ECO529: Modern Macro, Money and (International) Finance

Problem Set 1

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1. Solving Differntial Equations

Two approaches to solving differential equations:

1. Constructing the function starting from boundary condition
2. Updating the function starting from initial guess

1. Solving Differntial Equations

$$y' = g(x, y)$$

$$y' = y^{-19}, \quad y' = x \cos\left(x^2\right) y^2, \quad y'' = -y$$

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- Idea:
  - Construct the function iteratively, using differential equation to move from one grid point to another
- Requires:
  - Interval $[0, 10]$
  - Initial condition

1. Solving Differntial Equations

- Approximate the derivative by finite difference:

$$\frac{y_i - y_{i-1}}{x_i - x_{i-1}} = g(x, y)$$

- Explicit method

$$\frac{y_i - y_{i-1}}{x_i - x_{i-1}} = g(x_{i-1}, y_{i-1})$$

- Implicit method

$$\frac{y_i - y_{i-1}}{x_i - x_{i-1}} = g(x_i, y_i)$$

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- Explicit method: evaluate function once, faster, less stable
- Implicit method: multiple function evaluations, slower, more stable

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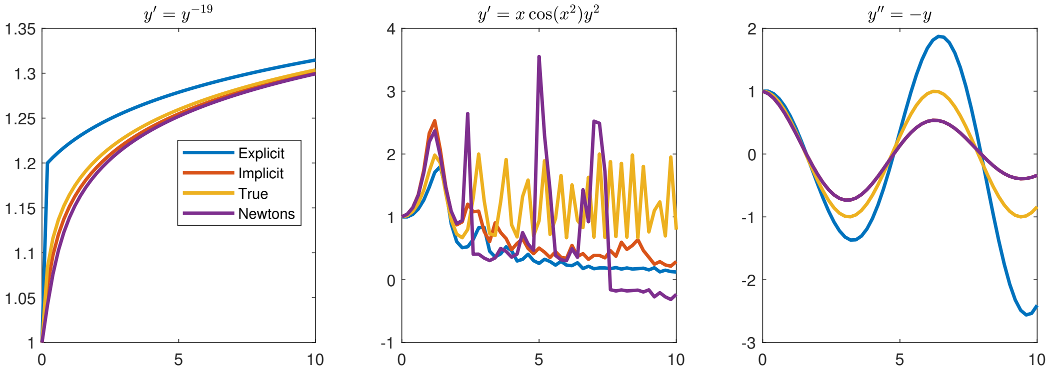
- Explicit method: evaluate function once, faster, less stable
- Implicit method: multiple function evaluations, slower, more stable
- Compromise: adjsut implicit method
  - Do not search for the root until convergence
  - Perform one step of Newton's method

$$F_i(y_i) := \frac{y_i - y_{i-1}}{x_i - x_{i-1}} - g(x_i, y_i)$$

$$y_i = y_{i-1} - (J^{i-1})^{-1} F_i(y_{i-1})$$

N = 51



$y' = y^{-19}$
1.35
1.3
1.25
1.2
1.15
1.1
1.05
1
0
5
10
Explicit
Implicit
True
Newtons
$y' = x\cos(x^2)y^2$
4
3
2
1
0
-1
0
5
10
$y'' = -y$
2
1
0
-1
-2
-3
0
5
10

2. Basak-Cuoco with Heterogeneous Discount Rates

- Solve the Basak-Cuoco model with $\rho^h < \rho^e$
  1. Non-degenerate stationary distribution of η
  2. Capital price q depends on η

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- What changes?
  0. Postulate processes, obtain returns
  1. Given C/N and SDF:
     - Investment decision
     - Portfolio decision
     - Market clearing conditions
  2. Evolution of η
  3. Value functions $\rightarrow$ C/N and price of risk

2. Basak-Cuoco with Heterogeneous Discount Rates

- Goods market clearing

$$C = C^e + C^h = \underbrace{(\rho^e \eta + \rho^h (1-\eta))}_{\hat{\rho}(\eta)} qK = (a - \iota) K$$

- Optimal investment choice: $q = 1 + \phi\iota$

$$q(\eta) = \frac{1 + \phi a}{1 + \phi \hat{\rho}(\eta)}$$
$$\iota(\eta) = \frac{a - \hat{\rho}(\eta)}{1 + \phi \hat{\rho}(\eta)}$$

2. Basak-Cuoco with Heterogeneous Discount Rates

- Law of motion for η from LOMs of N^e and qK:

$$\frac{d\eta}{\eta} = \left(\frac{a-\iota^e}{q} - \rho^e + \theta^e(\sigma+\sigma^q-\varsigma^e)(\sigma+\sigma^q)\right)dt - \theta^e(\sigma+\sigma^q)dZ$$

- Capital market clearing ($\theta^e = -\frac{1-\eta}{\eta}$) + log-utility ($\varsigma^e = \sigma^{n^e} = (1-\theta^e)(\sigma+\sigma^q)$) and $(a-\iota^e) = \hat{\rho}(\eta)q$:

$$\frac{d\eta}{\eta} = \underbrace{\left((1-\eta)(\rho^h-\rho^e) + \left(\frac{1-\eta}{\eta}\right)^2(\sigma+\sigma^q(\eta))^2\right)}_{\mu^\eta(\eta)}dt + \underbrace{\frac{1-\eta}{\eta}(\sigma+\sigma^q(\eta))}_{\sigma^\eta(\eta)}dZ$$

2. Basak-Cuoco with Heterogeneous Discount Rates

- Apply Ito's formula to $q(\eta)$:

$$dq(\eta) = \underbrace{\left(q'(\eta)\mu^{\eta}(\eta)\eta + \frac{(\sigma^{\eta}(\eta)\eta)^2}{2}q''(\eta)\right)}_{\mu^q(\eta)q(\eta)} dt + \underbrace{q'(\eta)\sigma^{\eta}(\eta)\eta}_{\sigma^q(\eta)q(\eta)} dZ$$

- Combine:

$$q(\eta) = \frac{1+\phi a}{1+\phi\hat{\rho}(\eta)}, \qquad \sigma^q(\eta)q(\eta) = q'(\eta)\sigma^{\eta}(\eta)\eta, \qquad \sigma^{\eta}(\eta) = \frac{1-\eta}{\eta}(\sigma+\sigma^q(\eta))$$

- Obtain:

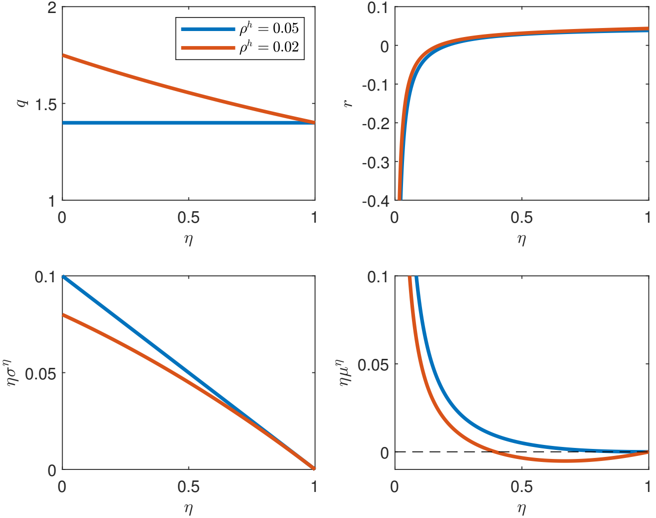
$$\sigma^q(\eta) = -\frac{\phi(\rho^e-\rho^h)(1-\eta)}{1+\phi\rho^e}\sigma, \qquad \sigma^{\eta}(\eta) = \frac{1-\eta}{\eta}\frac{1+\phi\hat{\rho}(\eta)}{1+\phi\rho^e}\sigma$$

2. Basak-Cuoco with Heterogeneous Discount Rates

- Get risk-free rate from experts portfolio choice:

$$r(\eta) = \frac{a - \iota(\eta)}{q(\eta)} + \Phi(\iota(\eta)) - \delta + \underbrace{\mu^q(\eta)}_{\neq 0} + \sigma \underbrace{\sigma^q(\eta)}_{\neq 0} - \varsigma^e(\eta)(\sigma + \underbrace{\sigma^q(\eta)}_{\neq 0})$$

2. Basak-Cuoco with Heterogeneous Discount Rates



2. Basak-Cuoco with Heterogeneous Discount Rates

- Total risk:

$$\sigma + \sigma^q(\eta) = \frac{\sigma}{1 - \frac{1-\eta}{\eta} \frac{q'(\eta)}{q/\eta}} = \left(1 - \frac{\phi(\rho^e - \rho^h)(1-\eta)}{1 + \phi\rho^e}\right)\sigma \; < \; \sigma \text{ if } \phi > 0$$

- Goods clearing condition:

$$\underbrace{\hat{\rho}(\eta)}_{C/N}\underbrace{q(\eta)K}_{N} + \iota(\eta)K = aK$$

- Suppose $K \downarrow$, $\eta \downarrow$, $C/N \downarrow \Longrightarrow q(\eta)$ and/or $\iota(\eta)$ must go up
- A drop in K is compensated by an increase in q, stabilizes qK
- If $\phi = 0$, then $\iota(\eta)$ adjusts alone

2. Stationary Distribution

- Stationary KFE:

$$0 = -\frac{d}{d\eta}\left(\mu_\eta(\eta)g(\eta)\right) + \frac{1}{2}\frac{d^2}{d\eta^2}\left(\sigma_\eta(\eta)^2 g(\eta)\right)$$

- Define $D(\eta) = \sigma_\eta(\eta)^2 g(\eta)$:

$$D'(\eta) = 2\frac{\mu_\eta(\eta)}{\sigma_\eta(\eta)^2}D(\eta)$$

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- Domain? $[0,1]$ / $[0,1)$ / $(0,1]$ / $(0,1)$

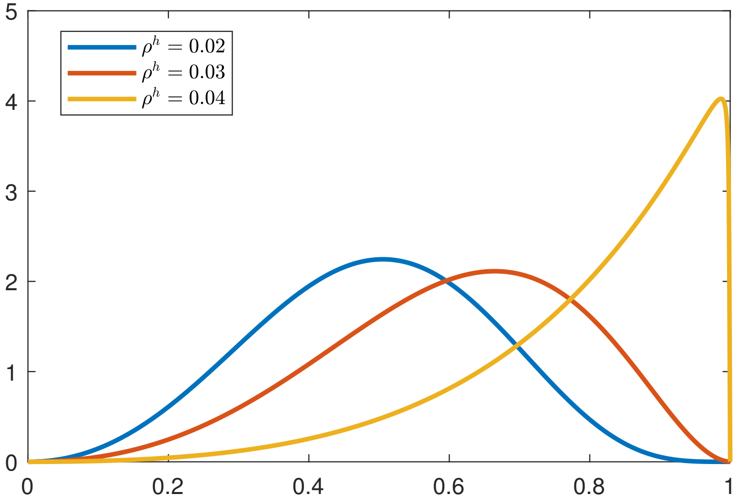
2. Stationary Distribution

- Solve $D'(\eta) = 2\frac{\mu_\eta(\eta)}{\sigma_\eta(\eta)^2}D(\eta)$ on $[0.0001, 0.9999]$
- No initial condition! But:

$$\int_0^1 g(\eta)d\eta = 1 \tag{1}$$

- Set $D(0.0001) = \mathit{anything}(\in \mathbb{R})$ and then rescale using (1)

2. Stationary distribution



3. Stochastic Maximum Principle

- Alternative to HJB, background for Martingale Method
  - Keep the stochastic structure as long as possible
  - Advantage: no state-space structure $\Longrightarrow$ economize on notation
- We will use for deriving:
  - Martingale pricing conditions
  - Equilibrium allocations

3. Stochastic Maximum Principle

- Step 1: set up Hamiltonian (one for each agent type)
  - Experts' problem:

$$
\max_{c^e,\theta^e,\iota^e} \mathbb{E}\left[\int_0^\infty e^{-\rho^e t}\log(c_t^e)dt\right]
$$

$$
\text{s.t} \quad \frac{dn_t^e}{n_t^e} = -\frac{c_t^e}{n_t^e}dt + r_t dt + (1-\theta_t^e)\left(dr_t^K(\iota_t^e) - r_t dt\right)
$$

$$
dr_t^K(\iota_t^e) = r_t^K(\iota_t^e)dt + \sigma_t^{r,K} dZ_t
$$

  - Rewrite LOM of n_t^e as:

$$
dn_t^e = \underbrace{\left[-c_t^e + n_t^e\left(r_t + (1-\theta_t^e)\left(r_t^K(\iota_t^e) - r_t\right)\right)\right]}_{\mu_t^{n^e} n_t^e} dt + \underbrace{n_t^e(1-\theta_t^e)\sigma_t^{r,K}}_{\sigma_t^{n^e} n_t^e} dZ_t
$$

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  - Co-state process:

$$d\xi_t^e = \mu_t^{\xi^e}\xi_t^e dt + \sigma_t^{\xi^e}\xi_t^e dZ_t$$

  - Hamiltonian:

$$H_t^e = e^{-\rho^e t}\log(c_t^e) + \xi_t^e \mu_t^{n^e} n_t^e + \sigma_t^{\xi^e}\xi_t^e \sigma_t^{n^e} n_t^e$$

3. Stochastic Maximum Principle

- Step 1: set up Hamiltonian (one for each agent type)

$$H_t^e = e^{-\rho^e t} \log(c_t^e) + \xi_t^e \left[-c_t^e + n_t^e \left(r_t + (1-\theta_t^e)\left(r_t^K(\iota_t^e) - r_t\right)\right)\right] + \sigma_t^{\xi^e} \xi_t^e n_t^e (1-\theta_t^e) \sigma_t^{r,K}$$

$$r_t^K(\iota_t^e) = \frac{a - \iota_t^e}{q_t} + \Phi(\iota_t^e) - \delta + \mu_t^q + \sigma \sigma_t^q, \qquad \sigma_t^{r,K} = \sigma + \sigma_t^q$$

3. Stochastic Maximum Principle

- Step 1: set up Hamiltonian (one for each agent type)

$$H_t^e = e^{-\rho^e t}\log(c_t^e) + \xi_t^e\left[-c_t^e + n_t^e\left(r_t + (1-\theta_t^e)\left(r_t^K(\iota_t^e) - r_t\right)\right)\right] + \sigma_t^{\xi^e}\xi_t^e n_t^e(1-\theta_t^e)\sigma_t^{r,K}$$

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- Step 2: FOCs and co-state equation:

$$\frac{\partial H_t^e}{\partial c_t^e} = \frac{\partial H_t^e}{\partial \theta_t^e} = \frac{\partial H_t^e}{\partial \iota_t^e} = 0$$

$$d\xi_t^e = -\frac{\partial H_t^e}{\partial n_t^e}dt + \sigma_t^{\xi^e}\xi_t^e dZ_t$$

3. Stochastic Maximum Principle

- Step 2
  - FOCs:

$$\xi_t^e = e^{-\rho^e t}\frac{1}{c_t}$$

$$r_t^K(\iota_t^e) - r_t = -\sigma_t^{\xi^e}\sigma_t^{r,K}$$

$$\Phi'(\iota_t^e) = \frac{1}{q_t}$$

  - Co-state equation:

$$d\xi_t^e = -\left[r_t + (1-\theta_t^e)\left(r_t^K(\iota_t^e) - r_t + \sigma_t^{\xi^e}\sigma^{r,K}\right)\right]\xi_t^e dt + \sigma_t^{\xi^e}\xi_t^e dZ_t$$

$$= -r_t\xi_t^e dt + \sigma_t^{\xi^e}\xi_t^e dZ_t$$

3. Stochastic Maximum Principle

- What if there is a constraint?
- Suppose $\theta_t^e \geq -0.5$

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- What if there is a constraint?
- Suppose $\theta_t^e \geq -0.5$
- Hamiltonian:

$$H_t^e = e^{-\rho^e t} \log(c_t^e) + \xi_t^e \mu_t^{n^e} n_t^e + \sigma_t^{\xi^e} \xi_t^e \sigma_t^{n^e} n_t^e + \lambda_t \left(\theta_t^e + 0.5\right)$$

- FOC wrt θ_t^e:

$$r_t^K(\iota_t^e) - r_t = -\sigma_t^{\xi^e} \sigma_t^{r,K} + \frac{\lambda_t}{\xi_t^e n_t^e}$$

$$\lambda_t \left(\theta_t^e + 0.5\right) = 0$$

- Other FOCs and co-state equation unchanged