0. Outside Equity vs Debt

\[ r_{OE} - r_t = \sigma_e \eta e_t \]

\[ K_t = (\theta e_t, K_t + \theta e_t, OE_t)(\sigma_r, K) \]

\[ 2 = (1 - \theta e_t, D_t)(\sigma_r, K) \]

\[ r_{OE} - r_t = \sigma_h \eta h_t \]

\[ K_t = (\theta h_t, K_t + \theta h_t, OE_t)(\sigma_r, K) \]

\[ 2 = (1 - \theta h_t, D_t)(\sigma_r, K) \]

Must be that \( \eta e_t = \eta h_t \)

If experts issue a little bit of debt:

- \( (1 - \theta e_t, D_t) \) increases, \( (1 - \theta h_t, D_t) \) decreases, and \( \eta e_t \) increases, \( \eta h_t \) decreases.

Experts demand lower interest rate on debt.

Households demand higher interest rate on debt.

Market does not clear.
0. Outside Equity vs Debt

- Suppose experts are unconstrained in OE issuance:

\[ r_t^{OE} - r_t = \varsigma_t^e \sigma_t^{r,K} = (\theta_t^{e,K} + \theta_t^{e,OE})(\sigma_t^{r,K})^2 = (1 - \theta_t^{e,D})(\sigma_t^{r,K})^2 \]

\[ r_t^{OE} - r_t = \varsigma_t^h \sigma_t^{r,K} = (\theta_t^{h,K} + \theta_t^{h,OE})(\sigma_t^{r,K})^2 = (1 - \theta_t^{h,D})(\sigma_t^{r,K})^2 \]

- Must be that \( \varsigma_t^e = \varsigma_t^h \)
Outside Equity vs Debt

- Suppose experts are unconstrained in OE issuance:

\[ r_t^{OE} - r_t = \varsigma_t^e \sigma_t^{r,K} = (\theta_t^e K + \theta_t^{e,OE}) (\sigma_t^{r,K})^2 = (1 - \theta_t^e D) (\sigma_t^{r,K})^2 \]

\[ r_t^{OE} - r_t = \varsigma_t^h \sigma_t^{r,K} = (\theta_t^h K + \theta_t^{h,OE}) (\sigma_t^{r,K})^2 = (1 - \theta_t^h D) (\sigma_t^{r,K})^2 \]

- Must be that \( \varsigma_t^e = \varsigma_t^h \)

- If experts issue a little bit of debt:

  - \( (1 - \theta_t^{e,D}) \uparrow, (1 - \theta_t^{h,D}) \downarrow \), and \( \varsigma_t^e \uparrow, \varsigma_t^h \downarrow \)
  
  - Experts demand lower interest rate on debt
  
  - Households demand higher interest rate on debt
  
  - Market does not clear
1. Numerical Methods

\[
\mu_\eta(\eta) = \eta(1 - \eta_t) \left[ (s^e_t - \sigma - \sigma^q_t)(\sigma^\eta_t + \sigma + \sigma^q_t) - (s^h_t - \sigma - \sigma^q_t) \left( -\frac{\eta_t\sigma^\eta_t}{1 - \eta_t} + \sigma + \sigma^q_t \right) - \left( \frac{C^e_t}{N^e_t} - \frac{C^h_t}{N^h_t} \right) \right] \\
\sigma_\eta(\eta) = (\chi_t - \eta_t)(\sigma + \sigma^q_t)
\]
1. Numerical Methods

- Grid = \{\eta_1, \eta_2, \ldots, \eta_N\}
- Constructing the M matrix:
  - Ensure \(\mu_1 \geq 0, \mu_N \leq 0, \sigma_1 = \sigma_N = 0\)
  - Including \{0\} or \{1\}?
1. Numerical Methods

Compute stationary distribution:

1. Iteration method:
   - $g^{i+1} = (I - \Delta tM')^{-1} g^i$
   - Set $g^0$ to anything that integrates to 1
   - Depending on how M is constructed: avoid $g^0_N > 0$.

2. Yuliy's trick from the lecture (requires $\mu_N > 0$)

   
   $0 = M'g \iff \begin{bmatrix} M'_{11} & M'_{1,2:N} \\ M'_{2:N,1} & M'_{2:N,2:N} \end{bmatrix} \begin{bmatrix} g_1 \\ g_2:N \end{bmatrix} = 0 \iff g_{2:N} = -(M'_{2:N,2:N})^{-1}M'_{2:N,1}g_1$

3. Solve for kernel: $M'g = 0$, e.g. null(full(M'))
1. Numerical Methods

- Absorbing boundaries ($\eta_1 = 0$, $\eta_N = 1$, $\mu_1 = \mu_N = \sigma_1 = \sigma_N = 0$)

- Have a look at matrix $M$:

\[
\begin{bmatrix}
0 & 0 & 0 & 0 \\
1.62 & -6.63 & 5.01 & 0 \\
0 & 6.15 & -18.85 & 12.70 \\
\end{bmatrix}
\]

\[
\begin{bmatrix}
0.02 & -0.02 & 0 & 0 \\
0 & 0.01 & -0.01 & 0 \\
0 & 0 & 0 & 0 \\
\end{bmatrix}
\]
1. Numerical Methods

- Absorbing boundaries ($\eta_1 = 0, \eta_N = 1, \mu_1 = \mu_N = \sigma_1 = \sigma_N = 0$)

- Adjust matrix $M$ (reflecting boundaries):

\[
M_{1:3,1:4} = \begin{bmatrix}
-1 & 1 & 0 & 0 \\
1.62 & -6.63 & 5.01 & 0 \\
0 & 6.15 & -18.85 & 12.70
\end{bmatrix}
\]

\[
M_{N-2:N,N-3:N} = \begin{bmatrix}
0.02 & -0.02 & 0 & 0 \\
0 & 0.01 & -0.01 & 0 \\
0 & 0 & 1 & -1
\end{bmatrix}
\]
1. Numerical Methods
1. Numerical Methods

\[ v(x, t) = E \left[ \int_t^T e^{-\rho s} x_s ds \mid x_t = x \right] \]

\[ dx_t = -x_t dt + (1 - x_t^2) dZ_t \]

\[ \rho v(x, t) = \partial_t v(x, t) + x - x \partial_x v(x, t) + \frac{(1 - x^2)^2}{2} \partial_{xx} v(x, t) \]

\[ v(x, T) = 0 \]

- Grid \( x \in [-1, 1], \ t \in [0, 1] \)
1. Numerical Methods

• Discretize state space: \( v(x, t) \) is a vector \( v_t \)

• Explicit method:

\[
\frac{v_t - v_{t-\Delta t}}{\Delta t} = \rho v_t - x - Mv_t
\]

\[
v_{t-\Delta t} = \Delta tx + ((1 - \rho \Delta t)I_N + \Delta tM)v_t
\]

• Implicit method:

\[
\frac{v_t - v_{t-\Delta t}}{\Delta t} = \rho v_{t-\Delta t} - x - Mv_{t-\Delta t}
\]

\[
v_{t-\Delta t} = ((1 + \rho \Delta t)I_N - \Delta tM)^{-1}(\Delta tx + v_t)
\]
1. Numerical Methods

- Stability of explicit method:

\[
v_{t-\Delta t} = \Delta t \mathbf{x} + \left( (1 - \rho \Delta t) \mathbf{I}_N + \Delta t \mathbf{M} \right) v_t \]

\[
1 - \rho \Delta t - \Delta t \left[ \frac{|\mu|}{\Delta x} + \frac{\sigma^2}{(\Delta x)^2} \right] \geq 0
\]

- To ensure this in the limit as \( \Delta x \to 0 \):

\[
\lim_{\Delta x \to 0} \frac{\Delta t}{(\Delta x)^2} \leq C \implies \Delta t = O((\Delta x)^2)
\]
1. Numerical Methods

- $\Delta x = 0.1$, $\Delta t = 0.01$
1. Numerical Methods

- \( \Delta x = 0.01, \Delta t = 0.01 \)
2/3. Two Sector Model

- Solution algorithm
  1. Guess value functions $v^e, v^h \implies \varsigma^i, \frac{C^i}{N^i}$
  2. Solve inner loop for $q, \kappa, \chi, \nu, \sigma^q \implies \mu^j, \mu^\eta, \sigma^\eta$
  3. Update $v^e, v^h$ via time step

$$
\mu^j_i \nu^i_t = \partial_t \nu^i_t + \eta \mu^\eta \partial_\eta \nu^i_t + \frac{1}{2} (\eta \sigma^\eta)^2 \partial_{\eta \eta} \nu^i_t
$$
2/3. Two Sector Model

- Solution algorithm
  1. Guess value functions $v^e, v^h \Rightarrow \zeta^i, \frac{c^i}{n^i}$
  2. Solve inner loop for $q, \kappa, \chi, \nu, \sigma^q \Rightarrow \mu^v, \mu^\eta, \sigma^\eta$
  3. Update $v^e, v^h$ via time step

\[
\mu^v_t v^i_t = \partial_t v^i_t + \eta \mu^\eta \partial_\eta v^i_t + \frac{1}{2}(\eta \sigma^\eta)^2 \partial_{\eta\eta} v^i_t
\]

- CRRA: $\mu^v_t = \rho^i - \frac{c^i_t}{n^i_t} - (1 - \gamma) \left( \Phi(v^i_t) - \delta - \gamma \frac{\sigma^2}{2} + \sigma \sigma^v_t \right)$

\[
\rho^i v^i_t = \partial_t v^i_t + \left( \frac{c^i_t}{n^i_t} + (1 - \gamma) \left( \Phi(v^i_t) - \delta - \gamma \frac{\sigma^2}{2} + \sigma \sigma^v_t \right) \right) v^i_t + \eta \mu^\eta \partial_\eta v^i_t + \frac{1}{2}(\eta \sigma^\eta)^2 \partial_{\eta\eta} v^i_t
\]
2/3. Two Sector Model

- **CRRA**
  \[ \frac{C_i}{N_i} = \frac{(\eta^i q)^{\frac{1-\gamma}{\gamma}}}{(\nu^i)^{\frac{1}{\gamma}}} \]
  \[ \mathbb{E}\left[ \frac{dV_t^i}{V_t^i} \right] = \rho^i - \frac{C_t}{N_t}^i \]

- **Epstein-Zin (IES=1)**
  \[ \frac{C_i}{N_i} = \rho^i \]
  \[ \mathbb{E}\left[ \frac{dV_t^i}{V_t^i} \right] = -\frac{\partial f^i(c, U)}{\partial U} - \frac{C_t}{N_t}^i \]
2/3. Two Sector Model

- Epstein-Zin (IES=1):

\[
f^i(c, U) = (1 - \gamma)\rho^i U \left( \log(c) - \frac{1}{1 - \gamma} \log((1 - \gamma)\rho^i U) \right)
\]

\[
\frac{\partial f^i(c, U)}{\partial U} = (1 - \gamma)\rho^i \left( \log(c) - \frac{1}{1 - \gamma} \log((1 - \gamma)\rho^i U) \right) - \rho^i
\]

- Recall \( U = V^i = \frac{1}{\rho^i} \frac{(\omega^i n^i)^{1-\gamma}}{1-\gamma} \) and \( \omega^i = \frac{(\rho^i v^i)^{1-\gamma}}{\eta^i q} \)

\[
\frac{\partial f^i(c, U)}{\partial U} = (1 - \gamma)\rho^i \left( \log(\rho^i) - \log(\omega^i) \right) - \rho^i
\]

\[
= (1 - \gamma)\rho^i \log(\rho^i \eta^i q) - \rho^i \log(\rho^i v^i) - \rho^i
\]
2/3. Two Sector Model

Log Utility $\rightarrow$ Epstein-Zin $\rightarrow$ CRRA

<table>
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<th>Sector</th>
<th>$\gamma$</th>
<th>$\psi$</th>
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<tr>
<td>IES</td>
<td>$\psi = 1$</td>
<td>$\psi = 1$</td>
</tr>
</tbody>
</table>
2/3. Two Sector Model

- $q$
- $C/N$
- $\sigma^q$
- Debt ($D/qK$)
- $\eta\mu^n$
- $C^s/N^s - C^h/N^h$
2/3. Two Sector Model

Graphs showing:
- $q$ vs. $\log\ E_Z$
- $C/N$ vs. $\sigma^2$
- Debt ($D/qK$) vs. $\eta\mu^b$
- $C^*/N^* - C^h/N^h$ vs. $\rho$
2/3. Two Sector Model
2/3. Two Sector Model