Asset Pricing under Asymmetric Information
Strategic Market Order Models

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A Classification of Market Microstructure Models

- simultaneous submission of demand schedules
  - competitive rational expectation models
  - strategic share auctions
- sequential move models
  - screening models in which the market maker submits a supply schedule first
    - static
      - ◇ uniform price setting
      - ◇ limit order book analysis
    - dynamic sequential trade models with multiple trading rounds
  - strategic market order models where the market maker sets prices ex-post

Strategic Market Order Models - Overview

- Kyle (1985) model
  - static version
  - dynamic version (in discrete time)
    - Refresher in Dynamic Programming
  - continuous time version (Back 1992)
- Multi-insider Kyle (1985) version
- Other strategic market order models

Kyle 1985 Model

- Model Setup
  - asset return $v \sim \mathcal{N}(p_0, \Sigma_0)$
  - Agents (risk neutral)
    - Insider who knows v and submit market order of size x
    - Noise trader who submit market orders of exogenous aggregate size $u \sim \mathcal{N}(0, \sigma_u^2)$
    - Market maker sets competitive price after observing **net** order flow $X = x + u$
  - Timing (order of moves)
    - Stage 1: Insider & liquidity traders submit market orders
    - Stage 2: Market Maker sets the execution price
  - Repeated trading in dynamic version

Kyle 1985 Model — Static Version

Single informed trader	**(Competitive) Market Maker**
0) *Information*	**0)** *Information*
$v :=$ asset's payoff	$X = x + u$ batch net order flow
1) *Conjecture (price-rule)*	**1)** *Conjecture (insider trading rule)*
$p = \mu + \lambda(x + u)$	$x = \alpha + \beta v$
2) *No Updating*	**2)** *Updating* $E[v \mid x + u]$
3) *Optimal Demand*	**3)** *Price Setting Rule*
$\max_x E[(v - p) \mid v]x$	$p = E[v \mid x + u]$
$\max_x E[v - \mu - \lambda x \mid v]x$	$p = E[v] + \frac{Cov[v, x+u]}{Var[x+u]}\{x + u - E[x + u]\}$
FOC: $x = -\frac{\mu}{2\lambda} + \frac{1}{2\lambda}v$	$p = p_0 + \frac{\beta\Sigma_0}{\beta^2\Sigma_0 + \sigma_u^2}\{x + u - \alpha + \beta E[v]\}$
SOC: $\lambda > 0$	
4) *Correct Beliefs*	**4)** *Correct Beleifs*
$\alpha = -\frac{\mu}{2\lambda}, \ \beta = \frac{1}{2\lambda}$	$\mu = p_0$ Martingale, $\quad \lambda = \frac{\beta\Sigma_0}{\beta^2\Sigma_0 + \sigma_u^2}$

Kyle 1985 Model — Static Version

- solve for unknown coefficients
  - 4 unknown Greeks
  - 4 equations
- $\lambda = \frac{1}{2}\sqrt{\frac{\Sigma_0}{\sigma_u^2}}$
  - λ (illiquidity) decreases with noise trading, σ_u^2
  - Σ_0 reflect info advantage of insider

Dynamic Programming - A Refresher

- The Problem:
  max for several periods $t = 1, ..., T$ (discrete time)

$$\max_{u_t} E_t \left[\sum_{s=1}^{T} v_s \left(\tilde{x}_s, u_s \right) \right] \qquad \forall t \qquad \Rightarrow \text{(sequential rationality)}$$

  under the following law of motion

$$\tilde{x}_{t+1} = f_t \left(\tilde{x}_t, u_{t,} \tilde{\varepsilon}_t \right)$$

  $\tilde{x}_t$: vector of state variables (sufficient state space)
  u_t: vector of control variables
  ε_t: vector of random shocks

- Method
  - Backward Induction
  - Dynamic Programming

Dynamic Programming - A Refresher

- Define Value Function

$$V_t(x_t) := E_t\Big[\sum_{s=t}^{T} v_s(\tilde{x}_s, \underset{\substack{\uparrow \\ \text{optimal values}}}{u_s^*})\Big]$$

- $\Rightarrow$ Bellman Equation

$$\max_{u_t} E_t\left[v_t(x_t, u_t) + V_{t+1}(x_{t+1})\right]$$

- Start at final **date** T

$$V_{T+1}(\cdot) := 0$$

$\Rightarrow$ in $t = T$

$$\max_{u_T} E_T\left[v_T(x_T, u_T)\right]$$

$$\overset{\text{FOC}}{\Rightarrow} u_T^* = g_T(x_T)$$

$$\Rightarrow V_T(x_T) = E_T\left[v_T(x_T, u_T^*)\right]$$

Dynamic Programming - A Refresher

- at **date** $T-1$

$$\max_{u_{T-1},u_T} E_{T-1}\left[\sum_{s=T-1}^{T} v_s\left(\widetilde{x}_s, u_s\right)\right]$$

given $V_T\left(x_T\right)$

$\Leftrightarrow$ $\max_{u_{T-1}} E_{T-1}\left[v_{T-1}\left(x_{T-1}, u_{T-1}\right) + V_T\left(x_T\right)\right]$
given law of motion

$\Leftrightarrow$

$\max_{u_{T-1}} E_{T-1}\left[v_{T-1}\left(x_{T-1}, u_{T-1}\right) + V_T\left(f_{T-1}\left(x_{T-1}, u_{T-1}, \widetilde{\varepsilon}_{T-1}\right)\right)\right]$
$\Rightarrow u^*_{T-1} = g_{T-1}\left(x_{T-1}\right)$
$\Rightarrow V_{T-1} =$
$E_{T-1}\left[v_{T-1}\left(x_{T-1}, u^*_{T-1}\right) + V_T\left(f_{T-1}\left(x_{T-1}, u^*_{T-1}, \widetilde{\varepsilon}_{T-1}\right)\right)\right]$

- and so on for **date** $T-2$ etc. (and if they didn't die in the uncertainties they are still solving ...)
- This process is quite time consuming.

Dynamic Programming - A Refresher

Alternative way:

- **Step 1:** "Guess" the general form of the value function

$$V_{t+1}(x_{t+1}) = \underbrace{H_{t+1}(x_{t+1})}_{\text{e.g. } H_{t+1}(x_{t+1})=\alpha_{t+1}x_{t+1}^2}$$

- **Step 2:** Derive optimal level of current control

$$\max_{u_t} E_t\left[v_t(x_t, u_t) + H_{t+1}(\tilde{x}_{t+1})\right]$$

$$\max_{u_t} E_t\left[v_t(x_t, u_t) + H_{t+1}(f_t(x_t, u_t, \varepsilon_t))\right]$$

$$\Rightarrow \quad u_t^* = \cdots$$

- **Step 3:** Derive value function and check whether it coincides with general value function

$$V_t(x_t) = E_t\left[v_t(x_t, u_t^*) + H_{t+1}(f_t(x_t, u_t^*, \tilde{\varepsilon}_t))\right]$$

$$\stackrel{?}{=} H_t(x_t) = \alpha_t x_t^2$$

Kyle (1985) — Dynamic Version

Insider

- **Step 1:** Conjectured price setting strategy (pricing rule)

$$\begin{aligned} p_n &= p_{n-1} + \lambda_n \Delta X_n \\ &= p_{n-1} + \lambda_n (\Delta x_n + \Delta u_n) \qquad \left(\frac{1}{\lambda_t} \cong \text{Liquidity} \right) \end{aligned}$$

- **Step 2:** 'Guess' Value function for insider's profit pricing rule is linear → guess quadratic value function)

$$E[\pi_{n+1} | \underbrace{\tilde{p}_1, \cdots, p_n, v}_{\text{Information set up to } n}] = \alpha_n (v - p_n)^2 + \delta_n$$

(expected profit from time $n+1$ onwards)

$$\pi_n = E_n \left[\pi_{n+1} + (v - p_n) \Delta x_n^i \right]$$

Kyle (1985) — Dynamic Version

Insider ctd.

- **Step 3:** Write Bellman Equation

$$\max_{\Delta x_n^i} E\left[(v-p_n)\,\Delta x_n^i + \alpha_n (v-p_n)^2 + \delta_n \middle| \underbrace{p_1,\cdots,p_{n-1},v}_{I_{n-1}}\right]$$

- **Step 4:** Given insider's beliefs $p_n = p_{n-1} + \lambda_n \Delta X_n$

$$\max_{\Delta x_n^i} E\left[\begin{array}{c}\left(v-p_{n-1}-\lambda_n\Delta x_n^i-\lambda_n\Delta u_n\right)\Delta x_n^i \\ +\alpha_n\left(v-p_{n-1}-\lambda_n\Delta x_n^i-\lambda_n u_n\right)^2+\delta_n\end{array}\middle| I_n\right]$$

Take expectations

$$\max_{\Delta x_n^i}\left[\begin{array}{c}\left(v-p_{n-1}-\lambda_n\Delta x_n^i\right)\Delta x_n^i+\alpha_n(\underbrace{v-p_{n-1}}_{\text{state variable}}-\lambda_n\underbrace{\Delta x_n^i}_{\text{control}})^2 \\ +\delta_n+\alpha_n\lambda_n^2\underbrace{\sigma_u^2}_{u\Rightarrow p_n\text{ noisy}}\Delta t_n\end{array}\right]$$

Kyle (1985) — Dynamic Version

Insider ctd.

- **Step 5:** maximize

FOC: $(v - p_{n-1}) - 2\lambda_n \Delta x_n^i - 2\alpha_n \lambda_n (v - p_{n-1}) + 2\alpha_n \lambda_n^2 \Delta x_n^i =$

$$\Delta x_n^i = \underbrace{\frac{1 - 2\alpha_n \lambda_n}{2\lambda_n (1 - \alpha_n \lambda_n)}}_{:=\beta_n \Delta t_n} (v - p_{n-1})$$

$$\text{SOC:} \quad \lambda_n (1 - \alpha_n \lambda_n) > 0$$

- **Step 6:** Check whether 'guessed' value fcn is correct

$$E[\pi | I_{n-1}] = \max_{\Delta x_n^i} E\left[(v - p_n) \Delta x_n^i + \alpha_n (\widetilde{v} - \widetilde{p}_n)^2 + \delta_n | I_{n-1} \right.$$

$$= \alpha_{n-1} (v - p_{n-1})^2 + \delta_{n-1}, \text{where}$$

$$\alpha_{n-1} = \frac{1}{4\lambda_n (1 - \alpha_n \lambda_n)}, \quad \delta_{n-1} = \delta_n + \alpha_n (\lambda_n)^2 \sigma_u^2 \Delta t_n$$

Kyle (1985) — Dynamic Version

Market Maker (Filtering Problem)

- **Step 1:** MM's belief about insider's strategy

$$\begin{aligned} \Delta x_n^i &= \beta_n \Delta t_n (v - p_{n-1}) \\ \Delta X_n &= \beta_n \Delta t_n (v - p_{n-1}) + \underbrace{\Delta u_n}_{Var[\Delta u_n]=\sigma_u^2 \Delta t_n} \end{aligned}$$

- **Step 2:** MM's filtering problem
  By definition:

$$\begin{aligned} p_{n-1} &: = E[v|\Delta X_1, \cdots, \Delta X_{n-1}] \\ \Sigma_{n-1} &: = Var[v|\Delta X_1, \cdots, \Delta X_{n-1}] \end{aligned}$$

$$\begin{aligned} E[\Delta X_n|\Delta X_1, \cdots, \Delta X_{n-1}] &= \beta_n \Delta t_n E[(v - p_{n-1}) + \Delta u_n| \cdot \\ Var[\Delta X_n| \cdots] &= (\beta_n \Delta t_n)^2 \Sigma_{n-1} + \sigma_u^2 \Delta t_n \\ Cov[v, \Delta X_n| \cdots] &= E[v(\beta_n \Delta t_n (v - p_{n-1}) + \Delta u_r \\ &= \beta_n \Delta t_n \Sigma_{n-1} \end{aligned}$$

Kyle (1985) — Dynamic Version

Now we have all ingredients for the Projection Theorem

$$p_n = p_{n-1} + \underbrace{\frac{\beta_n \Delta t_n \Sigma_{n-1}}{(\beta_n \Delta t_n)^2 \Sigma_{n-1} + \Delta t \sigma_u^2}}_{:=\lambda_n} \Delta X_n$$

$$\sum\nolimits_n = V[v|\cdots\Delta X_n] = \Sigma_{n-1} - \frac{(\beta_n \Delta t_n)^2 \Sigma_{n-1}^2}{(\beta_n \Delta t_n)^2 \Sigma_{n-1} + \Delta t \sigma_u^2}$$

$$= \frac{\sigma_u^2 \Sigma_{n-1}}{(\beta_n)^2 \Delta t_n \Sigma_{n-1} + \sigma_u^2}$$

$\Rightarrow$

$$\lambda_n = \frac{\beta_n \Delta t_n \sum_{n-1}}{(\beta_n \Delta t_n)^2 \Sigma_{n-1} + \Delta t \sigma_u^2}$$

$$\sum\nolimits_n = (1 - \lambda_n \beta_n \Delta t_n) \Sigma_{n-1} = \frac{\sigma_u^2 \lambda_n}{\beta_n}$$

$$\Rightarrow \quad \lambda_n = \beta_n \Sigma_n / \sigma_u^2$$

Kyle (1985) — Dynamic Version

- **Step 3:** Equate coefficients $\alpha_n, \beta_n, \delta_n, \sum_n$

$$\left.\begin{array}{c} \beta_n \Delta t_n = \frac{1-2\alpha_n\lambda_n}{2\lambda_n(1-\alpha_n\lambda_n)} \\ \alpha_{n-1} = \frac{1}{4\lambda_n(1-\alpha_n\lambda_n)} \\ \delta_{n-1} = \delta_n + \alpha_n \lambda_n^2 \sigma_u^2 \Delta t_n \\ \left.\begin{array}{c} \sum_n = \sigma_u^2 \Sigma_{n-1} \\ \lambda_n = \ldots \end{array}\right\} \text{as above} \end{array}\right\} \text{Solve recursive system of equations.}$$

- Interpretation of Equilibrium
  - restrain from aggressive trading
    - price impact in current trading round
    - price impact in all future trading rounds
  - ...

Generalizations of Kyle (1985)

- Multiple Insiders
  - all have same information
  - all hold different information
  - information is correlated
    $\Rightarrow$ see Foster & Viswanathan, JF 51, 1437-1478
- Risk averse insiders
  - CARA utility
- etc. etc.