[ \frac{d\xi_t}{\xi_t} = \mu_t^{\xi} dt + \sigma_t^{\xi} dZ_t \equiv -r_t \]
    \[ \equiv -\xi_t \]
      \[ (c \text{ is numeraire}) \]
0. Postulate Aggregates and Processes

- Individual capital evolution:
  \[
  \frac{dk_{t}^{i,i}}{k_{t}^{i,i}} = (\Phi(i_{t}^{i}) - \delta)d\tau + \sigma dZ_t + d\Delta_{t}^{k,i,i}
  \]
  Where \(\Delta_{t}^{k,i,i}\) is the individual cumulative capital purchase process

- Capital aggregation:
  - Within sector \(i\): \(K_{t}^{i} \equiv \int k_{t}^{i,i} d\tau\)
  - Across sectors: \(K_{t} \equiv \sum K_{t}^{i}\)
  - Capital share: \(k_{t}^{i} \equiv K_{t}^{i}/K_{t}\)

- Net worth aggregation:
  - Within sector \(i\): \(N_{t}^{i} \equiv \int n_{t}^{i,i} d\tau\)
  - Across sectors: \(N_{t} \equiv \sum N_{t}^{i}\)
  - Wealth share: \(\eta_{t}^{i} \equiv N_{t}^{i}/N_{t}\)

- Value of capital stock: \(q_{t}K_{t}\)
  Postulate 
  \[
  dq_{t}/q_{t} = \mu_{t}^{q} d\tau + \sigma_{t}^{q} dZ_t
  \]
  Postulated SDF-process: 
  \[
  \frac{d\xi_{t}}{\xi_{t}} = \mu_{t}^{\xi} d\tau + \sigma_{t}^{\xi} dZ_t
  \]
  \(\equiv -r_{t} \equiv -\xi_{t}\) (\(c\) is numeraire)
0. Postulate Aggregates and Processes

- ... from price processes to return processes (using Ito)
- Use Ito product rule to obtain capital gain rate (in absence of purchases/sales)
  - Define \( \tilde{k}_t \): \( \frac{d\tilde{k}_t}{\tilde{k}_t} = \left( \Phi(l_t^i) - \delta \right) dt + \sigma dZ_t + d\Delta_{k_t} \) without purchases/sales

\[
\begin{align*}
    dr_t^k(l_t^i) &= \left( \frac{a^i - l_t^i}{q} + \Phi(l_t^i) - \delta + \mu_t + \sigma \sigma_t^q \right) dt \\
    &\quad + (\sigma + \sigma_t^q) dZ_t
\end{align*}
\]

For aggregate capital return, Replace \( a^i \) with \( A(\kappa) \)

- Postulate SDF-process: (Example: \( \xi_t^i = e^{-\rho t} V'(n_t^i) \).
  \[
  \frac{d\xi_t^i}{\xi_t^i} = -r_t^i dt - \zeta_t^i dZ_t
  \]

Recall discrete time \( e^{-r^F} = E[SDF] \)
0. Postulate Aggregates and Processes

- ... from price processes to return processes (using Ito)
- Use Ito product rule to obtain capital gain rate (in absence of purchases/sales)
  
  - Define $\tilde{\kappa}_t^i$: $\frac{d\tilde{\kappa}_t^i}{\tilde{\kappa}_t^i} = (\Phi(i_t^i) - \delta)dt + \sigma dZ_t + d\Delta_{\kappa}^i$ without purchases/sales

  Dividend yield: $d\kappa_t^i = \frac{\alpha^i - l_t^i}{q}dt + \Phi(i_t^i) - \delta + \mu^q_t + \sigma \sigma^q_t)dt$
  E[Capital gain rate] = $\frac{d(q_t \kappa_t^i)}{(q_t \kappa_t^i)}$

  For aggregate capital return, replace $\alpha^i$ with $A(\kappa)$

- Postulate SDF-process: (Example: $\xi_t^i = e^{-\rho t}V'(n_t^i)$.)

  $d\xi_t^i = -r_t^i dt - \varsigma_t^i dZ_t$

  Poll 14: Why does drift of SDF equal risk-free rate
  
  a) no idio risk
  b) $e^{-r^F} = E[SDF]*1$
  c) no jump in consumption

Recall discrete time $e^{-r^F} = E[SDF]$
The Big Picture

Allocation of physical assets

Output $A(\kappa)$

Consumption + investment

Net worth distribution $\eta$

Capital growth $K$ $= \Phi(i) - \delta$

Price of risk $\zeta$

Precautionary

Drift

Volatility

Backward equation

Forward equation

Drift

Volatility
Solving MacroModels Step-by-Step

0. Postulate aggregates, price processes & obtain return processes

1. For given $C/N$-ratio and SDF processes for each $i$ finance block
   a. Real investment $\iota$ + Goods market clearing (static)
      ▪ Toolbox 1: Martingale Approach, HJB vs. Stochastic Maximum Principle Approach
   b. Portfolio choice $\theta$ + Asset market clearing or Asset allocation $\kappa$ & risk allocation $\chi$
      ▪ Toolbox 2: “price-taking social planner approach” – Fisher separation theorem
   c. “Money evaluation equation” $\theta$
      ▪ Toolbox 3: Change in numeraire to total wealth (including SDF)

2. Evolution of state variable $\eta$ (and $K$) forward equation

3. Value functions backward equation
   a. Value fcn. as fcn. of individual investment opportunities $\omega$
      ▪ Special cases: log-utility investment opportunities
   b. Separating value fcn. $V^i(n^i; \eta, K)$ into $v^i(\eta)u(K)(n^i/n^i)^{1-\gamma}$
   c. Derive $C/N$-ratio and $\zeta$ price of risk

4. Numerical model solution
   a. Transform BSDE for separated value fcn. $v^i(\eta)$ into PDE
   b. Solve PDE via value function iteration

5. KFE: Stationary distribution, Fan charts
1a. Individual Agent Choice of $\iota$, $\theta$, $c$

- Choice of $\iota$ is static problem (and separable) for each $t$
- \[ \max d r_t^k (i_t) \]
  \[ = \max_{i_t} \left( \frac{a^i - i_t}{q_t} + \Phi(i_t) - \delta + \mu^q + \sigma \sigma^q \right) \]

- FOC: $\frac{1}{q_t} = \Phi'(i_t)$ Tobin’s $q$
  - All agents $i_t = \iota \Rightarrow \frac{dK_t}{K_t} = (\Phi(\iota_t) - \delta) \, dt + \sigma \, dZ_t$
  - Special functional form:
    - $\Phi(\iota) = \frac{1}{\phi} \log(\phi \iota + 1) \Rightarrow \phi \iota = q - 1$
  - Goods market clearing: $(a^e - \iota_t)K_t = C^e_t + C^h_t$. For aggregate capital return,
    Replace $a^i$ with $A(\kappa)$

\[
\max_{\{\nu_t, \theta_t, c_t\}_{t=0}^\infty} E \left[ \int_0^\infty e^{-\rho t} u(c_t) dt \right]
\]

s.t.
\[
\frac{dn_t}{nt} = -\frac{c_t}{nt} dt + \sum_j \theta^j_t dr^j_t + \text{labor income/endow/taxes}
\]
\(n_0\) given

- Portfolio Choice: Martingale Approach
  - Let \(x^A_t\) be the value of a “self-financing trading strategy” (reinvest dividends)
  - \(\xi_t x^A_t\) follows a Martingale, i.e. drift = 0.
  - Let \(\frac{dx^A_t}{x^A_t} = \mu^A_t dt + \sigma^A_t dZ_t\),
  - Recall \(\frac{d\xi^i_t x^A_t}{\xi^i_t} = -r^i_t dt - \xi^i_t dZ_t\)
  - By Ito product rule
    \[
    \frac{d(\xi^i_t x^A_t)}{\xi^i_t x^A_t} = \left( -r^i_t + \mu^A_t - \xi^i_t \sigma^A_t \right) dt + \text{volatility terms}
    \]

- Expected return: \(\mu^A_t = r^i_t + \xi^i_t \sigma^A_t\)
  - For risk-free asset, i.e. \(\sigma^A_t = 0\): \(r^f_t = r^i_t\)
  - Excess expected return to risky asset B: \(\mu^A_t - \mu^B_t = \xi^i_t (\sigma^A_t - \sigma^B_t)\)
1a. Optimal Portfolio Choice - *back to our model*

- Using $\mu_t^A - r_t^f = \zeta_t^i \sigma_t^A$ for capital return (instead of generic asset $A$) without equity issuance

$$\frac{a-t^e_t}{q_t} + \Phi(t^e_t) - \delta + \mu_t^q + \sigma \sigma_t^q - r_t^f = \zeta_t^e (\sigma + \sigma_t^q)$$

- Recall
  - $\theta_t$ portfolio share in risk-free bond (if negative = debt/short position)
  - $(1 - \theta_t)$ portfolio share in (physical) capital $k_t$

- Asset markets clearing:
  - Capital market
    $$1 - \theta_t^e = \frac{q_t K_t}{N_t^e} = 1/\eta_t$$
  - Debt/bond market (by Walras Law)
Solving MacroModels Step-by-Step

0. Postulate aggregates, price processes & obtain return processes

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   - Real investment $\text{Real} i$ + Goods market clearing (static)
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   - Portfolio choice $\text{Portfolio} \theta$ + Asset market clearing or Asset allocation $\kappa$ & risk allocation $\chi$
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   - “Money evaluation equation” $\vartheta$
     - Toolbox 3: Change in numeraire to total wealth (including SDF)

2. Evolution of state variable $\eta$ (and $K$) 
   - forward equation

3. Value functions
   - Value fcn. as fcn. of individual investment opportunities $\omega$
     - Special cases: log-utility investment opportunities
   - Separating value fcn. $V_i^{\text{tn}}(n_i^t; \eta, K)$ into $v_i(\eta)u(K)(n_i^n/n_i^t)^{1-\gamma}$
   - Derive $C/N$-ratio and $\zeta$ price of risk

4. Numerical model solution
   - Transform BSDE for separated value fcn. $v_i(\eta)$ into PDE
   - Solve PDE via value function iteration

5. KFE: Stationary distribution, Fan charts
2. GE: Markov States and Equilibria

- Equilibrium is a map

Histories of shocks \( \{Z_s, s \in [0, t]\} \) \( \rightarrow \) prices \( q_t, \zeta_t, \iota_t, \theta_t \)

- net worth distribution

\[
\eta_t^e = \frac{N_t^e}{q_tK_t} \in (0,1)
\] net worth share

- All agents maximize utility
  - Choose: portfolio, consumption, technology

- All markets clear
  - Consumption, capital, money, outside equity
Recall: Basics of Ito Calculus

- Geometric Ito Process: \( dX_t = \mu_t X_t dt + \sigma_t X_t dZ_t \)
- Ito’s Lemma:
  \[
  df(X_t) = f'(X_t)\mu_t X_t dt + \frac{1}{2} f''(X_t)(\sigma_t X_t)^2 dt + f'(X_t)\sigma_t X_t dZ_t
  \]
- Ito product rule:
  \[
  \frac{d(X_t Y_t)}{X_t Y_t} = (\mu_t^X + \mu_t^Y + \sigma_t^X \sigma_t^Y) dt + (\sigma_t^X + \sigma_t^Y) dZ_t
  \]
- Ito ratio/quotation rule:
  \[
  \frac{d(X_t/Y_t)}{X_t/Y_t} = (\mu_t^X - \mu_t^Y + \sigma_t^Y (\sigma_t^Y - \sigma_t^X)) dt + (\sigma_t^X - \sigma_t^Y) dZ_t
  \]
2. Law of Motion of Wealth Share $\eta_t$

- **Method 1:** Using Ito’s quotation rule $\eta_t = N^e_t / (q_t K_t)$

$$\frac{d N^e_t}{N^e_t} = \frac{d n^e_t}{n^e_t} = -\frac{c^e_t}{n^e_t} dt + r_t dt + (1 - \theta^e_t)[d r^K_t - r_t dt]$$

$$\frac{d N^e_t}{N^e_t} = -\rho dt + r_t dt + (1 - \theta^e_t) \left[ \left( \frac{a - \ell^e_t}{q_t} + \Phi(\ell^e_t) - \delta + \mu^q_t + \sigma \sigma^q_t - r_t \right) dt + (\sigma + \sigma^q_t) dZ_t \right]$$

$$\frac{dq_t K_t}{q_t K_t} = \left( \frac{\mu^q_t + \Phi(\ell^e_t) - \delta + \sigma \sigma^q_t}{q_t} \right) dt + (\sigma + \sigma^q_t) dZ_t$$

- Ito ratio rule:

$$\frac{d (X_t/Y_t)}{X_t/Y_t} = \left( \mu^X_t - \mu^Y_t + \sigma^Y_t (\sigma^Y_t - \sigma^X_t) \right) dt + (\sigma^X_t - \sigma^Y_t) dZ_t$$

$$\frac{d \eta_t}{\eta_t} = \left( \frac{a - \ell^e_t}{q_t} - \rho + \theta^e_t (\sigma + \sigma^q_t) - \sigma^e_t (\sigma + \sigma^q_t) \right) dt - \theta^e_t (\sigma + \sigma^q_t) dZ_t$$

- **Method 2:** Change of numeraire + Martingale (Lecture Notes)

Using portfolio choice equation
Solving MacroModels Step-by-Step

0. Postulate aggregates, price processes & obtain return processes

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2. Evolution of state variable $\eta$ (and $K$) 
   - Forward equation

3. Value functions
   a. Value fcn. as fcn. of individual investment opportunities $\omega$
   - Special cases: log-utility, constant investment opportunities
   b. Separating value Log-utility $\eta, K$ into $v^i(\eta) w(K)(n^i/n^i)^{1-\gamma}$
   c. Derive $C/N$-ratio and $\zeta$ price of risk

4. Numerical model solution
   a. Transform BSDE for separated value fcn. $v^i(\eta)$ into PDE
   b. Solve PDE via value function iteration

5. KFE: Stationary distribution, Fan charts
The Big Picture

- Allocation of physical assets
- Risk amplification
- Capital price of risk

Output $A(\kappa)$

Net worth distribution $\eta$

Precautionary

Drift

Volatility

Value function

Forward equation with expectations

Backward equation with expectations
3a. CRRA Value Function

- Martingale Approach: works best in endowment economy
- Here: mix Martingale approach with value function (envelop condition)

\[ V^i(n^i_t; \eta_t, K_t) \] for individuals \( i \)

- For CRRA/power utility \( u(c_t^i) = \frac{(c_t^i)^{1-\gamma} - 1}{1-\gamma} \)

\[ \Rightarrow \] increase net worth by factor, optimal \( c^i \) for all future states increases by this factor \( \Rightarrow \left( \frac{c_t^i}{n_t^i} \right) \)-ratio is invariant in \( n_t^i \)

\[ \Rightarrow \] value function can be written as

\[ V^i(n^i_t; \eta_t, K_t) = \frac{u(\omega_t^i(\eta_t, K_t)n_t^i)}{\rho^i} \]

- \( \omega_t^i \) Investment opportunity/ “net worth multiplier”
  - \( \omega_t^i(\eta_t, K_t) \)-function turns out to be independent of \( K_t \)
  - Change notation from \( \omega_t^i(\eta_t, K_t) \)-function to \( \omega_t^i \)-process
3a. CRRA Value Function: relate to $\omega$

- $\Rightarrow$ value function can be written as $u(\omega_t n_t)$, that is

$$
\frac{u(\omega_t n_t)}{\rho} = \frac{1}{\rho^i} \frac{(\omega_t n_t)^{1-\gamma} - 1}{1-\gamma} = \frac{1}{\rho^i} \frac{(\omega_t)^{1-\gamma} (n_t)^{1-\gamma} - 1}{1-\gamma}
$$

- $\frac{\partial V}{\partial n^i} = u'(c^i)$ by optimal consumption (if no corner solution)

$$
\frac{(\omega_t)^{1-\gamma} (n_t)^{-\gamma}}{\rho^i} = (c_t^i)^{-\gamma} \iff \frac{c_t^i}{n_t^i} = (\rho^i)^{1/\gamma} (\omega_t^{1-1/\gamma})
$$

Next step:

a) Special simple cases
b) replace $\omega_t$ with something scale invariant
### 3a. CRRA Value Function: Special Case \( \log -utility \)

\[
\frac{c_t^i}{n_t^i} = (\rho_i)^{1/\gamma}(\omega_t^i)^{1-1/\gamma}
\]

- Ito for volatility term: \( \sigma_t^c = \sigma_t^n + (1 - 1/\gamma)\sigma_t^\omega \)

- For \( \log \) utility \( \gamma = 1 \):
  \[
  \xi_t^i = e^{-\rho^t_i / c_t^i} = e^{-\rho^t_i / (\rho n_t^i)} \text{ for any } \omega_t^i \Rightarrow \sigma_t^n = \sigma_t^c = \zeta_t^i
  \]
  - Expected excess return: \( \mu_t^A - r_t^F = \sigma_t^n \sigma_t^A \)
  - Recall \( \frac{dn_t^i}{n_t^i} = -\frac{c_t^i}{n_t^i} dt + (1 - \theta^i)dr_t^K + \theta^i dr_t \)

- Consumption choice: \( c_t^i = \rho^i n_t^i \)
  - \( \omega_t \) does not matter \( \Rightarrow \) income and substitution effect cancel out

- Portfolio choice: myopic (no Mertonian hedging demand)
  - Volatility of investment of opportunity/net worth multiplier does not matter \( \Rightarrow \) Myopic price of risk \( \zeta_t^i = \sigma_t^n = \sigma_t^c \)
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   c. Derive $C/N$-ratio and $\zeta$ price of risk

4. Numerical model solution
   a. Transform BSDE for separated value fcn. $v^i(\eta)$ into PDE
   b. Solve PDE via value function iteration

5. KFE: Stationary distribution, Fan charts
Recall: Market Clearing

- Output good market

\[ C_t = (a - \iota_t^e)K_t \]
\[ \rho q_t K_t = (a - \iota_t^e(q_t))K_t \implies q_t = q, \forall t \]

\[ \rho q_t = (a - \iota_t^e(q_t)) \]

- Hence \( \iota_t^e = \iota^e \), \( \mu_t^q = \sigma_t^q = 0, \forall t \).

- Capital market

\[ 1 - \theta_t^e = \frac{q_t K_t}{N_t^e} = 1/\eta_t \]

- Debt/bond market (by Walras Law)
4b. Model Solution

- Using $\rho q_t = (a - \iota(q_t))$, $\phi t = q_t - 1$, for $\Phi(\iota) = \frac{1}{\phi} \log(\phi t + 1)$

$$q = \frac{1 + \phi a}{1 + \phi \rho}$$

- Using portfolio choice, goods & capital market clearing

$$r_t = \frac{a - \iota^e}{q_t} + \Phi(\iota^e) - \delta + \mu^q_t + \sigma^q_t - \zeta_t (\sigma + \sigma^q_t)$$

$$= \rho + \Phi(\iota^e) - \delta - (1 - \theta_t)\sigma^2$$

$$= \rho + \Phi(\iota^e) - \delta - \frac{\sigma^2}{\eta_t} \quad \text{from capital market clearing}$$

$$r_t = \rho + \frac{1}{\phi} \log\left(\frac{1 + \phi a}{1 + \phi \rho}\right) - \delta - \frac{\sigma^2}{\eta_t} \quad \text{risk-free rate}$$

- Goods & capital market clearing and $\eta$-evolution

$$\frac{d\eta_t}{\eta_t} = \frac{(1 - \eta_t)^2}{\eta_t^2} \sigma^2 dt + \frac{1 - \eta_t}{\eta_t} \sigma dZ_t$$
Numerical example

\[ a = 0.11, \rho = 5\%, \sigma = 0.1, \Phi(\iota) = \frac{\log(\phi \iota + 1)}{\phi}, \phi = 10 \]
Observation of Basak-Cuoco Model

- $\eta_t$ fluctuates with macro shocks, since experts are levered
- Price of risk, i.e. Sharpe ratio, is
  \[ \frac{\sigma}{\eta_t} = \frac{\rho + \Phi(i) - \delta - r_t}{\sigma} \]
  - Goes to $\infty$ as $\eta_t$ goes to zero
  - Achieved via risk-free rate
    \[ r_t = \rho + \Phi(i) - \delta - \sigma^2/\eta_t \rightarrow -\infty \]
  - Rather than depressing price of risky asset, $q_t = q \forall t$
- No endogenous risk $\sigma^q = 0$
  - No amplification
  - No volatility effects
- $\mu_t^\eta = \frac{(1-\eta_t)^2}{\eta_t^2} \sigma^2 > 0 \Rightarrow$ in the long run HH-net worth share vanishes
- Way out:
  - Different discount rates $\rho$ (KM)
  - Switching types (BGG)
  - 2 types of experts (BruSan)
Desired Model Properties

- Normal regime: stable around steady state
  - Experts are adequately capitalized
  - Experts can absorb macro shock

- Endogenous risk and price of risk
  - Fire-sales, liquidity spirals, fat tails
  - Spillovers across assets and agents
  - Market and funding liquidity connection
  - SDF vs. cash-flow news

- Volatility paradox

- Financial innovation less stable economy

("Net worth trap” double-humped stationary distribution)