Modern Macro, Money, and International Finance

Eco529

Lecture 04: A Simple Real Macro Model with Heterogenous Agents

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Course Overview

Real Macro-Finance Models with Heterogeneous Agents

1. A Simple Real Macro-finance Model
2. Endogenous (Price of) Risk Dynamics
3. A Model with Jumps due to Sudden Stops/Runs

Money Models

1. A Simple Money Model
2. Cashless vs. Cash Economy and "The I Theory of Money"
3. Welfare Analysis & Optimal Policy
    1. Fiscal, Monetary, and Macroprudential Policy

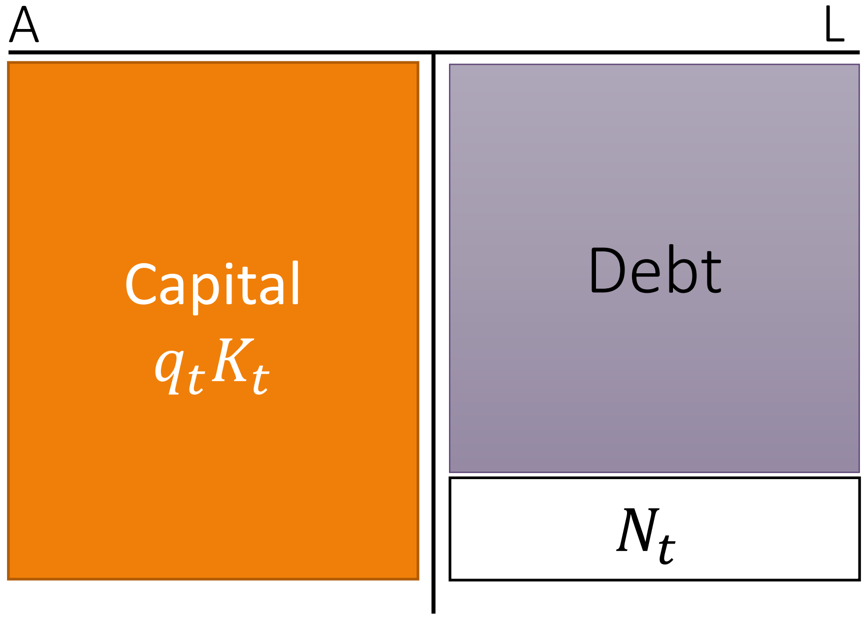
International Macro-Finance Models

1. International Financial Architecture

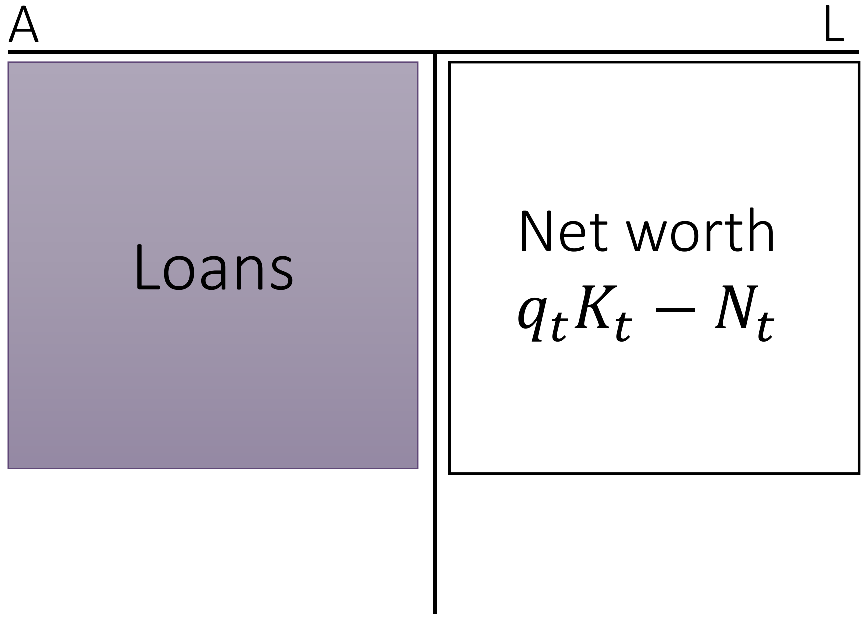
Digital Money

Simple Two Sector Model: Basak Cuoco (1998)

- Expert sector



Household sector



See Lecture Notes, Chapter 2 or
Handbook of Macroeconomics 2017, Chapter 18

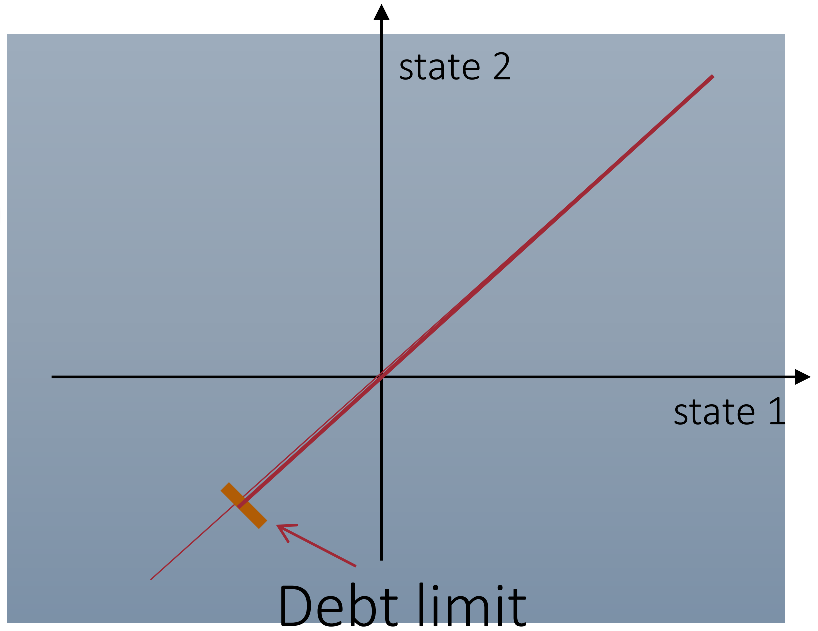
Financial Frictions and Distortions

- Belief distortions
  - Match “belief surveys”

- Incomplete markets
  - “natural” leverage constraint *(BruSan)*
  - Costly state verification *(BGG)*

- + Leverage constraints
  (no “liquidity creation”)
  - Exogenous limit *(Bewley/Ayagari)*
  - Collateral constraints
    - Next period’s price *(KM)*
      $$Rb_t \leq q_{t+1}k_t$$
    - Next periods volatility *(VaR, JG)*
    - Current price

- Search Friction *(DGP)*



Debt limit
can depend on
prices/volatility

Two Sector Model Setup

Expert sector

- Output: $y_t^e = ak_t^e$
- Consumption rate: c_t^e
- Investment rate: ι_t^e

$$\frac{dk_t^{e,\tilde{\imath}}}{k_t^{e,\tilde{\imath}}} = \left(\Phi\left(\iota_t^{e,\tilde{\imath}}\right) - \delta\right)dt + \sigma dZ_t + d\Delta_t^{k,\tilde{\imath},e}$$

agent $\tilde{\imath}$ of type i (expert, HH)

Household sector

- Consumption rate: c_t^h

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- $E_0[\int_0^\infty e^{-\rho t}\frac{(c_t^e)^{1-\gamma}}{1-\gamma}dt]$

Household sector

- Consumption rate: c_t^h

- $E_0[\int_0^\infty e^{-\rho t}\frac{(c_t^h)^{1-\gamma}}{1-\gamma}dt]$

Log-utility in Basak Cuoco 1998

Two Sector Model Setup

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- $E_0[\int_0^\infty e^{-\rho t} \frac{(c_t^e)^{1-\gamma}}{1-\gamma} dt]$

Friction: Can only issue

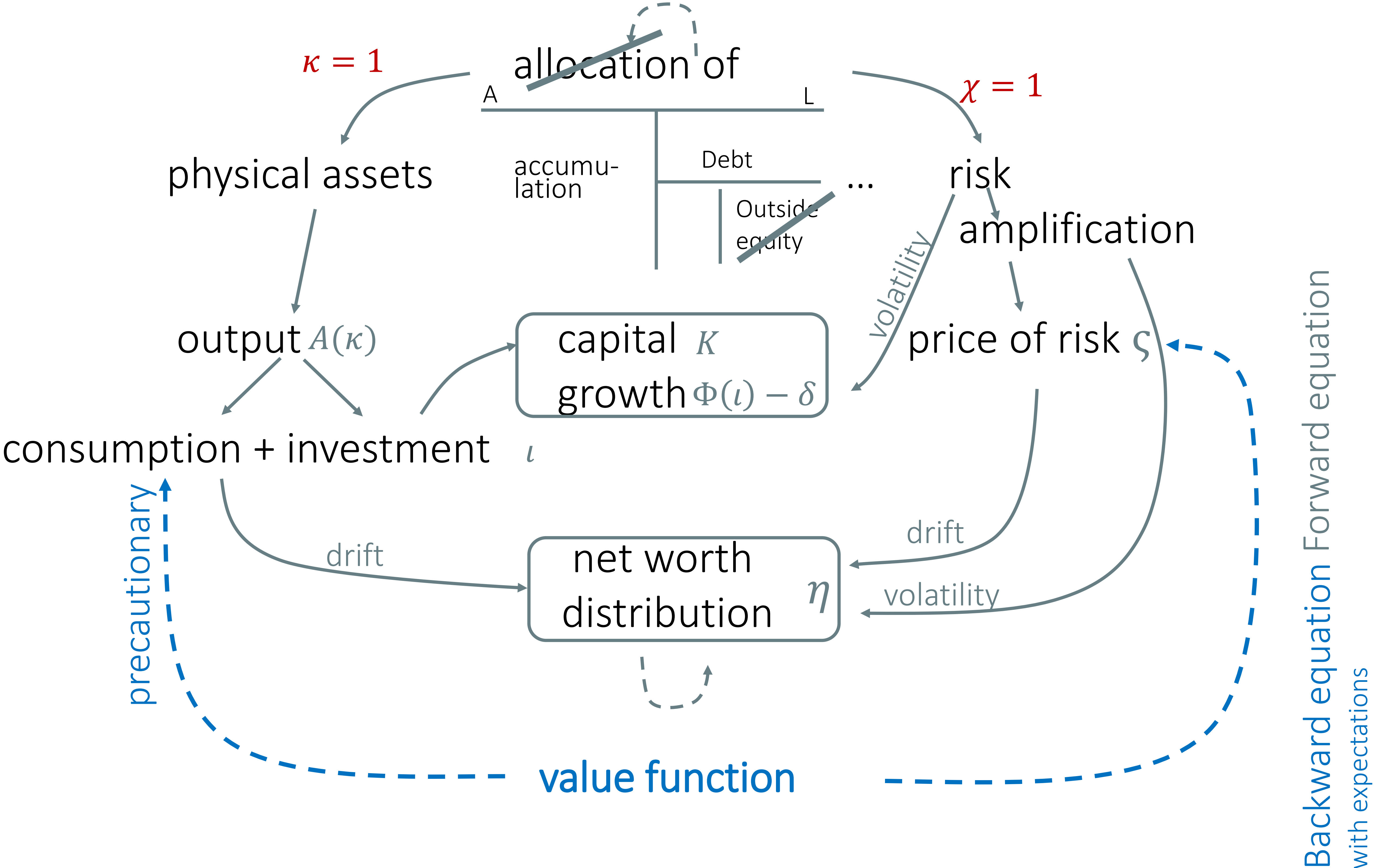
- Risk-free debt

Household sector

- Consumption rate: c_t^h

- $E_0[\int_0^\infty e^{-\rho t} \frac{(c_t^h)^{1-\gamma}}{1-\gamma} dt]$

The Big Picture



$\kappa = 1$

~~allocation~~ of

A

L

accumulation

Debt

~~Outside equity~~

…

$\chi = 1$

physical assets

risk

amplification

volatility

output $A(\kappa)$

capital K

growth $\Phi(\iota) - \delta$

price of risk ς

consumption + investment ι

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net worth distribution η

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value function

Forward equation

Backward equation

with expectations

Solving MacroModels Step-by-Step

0. Postulate aggregates, price processes & obtain return processes
1. For given C/N-ratio and SDF processes for each i *finance block*
   a. Real investment ι + Goods market clearing *(static)*
   - *Toolbox 1:* Martingale Approach, ~~HJB vs. Stochastic Maximum Principle Approach~~

   b. Portfolio choice θ + Asset market clearing
   ~~Asset allocation κ & risk allocation χ~~
   - ~~*Toolbox 2:* “price-taking social planner approach” – Fisher separation theorem~~

   c. ~~“Money evaluation equation” ϑ~~
   - ~~*Toolbox 3:* Change in numeraire to total wealth (including SDF)~~
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3. Value functions *backward equation*
   a. Value fcn. as fcn. of individual investment opportunities ω
   - *Special cases: log-u* Log-utility *nt investment ~~opportunities~~*

   ~~b. Separating value fcn. $V^i(n^{\tilde{i}};\eta,K)$ into $v^i(\eta)u(K)(n^{\tilde{i}}/n^i)^{1-\gamma}$~~
   c. Derive $\check{\rho} = C/N$-ratio and $\varsigma, \tilde{\varsigma}$ prices of risks
4. Numerical model solution
   a. Transform BSDE ~~for separated value fcn. $v^{\tilde{i}}(\eta)$~~ into PDE
   ~~b. Solve PDE via value function iteration~~
5. KFE: Stationary distribution, Fan charts

0. Postulate Aggregates and Processes

- Individual capital evolution:

$$\frac{dk_t^{\tilde{\imath},i}}{k_t^{\tilde{\imath},i}} = \left(\Phi\left(\iota^{\tilde{\imath},i}\right) - \delta\right)dt + \sigma dZ_t + d\Delta_t^{k,\tilde{\imath},i}$$

  - Where $\Delta_t^{k,\tilde{\imath},i}$ is the individual cumulative capital purchase process
- Capital aggregation:
  - Within sector i: $K_t^i \equiv \int k_t^{\tilde{\imath},i} d\tilde{\imath}$
  - Across sectors: $K_t \equiv \sum_i K_t^i$
  - Capital share: $\kappa_t^i \equiv K_t^i / K_t$

$$\frac{dK_t}{K_t} = \left(\Phi\left(\iota_t^i\right) - \delta\right)dt + \sigma dZ_t$$

- Net worth aggregation:
  - Within sector i: $N_t^i \equiv \int n_t^{\tilde{\imath},i} d\tilde{\imath}$
  - Across sectors: $N_t \equiv \sum_i N_t^i$
  - Wealth share: $\eta_t^i \equiv N_t^i / N_t$
- Value of capital stock: $q_t K_t$

  Postulate $dq_t / q_t = \mu_t^q dt + \sigma_t^q dZ_t$

Same Brownian

- Postulated SDF-process: $\frac{d\xi_t^i}{\xi_t^i} = \underbrace{\mu_t^{\xi}}_{\equiv -r_t} + \underbrace{\sigma_t^{\xi^i}}_{\equiv -\varsigma_t^i} dZ_t$ (c is numeraire)

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0. Postulate Aggregates and Processes

- … from price processes to return processes (using Ito)
  - Use Ito product rule to obtain capital gain rate (in absence of purchases/sales)
    - Define $\check{k}_t^{\tilde{\imath}}$: $\frac{d\check{k}_t^{\tilde{\imath},i}}{\check{k}_t^{\tilde{\imath},i}} = \left(\Phi\left(\iota_t^{\tilde{\imath},i}\right) - \delta\right)dt + \sigma dZ_t + \cancel{d\Delta_t^{k,\tilde{\imath},i}}$ without purchases/sales

$$dr_t^k\left(\iota_t^{\tilde{\imath},i}\right) = \left(\overbrace{\frac{a^i - \iota_t^i}{q}}^{\text{Dividend yield}} + \overbrace{\Phi\left(\iota_t^i\right) - \delta + \mu_t^q + \sigma\sigma_t^q}^{\text{E[Capital gain rate]}= \frac{d(q_t \check{k}^i{}_t)}{(q_t \check{k}_t^i)}}\right)dt + \left(\sigma + \sigma_t^q\right)dZ_t$$

For aggregate capital return,
Replace a^i with $A(\kappa)$

- Postulate SDF-process: (Example: $\xi_t^i = e^{-\rho t}V'(n_t^i)$.)

$$\frac{d\xi_t^i}{\xi_t^i} = -r_t^i dt - \underset{\substack{\uparrow \\ \text{Price of risk}}}{\varsigma_t^i} dZ_t$$

Recall discrete time $e^{-r^F} = E[SDF]$

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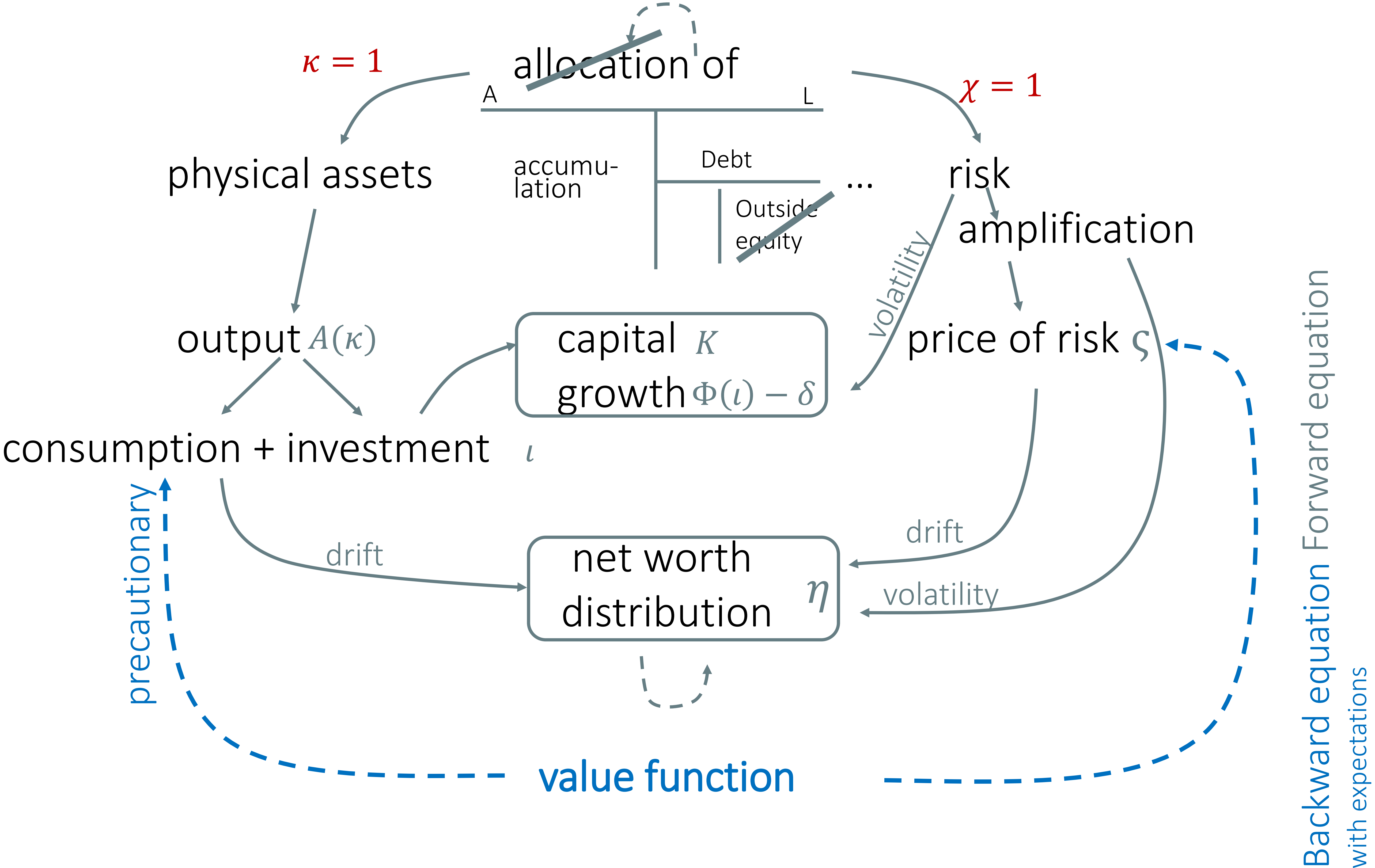
Poll 14: Why does drift of SDF equal risk-free rate

a) no idio risk

b) $e^{-r^F} = E[SDF]*1$

c) no jump in consumption

The Big Picture



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1a. Individual Agent Choice of ι, θ, c

- Choice of ι is static problem (and separable) for each t
- $\max_{\iota_t^i} dr_t^k(\iota_t^i)$

$$= \max_{\iota_t^i} \left(\frac{a^i - \iota_t^i}{q_t} + \Phi(\iota_t^i) - \delta + \mu^q + \sigma\sigma^q \right)$$

For aggregate capital return,
Replace a^i with $A(\kappa)$

- FOC: $\frac{1}{q_t} = \Phi'(\iota_t^i)$ Tobin's q
  - All agents $\iota_t^i = \iota_t \Rightarrow \frac{dK_t}{K_t} = (\Phi(\iota_t) - \delta)\, dt + \sigma dZ_t$
  - Special functional form:
    - $\Phi(\iota) = \frac{1}{\phi}\log(\phi\iota + 1) \Rightarrow \phi\iota = q - 1$
- Goods market clearing: $(a^e - \iota_t)K_t = C_t^e + C_t^h$.

1a. Method 3: Martingale Approach – Cts. Time

$$\max_{\{\iota_t,\boldsymbol{\theta}_t,c_t\}_{t=0}^{\infty}} E\left[\int_0^{\infty} e^{-\rho t} u(c_t)dt\right]$$

s.t. $\frac{dn_t}{n_t} = -\frac{c_t}{n_t}dt + \sum_j \theta_t^j dr_t^j$ + labor income/endow/taxes

n_0 given

- Portfolio Choice: Martingale Approach
  - Let x_t^A be the value of a "self-financing trading strategy"(reinvest dividends)
- $\xi_t x_t^A$ follows a Martingale, i.e. drift $= 0$.
  - Let $\frac{dx_t^A}{x_t^A} = \mu_t^A dt + \sigma_t^A dZ_t,$
  - Recall $\frac{d\xi_t^i}{\xi_t^i} = -r_t^i dt - \varsigma_t^i dZ_t$
  - By Ito product rule

$$\frac{d(\xi_t^i x_t^A)}{\xi_t^i x_t^A} = \Big(\underbrace{-r_t^i + \mu_t^A - \varsigma_t^i \sigma_t^A}_{=0}\Big)dt \text{ +volatility terms}$$

  - Expected return: $\boxed{\mu_t^A = r_t^i + \varsigma_t^i \sigma_t^A}$
    - For risk-free asset, i.e. $\sigma_t^A = 0$: $r_t^f = r_t^i$
    - Excess expected return to risky asset B: $\mu_t^A - \mu_t^B = \varsigma_t^i(\sigma_t^A - \sigma_t^B)$

1a. Optimal Portfolio Choice - *back to our model*

- Using $\mu_t^A - r_t^f = \varsigma_t^i \sigma_t^A$ for capital return (instead of generic asset A)

without equity issuance

$$\frac{a-\iota_t^e}{q_t} + \Phi(\iota_t^e) - \delta + \mu_t^q + \sigma\sigma_t^q - r_t^f = \varsigma_t^e\left(\sigma + \sigma_t^q\right)$$

- Recall
  - θ_t portfolio share in risk-free bond (if negative = debt/short position)
  - $(1-\theta_t)$ portfolio share in (physical) capital k_t
- Asset markets clearing:
  - Capital market $$1-\theta_t^e = \underbrace{\frac{q_t K_t}{N_t^e}}_{=1/\eta_t}$$
  - Debt/bond market (by Walras Law)

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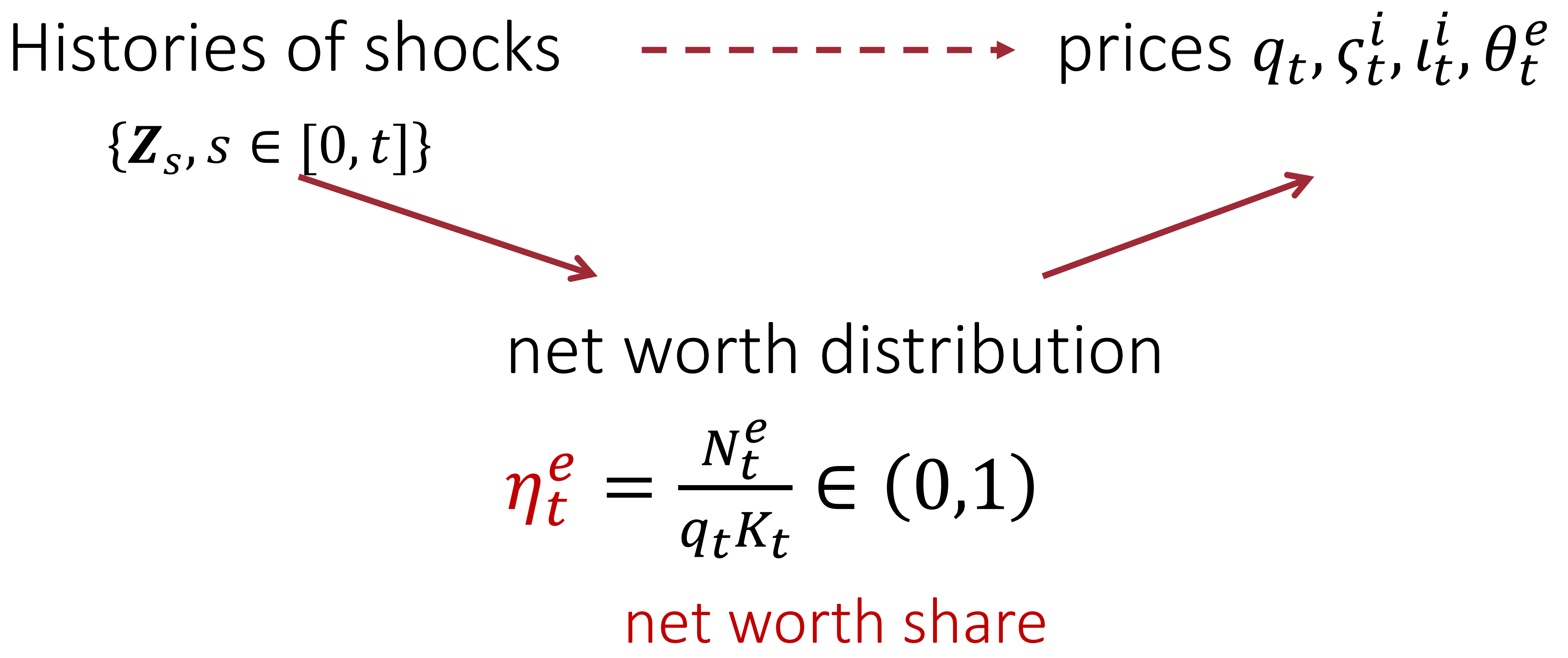
2. GE: Markov States and Equilibria

- Equilibrium is a map



Histories of shocks $\{\boldsymbol{Z}_s, s \in [0,t]\}$

prices $q_t, \varsigma_t^i, \iota_t^i, \theta_t^e$

net worth distribution

$$\eta_t^e = \frac{N_t^e}{q_t K_t} \in (0,1)$$

net worth share

- All agents maximize utility
  - Choose: portfolio, consumption, technology
- All markets clear
  - Consumption, capital, money, outside equity

Recall: Basics of Ito Calculus

- Geometric Ito Process $dX_t = \mu_t^X X_t dt + \sigma_t^X X_t dZ_t$
- Ito's Lemma:

$$df(X_t) = f'(X_t)\mu_t^X X_t dt + \frac{1}{2} f''(X_t)(\sigma_t^X X_t)^2 dt + f'(X_t)\sigma_t^X X_t dZ_t$$

- Ito product rule:

$$\frac{d(X_t Y_t)}{X_t Y_t} = (\mu_t^X + \mu_t^Y + \sigma_t^X \sigma_t^Y)dt + (\sigma_t^X + \sigma_t^Y)dZ_t$$

- Ito ratio/quotation rule:

$$\frac{d(X_t/Y_t)}{X_t/Y_t} = \left(\mu_t^X - \mu_t^Y + \sigma_t^Y(\sigma_t^Y - \sigma_t^X)\right)dt + (\sigma_t^X - \sigma_t^Y)dZ_t$$

2. Law of Motion of Wealth Share η_t

- **Method 1:** Using Ito's quotation rule $\eta_t = N_t^e/(q_t K_t)$
- $\frac{dN_t^e}{N_t^e} = \frac{dn_t^e}{n_t^e} = -\frac{c_t^e}{n_t^e}dt + r_t dt + (1-\theta_t^e)[dr_t^K - r_t dt]$

$$\frac{dN_t^e}{N_t^e} = -\rho dt + r_t dt + (1-\theta_t^e)\left[\underbrace{\left(\frac{a-\iota_t^e}{q_t} + \Phi(\iota_t^e) - \delta + \mu_t^q + \sigma\sigma_t^q - r_t\right)}_{=\varsigma_t^e(\sigma+\sigma^q)} dt + \left(\sigma + \sigma_t^q\right) dZ_t\right]$$

$$\frac{dq_t K_t}{q_t K_t} = \underbrace{\left(\mu_t^q + \Phi(\iota_t^e) - \delta + \sigma\sigma_t^q\right)}_{=r_t - \frac{a-\iota_t^e}{q_t} + \varsigma_t^e(\sigma+\sigma^q)} dt + \left(\sigma + \sigma_t^q\right) dZ_t$$

Using portfolio choice equation

- Ito ratio rule:

$$\frac{d(X_t/Y_t)}{X_t/Y_t} = \left(\mu_t^X - \mu_t^Y + \sigma_t^Y(\sigma_t^Y - \sigma_t^X)\right) dt + (\sigma_t^X - \sigma_t^Y) dZ_t$$

- $\frac{d\eta_t}{\eta_t} = \left(\frac{a-\iota_t^e}{q_t} - \rho + \theta_t^e\left((\sigma + \sigma_t^q) - \varsigma_t^e\right)\left(\sigma + \sigma_t^q\right)\right) dt - \underbrace{\theta_t^e}_{<0} \left(\sigma + \sigma_t^q\right) dZ_t$

- **Method 2:** Change of numeraire + Martingale (Lecture Notes)

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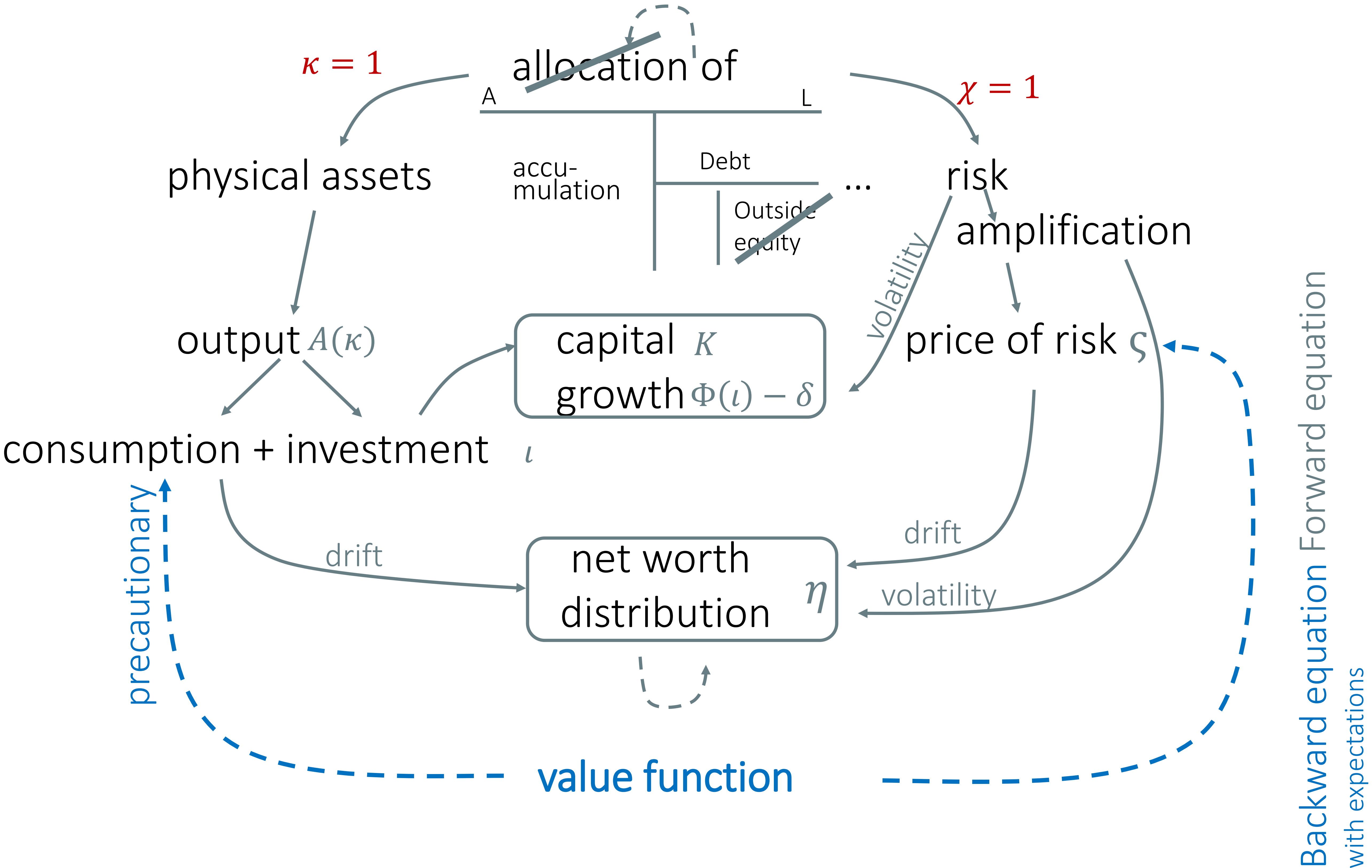

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3a. CRRA Value Function

Applies separately for each type of agent

- Martingale Approach: works best in endowment economy
- Here: mix Martingale approach with value function (envelop condition)

- $V^i(n_t^i; \boldsymbol{\eta}_t, K_t)$ for individuals i
- For CRRA/power utility $u\left(c_t^i\right) = \frac{(c_t^i)^{1-\gamma}-1}{1-\gamma}$

$\Rightarrow$ increase net worth by factor, optimal c^i for all future states increases by this factor $\Rightarrow \left(\frac{c_t^i}{n_t^i}\right)$-ratio is invariant in n_t^i

- $\Rightarrow$ value function can be written as $V^i\left(n_t^i; \boldsymbol{\eta}_t, K_t\right) = \frac{u(\omega^i(\boldsymbol{\eta}_t, K_t) n_t^i)}{\rho^i}$
- ω_t^i Investment opportunity/ "net worth multiplier"
  - $\omega^i(\boldsymbol{\eta}_t, K_t)$-function turns out to be independent of K_t
  - Change notation from $\omega^i(\boldsymbol{\eta}_t, K_t)$-function to ω_t^i-process

3a. CRRA Value Function: relate to ω

- $\Rightarrow$ value function can be written as $\frac{u(\omega_t^i n_t^i)}{\rho}$, that is

$$= \frac{1}{\rho^i} \frac{\left(\omega_t^i n_t^i\right)^{1-\gamma} - 1}{1-\gamma} = \frac{1}{\rho^i} \frac{\left(\omega_t^i\right)^{1-\gamma} \left(n_t^i\right)^{1-\gamma} - 1}{1-\gamma}$$

- $\frac{\partial V}{\partial n^i} = u'(c^i)$ by optimal consumption (if no corner solution)

$$\frac{\left(\omega_t^i\right)^{1-\gamma} \left(n_t^i\right)^{-\gamma}}{\rho^i} = (c_t^i)^{-\gamma} \Leftrightarrow \boxed{\frac{c_t^i}{n_t^i} = (\rho^i)^{1/\gamma} (\omega_t^i)^{1-1/\gamma}}$$

Next step:

a) Special simple cases

b) replace ω_t with something scale invariant

3a. CRRA Value Function: Special Case $\log -$utility

$$\boxed{\frac{c_t^i}{n_t^i} = (\rho^i)^{1/\gamma}(\omega_t^i)^{1-1/\gamma}}$$

- Ito for volatility term: $\sigma_t^{c^i} = \sigma_t^{n^i} + (1 - 1/\gamma)\sigma_t^{\omega^i}$
- For log utility $\gamma = 1$:
  $\xi_t^i = e^{-\rho^i t}/c_t^i = e^{-\rho^i t}/(\rho n_t^i)$ for any $\omega_t^i \Rightarrow \sigma_t^{n^i} = \sigma_t^{c^i} = \varsigma_t^i$
  - Expected excess return: $\mu_t^A - r_t^F = \sigma_t^{n^i}\sigma_t^A$
  - Recall $\frac{dn_t^i}{n_t^i} = -\frac{c_t^i}{n_t^i}dt + \left(1 - \theta^i\right)dr_t^K + \theta^i dr_t$

  - Consumption choice: $c_t^i = \rho^i n_t^i$
    - ω_t does not matter $\Rightarrow$ income and substitution effect cancel out

  - Portfolio choice: myopic (no Mertonian hedging demand)
    - Volatility of investment of opportunity/net worth multiplier does not matter $\Rightarrow$ Myopic price of risk $\varsigma_t^i = \sigma_t^{n^i} = \sigma_t^{c^i}$

Solving MacroModels Step-by-Step

0. Postulate aggregates, price processes & obtain return processes
1. For given C/N-ratio and SDF processes for each i *finance block*
   a. Real investment ι + Goods market clearing *(static)*
   - *Toolbox 1:* Martingale Approach, ~~HJB vs. Stochastic Maximum Principle Approach~~

   b. Portfolio choice θ + Asset market clearing or ~~Asset allocation κ & risk allocation χ~~
   - ~~*Toolbox 2:* "price-taking social planner approach" – Fisher separation theorem~~

   c. "Money evaluation equation" ϑ
   - ~~*Toolbox 3:* Change in numeraire to total wealth (including SDF)~~
2. Evolution of state variable η (and K) *forward equation*
3. Value functions *backward equation*
   a. Value fcn. as fcn. of individual investment opportunities ω
   - ~~*Special cases: log-uti*~~ Log-utility ~~*t investment opportunities*~~

   b. ~~Separating value fcn. $V^i(n^{\tilde{i}};\eta,K)$~~ into $v^i(\eta)u(K)(n^{\tilde{i}}/n^i)^{1-\gamma}$

   c. Derive C/N-ratio and ς price of risk
4. Numerical model solution
   a. ~~Transform BSDE for separated value fcn. $v^i(\eta)$~~ into PDE

   b. ~~Solve PDE via value function iteration~~
5. KFE: Stationary distribution, Fan charts

Recall: Market Clearing

- Output good market

$$C_t = (a - \iota_t^e)K_t$$

$$\rho q_t K_t = \big(a - \iota_t^e(q_t)\big)K_t \Rightarrow q_t = q, \forall t$$

$$\rho q_t = (a - \iota_t^e(q_t))$$

  - Hence $\iota_t^e = \iota^e,\ \mu_t^q = \sigma_t^q = 0, \forall t.$

- Capital market

$$1 - \theta_t^e = \frac{q_t K_t}{\underbrace{N_t^e}_{=1/\eta_t}}$$

- Debt/bond market (by Walras Law)

4b. Model Solution

- Using $\rho q_t = (a - \iota(q_t))$, $\phi \iota_t = q_t - 1$, for $\Phi(\iota) = \frac{1}{\phi}\log(\phi\iota + 1)$

$$q = \frac{1 + \phi a}{1 + \phi \rho}$$

- Using portfolio choice, goods & capital market clearing

$$r_t = \frac{a - \iota^e}{q_t} + \Phi(\iota^e) - \delta + \cancel{\mu_t^q} + \sigma \cancel{\sigma_t^q} - \varsigma_t(\sigma + \cancel{\sigma_t^q})$$

$$= \rho + \Phi(\iota^e) - \delta - (1 - \theta_t)\sigma^2$$

$$= \rho + \Phi(\iota^e) - \delta - \frac{\sigma^2}{\eta_t}$$

from capital market clearing

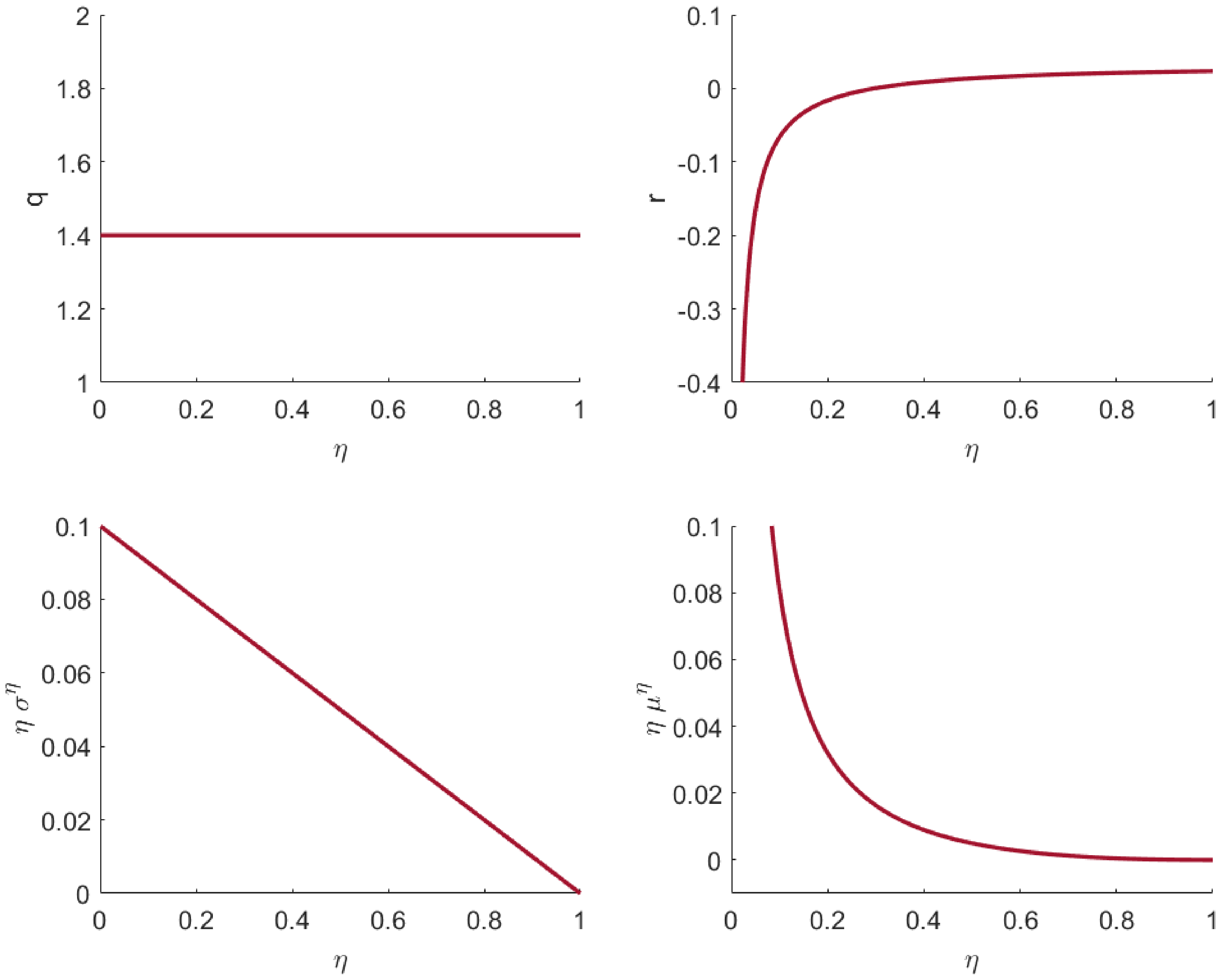
$$r_t = \rho + \frac{1}{\phi}\log(\frac{1 + \phi a}{1 + \phi \rho}) - \delta - \frac{\sigma^2}{\eta_t}$$

risk-free rate

- Goods & capital market clearing and η-evolution

$$\frac{d\eta_t}{\eta_t} = \frac{(1 - \eta_t)^2}{\eta_t^2}\sigma^2 dt + \frac{1 - \eta_t}{\eta_t}\sigma dZ_t$$

Numerical example



$$a = .11, \rho = 5\%, \sigma = .1, \Phi(\iota) = \frac{\log(\phi\iota + 1)}{\phi}, \phi = 10$$

Observation of Basak-Cuoco Model

- η_t fluctuates with macro shocks, since experts are levered
- Price of risk, i.e. Sharpe ratio, is

$$\frac{\sigma}{\eta_t} = \frac{\rho + \Phi(\iota) - \delta - r_t}{\sigma}$$

  - Goes to ∞ as η_t goes to zero
  - Achieved via risk-free rate

$$r_t = \rho + \Phi(\iota) - \delta - \sigma^2/\eta_t \to -\infty$$

  - Rather than depressing price of risky asset, $q_t = q\ \forall t$
- No endogenous risk $\sigma^q = 0$
  - No amplification
  - No volatility effects
- $\mu_t^\eta = \frac{(1-\eta_t)^2}{\eta_t^2}\sigma^2 > 0 \Rightarrow$in the long run HH-net worth share vanishes
  - Way out:
    - Different discount rates ρ (KM)
    - Switching types (BGG)
    - 2 types of experts (BruSan)

Desired Model Properties

- Normal regime: stable around steady state
  - Experts are adequately capitalized
  - Experts can absorb macro shock
- Endogenous risk and price of risk
  - Fire-sales, liquidity spirals, fat tails
  - Spillovers across assets and agents
  - Market and funding liquidity connection
  - SDF vs. cash-flow news
- Volatility paradox
- Financial innovation less stable economy
- (“Net worth trap” double-humped stationary distribution)