1. Solving Differential Equations

Two approaches to solving differential equations:

1. Constructing the function starting from boundary condition
2. Updating the function starting from initial guess
1. Solving Differential Equations

\[ y' = g(x, y) \]
\[ y' = y^{-19}, \quad y' = x \cos(x^2) y^2, \quad y'' = -y \]
1. Solving Differential Equations

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- Idea:
  - Construct the function iteratively, using differential equation to move from one grid point to another

- Requires:
  - Interval \([0, 10]\)
  - Initial condition
1. Solving Differential Equations

- Approximate the derivative by finite difference:

\[
\frac{y_i - y_{i-1}}{x_i - x_{i-1}} = g(x, y)
\]

- Explicit method

\[
\frac{y_i - y_{i-1}}{x_i - x_{i-1}} = g(x_{i-1}, y_{i-1})
\]

- Implicit method

\[
\frac{y_i - y_{i-1}}{x_i - x_{i-1}} = g(x_i, y_i)
\]
1. Solving Differential Equations

- Explicit method: evaluate function once, faster, less stable
- Implicit method: multiple function evaluations, slower, more stable

\[
F_i(y_i) := y_i - y_{i-1} - x_i - x_{i-1} - g(x_i, y_i)
\]

\[
y_i = y_{i-1} - (J_{i-1})^{-1} F_i(y_{i-1})
\]
1. Solving Differential Equations

- Explicit method: evaluate function once, faster, less stable
- Implicit method: multiple function evaluations, slower, more stable
- Compromise: adjust implicit method
  - Do not search for the root until convergence
  - Perform one step of Newton’s method

\[
F_i(y_i) := \frac{y_i - y_{i-1}}{x_i - x_{i-1}} - g(x_i, y_i)
\]

\[
y_i = y_{i-1} - (J^{i-1})^{-1}F_i(y_{i-1})
\]
\[ y' = y^{-19} \]

\[ y' = x \cos(x^2)y^2 \]

\[ y'' = -y \]
2. Basak-Cuoco with Heterogeneous Discount Rates

- Solve the Basak-Cuoco model with $\rho^h < \rho^e$
  1. Non-degenerate stationary distribution of $\eta$
  2. Capital price $q$ depends on $\eta$
2. Basak-Cuoco with Heterogeneous Discount Rates

- Solve the Basak-Cuoco model with $\rho^h < \rho^e$
  1. Non-degenerate stationary distribution of $\eta$
  2. Capital price $q$ depends on $\eta$

- What changes?
  0. Postulate processes, obtain returns
  1. Given $C/N$ and SDF:
     - Investment decision
     - Portfolio decision
     - Market clearing conditions
  2. Evolution of $\eta$
  3. Value functions $\rightarrow C/N$ and price of risk
2. Basak-Cuoco with Heterogeneous Discount Rates

- Goods market clearing

\[ C = C^e + C^h = (\rho^e\eta + \rho^h(1 - \eta)) \hat{\rho}(\eta) qK = (a - \iota)K \]

- Optimal investment choice: \( q = 1 + \phi\iota \)

\[ q(\eta) = \frac{1 + \phi a}{1 + \phi \hat{\rho}(\eta)} \]

\[ \iota(\eta) = \frac{a - \hat{\rho}(\eta)}{1 + \phi \hat{\rho}(\eta)} \]
2. Basak-Cuoco with Heterogeneous Discount Rates

Law of motion for $\eta$ from LOMs of $N^e$ and $qK$:

$$\frac{d\eta}{\eta} = \left( \frac{a - \iota^e}{q} - \rho^e + \theta^e (\sigma + \sigma^q - \varsigma^e) (\sigma + \sigma^q) \right) dt - \theta^e (\sigma + \sigma^q) dZ$$

Capital market clearing ($\theta^e = -\frac{1-\eta}{\eta}$) + log-utility

$(\varsigma^e = \sigma^{n^e} = (1 - \theta^e)(\sigma + \sigma^q))$ and $(a - \iota^e) = \hat{\rho}(\eta) q$:

$$\frac{d\eta}{\eta} = \left( (1 - \eta) (\rho^h - \rho^e) + \left( \frac{1-\eta}{\eta} \right)^2 (\sigma + \sigma^q(\eta))^2 \right) dt + \frac{1-\eta}{\eta} (\sigma + \sigma^q(\eta)) dZ$$
2. Basak-Cuoco with Heterogeneous Discount Rates

- Apply Ito’s formula to $q(\eta)$:

$$
 dq(\eta) = \left( q'(\eta)\mu^\eta(\eta)\eta + \frac{(\sigma^\eta(\eta)\eta)^2}{2}q''(\eta) \right) dt + q'(\eta)\sigma^\eta(\eta)\eta dZ
$$

- Combine:

$$
 q(\eta) = \frac{1 + \phi a}{1 + \phi \hat{\rho}(\eta)}, \quad \sigma^q(\eta)q(\eta) = q'(\eta)\sigma^\eta(\eta)\eta, \quad \sigma^\eta(\eta) = \frac{1 - \eta}{\eta}(\sigma + \sigma^q(\eta))
$$

- Obtain:

$$
 \sigma^q(\eta) = -\frac{\phi(\rho^e - \rho^h)(1 - \eta)}{1 + \phi \rho^e} \sigma, \quad \sigma^\eta(\eta) = \frac{1 - \eta}{\eta} \frac{1 + \phi \hat{\rho}(\eta)}{1 + \phi \rho^e} \sigma
$$
2. Basak-Cuoco with Heterogeneous Discount Rates

- Get risk-free rate from experts portfolio choice:

\[
\begin{align*}
    r(\eta) &= \frac{a - \nu(\eta)}{q(\eta)} + \Phi(\nu(\eta)) - \delta + \mu^q(\eta) + \sigma^q(\eta) - \zeta^e(\eta)(\sigma + \sigma^q(\eta)) \\
    &\quad + \left\{ \begin{array}{c}
        \sigma^q(\eta) \\
        \frac{\varsigma^q(\eta)}{\sigma} \\
        \frac{\varsigma^q(\eta)}{\sigma^q(\eta)} \\
        \frac{\varsigma^q(\eta)}{\sigma^q(\eta)} \\
        \frac{\varsigma^q(\eta)}{\sigma^q(\eta)} \\
    \end{array} \right\} \\
    &\quad + \sigma \left\{ \begin{array}{c}
        \frac{\varsigma^q(\eta)}{\sigma} \\
        \frac{\varsigma^q(\eta)}{\sigma^q(\eta)} \\
        \frac{\varsigma^q(\eta)}{\sigma^q(\eta)} \\
        \frac{\varsigma^q(\eta)}{\sigma^q(\eta)} \\
        \frac{\varsigma^q(\eta)}{\sigma^q(\eta)} \\
    \end{array} \right\} \\
    &\quad + \varsigma e(\eta) \left\{ \begin{array}{c}
        \sigma \left\{ \begin{array}{c}
            \frac{\varsigma^q(\eta)}{\sigma} \\
            \frac{\varsigma^q(\eta)}{\sigma^q(\eta)} \\
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            \frac{\varsigma^q(\eta)}{\sigma^q(\eta)} \\
        \end{array} \right\} \\
    \end{array} \right\} \\
    \end{align*}
\]
2. Basak-Cuoco with Heterogeneous Discount Rates
2. Basak-Cuoco with Heterogeneous Discount Rates

▶ Total risk:

\[
\sigma + \sigma^q(\eta) = \frac{\sigma}{1 - \frac{1-\eta}{\eta} q'(\eta)} = \left(1 - \frac{\phi(\rho^e - \rho^h)(1 - \eta)}{1 + \phi \rho^e}\right) \sigma < \sigma \text{ if } \phi > 0
\]

▶ Goods clearing condition:

\[
\hat{\rho}(\eta) q(\eta)K + \iota(\eta)K = aK
\]

▶ Suppose \(K \downarrow, \eta \downarrow, C/N \downarrow \implies q(\eta) \text{ and/or } \iota(\eta) \text{ must go up}

▶ A drop in \(K\) is compensated by an increase in \(q\), stabilizes \(qK\)

▶ If \(\phi = 0\), then \(\iota(\eta)\) adjusts alone
2. Stationary Distribution

▶ Stationary KFE:

\[
0 = - \frac{d}{d\eta} (\mu_\eta(\eta) g(\eta)) + \frac{1}{2} \frac{d^2}{d\eta^2} (\sigma_\eta(\eta)^2 g(\eta))
\]

▶ Define \( D(\eta) = \sigma_\eta(\eta)^2 g(\eta) \):

\[
D'(\eta) = 2 \frac{\mu_\eta(\eta)}{\sigma_\eta(\eta)^2} D(\eta)
\]
2. Stationary Distribution

- Stationary KFE:

\[ 0 = -\frac{d}{d\eta} (\mu_\eta(\eta) g(\eta)) + \frac{1}{2} \frac{d^2}{d\eta^2} (\sigma_\eta(\eta)^2 g(\eta)) \]

- Define \( D(\eta) = \sigma_\eta(\eta)^2 g(\eta) \):

\[ D'(\eta) = 2 \frac{\mu_\eta(\eta)}{\sigma_\eta(\eta)^2} D(\eta) \]

- Domain? \([0, 1] / [0, 1) / (0, 1] / (0, 1)\)
2. Stationary Distribution

- Solve $D'(\eta) = 2 \frac{\mu_\eta(\eta)}{\sigma_\eta(\eta)^2} D(\eta)$ on $[0.0001, 0.9999]$

- No initial condition! But:

\[
\int_0^1 g(\eta) d\eta = 1 \tag{1}
\]

- Set $D(0.0001) = \text{anything} \in \mathbb{R}$ and then rescale using (1)
2. Stationary distribution

\[ \text{Diagram showing stationary distribution with curves for different } \rho^h. \]
3. Stochastic Maximum Principle

- Alternative to HJB, background for Martingale Method
  - Keep the stochastic structure as long as possible
  - Advantage: no state-space structure $\implies$ economize on notation
- We will use for deriving:
  - Martingale pricing conditions
  - Equilibrium allocations
3. Stochastic Maximum Principle

- Step 1: set up Hamiltonian (one for each agent type)
  - Experts’ problem:
    \[
    \max_{c^e, \theta^e, \iota^e} \mathbb{E} \left[ \int_0^\infty e^{-\rho^e t} \log(c^e_t) \, dt \right]
    \]
    \[
    \text{s.t} \quad \frac{dn^e_t}{n^e_t} = -\frac{c^e_t}{n^e_t} \, dt + r_t \, dt + (1 - \theta^e_t) \left( dr^K_t(\iota^e_t) - r_t \, dt \right)
    \]
    \[
    dr^K_t(\iota^e_t) = r^K_t(\iota^e_t) \, dt + \sigma^{r,K}_t \, dZ_t
    \]
  - Rewrite LOM of \( n^e_t \) as:
    \[
    dn^e_t = \left[ -c^e_t + n^e_t \left( r_t + (1 - \theta^e_t) \left( r^K_t(\iota^e_t) - r_t \right) \right) \right] \, dt + n^e_t(1 - \theta^e_t)\sigma^{r,K}_t \, dZ_t
    \]
3. Stochastic Maximum Principle

Step 1: set up Hamiltonian (one for each agent type)

Experts’ problem:

\[
\max_{c^e, \theta^e, \iota^e} \mathbb{E} \left[ \int_0^\infty e^{-\rho^e t} \log(c^e_t) \, dt \right]
\]

\[s.t \quad dn_t^e = \underbrace{-c_t^e + n_t^e \left( r_t + (1 - \theta_t^e) \left( r^K_t (\iota_t^e) - r_t \right) \right)}_{\mu_t^e n_t^e} \, dt + \underbrace{n_t^e (1 - \theta_t^e) \sigma_t^e r^K_t \, dZ_t}_{\sigma_t^e n_t^e} \]
3. Stochastic Maximum Principle

▶ Step 1: set up Hamiltonian (one for each agent type)

▶ Experts’ problem:

\[
\max_{c^e, \theta^e, \iota^e} \mathbb{E} \left[ \int_0^\infty e^{-\rho t} \log(c_t^e) dt \right]
\]

s.t. \( dn_t^e = \left[ -c_t^e + n_t^e \left( r_t + (1 - \theta_t^e) \left( r^K_t (\iota_t^e) - r_t \right) \right) \right] dt + n_t^e (1 - \theta_t^e) \sigma_t^K dZ_t
\]

▶ Co-state process:

\[
d\xi_t^e = \mu_t^e \xi_t^e dt + \sigma_t^e \xi_t^e dZ_t
\]

▶ Hamiltonian:

\[
H_t^e = e^{-\rho t} \log(c_t^e) + \xi_t^e \mu_t^e n_t^e + \sigma_t^e \xi_t^e \sigma_t^e n_t^e
\]
3. Stochastic Maximum Principle

- Step 1: set up Hamiltonian (one for each agent type)

\[
H^e_t = e^{-\rho^e t} \log(c^e_t) + \xi^e_t \left[ -c^e_t + n^e_t (r_t + (1 - \theta^e_t) (r^K_t(\nu^e_t) - r_t)) \right] + \sigma^e_t \xi^e_t n^e_t (1 - \theta^e_t) \sigma^{r,K}_t
\]

\[
r^K_t(\nu^e_t) = \frac{a - \nu^e_t}{q_t} + \Phi(\nu^e_t) - \delta + \mu^q_t + \sigma \sigma^q_t, \quad \sigma^{r,K}_t = \sigma + \sigma^q_t
\]
3. Stochastic Maximum Principle

▶ Step 1: set up Hamiltonian (one for each agent type)

\[ H^e_t = e^{-\rho^e_t} \log(c^e_t) + \xi^e_t \left[ -c^e_t + n^e_t \left( r_t + (1 - \theta^e_t) \left( r^K_t(\nu^e_t) - r_t \right) \right) \right] + \sigma^e_t \xi^e_t n^e_t (1 - \theta^e_t) \sigma^{r,K}_t \]

\[ r^K_t(\nu^e_t) = \frac{a - \nu^e_t}{q_t} + \Phi(\nu^e_t) - \delta + \mu^q_t + \sigma \sigma^q_t, \quad \sigma^{r,K}_t = \sigma + \sigma^q_t \]

▶ Step 2: FOCs and co-state equation:

\[ \frac{\partial H^e_t}{\partial c^e_t} = \frac{\partial H^e_t}{\partial \theta^e_t} = \frac{\partial H^e_t}{\partial \nu^e_t} = 0 \]

\[ d\xi^e_t = -\frac{\partial H^e_t}{\partial n^e_t} dt + \sigma^e_t \xi^e_t dZ_t \]
3. Stochastic Maximum Principle

▶ Step 2

▶ FOCs:

\[ \xi_t^e = e^{-\rho_t^e} \frac{1}{c_t} \]

\[ r_t^K(l_t^e) - r_t = -\sigma_t^e \sigma_t^{r,K} \]

\[ \Phi'(l_t^e) = \frac{1}{q_t} \]

▶ Co-state equation:

\[ d\xi_t^e = - \left[ r_t + (1 - \theta_t^e) \left( r_t^K(l_t^e) - r_t + \sigma_t^e \sigma_t^{r,K} \right) \right] \xi_t^e dt + \sigma_t^e \xi_t^e dZ_t \]

\[ = -r_t \xi_t^e dt + \sigma_t^e \xi_t^e dZ_t \]
3. Stochastic Maximum Principle

► What if there is a constraint?
► Suppose $\theta_t^e \geq -0.5$
3. Stochastic Maximum Principle

- What if there is a constraint?
- Suppose $\theta_t^e \geq -0.5$
- Hamiltonian:

$$H_t^e = e^{-\rho t} \log(c_t^e) + \xi_t^e \mu_t^e n_t^e + \sigma_t^e \xi_t^e \sigma_t^e n_t^e + \lambda_t (\theta_t^e + 0.5)$$

- FOC wrt $\theta_t^e$:

$$r^K_t(t^e_t) - r_t = -\sigma_t^e \sigma_t^r, K + \frac{\lambda_t}{\xi_t^e n_t^e}$$

$$\lambda_t (\theta_t^e + 0.5) = 0$$

- Other FOCs and co-state equation unchanged