Macro, Money and (International) Finance – Problem Set 1

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Prepared by Andrey Alexandrov (alexandrov@princeton.edu), based on problem sets by Sebastian Merkel. Please let me know if any tasks are unclear or you find mistakes in the problem descriptions.

Please submit to `alexandrov@princeton.edu` by 1pm on Monday, September 19 (Princeton time).

1 Solving Differential Equations

1. Read Section 1 of Sebastian's notes on differential equations.
2. Solve the following ODEs

$$y' = y^{-19} \tag{1}$$
$$y' = x\cos\left(x^2\right)y^2 \tag{2}$$
$$y'' = -y \tag{3}$$

on the interval $[0, 10]$ with the initial condition $y(0) = 1$ for all three equations and an additional initial condition $y'(0) = 0$ for equation (3) using the following three methods:

(a) explicit Euler method (Section 1.2.1);

(b) implicit Euler method (Section 1.2.2), using a built-in root-finder of your numerical software;

(c) a built-in ODE solver of your numerical software.

Compare the accuracy of explicit and implicit methods across different grid sizes (N = 11, 51, 501, 10001). For each of the three equations find the grid size that you like the most and plot the results from the three approximation methods together with the respective true solution. These are given by:

$$y(x) = (20x + 1)^{\frac{1}{20}}, \qquad y(x) = \frac{1}{1 - \sin x^2/2}, \qquad y(x) = \cos(x)$$

3. Now consider a variation of the implicit method: instead of using a built-in root-finder, perform <u>one</u> step of the Newton's method. It is an iterative root-finding algorithm that solves $F(y) = 0$ starting from an initial guess y^0 and updating via:

$$y^{n+1} = y^n - (J^n)^{-1}F(y^n) \tag{4}$$

where J^n is the Jacobian of $F(y^n)$, so that $J^n_{ij} = \partial F_i(y^n)/\partial y^n_j$ for the multivariate case. The idea is to compute the tangent of $F(\cdot)$ at y^n and find the point y^{n+1} where this tangent intersects zero. Instead of iterating the algorithm until convergence, we can make a single step and hopefully save some time without loosing a lot of accuracy. For our purpose, define $F_i(\cdot)$ at every grid point as follows:

$$F_i(y) := \frac{y - y_{i-1}}{x_i - x_{i-1}} - g(x_i, y) \tag{5}$$

where $g(x_i, y)$ is the RHS of the explicitely written ODE (see equation (2) in Sebastian's notes). At each step, use y_{i-1} as the initial guess and compute y_i via one step of the Newton's method.

Compare the results with the 'fully-fledged' implicit method. When does the variation work well and when does it fail?

2 The Basak-Cuoco Model with Heterogeneous Discount Rates

Consider the model from Lecture 4 (with log utility and $\Phi(\iota) = \frac{1}{\phi}\log(1 + \phi\iota)$), but unlike there assume that households are more patient than experts, i.e. they have a discount rate $\rho^h < \rho^e$. This is the simplest way to generate both a nondegenerate stationary distribution and some endogenous capital price dynamics.

1. Derive closed-form expressions for ι, q, σ^q, μ^η and σ^η as a function of η and model parameters:
   (a) Start with goods market clearing condition and use $\hat{\rho}(\eta) = \rho^e\eta + \rho^h(1 - \eta)$ to ease notation. Derive $q(\eta)$ and $\iota(\eta)$.
   (b) Use $q(\eta)$ and the law of motion for η to find $\sigma^q(\eta)$ and $\sigma^\eta(\eta)$.
   (c) Derive $\mu^\eta(\eta)$.
2. Replicate the figures from slide 31, setting $\delta = 0.035$, then add to each plot the corresponding line for the model with $\rho^e = 5\%$ and $\rho^h = 2\%$ (and all other parameters as before).
3. Assume $\phi > 0$. Show that in this model asset price movements mitigate exogenous risk (i.e. $\sigma^q + \sigma < \sigma$). Explain economically why this happens and why the effect disappears if $\phi = 0$.
4. Argue that the model must have a nondegenerate stationary distribution (just give some intuition, not a formal proof). Read the section on Kolmogorov Forward Equations in Yuliy's notes (page 15 onward). Compute the stationary density of η by numerically solving the ODE stated on page 16 (for $D(x)$) using the same parameters as in part 2. Choose your favourite method from the previous exercise and think about appropriate initial conditions. Remember to use the fact that density integrates to one!

3 Stochastic Maximum Principle

Read the section on Stochastic Maximum Principle in Yuliy's notes (page 20 onward) and slides 32-35 of Lecture 3.