A Monetary Model with Money and Safe Assets*

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Abstract

This paper sets forth a tractable incomplete markets monetary model with money and other nominal safe assets including a long-maturity government bond. Money serves as a medium of exchange whereas all safe assets allow agents to self-insure against risks. Portfolio demand for these assets can serve as a nominal anchor that determines the price level. The paper examines and contrasts monetary transmission mechanisms involving different monetary policy instruments and links it to classic monetary results, including the Tobin effect, the Friedman rule, Sargent-Wallace unpleasant monetarist arithmetic, Wallace-Neutrality of open market operations, and Sims' stepping on a rake effect.

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1 Introduction

Monetary economics identifies three fundamental roles of money. Money serves as a medium of exchange, a store of value, and a unit of account. Over the past decades, the mainstream of monetary macroeconomics literature, which is dominated by New Keynesian models, has almost exclusively focused on the unit of account role. Following Woodford (1998, 2000), it has largely removed money from monetary models. Outside of this mainstream, there is a rich literature, both classic and recent, that emphasizes the remaining two roles of money, which are both closely linked to the nature of money as a financial asset. This literature offers a wealth of insights for monetary policy analysis beyond New Keynesian conclusions, typically illustrated through specific models emphasizing a specific role of money and tailored to specific transmission channels and effects of monetary policy.[1] In models of money as a medium of exchange, money overcomes (intratemporal) frictions in exchange ("monetary frictions") such as the double-coincidence of wants problem. Models of money as a store of value come in different flavors. A strand of literature called the Fiscal Theory of the Price Level (FTPL) emphasizes money as a fundamentally valued asset due to fiscal backing in frictionless settings. Another strand models unbacked fiat money as an asset that helps to overcome frictions in intertemporal exchange. For example, in macrofinance models with idiosyncratic and aggregate risk, safe assets (which includes money and typically government bonds) help to overcome financial frictions.

The objective of this paper is to develop a unified and tractable framework that incorporates both types of frictions, monetary and financial, as well as possibly fiscal backing. Within this setting, money plays a role in mitigating monetary frictions, while money and government debt serve as safe assets that alleviate financial frictions arising from incomplete markets. Idiosyncratic risk remains uninsurable, whereas aggregate risk can be traded, and the supply of safe assets is strictly positive. This framework has rich implications for monetary policy, even in the absence New Keynesian price setting frictions. The model further emphasizes the interlinkages between monetary and fiscal policy. An additional aim of this paper is to replicate, within a single tractable environment, a broad spectrum of classic, recent, and novel results. These include the Tobin effect, Wallace neutrality of open market operations, Sims' "stepping on a rake" effect, the Friedman rule, Sargent–Wallace unpleasant monetarist arithmetic. More broadly,

[1] Lagos (2025) recently re-examined Woodford's cashless limit result.

our approach helps bridge diverse strands of monetary theory. Our framework not only provides analytical clarity and identifies key conditions but also establishes conceptual linkages among these results. It can serve both as a building block for theoretical and quantitative work and as an effective pedagogical tool for teaching monetary (macro)economics.

More specifically, we consider a continuous-time dynamic environment in which each agent operates physical capital subject to uninsurable idiosyncratic risk, the magnitude of which may vary over time. Agents choose the degree of capital utilization, which, analogous to labor effort, entails a disutility cost. Production requires sufficient nominal money holdings, introducing a monetary friction. The government issues both money and government bonds, potentially across different maturities. A key reason for the tractability of our framework is that most real variables can be expressed as static functions of two endogenous equilibrium quantities—the fraction of nominal wealth of total wealth and the share of money as a fraction of nominal wealth—alongside an exogenous process that may be subject to aggregate shocks. This structure allows for analytical clarity while preserving economic richness.

We replicate and derive numerous results. The real allocations are determined separately from nominal quantities, i.e., classic dichotomy arises as long as higher inflation is compensated with higher nominal interest rates and price level jumps do not alter market completeness. This occurs even though, in our framework, money and nominal bonds—both supplied in strictly positive amounts—affect real allocations by mitigating monetary and financial frictions. A central result is the government debt maturity irrelevance theorem, which states that the maturity structure of government debt does not influence real variables as long as it does not affect the degree of market incompleteness. Consequently, the model can be solved for any given maturity structure in two steps: first, determine the real side of the economy, and then solve for nominal dynamics, which may depend on the maturity structure.

The real value of government liabilities can be decomposed into three components. The first derives from fiscal backing through future primary surpluses, consistent with the FTPL. The second reflects the future provision of safe-asset services: holding and retrading safe assets partially alleviates incomplete-market frictions. The retrading strategy yields payoffs linked to individual idiosyncratic risk. The third component arises from medium-of-exchange services: money holdings (but not bonds) relax monetary frictions, making money more valuable than short-maturity bonds by an amount corresponding to the marginal benefit of liquidity. The discounted stream of interest rate spreads between money and bonds constitutes the third value component.

Importantly, the model enables a systematic classification of monetary policy measures and their fiscal implications in an economy with a positive supply of government debt and money—rather than a cashless setting. We first examine interest rate policies, then quantity and balance sheet measures, and finally turn to optimal policy considerations. Interest rate policy involves more than one rate: the nominal rate on bonds with instantaneous maturity, the rate on money—which is lower by a spread because money relaxes the monetary constraint at the margin—and the average rate the government pays, which is a weighted average of the two based on the shares of money and bonds. Focusing initially on this average nominal rate while holding the spread fixed, a rate increase funded by debt expansion does not alter real allocations; inflation rises one-for-one with nominal rates, consistent with superneutrality and the neo-Fisherian effect. In contrast, if higher interest payments are financed through fiscal tightening rather than debt issuance, investment and capital utilization decline, causing an initial drop in the price level. Thus, whether an interest rate hike raises or lowers the price level depends critically on the fiscal response.

Next, we consider an increase in the spread between the rates on bonds and money that leaves the average interest rate unchanged. This can be achieved by reducing the money share of nominal liabilities—making money scarcer—and lowering the rate paid on money. A higher spread raises the nominal wealth share while reducing investment and utilization, and initially lowers the price level. This monetary contraction resembles a tax-funded interest rate hike but differs in two key respects: unlike the latter, it requires no fiscal contraction but it entails a larger output contraction.

To obtain a price-level decline from an interest rate hike without a rise in the real value of nominal assets, one must introduce long-term, non-floating-rate government bonds. This mechanism reflects Sims (2011)' "stepping on a rake" insight: announcing a rate hike reduces the nominal value of long-term bonds. Since government liabilities remain backed by the same real fiscal surpluses, safe-asset services, and monetary services, the price level must fall to match the lower nominal value of bonds. Consequently, an interest rate hike exerts an initial deflationary force. But its impact on the price level is transitory, and eventually inflation is larger than with short-term bonds.

The theme that monetary tightening may lead to more inflation in the long run is a recurrent thread in monetary economics and has also been emphasized by Sargent and Wallace (1981), who emphasize a different channel that operates via seigniorage financing of fiscal deficits. Our analysis clarifies their "unpleasant monetarist arithmetic" that a monetary authority seeking lower inflation must eventually accept higher inflation if the fiscal authority refuse to cooperate. This arithmetic holds precisely only under special conditions, including a technologically fixed nominal rate on money, no financial frictions, no output effects from the monetary friction and no initial revaluation of nominal government debt. More generally, even if the unpleasant arithmetic holds, seigniorage from money creation does not necessarily play the same pivotal role that it plays in Sargent and Wallace (1981).

Monetary policy can also operate through open market operations and quantitative easing (QE) or tightening (QT), which alter the money share of nominal assets. QE involving purchases of short-term bonds in exchange for money. This lowers the spread between the bond and money rate, affecting seigniorage income. QE targeting long-term bonds changes the duration of government liabilities, influencing the impact of subsequent interest rate moves via the stepping on the rake effect. More broadly, Wallace (1981) neutrality holds: asset purchases are neutral (for real allocations) if they do not affect fiscal policy, idiosyncratic risk exposures, or real money balances. Nevertheless, asset purchases affect nominal price level dynamics and this may allow the central bank to exert a greater level of control over nominal prices than with rate policy alone.

Having analyzed interest rate and quantity-based monetary policy measures, we turn to the question of optimal, welfare-maximizing monetary policy. In our tractable framework, total welfare can be decomposed into two components: one that depends solely on capital utilization and another that depends only on the nominal asset wealth share. Equilibrium capital utilization falls short of the constrained-efficient level for two reasons. First, when the monetary friction binds, agents economize on money holdings. Second, the wealth effect from nominal assets crowds out utilization effort. Whether the nominal asset share in equilibrium is higher or lower than in the Ramsey planner's solution depends on parameter values. A "Modified Friedman Rule" emerges in this setting: it is inefficient to induce agents to economize on money holdings, so long as the government could replace bonds with money, implying that a government should fund itself exclusively with money, except if the spread between money and bond rates is zero. However, optimality of this rule may not survive the introduction of additional tax distortions that are absent from our model, as then seigniorage revenue may substitute for distortionary taxes.

Related Literature. Safe assets and money play a distinctive role in economies characterized by frictions. The retrading of safe assets helps alleviate intertemporal financing constraints (Brunnermeier et al., 2024a). In overlapping-generations (OLG) models, such as Samuelson (1958) and Asriyan et al. (2020), safe assets enable trade across generations. In Bewley (1977, 1980), and Brunnermeier and Sannikov (2016b), retrading partially mitigates incomplete-market frictions. Technically, our model builds on Brunnermeier and Sannikov (2016a), Merkel (2020), and Brunnermeier et al. (2024a).

Intratemporal monetary frictions emphasize the transactional role of money as a special asset that facilitates exchange. Like Clower (1967); Lucas (1982); Lucas and Stokey (1987) we opt for the cash-in-advance constraints approach. Alternative approaches include models with transaction costs, money-in-the-utility-function (as, e.g., in Sidrauski, 1967; Di Tella, 2020), shopping-time models (Lucas, 1980), and search-based models (e.g., Kiyotaki and Wright, 1989; Lagos and Wright, 2005).

The FTPL literature models money as a store of value that has fundamental value due to fiscal backing (e.g., Leeper, 1991; Sims, 1994, 2013; Woodford, 1995; Cochrane, 2023). This literature typically abstracts from financial frictions and, in more recent contributions, also from monetary frictions.

The results in this paper are linked to many classical findings in economics, with references provided in the main sections of the paper.

2 Model

2.1 Setup

Overview. The model is based on the safe asset framework of Brunnermeier et al. (2024b),[2] augmented by a role for money as a medium of exchange as in Merkel (2020) and Di Tella (2020). The aggregate production technology is linear in a single production factor, capital. But for individual agents, operating capital is subject to two key frictions that give rise to demand for monetary assets. Agents face uninsurable id-

[2] See also Angeletos (2007) and Brunnermeier and Sannikov (2016a,b) for foundational contributions on this specific framework.

iosyncratic return risk, which generates demand for a safe store of value, in the spirit of Bewley (1980)'s model of money, and a monetary friction in production, which generates demand for money as a medium of exchange.

Despite the richness of the framework, the model remains analytically tractable in the presence of ex-post heterogeneity of agents and a rich structure of aggregate shocks. This is achieved by specific assumptions about the environment that ensure aggregation of individual decision rules. The price to pay for this tractability is that the wealth distribution is irrelevant for the evolution of aggregates, in contrast to a large strand of incomplete markets literature based on uninsurable income risk and borrowing constraints a la Huggett (1993), Aiyagari (1994), and Krusell and Smith (1998).

Environment. Time is continuous. The economy consists of a continuum of agents, indexed by $i \in \mathbb{I} := [0,1]$, with preferences over consumption $\{c_t^i\}$ and entrepreneurial effort $\{u_t^i\}$ ("capital utilization") described by the utility function

$$V_0^i := \mathbb{E}\left[\int_0^\infty e^{-\rho t}\left(\log c_t^i - \frac{(u_t^i)^{1+\varphi}}{1+\varphi}\right) dt\right]$$

with parameters $\varphi, \rho > 0$. Each agent faces a portfolio choice between productive capital and two nominal assets issued by the government, money and nominal bonds.

Physical capital k_t^i operated by agent $i \in \mathbb{I}$ generates an output flow of $a_t u_t^i k_t^i \mathrm{d}t$ units of output goods, where a_t is an exogenous productivity process common to all agents and u_t^i is the agent's capital utilization choice.[3] In the aggregate, the output flow is

$$Y_t \mathrm{d}t := a_t \int_{\mathbb{I}} u_t^i k_t^i \mathrm{d}i \mathrm{d}t.$$

In order to produce, each agent $i \in \mathbb{I}$ must have sufficient nominal money holdings, m_t^i, to satisfy the cash constraint

$$\mathcal{P}_t a_t u_t^i k_t^i \leq \nu_t m_t^i, \tag{1}$$

where $\mathcal{P}_t$ denotes the nominal goods price and ν_t is an exogenous velocity process. The constraint (1) is a reduced-form device to model the medium of exchange role of

[3]We borrow this formulation of variable capital utilization from Li and Merkel (2024). It is particularly tractable as it results in all agents choosing the same utilization rate in equilibrium.

money. Our specific approach may be interpreted as capturing payment frictions in an unmodeled supply chain.[4]

Capital holdings of agent i evolve according to

$$\mathrm{d}k_t^i = \left(\iota_t^i - \delta\right) k_t^i \mathrm{d}t + \tilde{\sigma}_t k_t^i \mathrm{d}\tilde{Z}_t^i + \mathrm{d}\Delta_t^{K,i}. \tag{2}$$

The dt-term in this equation describes physical investment and capital depreciation. The flow of investment expenditures, measured in output goods, is $\iota_t^i k_t^i \mathrm{d}t$. The parameter δ denotes the capital depreciation rate. The $\mathrm{d}\tilde{Z}_t^i$-term captures idiosyncratic capital risk: $\tilde{Z}_t^i$ is a Brownian motion specific to agent i and independent across different $i \in \mathbb{I}$. The scaling factor $\tilde{\sigma}_t$ describes the volatility of idiosyncratic shocks and is an exogenous stochastic process common for all agents. Importantly, asset markets are incomplete, so that agents are not able to share the idiosyncratic risk from operating physical capital. This feature introduces a demand for safe assets. Finally, installed capital can be (frictionlessly) traded between agents. The $\mathrm{d}\Delta_t^{K,i}$-term captures these capital trades. In the aggregate, $\int_{\mathbb{I}} \mathrm{d}\Delta_t^{K,i} \mathrm{d}i = 0$.

Nominal assets, money and bonds, are issued by the government. We denote by $\mathcal{M}_t \geq 0$ and $\mathcal{B}_t \geq 0$ the outstanding nominal quantities of money and bonds, respectively. For now, we assume that both money and bonds have infinitesimal duration, that is they make floating nominal interest payments, at rates $i_t^{\mathcal{M}}$ and $i_t^{\mathcal{B}}$, respectively, while their nominal market value remains always stable at 1. The government funds interest payments from two sources, by issuing additional liabilities and by imposing proportional capital taxes $\tau_t k_t^i \mathrm{d}t$ on all agents $i \in \mathbb{I}$. The government's flow budget constraint is

$$\mathrm{d}\underbrace{\left(\mathcal{M}_t + \mathcal{B}_t\right)}_{:=\mathcal{MB}_t} = \left(i_t^{\mathcal{M}} \mathcal{M}_t + i_t^{\mathcal{B}} \mathcal{B}_t - \mathcal{P}_t \tau_t K_t\right) \mathrm{d}t. \tag{3}$$

Here, $K_t := \int_{\mathbb{I}} k_t^i \mathrm{d}i$ denotes the aggregate capital stock and, as before, $\mathcal{P}_t$ the nominal price level. Denoting by $\mu_t^{\mathcal{MB}} := \frac{\mathrm{d}\mathcal{MB}_t}{\mathcal{MB}_t \mathrm{d}t}$ the growth rate of nominal government liabilities and by $q_t^{\mathcal{MB}} K_t := \frac{\mathcal{MB}_t}{\mathcal{P}_t}$ their real value, we can rewrite equation (3) equivalently in

[4] The specific formulation chosen here, following Merkel (2020), of adding the constraint to the production side instead of the consumption side offers advantages in terms of tractability because it only distorts the agent's portfolio choice but not the consumption-saving choice. Aside from this detail, implications are the same as for other common approaches to reduced-form monetary frictions.

the form

$$\tau_t = -\underbrace{\left(\mu_t^{\mathcal{MB}} - \frac{i_t^{\mathcal{M}}\mathcal{M}_t + i_t^{\mathcal{B}}\mathcal{B}_t}{\mathcal{M}_t + \mathcal{B}_t}\right)}_{=:\breve{\mu}_t^{\mathcal{MB}}} q_t^{\mathcal{MB}}, \tag{4}$$

where the variable $\breve{\mu}_t^{\mathcal{MB}}$ represents the growth rate of nominal liabilities in excess of what is required to make interest payments out of new issuances.

The exogenous processes $a_t, \nu_t, \tilde{\sigma}_t$ are assumed to satisfy

$$a_t = a(X_t), \qquad \nu_t = \nu(X_t), \qquad \tilde{\sigma}_t = \tilde{\sigma}(X_t), \tag{5}$$

where $a, \nu, \tilde{\sigma}$ are twice continuously differentiable functions and X_t is an exogenous state process that takes values in a set $\mathbb{X} \subset \mathbb{R}^{d_X}$. We assume further that X_t is a Markov diffusion process with evolution

$$\mathrm{d}X_t = \mu_X(X_t)\mathrm{d}t + \Sigma_X(X_t)\mathrm{d}Z_t, \tag{6}$$

where Z_t is a d_Z-dimensional Brownian motion and $\mu_d : \mathbb{X} \to \mathbb{R}^{d_X}$, $\Sigma_X : \mathbb{X} \to \mathbb{R}^{d_X \times d_Z}$ are continuous functions that satisfy sufficient regularity assumptions such that equation (6) has a unique solution for any initial condition $X_0 = x \in \mathbb{X}$.

Agent Decision Problem. Let q_t^K be the market price of capital in units of the output good. Denote by

$$n_t^i := q_t^K k_t^i + \frac{m_t^i + \mathcal{b}_t^i}{\mathcal{P}_t}$$

the net worth of agent $i \in \mathbb{I}$. Here, $m_t^i, \mathcal{b}_t^i \geq 0$ denote nominal money and bond holdings, respectively. Net worth evolves according to

$$\mathrm{d}n_t^i = -c_t^i\mathrm{d}t + n_t^i\left(\theta_t^i\left(\theta_t^{\mathcal{M},i}\mathrm{d}r_t^{\mathcal{M}} + (1-\theta_t^{\mathcal{M},i})\mathrm{d}r_t^{\mathcal{B}}\right) + (1-\theta_t^i)\mathrm{d}r_t^{K,i}(\iota_t^i, u_t^i)\right), \tag{7}$$

where

$$\theta_t^i := \frac{(m_t^i + \mathcal{b}_t^i)/\mathcal{P}_t}{n_t^i}, \qquad \theta_t^{\mathcal{M},i} := \frac{m_t^i}{m_t^i + \mathcal{b}_t^i}$$

are the fraction of wealth allocated to nominal asset and the share of money holdings in the nominal asset portfolio, respectively, and $\mathrm{d}r_t^{\mathcal{M}}$, $\mathrm{d}r_t^{\mathcal{B}}$, and $\mathrm{d}r_t^{K,i}(\iota, u)$ denote the real returns on money, bonds, and capital (conditional on choosing $u_t^i = u$, $\iota_t^i = \iota$), respectively.

The return on capital for agent i is given by

$$dr_t^{K,i}(\iota,u) = \frac{ua_t - \tau_t - \iota}{q_t^K}\mathrm{d}t + \frac{\mathrm{d}(q_t^K(k_t^i - \Delta_t^{K,i})) + \Delta_t^{K,i}\mathrm{d}q_t^K}{q_t^K k_t^i}$$
$$= \left(\frac{ua_t - \iota}{q_t^K} + \frac{\breve{\mu}_t^{\mathcal{MB}} q_t^{\mathcal{MB}}}{q_t^K} + \iota - \delta\right)\mathrm{d}t + \frac{\mathrm{d}q_t^K}{q_t^K} + \tilde{\sigma}_t \mathrm{d}\tilde{Z}_t^i, \tag{8}$$

where the second line uses the capital evolution (2) and the government budget constraint (4). The (real) return on money and bonds are

$$\mathrm{d}r_t^{\mathcal{M}} = i_t^{\mathcal{M}}\mathrm{d}t + \frac{\mathrm{d}(1/\mathcal{P}_t)}{1/\mathcal{P}_t} = \left(i_t^{\mathcal{M}} - \mu_t^{\mathcal{MB}}\right)\mathrm{d}t + \frac{\mathrm{d}(q_t^{\mathcal{MB}}K_t)}{q_t^{\mathcal{MB}}K_t}, \tag{9}$$

$$\mathrm{d}r_t^{\mathcal{B}} = i_t^{\mathcal{B}}\mathrm{d}t + \frac{\mathrm{d}(1/\mathcal{P}_t)}{1/\mathcal{P}_t} = \left(i_t^{\mathcal{B}} - \mu_t^{\mathcal{MB}}\right)\mathrm{d}t + \frac{\mathrm{d}(q_t^{\mathcal{MB}}K_t)}{q_t^{\mathcal{MB}}K_t}, \tag{10}$$

respectively. In both lines, the last equality follows from $1/\mathcal{P}_t = \frac{q_t^{\mathcal{MB}}K_t}{\mathcal{MB}_t}$.

Using the definitions of the portfolio weights θ_t^i, $\theta_t^{\mathcal{M},i}$, we can rewrite the cash constraint (1) equivalently as

$$\frac{a_t u_t^i}{q_t^K}(1-\theta_t^i) \leq \nu_t \theta_t^{\mathcal{M},i}\theta_t^i. \tag{11}$$

The decision problem of agent $i \in \mathbb{I}$ is to choose consumption c_t^i, investment ι_t^i, capital utilization u_t^i, and portfolio weights $\theta_t^i, \theta_t^{\mathcal{M},i}$ to maximize utility V_0^i subject to the net worth evolution (7), the return expressions (8), (9), and (10), the cash constraint (11), and a solvency constraint $n_t^i \geq 0$ that rules out Ponzi schemes. We remark that the choice of the capital trading process $\mathrm{d}\Delta_t^{K,\tilde{i}}$ from equation (2) is implicit in the portfolio weight θ_t^i and can be backed out ex post.

Market Clearing. The goods market clearing condition is

$$\int_{\mathbb{I}} c_t^i \mathrm{d}i + \int_{\mathbb{I}} \iota_t^i k_t^i \mathrm{d}i = Y_t = a_t \int_{\mathbb{I}} u_t^i k_t^i \mathrm{d}i. \tag{12}$$

The market for all government liabilities (the $\mathcal{MB}_t$-aggregate) clears if

$$\int_{\mathbb{I}} \theta_t^i n_t^i \mathrm{d}i = \frac{\mathcal{MB}_t}{\mathcal{P}_t} = q_t^{\mathcal{MB}} K_t \tag{13}$$

and the money market clears if

$$\int_{\mathbb{I}} \theta_t^{\mathcal{M},i} \theta_t^i n_t^i \mathrm{d}i = \frac{\mathcal{M}_t}{\mathcal{P}_t} := q_t^{\mathcal{M}} K_t. \tag{14}$$

When these conditions are satisfied, the capital market clears by Walras' law.

Equilibrium. A competitive equilibrium is defined in the usual way as a set of allocations and prices such that all households maximize utility and all markets clear. Here, prices and aggregate variables may depend only on aggregate exogenous histories, which are summarized by the filtration $\mathcal{F}_t$ generated by the aggregate Brownian motion Z_t. In contrast, individual outcomes for household $i \in \mathbb{I}$ can depend on both aggregate and individual idiosyncratic histories, that is on any information in the finer filtration $\tilde{\mathcal{F}}_t^i$ that is generated by both $\mathcal{F}_t$ and the idiosyncratic Brownian motion $\tilde{Z}_t^i$. Throughout this paper, we make the additional technical assumption that all stochastic processes in an equilibrium are Ito processes, unless explicitly stated otherwise.[5] Formally, we define an equilibrium as follows.[6]

Definition 1 (Equilibrium)**.** *Given an exogenous process X_t, functions as in* (5)*, initial stocks of capital and nominal assets, K_0, $\mathcal{MB}_0$, and an initial cross-sectional wealth distribution $\{\eta_0^i\}_{i\in\mathbb{I}}$ satisfying $\int_{\mathbb{I}} \eta_0^i di = 1$, a* competitive equilibrium *consists of aggregate (Ito) stochastic processes K_t, $\mathcal{MB}_t$, $\mathcal{P}_t$, $q_t^{\mathcal{M}}$, $q_t^{\mathcal{MB}}$, q_t^K, $\breve{\mu}_t^{\mathcal{MB}}$, $i_t^{\mathcal{M}}$, $i_t^{\mathcal{B}}$ adapted to $\mathcal{F}_t$ and, for each $i \in \mathbb{I}$, individual (Ito) stochastic processes c_t^i, ι_t^i, u_t^i, θ_t^i, $\theta_t^{\mathcal{M},i}$, n_t^i adapted to $\tilde{\mathcal{F}}_t^i$, such that*

1. *For each agent $i \in \mathbb{I}$, c_t^i, ι_t^i, u_t^i, θ_t^i, $\theta_t^{\mathcal{M},i}$ solve i's optimization problem for initial wealth $n_0^i = \eta_0^i(q_0^K + q_0^{\mathcal{MB}})K_0$ and n_t^i is as implied by the net worth evolution* (7)*. In the optimization problem, $\mu_t^{\mathcal{MB}}$ is defined as $\mu_t^{\mathcal{MB}} := \breve{\mu}_t^{\mathcal{MB}} + i_t^{\mathcal{B}} + \frac{q_t^{\mathcal{M}}}{q_t^{\mathcal{MB}}}(i_t^{\mathcal{M}} - i_t^{\mathcal{B}})$.*

[5]Formally, an Ito process A_t adapted to $\tilde{\mathcal{F}}_t^i$ is a stochastic process that has a differential representation

$$\mathrm{d}A_t = \mu_{A,t}\mathrm{d}t + \sigma_{A,t}\mathrm{d}Z_t + \tilde{\sigma}_{A,t}\mathrm{d}\tilde{Z}_t^i,$$

where $\mu_{A,t}$, $\sigma_{A,t}$, $\tilde{\sigma}_{A,t}$ are $\tilde{\mathcal{F}}_t^i$-progressively measurable processes that satisfy suitable integrability conditions for the stochastic integrals to be defined. The requirement of an Ito process is not particularly restrictive. Because the filtrations $\mathcal{F}_t$ and $\tilde{\mathcal{F}}_t^i$ are generated by Brownian motions, the martingale representation theorem implies that all (sufficiently regular) martingales can be written as stochastic integrals with respect to Brownian motions. Therefore, the added restriction of the Ito process assumption is primarily that the expected components (compensators) of processes must be absolutely continuous.

[6]We do not include taxes τ_t in the equilibrium definition as we have substituted them out from all equations. τ_t can be recovered from equation (4).

2. *K_t satisfies the aggregate capital evolution*[7]

$$dK_t = \left(\int_{\mathbb{I}} \iota_t^i k_t^i di - \delta K_t \right) dt$$

with the given initial condition K_0.

3. *$\mathcal{MB}_t$ satisfies the evolution*

$$d\mathcal{MB}_t = \left(\breve{\mu}_t^{\mathcal{MB}} + i_t^{\mathcal{B}} + \frac{q_t^{\mathcal{M}}}{q_t^{\mathcal{MB}}} (i_t^{\mathcal{M}} - i_t^{\mathcal{B}}) \right) \mathcal{MB}_t dt$$

with the given initial condition $\mathcal{MB}_0$.

4. *$q_t^{\mathcal{MB}} K_t$ is the real value of government liabilities: for all t,*

$$q_t^{\mathcal{MB}} K_t = \frac{\mathcal{MB}_t}{\mathcal{P}_t}.$$

5. *All asset values are nonnegative, $q_t^{\mathcal{M}}, q_t^{\mathcal{B}} := q_t^{\mathcal{MB}} - q_t^{\mathcal{M}}, q_t^K \geq 0$ for all t.*

6. *All markets clear: for all t, equations* (12), (13), *and* (14) *hold.*

2.2 Characterizing Equilibrium

In this subsection, we provide a characterization of competitive equilibria. Throughout, we focus on economic interpretations and the logical steps in the model solution procedure. Formal details, algebraic derivations, and all proofs can be found in Appendix A.1.

Solving the Agent Problem. Our setup leads to a tight analytical characterization of the solution to the agent decision problem.

Lemma 1 (Agent Choices)**.** *The optimal choices c_t^i, ι_t^i, u_t^i, $\theta_t^{\mathcal{M},i}$, θ_t^i of agent $i \in \mathbb{I}$ solve*

$$c_t^i : \qquad c_t^i = \rho n_t^i \tag{15}$$

$$\iota_t^i : \qquad q_t^K = 1 \tag{16}$$

[7] This equation follows from the individual evolution (2) by integrating over all $i \in \mathbb{I}$ and using the fact that both idiosyncratic shocks and capital trades average out in the aggregate.

$$u_t^i: \qquad \frac{\rho(u_t^i)^\varphi}{1-\theta_t^i} = (1-\lambda_t^{\mathcal{M},i})\frac{a_t}{q_t^K} \tag{17}$$

$$\theta_t^{\mathcal{M},i}: \qquad i_t^{\mathcal{B}} - i_t^{\mathcal{M}} = \lambda_t^{\mathcal{M},i}\nu_t \tag{18}$$

$$\theta_t^i: \quad \frac{\mathbb{E}_t\left[dr_t^{K,i} - dr_t^{\mathcal{MB},i}\right]}{dt} = \underbrace{\frac{\mathbb{E}_t\left[\frac{d\langle n_t^i, q_t^K\rangle}{q_t^K} - \frac{d\langle n_t^i, q_t^{\mathcal{MB}}\rangle}{q_t^{\mathcal{MB}}}\right]}{n_t^i dt}}_{\text{aggr. risk premium}} + \underbrace{(1-\theta_t^i)\tilde{\sigma}_t^2}_{\text{idio. risk premium}} + \underbrace{\frac{\theta_t^{\mathcal{M},i}}{1-\theta_t^i}\lambda_t^{\mathcal{M},i}\nu_t}_{\text{liquidity premium}} \tag{19}$$

where $\lambda_t^{\mathcal{M},i} \geq 0$ is a Lagrange multiplier that satisfies the complementary slackness condition

$$\lambda_t^{\mathcal{M},i} = 0 \text{ or constraint (11) holds with equality}$$

and $dr_t^{K,i}$, $dr_t^{\mathcal{MB},i}$ are defined by

$$dr_t^{K,i} := dr_t^{K,i}(\iota_t^i, u_t^i), \qquad dr_t^{\mathcal{MB},i} := \theta_t^{\mathcal{M},i} dr_t^{\mathcal{M}} + (1-\theta_t^{\mathcal{M},i}) dr_t^{\mathcal{B}}.$$

These conditions have intuitive economic interpretations. Equation (15) is the standard permanent income consumption rule for logarithmic utility. Equation (16) is a Tobin's q relationship between the market value of capital and the marginal cost of capital investments. The utilization effort choice, equation (17), is akin to an optimal labor supply condition in models with labor. If $\lambda_t^{\mathcal{M},i} = 0$, it equates the resource value of the marginal disutility from utilization per unit of capital investment to the marginal effect of higher utilization on the return on capital. More generally, the right-hand side is multiplied by a wedge $1 - \lambda_t^{\mathcal{M},i}$ that induces lower utilization when the cash constraint (11) is binding. Equation (18) relates the Lagrange multiplier on the binding cash constraint to the opportunity cost of holdings money instead of bonds, $i_t^{\mathcal{B}} - i_t^{\mathcal{M}}$. Finally, equation (19) captures the optimal portfolio choice between capital assets and nominal assets. It equates the expected premium earned on capital to the sum of three required premia, an aggregate risk premium, an idiosyncratic risk premium, and a liquidity premium. These required premia vary with the agent's portfolio weight θ_t^i: reducing θ_t^i (i) shifts the agent's risk exposure from $q_t^{\mathcal{MB}}$ price risk towards q_t^K price risk, (ii) increases the agent's idiosyncratic risk exposure, and (iii) tightens the cash constraint by simultaneously raising the need to hold money and reducing money holdings.

The choice conditions (15)–(19) have a symmetric solution for which all agents choose identical consumption-wealth ratios, investment and utilization rates, and portfolio weights. In the following, we will always restrict attention to such symmetric equilibria and drop i-superscripts from now on.[8] In particular, agents are homogeneous up to scale, which allows us to aggregate individual decision rules analytically.

Static Equilibrium Conditions. We show next that we can represent most real variables of interest as static functions of a limited set of endogenous equilibrium quantities. Recall that $q_t^K K_t$, $q_t^{\mathcal{M}} K_t$, $q_t^{\mathcal{B}} K_t$ represent the three components of aggregate net wealth: capital wealth, money wealth, and bond wealth. Recall also that $q_t^{\mathcal{MB}} = q_t^{\mathcal{M}} + q_t^{\mathcal{B}}$ denotes total wealth from nominal assets per unit of capital in the economy. We define $q_t := q_t^{\mathcal{MB}} + q_t^K$ as total wealth per unit of capital and the two ratios

$$\vartheta_t := \frac{q_t^{\mathcal{MB}}}{q_t}, \qquad \vartheta_t^{\mathcal{M}} := \frac{q_t^{\mathcal{M}}}{q_t^{\mathcal{MB}}} = \frac{\mathcal{M}_t}{\mathcal{MB}_t},$$

of nominal wealth as a fraction of total wealth and of money wealth as a fraction of nominal wealth. The shares ϑ_t and $\vartheta_t^{\mathcal{M}}$ will play a key role in the following.

By aggregating individual consumption and investment choices and using that $c_t^i / n_t^i = \rho$, $\iota_t^i = \iota_t$ do not depend on i, we obtain for aggregate consumption and investment,

$$\begin{aligned} C_t &= \int_{\mathbb{I}} c_t^i \mathrm{d}i = \int_{\mathbb{I}} \rho n_t^i \mathrm{d}i = \rho q_t K_t \\ I_t &:= \int_{\mathbb{I}} \iota_t^i k_t^i \mathrm{d}i = \iota_t \int_{\mathbb{I}} k_t^i \mathrm{d}i = \iota_t K_t. \end{aligned}$$

Using further that all agents choose the same capital utilization rate, $u_t^i = u_t$, aggregate supply simplifies to

$$Y_t = a_t \int_{\mathbb{I}} u_t k_t^i \mathrm{d}i = a_t u_t K_t.$$

Substituting these expressions into goods market clearing (12) and canceling K_t yields

$$\rho q_t + \iota_t = a_t u_t.$$

In addition, the optimal investment condition (16) implies

$$(1 - \vartheta_t) q_t = q_t^K = 1.$$

[8]Because agents are indifferent to any investment rate choice ι_t^i, there are certainly also asymmetric solutions to these equations. But these solutions feature the same consumption allocation and behavior of aggregates.

Solving these two equations for q_t and ι_t, we can express aggregate wealth, consumption, and investment as a function of utilization u_t and the nominal wealth share ϑ_t:

Lemma 2. *In equilibrium,*

$$q_t = \frac{1}{1-\vartheta_t}, \qquad \frac{C_t}{K_t} = \rho q_t, \qquad \iota_t = \frac{I_t}{K_t} = \frac{(1-\vartheta_t)a_t u_t - \rho}{1-\vartheta_t}.$$

Next, we will express also u_t in terms of ϑ_t (and the money share $\vartheta_t^{\mathcal{M}}$). To do so, we turn to the optimal utilization condition (17). By market clearing for government liabilities, equation (13), and the fact that all agents choose identical portfolios, we obtain $\theta_t^i = \vartheta_t$. In addition, condition (18) tells us that the Lagrange multiplier $\lambda_t^{\mathcal{M},i}$ is not i-dependent. These observations permit us to simplify condition (17) as follows:

$$\rho u_t^{\varphi} = (1-\lambda_t^{\mathcal{M}})\frac{a_t}{q_t}.$$

Analogous replacements in the cash constraint (11) lead us to the conclusion that this inequality is in equilibrium equivalent to

$$\frac{a_t u_t}{q_t} \leq \nu_t \vartheta_t^{\mathcal{M}} \vartheta_t.$$

Next, we can use Lemma 2 to replace q_t in the previous two conditions as a function of ϑ_t and u_t and combine these conditions with the complementary slackness condition from Lemma 1 to express u_t and $\lambda_t^{\mathcal{M}}$ in equilibrium as functions of ϑ_t and $\vartheta_t^{\mathcal{M}}$:

Lemma 3. *The equilibrium utilization rate u_t and liquidity premium multiplier $\lambda_t^{\mathcal{M}}$ satisfy*

$$u_t = \min\{u^0(\vartheta_t; X_t), u^c(\vartheta_t, \vartheta_t^{\mathcal{M}}; X_t)\}, \qquad \lambda_t^{\mathcal{M}} = 1 - \frac{\rho u_t^{\varphi}}{a_t}\frac{1}{1-\vartheta_t},$$

where

$$u^0(\vartheta; X) := \left(\frac{(1-\vartheta)a(X)}{\rho}\right)^{1/\varphi},$$
$$u^c(\vartheta, \vartheta^{\mathcal{M}}; X) := \frac{\nu(X)}{a(X)}\frac{\vartheta^{\mathcal{M}}\vartheta}{1-\vartheta}.$$

Lemmas 2 and 3 express most equilibrium variables of interest as static functions of

ϑ_t, $\vartheta_t^{\mathcal{M}}$, and the exogenous process X_t. All remaining real variables can be backed out easily from static conditions stated previously in the model setup.

Characterizing ϑ-Dynamics. Due to asset market clearing, $\vartheta_t = \theta_t$, so the dynamics of the nominal wealth share are determined by the portfolio choice of agents between capital and nominal assets. Substituting the explicit return expressions, equations (8), (9), and (10), and the net worth evolution (7) into the portfolio choice condition (19), imposing market clearing and rearranging, we arrive at the following dynamic equation for the evolution of ϑ_t:

Lemma 4 (Evolution of the Nominal Wealth Share)**.** *In equilibrium, ϑ_t must satisfy the equation*

$$\mathbb{E}_t[d\vartheta_t] = \left(\rho + \check{\mu}_t^{\mathcal{MB}} - (1-\vartheta_t)^2\tilde{\sigma}_t^2 - \vartheta_t^{\mathcal{M}}\lambda_t^{\mathcal{M}}\nu_t\right)\vartheta_t dt. \tag{20}$$

Mathematically, equation (20) is a backward stochastic differential equation (BSDE), an equation that restricts only the expected change of a process with a free initial condition. Economically, this means that equation (20) is forward looking and determines the current nominal wealth share ϑ_t as a function of the expected future evolution of the terms on the right-hand side. We defer a more in-depth discussion of this equation to Section 3, where we provide a detailed analysis of the sources of the value of money and bonds in this economy.

We remark that the lemmas stated in this subsection fully characterize the restrictions imposed on real variables from the behavior of the private sector. The remaining unknowns, $\mu_t^{\mathcal{MB}}$ and $\vartheta_t^{\mathcal{M}}$, that appear in some of the equations are tightly linked to the government's policy instruments and, hence, determined by government policy.

2.3 Classical Dichotomy

Because prices are flexible, our model exhibits a form of classical dichotomy whereby real allocations are determined separately from nominal quantities. We first formulate a precise statement in the absence of aggregate shocks:

Proposition 1 (Classical Dichotomy, Deterministic Case)**.** *Suppose that X_t is deterministic and consider a competitive equilibrium with nominal price level path $\mathcal{P}_t$. Let $\mathcal{P}'_t$ be any arbitrary positive and absolutely continuous path that satisfies $\mathcal{P}'_0 = \mathcal{P}_0$. Then there is a second equilibrium for the same initial conditions and with the same real allocations (all variables in*

Definition 1 other than $\mathcal{MB}$, $\mathcal{P}$, $i^{\mathcal{M}}$, and $i^{\mathcal{B}}$) as the original equilibrium but with price level path $\mathcal{P}'$. The nominal interest rate paths in this second equilibrium, $i_t^{\mathcal{M}\prime}$, $i_t^{\mathcal{B}\prime}$, are related to the those in the original equilibrium, $i_t^{\mathcal{M}}$, $i_t^{\mathcal{B}}$, as follows:

$$i_t^{\mathcal{M}} - \frac{\mathbb{E}_t[d\mathcal{P}_t]}{\mathcal{P}_t dt} = i_t^{\mathcal{M}\prime} - \frac{\mathbb{E}_t[d\mathcal{P}'_t]}{\mathcal{P}'_t dt}, \qquad i_t^{\mathcal{B}} - \frac{\mathbb{E}_t[d\mathcal{P}_t]}{\mathcal{P}_t dt} = i_t^{\mathcal{B}\prime} - \frac{\mathbb{E}_t[d\mathcal{P}'_t]}{\mathcal{P}'_t dt}.$$

The intuition for this result is simple. First, due to $\mathcal{P}_0 = \mathcal{P}'_0$, there is no revaluation of nominal assets at the initial date, $\mathcal{MB}_0/\mathcal{P}_0 = \mathcal{MB}_0/\mathcal{P}'_0$, compared to the original equilibrium. So changing the price level path to $\mathcal{P}'$ at most changes the dynamics of (expected) inflation. Second, adjusting nominal interest rates to $i_t^{\mathcal{M}\prime}$, $i_t^{\mathcal{B}\prime}$ as in the proposition ensures that the real return on nominal asset is not affected by the change in inflation dynamics. This neutralizes the effect of inflation changes on agent's portfolio demand for nominal assets. Finally, there are also no fiscal side effects from nominal interest rate changes if these changes are fully funded by altering the growth rate of the nominal asset stock $\mathcal{MB}_t$.

A similar result also holds in the presence of aggregate shocks. However, in this case we do not only need to rule out initial revaluations of nominal assets but also changes in inflation/deflation surprises in response to aggregate shocks across equilibria, as these would affect the real return on nominal assets in a way that cannot be neutralized by adjusting nominal interest rates.

Proposition 2 (Classical Dichotomy, General Case)**.** *Consider a competitive equilibrium with nominal price level path $\mathcal{P}_t$. Let $\mathcal{P}'_t$ be any arbitrary positive Ito process that satisfies $\mathcal{P}'_0 = \mathcal{P}_0$ and, for all $t > 0$,*

$$\frac{d\mathcal{P}_t - \mathbb{E}_t[d\mathcal{P}_t]}{\mathcal{P}_t} = \frac{d\mathcal{P}'_t - \mathbb{E}_t[d\mathcal{P}'_t]}{\mathcal{P}'_t},$$

i.e., both price level processes feature identical inflation surprises. Then the conclusions from Proposition 1 continue to hold.

We remark that the previous propositions do not imply that money does not matter for real allocations or that money is necessarily neutral. In general, the presence of money and nominal bonds always affects real allocations. This is evident from the presence of the nominal wealth share ϑ_t in many equilibrium relationships derived in the previous section. Furthermore, the previous propositions show that also the dynamics of nominal prices $\mathcal{P}_t$ matter to the extent that they affect the real returns on nominal assets. In particular, expected inflation is only neutral if all nominal rates, including the rate on money $i_t^{\mathcal{M}}$, are adjusted to neutralize its effect on real rates. If this rate is fixed for technological or institutional reasons (e.g., $i^{\mathcal{M}} \equiv 0$ if money is assumed to be cash), then changes in expected inflation do have real effects.

2.4 Extension: Long-Duration Bonds

Our baseline model assumes that government bonds have infinitesimal duration. It is relatively straightforward, albeit notationally involved, to generalize the model to allow for arbitrary debt duration. In Appendix B.1, we outline such a generalized model in detail. Here, we provide a summary and show that the extended model is, in a certain sense, equivalent to our baseline model.

To keep the notation simple, we present here only a special case of the model without money (and no monetary friction), so that $\mathcal{MB}_t = \mathcal{B}_t$. Appendix B.1 presents the full setup with money. We remark that all conclusions remain valid in the general model.

We assume that the government bond stock consists of zero coupon bonds of arbitrary duration $\Delta \geq 0$. For each $\Delta \geq 0$, denote by $\mathcal{X}_t(\Delta)$ the face value of zero coupon bonds outstanding with time to maturity Δ. We define the nominal zero coupon bond price for a bond with time to maturity Δ as

$$\mathcal{P}_t^{\mathcal{B}}(\Delta) := \mathbb{E}_t\left[\frac{\xi_{t+\Delta}/\xi_t}{\mathcal{P}_{t+\Delta}/\mathcal{P}_t}\right], \tag{21}$$

where ξ_t is any real SDF that prices aggregate claims. The nominal value of all outstanding government bonds at time t is therefore

$$\mathcal{B}_t = \int_0^\infty \mathcal{P}_t^{\mathcal{B}}(\Delta)\mathcal{X}_t(\Delta)\mathrm{d}\Delta. \tag{22}$$

Over a small time interval $[t, t+dt]$, the value of bonds outstanding at t, inclusive of cash flows bond holders receive from bonds maturing in $[t, t+dt]$, changes by

$$\int_0^\infty \mathcal{P}_{t+\mathrm{d}t}^{\mathcal{B}}(\Delta-\mathrm{d}t)\mathcal{X}_t(\Delta)\mathrm{d}\Delta - \mathcal{B}_t = \int_0^\infty \mathcal{X}_t(\Delta)\underbrace{\left(\mathcal{P}_{t+\mathrm{d}t}^{\mathcal{B}}(\Delta-\mathrm{d}t) - \mathcal{P}_t^{\mathcal{B}}(\Delta)\right)}_{\approx \mathrm{d}\mathcal{P}_t^{\mathcal{B}}(\Delta) - \mathcal{P}_t^{\mathcal{B}\prime}(\Delta)\mathrm{d}t \text{ for small } \mathrm{d}t}\mathrm{d}\Delta,$$

where we use the convention $\mathcal{P}_t^{\mathcal{B}}(\Delta) = 1$ for $\Delta \leq 0$ to include principal repayments from maturing bonds. In general, $\mathcal{B}_{t+\mathrm{d}t} - \mathcal{B}_t$ differs from the previous expression due to bond issuances and repurchases. In total, the flow budget constraint of the government is therefore

$$\underbrace{\mathrm{d}\mathcal{B}_t - \int_0^\infty \mathcal{X}_t(\Delta) \left(\mathrm{d}\mathcal{P}_t^{\mathcal{B}}(\Delta) - \mathcal{P}_t^{\mathcal{B}\prime}(\Delta)\mathrm{d}t\right) \mathrm{d}\Delta}_{\text{new debt issuance}} = -\mathcal{P}_t \tau_t K_t \mathrm{d}t.$$

The previous equation replaces equation (3) from the baseline model. Recall that we have set $\mathcal{M}_t \equiv 0$ for the presentation in this section. To connect this model with the baseline model, we continue to denote by $i_t^{\mathcal{B}}$ the nominal short rate, that is the nominal interest rate on a hypothetical zero net supply nominal bond with infinitesimal duration. This rate can be formally defined by $i_t^{\mathcal{B}} := -\mathcal{P}_t^{\mathcal{B}\prime}(0)$.[9]

The remaining model setup is unchanged, except that the return on bonds is no longer given by equation (10), but instead by

$$\mathrm{d}r_t^{\mathcal{MB}} = \mathrm{d}r_t^{\mathcal{B}} = \frac{\int_0^\infty \mathcal{X}_t(\Delta) \left(\mathrm{d}(\mathcal{P}_t^{\mathcal{B}}(\Delta)/\mathcal{P}_t) - \mathcal{P}_t^{\mathcal{B}\prime}(\Delta)/\mathcal{P}_t \mathrm{d}t\right) \mathrm{d}\Delta}{\mathcal{B}_t/\mathcal{P}_t}.$$

We can bring this return expression into a form that more closely mirrors the return on bonds in the baseline model. Specifically, we show in the appendix that the government budget constraint and Ito's lemma imply:

Lemma 5. *The return on nominal assets satisfies in any equilibrium*

$$dr_t^{\mathcal{MB}} = -\breve{\mu}_t^{\mathcal{MB}} dt + \frac{d(q_t^{\mathcal{MB}} K_t)}{q_t^{\mathcal{MB}} K_t},$$

where the notation $\breve{\mu}_t^{\mathcal{MB}} := -\frac{\tau_t}{q_t^{\mathcal{MB}}}$ *is defined in a way that the government budget equation* (4) *from the baseline model continues to hold.*

Observe that the maturity structure $\{\mathcal{X}_t(\Delta)\}_{\Delta \geq 0}$ of government bonds does not enter the previous expression for the real return on the nominal asset portfolio $\mathrm{d}r_t^{\mathcal{MB}}$. Indeed, the exact same equation also holds in the baseline model with short-term bonds.

[9]The nominal yield of a Δ-bond, $i_t^{\mathcal{B}}(\Delta)$, is related to the bond price by the equation $e^{-i_t^{\mathcal{B}}(\Delta)\Delta} = \mathcal{P}_t^{\mathcal{B}}(\Delta)$, so $\mathcal{P}_t^{\mathcal{B}\prime}(\Delta) = -e^{-i_t^{\mathcal{B}}(\Delta)\Delta}(i_t^{\mathcal{B}\prime}(\Delta)\Delta + i_t^{\mathcal{B}}(\Delta))$. Evaluating this at $\Delta = 0$ yields $i_t^{\mathcal{B}} := i_t^{\mathcal{B}}(0) = -\mathcal{P}_t^{\mathcal{B}\prime}(0)$.

Because the remaining model equations are also the same, we arrive at the following irrelevance result.

Proposition 3 (Irrelevance of Maturity Structure)**.** *For any equilibrium in the extended model with long-term bonds, there is a unique equilibrium in the baseline model that features the same real allocation (all variables in Definition 1 other than $\mathcal{MB}$, $\mathcal{P}$, $i^{\mathcal{M}}$, and $i^{\mathcal{B}}$) and the same nominal short rates on money and bonds, $i^{\mathcal{M}}$ and $i^{\mathcal{B}}$. In addition, Lemmas 2–4 remain valid in the extended model.*

This proposition is akin to the Modigliani and Miller (1958) capital structure irrelevance theorem, but for government debt maturity structure instead of the capital structure of a private company. The intuition is the same: the fundamental sources of value of nominal assets (cash and service flows), which we will discuss in detail in the next section, do not depend on the nominal payoff structure, so that it is irrelevant how they are packaged into individual financial claims.[10] Proposition 3 allows us, without loss of generality (w.l.o.g.), to restrict attention to the baseline model for the purpose of studying the real effects of money and monetary policy.

That being said, the proposition does not apply to *nominal* variables, whose joint evolution typically depends on the maturity structure of debt. Specifically, the nominal price level is $\mathcal{P}_t = \frac{\mathcal{MB}_t}{q_t^{\mathcal{MB}} K_t}$. Whereas the denominator, $q_t^{\mathcal{MB}} K_t$, is unaffected by the maturity structure,[11] the joint evolution of the numerator, $\mathcal{MB}_t = \mathcal{B}_t + \mathcal{M}_t$, and nominal interest rates does depend on the maturity structure. Specifically, we show in Appendix B.1 that the nominal bond prices $\mathcal{P}_t^{\mathcal{B}}(\Delta)$ satisfy

$$\mathcal{P}_t^{\mathcal{B}}(\Delta) = \mathbb{E}_t \left[\frac{\Theta_{t+\Delta}}{\Theta_t} \exp\left(-\int_t^{t+\Delta} i_{t'}^{\mathcal{B}} \mathrm{d}t' \right) \right], \tag{23}$$

$$\Theta_t := \exp\left(\int_0^t \left(\sigma_{t'}^{\vartheta} - \sigma_{t'}^{\mathcal{MB}} \right) \mathrm{d}Z_{t'} - \frac{1}{2} \int_0^t \left(\sigma_{t'}^{\vartheta} - \sigma_{t'}^{\mathcal{MB}} \right)^2 \mathrm{d}t' \right).$$

Without aggregate shocks, $\mathcal{P}_t^{\mathcal{B}}(\Delta)$ simply depends inversely on the path of future nominal short rates over the interval $[t, t+\Delta]$, in line with the expectations hypothesis.

[10] In the full model with money, the maintained assumption is that bonds can never be used for payments, regardless of their maturity. If short-term bonds could be a substitute for money in some transactions but not long-term bonds, the irrelevance proposition would break down, of course.

[11] This also implies that any variation in nominal bond prices is offset by variation in the price level. It clarifies that Proposition 3 relies on *nominal* debt and flexible prices. With real debt, as in Angeletos (2002), the maturity structure would affect the asset span and Proposition 3 would no longer hold.

More generally, covariation with Θ_t may generate term premia, but keeps the inverse relationship between the $i_t^{\mathcal{B}}$-path and bond prices intact. Note that Θ_t is a martingale, $\mathbb{E}_t[\Theta_{t+\Delta}/\Theta_t] = 1$, so that $\mathcal{P}_t^{\mathcal{B}}(\Delta)$ is only affected by its presence in equation (23) if $\Theta_{t+\Delta}/\Theta_t$ covaries with $\exp\left(-\int_t^{t+\Delta} i_{t'}^{\mathcal{B}} \mathrm{d}t'\right)$.

3 Understanding the Value of Nominal Assets

One of the most fundamental issues in monetary economics concerns the question what determines the value of money, or, equivalently, the general price level. This question has two interrelated aspects. The first concerns the valuation of nominal assets in a given equilibrium. Specifically, what are the motives for individual agents in this equilibrium for holding these assets and how do these motives quantitatively determine the equilibrium value of nominal assets? In this section, we analyze this first aspect by characterizing the determinants of the portfolio demand for nominal government liabilities. We show that this demand arises from three sources, (i) fiscal backing, (ii) safe asset services, and (iii) medium of exchange services, and provide a decomposition of the total value into these three components.

The second aspect is concerned with determinacy of the monetary equilibrium: under which condition is there a unique equilibrium or at least a unique prediction for the value of money across equilibria? This aspect is related to the first in that each of the sources of value of nominal assets may impose restrictions on the equilibrium dynamics for the value of money. Whether these restrictions are sufficient to determine the latter uniquely depends on a number of other considerations, including on how government policy is adjusted off-equilibrium. A comprehensive determinacy analysis is technically involved and beyond the scope of this paper. We pick up this issue in a companion paper, Brunnermeier and Merkel (2025). To keep the present paper self-contained, we summarize a simple uniqueness result from that paper in Section 3.3.

3.1 Key Valuation Equation

Recall that $\vartheta_t = \frac{q_t^{\mathcal{MB}} K_t}{q_t K_t}$ expresses the value of nominal government liabilities as a fraction of total wealth. Our key equation decomposes ϑ_t into three components:

Proposition 4. *In any equilibrium, the nominal wealth share ϑ_t satisfies the equation*

$$\vartheta_t = \overbrace{\mathbb{E}_t\left[\int_t^\infty e^{-\rho(t'-t)}\hat{\tau}_{t'}dt'\right]}^{\text{tax backing}} + \overbrace{\mathbb{E}_t\left[\int_t^\infty e^{-\rho(t'-t)}(1-\vartheta_{t'})^2\tilde{\sigma}_{t'}^2\vartheta_{t'}dt'\right]}^{\text{safe asset services}}$$
$$+ \underbrace{\mathbb{E}_t\left[\int_t^\infty e^{-\rho(t'-t)}\vartheta_{t'}^{\mathcal{M}}\lambda_{t'}^{\mathcal{M}}\nu_{t'}\vartheta_{t'}dt'\right]}_{\text{medium of exchange services}}, \tag{24}$$

where $\hat{\tau}_t := \frac{\tau_t K_t}{q_t K_t}$ denotes the ratio of tax revenues to total wealth.

Proof. The equation follows from integrating equation (20) for the evolution of ϑ_t forward in time and using that $\lim_{T\to\infty} e^{-\rho(T-t)}\vartheta_T = 0$ because ϑ_T is bounded. Additional details are provided in Appendix A.3. □

Equation (24) can be thought of as an asset pricing or valuation equation that expresses the value of all nominal assets in the economy, as a fraction of total wealth, in terms of three sources of value. At this stage, the three labels are merely suggestive. In the following, we utilize the dynamic trading perspective to asset valuation developed in Brunnermeier et al. (2024b) to discuss the precise meaning of each of the terms in equation (24).

3.2 Intuition: Dynamic Trading Perspective

To form intuition, let us for a moment switch to discrete time notation, denoting the discrete time step by $\mathrm{d}t > 0$. Consider an individual agent i who holds at time t a fraction $\eta_t^{\mathcal{MB},i}$ of the outstanding stocks of nominal liabilities. In the period from t to $t + \mathrm{d}t$, these holdings generate cash flows from three sources.

First, the holdings entitle the agent to a fraction of the total payments that the government makes to its liability holders in the period, which equals the primary surplus $\tau_t K_t \mathrm{d}t$ (by the government budget constraint (3)).[12] The agent is entitled to a fraction $\eta_t^{\mathcal{MB},i}$ of these payments, so receives a cash flow $\eta_t^i \tau_t K_t \mathrm{d}t$ from the government.

[12] Total payments to government liability holders as a collective equal interest payments net of funds collected from issuing additional government liabilities, as the latter represent a negative cash flow to government liability holders. That net payment is precisely the primary surplus.

Second, the agent may choose to change holdings at the end of the period to a fraction $\eta_{t+\mathrm{d}t}^{\mathcal{MB},i}$ of the outstanding stock, not necessarily equal to the original holdings $\eta_t^{\mathcal{MB},i}$. In this case, the agent also realizes a trading cash flow $(\eta_t^{\mathcal{MB},i} - \eta_{t+\mathrm{d}t}^{\mathcal{MB},i})q_{t+\mathrm{d}t}^{\mathcal{MB}}K_{t+\mathrm{d}t}$ from selling government liabilities to other agents.

Third, in addition to these direct cash flows from the nominal asset portfolio itself, there is an indirect cash flow that accrues to the capital return but arises from the contribution of money to production via the cash constraint (1). Specifically, a fraction $\vartheta_t^{\mathcal{M}}$ of the agent's nominal asset holdings take the form of money and relax this constraint, which increases productivity of the agent's capital holdings. Let us define the "shadow dividend yield" that money earns in production as the marginal contribution of money holdings to production, that is

$$\frac{\partial(u_t^i a_t k_t^i)}{\partial(\mathfrak{m}_t^i/\mathcal{P}_t)} = \lambda_t^{\mathcal{M}}\nu_t,$$

where the right-hand expression is a consequence of the envelope theorem. The agent's total money holdings at date t are $\eta_t^{\mathcal{MB},i}\vartheta_t^{\mathcal{M}}q_t^{\mathcal{MB}}K_t$, so the total "shadow cash flow" from relaxation of the cash constraint is $\eta_t^{\mathcal{MB},i}\vartheta_t^{\mathcal{M}}\lambda_t^{\mathcal{M}}\nu_t q_t^{\mathcal{MB}}K_t\mathrm{d}t$.

In sum, the total cash flows, both direct and indirect, generated by agent i's nominal asset holdings over the period from t to $t+\mathrm{d}t$ are

$$\underbrace{\eta_t^{\mathcal{MB},i}\tau_t K_t\mathrm{d}t}_{\text{direct CF from government}} + \underbrace{(\eta_t^{\mathcal{MB},i} - \eta_{t+\mathrm{d}t}^{\mathcal{MB},i})q_{t+\mathrm{d}t}^{\mathcal{MB}}K_{t+\mathrm{d}t}}_{\text{direct CF from trading}} + \underbrace{\eta_t^{\mathcal{MB},i}\vartheta_t^{\mathcal{M}}\lambda_t^{\mathcal{M}}\nu_t q_t^{\mathcal{MB}}K_t\mathrm{d}t}_{\text{indirect "shadow dividend" from money}}.$$

Denote by $\xi_t^i = e^{-\rho t}\frac{c_0^i}{c_t^i}$ the stochastic discount factor that individual agent i uses to price future cash flows. Assuming a transversality condition, which is a necessary condition of optimal behavior, the value that agent i attaches to her own holdings of nominal assets at time t, denoted by v_t^i, is precisely the expected present value of these direct and indirect cash flows, discounted at the SDF ξ_t^i:

$$v_t^i = v_t^{\text{gov. CF},i} + v_t^{\text{trading CF},i} + v_t^{\text{money dividend},i},$$

where

$$\xi_t^i v_t^{\text{gov. CF},i} := \mathbb{E}_t\left[\sum_{j=0}^{\infty} \xi^i_{t+(j+1)\mathrm{d}t}\eta^{\mathcal{MB},i}_{t+j\mathrm{d}t}\tau_{t+j\mathrm{d}t}K_{t+j\mathrm{d}t}\mathrm{d}t\right],$$

$$\xi_t^i v_t^{\text{trading CF},i} := \mathbb{E}_t\left[\sum_{j=0}^{\infty} \xi^i_{t+(j+1)\mathrm{d}t}(\eta^{\mathcal{MB},i}_{t+j\mathrm{d}t} - \eta^{\mathcal{MB},i}_{t+(j+1)\mathrm{d}t})q^{\mathcal{MB}}_{t+(j+1)\mathrm{d}t}K_{t+(j+1)\mathrm{d}t}\right],$$

$$\xi_t^i v_t^{\text{money dividend},i} := \mathbb{E}_t\left[\sum_{j=0}^{\infty} \xi^i_{t+(j+1)\mathrm{d}t}\eta^{\mathcal{MB},i}_{t+j\mathrm{d}t}\vartheta^{\mathcal{M}}_{t+j\mathrm{d}t}\lambda^{\mathcal{M}}_{t+j\mathrm{d}t}\nu_{t+j\mathrm{d}t}q^{\mathcal{MB}}_{t+j\mathrm{d}t}K_{t+j\mathrm{d}t}\mathrm{d}t\right].$$

If we aggregate over the holdings of all agents i, then, in equilibrium, the total valuation of all agents should equal the market value of government liabilities, that is,

$$q_t^{\mathcal{MB}}K_t = \int v_t^i \mathrm{d}i = \int v_t^{\text{gov. CF},i}\mathrm{d}i + \int v_t^{\text{trading CF},i}\mathrm{d}i + \int v_t^{\text{money dividend},i}\mathrm{d}i. \tag{25}$$

While the previous discussion and notation were kept suggestive to highlight the economic ideas, everything can be made formally precise in our continuous-time environment at the expense of heavier notation. We do so in Appendix A.3. There, we provide rigorous continuous-time definitions of v_t^i and its three subcomponents and show that equation (25) is indeed satisfied.

It turns out that the decomposition in equation (25) is precisely the decomposition in our key valuation equation (24), once we divide both sides of the former equation by aggregate wealth q_tK_t:

Proposition 5. *The terms in the decomposition of Proposition 4 are related to the terms in equation (25) as follows:*

$$\text{“tax backing” term} = \frac{\int v_t^{\text{gov. CF},i}di}{q_tK_t},$$

$$\text{“safe asset services” term} = \frac{\int v_t^{\text{trading CF},i}di}{q_tK_t},$$

$$\text{“medium of exchange services” term} = \frac{\int v_t^{\text{money dividend},i}di}{q_tK_t}.$$

The previous proposition justifies the labels we have chosen in the decomposition of Proposition 4 and allows us to attach a precise meaning to each of the terms. The first term, "tax backing", is the present value of future primary surpluses, the tax backing of nominal assets, valued from the perspective of individual holders of these assets. The second term, "safe asset services", is the aggregate present value of the trading cash flows agents receive from trading nominal assets with each other. As discussed in Brunnermeier et al. (2024b), this value is related to the role that government debt plays as a safe asset. The third term, "medium of exchange services", is the aggregate value of the liquidity services that money provides to agents by relaxing the cash constraint in production.

Let us conclude this discussion with a remark on the second term that captures the value of equilibrium trades. The presence of this term may seem puzzling to some readers because, in the aggregate, trades among agents wash out, $\int (\eta^{\mathcal{MB},i}_{t+j\mathrm{d}t} - \eta^{\mathcal{MB},i}_{t+(j+1)\mathrm{d}t})\mathrm{d}i = 0$ for all j. However, the *value* of these trades for individual agents does not need to wash out if buyers and sellers have different marginal utilities. For example, if a seller, who realizes a positive cash flow, has a higher marginal utility than the buyer, who realizes a negative cash flow, then the net value of a marginal trade is positive. In complete markets, this is impossible because the marginal gains from trading assets are zero. Formally, all individual SDFs are aligned when markets are complete, $\xi^{i_1} = \xi^{i_2}$ for any $i_1, i_2 \in \mathbb{I}$. Indeed, if this was the case, then we could conclude that $\int v_t^{\text{trading CF},i}\mathrm{d}i = 0$. However, when markets are incomplete, then individual SDFs are not aligned and the aggregate present value of equilibrium asset trades in government liabilities can be strictly positive.[13]

3.3 The Value of Nominal Assets and Determinacy

Monetary models are often plagued with indeterminacy, of nominal variables such as the price level but possibly also of real equilibrium quantities. Indeterminacy is an issue for analyzing the transmission of shocks and (monetary) policy actions. In models with multiple equilibria, it is generally difficult to disentangle the effects arising from

[13] This remark also clarifies why only *idiosyncratic* risk appears in the second term of the decomposition in Proposition 4. In this model, all agents are symmetrically exposed to aggregate shocks, so that aggregate risk markets are essentially complete. In a richer model that features market incompleteness with respect to aggregate shocks, such as Brunnermeier and Sannikov (2014, 2016a), Merkel (2020), Alexandrov and Brunnermeier (2025), also aggregate risk terms would appear in the safe asset services term.

the internal propagation of the model from those arising from, possibly inadvertent, changes in the equilibrium selection.

It turns out that all three sources of portfolio demand for nominal government liabilities uncovered in Proposition 4 can provide a nominal anchor that leads to price level determinacy, as well as uniqueness of the equilibrium more broadly. We provide a rigorous technical foundation and economic discussion of this claim in a companion paper, Brunnermeier and Merkel (2025). For the purposes of this paper, the following uniqueness proposition is sufficient (compare Brunnermeier and Merkel, 2025, Proposition 8): For any given exogenous processes for taxes as a proportion of aggregate wealth, $\hat{\tau}_t \in [0, \rho)$, interest paid on money, $i_t^{\mathcal{M}} \in \mathbb{R}$, and the money share of nominal liabilities, $\vartheta_t^{\mathcal{M}} \in [0, 1]$, there is at most one asymptotically monetary equilibrium. An asymptotically monetary equilibrium is an equilibrium according to Definition 1 with the additional property that ϑ_t remains bounded away from zero for large t. From now on, we will always restrict attention to asymptotically monetary equilibria and assume that policy variables $\hat{\tau}_t$, $i_t^{\mathcal{M}}$, and $\vartheta_t^{\mathcal{M}}$ are exogenously specified to ensure uniqueness.

We remark that the requirement that $\hat{\tau}_t$, $i_t^{\mathcal{M}}$, and $\vartheta_t^{\mathcal{M}}$ are exogenously given is, in a certain sense, without loss of generality for the analysis of the effects of shocks and monetary policy. There may be good reasons in some contexts to specify government actions in terms of feedback rules that react to endogenous variables instead of an exogenous specification. However, given any equilibrium that arises under such a feedback rule, the very same equilibrium also arises under an exogenous policy rule that simply sets each policy variable to its value in that given equilibrium irrespective of other endogenous variables. We therefore do not restrict the set of possible equilibria in any way by restricting attention to policies that specify $\hat{\tau}_t$, $i_t^{\mathcal{M}}$, and $\vartheta_t^{\mathcal{M}}$ exogenously.

4 Model Dynamics and Shock Transmission

4.1 Steady State

Let us suppose that exogenous processes $\tilde{\sigma}_t = \tilde{\sigma}$, $\nu_t = \nu$, and $a_t = a$ are constant and that the economy is in a steady state with constant ϑ_t, $\vartheta_t^{\mathcal{M}}$, and $\hat{\tau}_t$. Imposing a steady state in equation (20) leads to the equation $\mathbb{E}_t[d\vartheta_t] = 0$, which can be reduced to an equation in the single unknown ϑ by substituting in the equations from Lemma 3. The resulting equation may have more than one solution. However, if we restrict attention to *monetary equilibria*, i.e., equilibria in which nominal assets have a positive real value, then the steady state is unique. For the following proposition, we parameterize the stance of fiscal policy by $\breve{\mu}^{\mathcal{MB}} = -\hat{\tau}/\vartheta$, which must also be constant in steady state, instead of $\hat{\tau}$ as this results in a cleaner existence condition.

Proposition 6 (Steady State). *For given $\tilde{\sigma}$, ν, a, $\vartheta^{\mathcal{M}}$, and $\breve{\mu}^{\mathcal{MB}}$, there is a monetary steady state equilibrium if and only if*

$$\tilde{\sigma}^2 + \vartheta^{\mathcal{M}}\nu > \rho + \breve{\mu}^{\mathcal{MB}} > 0.$$

If this condition is satisfied, the monetary steady-state value for ϑ_t is unique and given by

$$\vartheta = \max\{\vartheta^0, \vartheta^c\},$$

where ϑ^0 and ϑ^c are the unique solutions in $[0,1]$ to the equations

$$\rho + \breve{\mu}^{\mathcal{MB}} = (1-\vartheta^0)^2\tilde{\sigma}^2,$$
$$\rho + \breve{\mu}^{\mathcal{MB}} + \rho(\vartheta^c)^{\varphi}\left(\frac{\nu}{a}\frac{\vartheta^{\mathcal{M}}}{1-\vartheta^c}\right)^{1+\varphi} = (1-\vartheta^c)^2\tilde{\sigma}^2 + \vartheta^{\mathcal{M}}\nu.$$

The two cases in the proposition, (i) $\vartheta = \vartheta^0$ and (ii) $\vartheta = \vartheta^c$, correspond to the two cases in Lemma 3 for the utilization rate equation. Case (i) obtains when the cash constraint is slack, so that there are no medium-of-exchange services from money. The value of nominal assets derives then fully from their role as a store of value, either due to tax backing or safe asset services. We therefore call this case the *store-of-value regime*. In case (ii), in contrast, the medium-of-exchange role of money plays an important role. While tax backing and safe asset services are still present, we will see in the following, in the context of numerical examples, that demand for nominal assets is considerably less sensitive to variation in these sources of the value of government liabilities.[14] Instead, it is mainly determined by the monetary friction. We therefore call this case the *medium-of-exchange regime*.

[14] In this paper, we content ourselves with this somewhat unsharp observation based on a numerical example. For a related but slightly different specification of money demand, Merkel (2020) proves a theoretically sharp dichotomy across the two regimes. We can approach this dichotomy here in the limit $\varphi \to \infty$. See also Brunnermeier et al. (2020, Chapter 8) for additional details.

Figure 1 depicts comparative statics with respect to changes in $\hat{\tau}$, $\tilde{\sigma}$, ν, and a. Solid lines in the first and second columns show how the nominal wealth share ϑ and utilization u, respectively, depend on the four parameters. In the parameter range that falls into the store-of-value regime, lines are colored blue. The red line segments correspond to the medium-of-exchange regime. The third column shows the contribution of each of the three components in our key equation (24), fiscal backing, safe asset services, and medium of exchange services, to the total value of nominal assets.

Comparative statics for ϑ (first column) are qualitatively as one would expect from equation (24): the nominal wealth share is strictly increasing in tax backing ($\hat{\tau}$) and idiosyncratic risk ($\tilde{\sigma}$), which raises safe asset services, and it is weakly increasing in any variable that tightens the cash constraint (1) and therefore increases medium-of-exchange services. The latter means that ϑ is weakly decreasing in velocity ν and weakly increasing in productivity a, as both a lower ν and higher a expand the need for transaction media. These relationships are strict in the medium-of-exchange regime, when the constraint binds, while the parameters ν and a have locally no impact on ϑ in the store-of-value regime, when the constraint is slack. Even for parameters $\hat{\tau}$ and $\tilde{\sigma}$, in which ϑ is strictly increasing, there is nevertheless a notable change in the sensitivity of ϑ to changes in these variables across regimes. ϑ is considerably more sensitive to changes in these variables in the store-of-value than in the medium-of-exchange regime. We therefore observe an approximate dichotomy that we have hinted at previously: parameters that affect the medium-of-exchange role of money (ν and a) only affect the value of nominal assets in the medium-of-exchange regime, whereas variables that affect components of the value of nominal assets that arise from the store of value role ($\hat{\tau}$ and $\tilde{\sigma}$), almost only affect ϑ in the store-of-value regime.

Comparative statics for ϑ translate into comparative statics for utilization u according to the formulas from Lemma 3. The second column of Figure 1 visualizes this. Parameter changes in the first two rows do not affect the optimal utilization decision directly but only via their implication on ϑ. In the store-of-value regime, u moves inversely to ϑ. We will explain the economics of this in more detail in Section 4.3. In the medium-of-exchange regime, in contrast, u changes in the same direction as ϑ because a larger real value of nominal assets, expands available real money balances. The third row concerns a change in maximum velocity ν, which is only relevant in the medium-of-exchange regime and moves ϑ and u in opposite direction for obvious reasons. The final row shows that utilization is globally increasing in productivity, but the increase

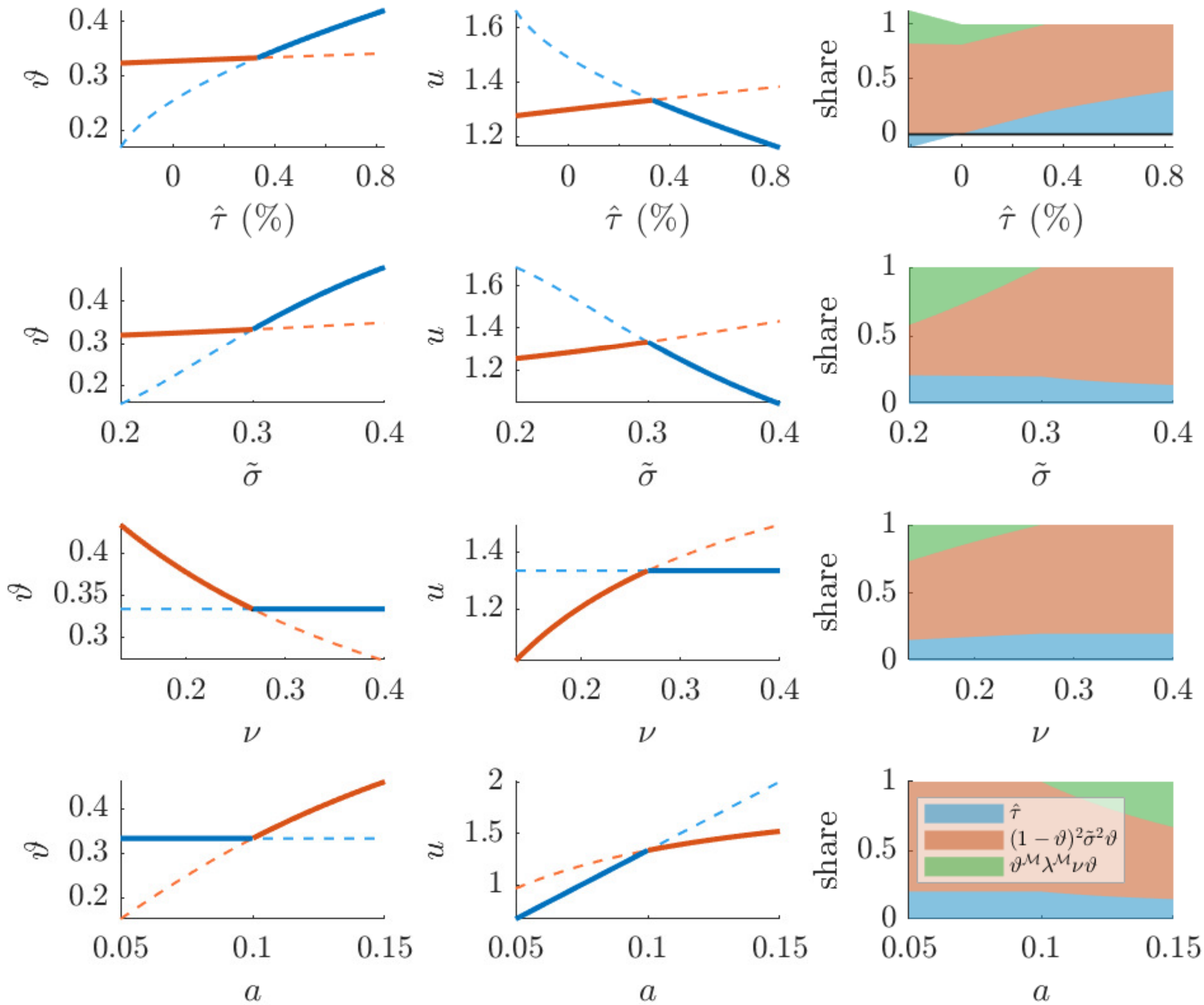

Figure 1: Steady-state values of nominal wealth share ϑ (left column), utilization u (middle column), and decomposition of nominal asset value (right column) as a function of parameter values ($\hat{\tau}$, $\tilde{\sigma}$, ν, a). In the left and middle columns, solid lines depict equilibrium quantities. For blue line segments, the economy is in the store-of-value regime, for red line segments, it is in the medium-of-exchange regime. Blue dashed lines depict the solutions ϑ^0, u^0 conditional on a non-binding cash constraint. Red dashed lines depict the solutions ϑ^c, u^c conditional on a binding cash constraint.

is somewhat muted in the medium-of-exchange regime as agents try to economize on the need for additional media of exchange when there is more production.

The final column in Figure 1 shows how the relevance of each of the three components of the value of nominal assets according to equation (24) shifts with parameters. The picture that emerges is largely what one would expect from the equation and the previous discussion. Changes in taxes $\hat{\tau}$ affect primarily the share of the total value of nominal assets that is due to tax backing. For all other parameter changes, the share of tax backing is inversely related to ϑ but quantitatively almost constant. Instead, these parameter changes primarily affect the relative contribution of the two types of service flows to the overall value of nominal assets.

4.2 Shock Transmission

Figures 2 and 3 depict impulse response functions in response to a one-time mean-reverting shock to a_t ("supply shock"), $\tilde{\sigma}_t$ ("safe asset demand shock"), and ν_t ("money demand shock"). Parameters were chosen in a way such that the economy is always in the store-of-value regime in Figure 2 and always in the medium-of-exchange regime in Figure 3 but that the steady states in both model variants are otherwise comparable quantitatively (same ϑ, very similar u and ι). In all experiments, we assume that the tax-wealth ratio $\hat{\tau}_t$, the money share of nominal liabilities $\vartheta_t^{\mathcal{M}}$, and the reserve rate $i_t^{\mathcal{M}}$ are held constant throughout. For simplicity, the figures have been created under the assumption that the exogenous state X evolves deterministically. The picture that emerges in the presence of aggregate risk would be qualitatively similar.

The first column in Figure 2 shows the response of the nominal wealth share ϑ_t, the capital utilization rate u_t, the investment rate ι_t, and the nominal price level $\mathcal{P}_t$ to a positive supply shock (an increase in productivity a_t) in the store-of-value regime. As expected from the comparative statics discussed in the previous section, an increase in productivity does not affect portfolio demand ϑ_t and leads to a strong increase in capital utilization in this regime. Investment also increases, so that capital and real output grow at a faster rate. Because our assumption on policy implies here that the growth rate of nominal assets $\mu^{\mathcal{MB}}$ is not affected by the shock,[15] there is a transitory reduction in inflation, so that the price level eventually settles at a lower trend.

[15]By the government budget constraint (4), $\mu_t^{\mathcal{MB}} = i_t^{\mathcal{M}} - \frac{\hat{\tau}_t}{\vartheta_t}$ in the store-of-value regime and none of

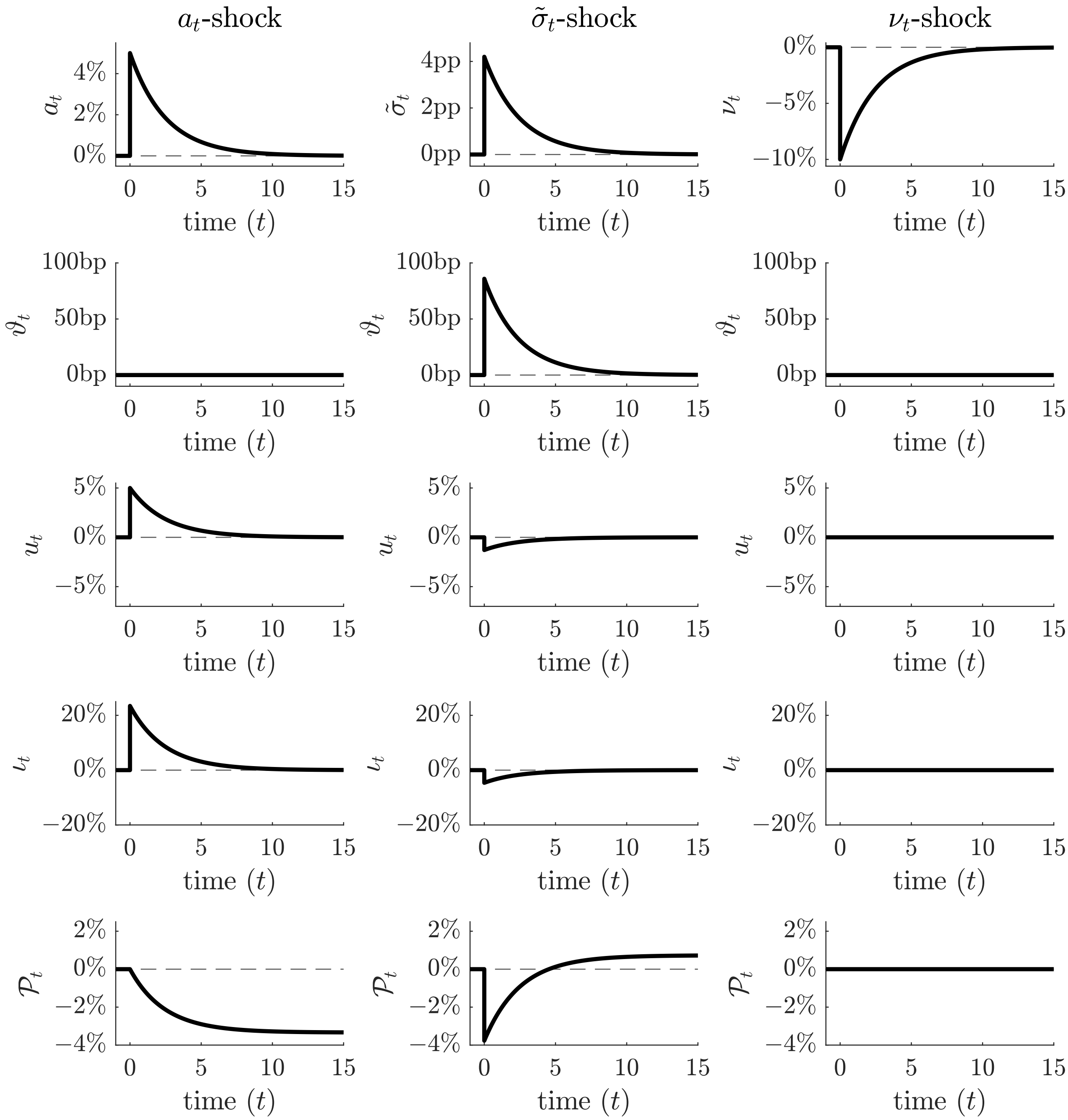

Figure 2: Impulse response functions in response to supply (left), safe asset demand (middle), and money demand (right) shocks when the economy is in the *store-of-value regime*. The first row in each column depicts the exogenous path fed into the model for each of the three experiments. All other variables are held fixed at steady state. Panels for a_t, ν_t, u_t, ι_t, $\mathcal{P}_t$ show percentage deviations of these variables from steady state, panels for $\tilde{\sigma}_t$ and ϑ_t show absolute deviations from steady state.

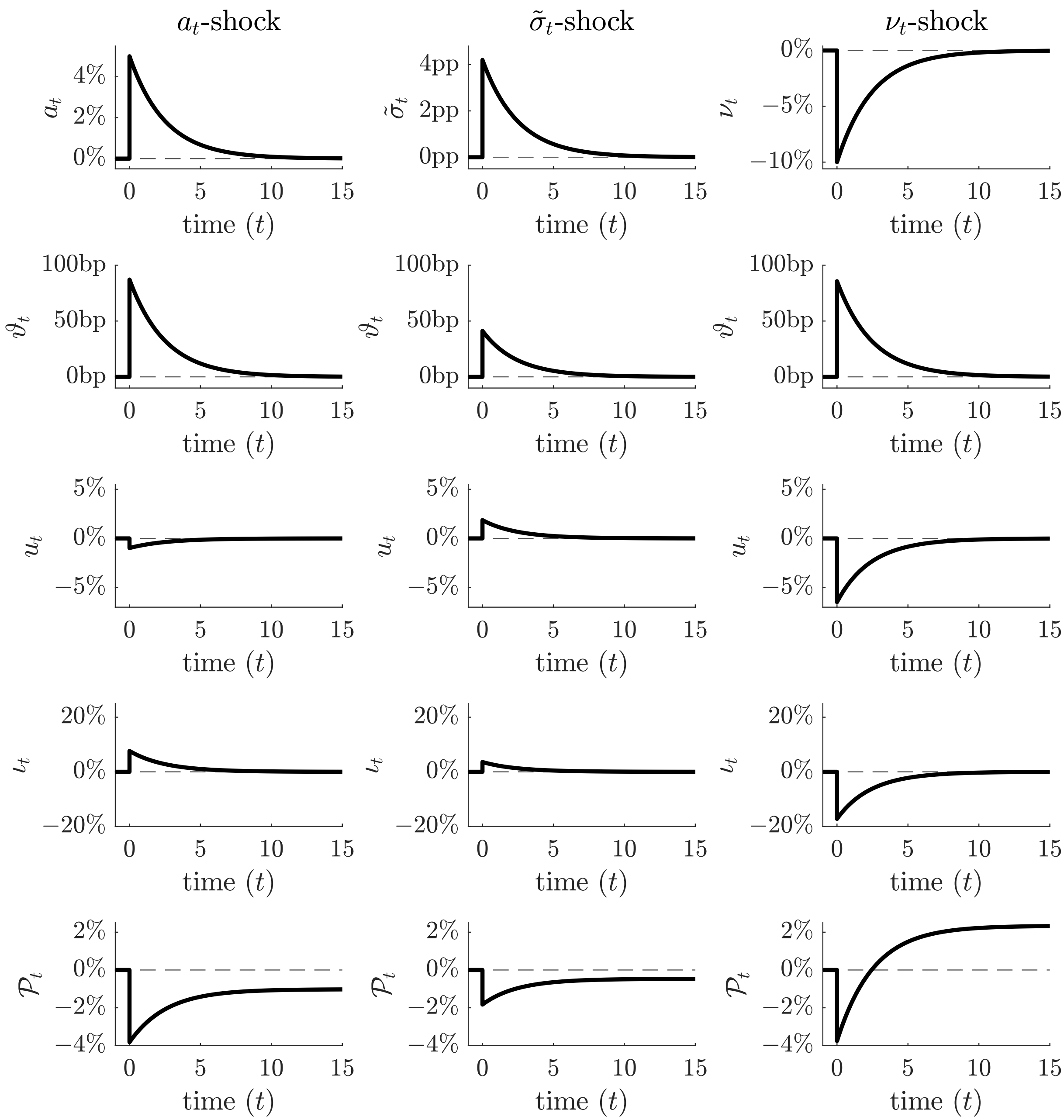

Figure 3: Impulse response functions in response to supply (left), safe asset demand (middle), and money demand (right) shocks when the economy is in the *medium-of-exchange regime*. The first row in each column depicts the exogenous path fed into the model for each of the three experiments. All other variables are held fixed at steady state. Panels for a_t, ν_t, u_t, ι_t, $\mathcal{P}_t$ show percentage deviations of these variables from steady state, panels for $\tilde{\sigma}_t$ and ϑ_t show absolute deviations from steady state.

The first column in Figure 3 depicts responses to the same positive supply shock in the medium-of-exchange regime. In this case, portfolio demand for nominal assets increases, there is a drop instead of a rise in capital utilization, and the positive response of investment is substantially attenuated. This is because the additional supply aggravates the shortage of media of exchange, so that agents both demand more money and economize on the need for it by reducing capital utilization. As a result, the price level jumps down immediately, but eventually recovers as the money shortage eases. Once again, because the overall impact on investment and capital growth is still positive, the price level eventually settles at a lower trend.

The second columns in Figures 2 and 3 show dynamics in response to a positive safe asset demand shock due to an increase in idiosyncratic risk $\tilde{\sigma}_t$ in the store-of-value and medium-of-exchange regime, respectively. As expected from the comparative statics analysis in the previous section, such a shock strongly increases portfolio demand for nominal assets (ϑ_t) when the economy is in the store-of-value regime but the response is substantially weaker in the medium-of-exchange regime. Utilization and the investment rate decrease in the store-of-value regime, but they increase in the medium-of-exchange regime. The reduction of u_t and ι_t in the store-of-value regime is due to the Tobin Effect, that we discuss in the following Section 4.3. While this effect is also present in the medium-of-exchange regime, there is then an additional force arising from the monetary friction: an increase in the real value of money eases the shortage of media of exchange. This additional force always overcompensates the effect of the Tobin Effect on utilization, so that u_t unambiguously rises in response to larger safe asset demand in the medium-of-exchange regime. In contrast, the effect on investment ι_t is ambiguous. Investment can rise, as depicted here, but it is also possible to construct examples in which ι_t falls. In either regime, the increase in portfolio demand for nominal assets leads to a reduction in the nominal price level $\mathcal{P}_t$ on impact that is followed by gradual inflation as ϑ_t reverts back to its steady-state level. The two regimes differ with respect to the magnitude of the impact effect, which is bigger in the store-of-value regime, and with respect to the eventual trend path on which $\mathcal{P}_t$ settles on, which can be traced back to the different effects on capital accumulation across the two regimes.

The final columns in Figures 2 and 3 depict dynamics in response to a positive money demand shock due to a reduction in velocity ν_t. This shock has only an effect on the economy in the medium-of-exchange regime, as otherwise money is abundant

the variables on the right-hand side of this equation react to the shock.

and marginal changes in velocity do not affect anything.[16] Let us therefore focus on the impulse responses depicted in Figure 3. As in the steady-state analysis, lower velocity tightens the cash constraint, which raises demand for nominal assets (ϑ_t) and reduces capital utilization (u_t) as agents economize on transactions. This is accompanied by a sharp contraction in investment activity (ι_t). The nominal price level ($\mathcal{P}_t$) falls on impact to accommodate the increased demand for nominal assets, but eventually overshoots.

4.3 Money and Growth: the Tobin Effect

Our model has interesting implications for the interaction between demand for nominal assets and real investment, which is the driver of economic growth in this model economy. Specifically, we observe from Lemma 2 that, given resource utilization u_t, the physical investment rate ι_t is strictly decreasing in the nominal wealth share ϑ_t. The intuition here is that, at least for given u_t, a wealth effect from nominal asset wealth raises consumption demand and crowds out real investment.

By goods market clearing (equation (12)), investment is the difference between goods supply and consumption demand,

$$I_t = Y_t - C_t = a u_t K_t - \rho q_t K_t.$$

An increase in the portfolio demand for nominal assets (ϑ_t) does not affect goods supply, except via changes in the incentives to utilize the capital stock, from which we abstract for the moment by keeping u_t fixed. But it increases private consumption demand via a wealth effect: for fixed capital price q_t^K, a higher ϑ_t raises total wealth per unit of capital $q_t = q_t^K + q_t^{\mathcal{MB}}$. Market clearing can be achieved by two margins of adjustment: (i) a fall in investment, or (ii) a fall in the capital price, which dampens the increase in wealth q_t and consumption demand. Only the first margin of adjustment is possible in equilibrium as the Tobin's q condition (16) prevents a fall in the capital price.[17]

[16] Large reductions in velocity may of course cause a temporary transition into the medium-of-exchange regime. The parameters in Figure 2 have been chosen deliberately to avoid this outcome.

[17] If we were to contemplate such a fall in q^K, agents would aggressively react by lowering investment activity ι_t due to the frictionless linear investment technology, which pushes q_t^K back to its equilibrium value 1 and all the adjustment pressure into margin (i). In a model extension with capital adjustment costs, some limited capital price adjustment would be possible, but marginal (i) would still be operative except in the limit where capital cannot be adjusted at all.

Our model thus formalizes an argument made by Tobin (1965) that portfolio choice between monetary and capital assets is a key determinant of real investment. This link between demand for monetary assets and investment is called the *Tobin effect*.[18]

While we have held capital utilization u_t fixed so far, a similar wealth effect logic applies to utilization effort itself, magnifying the Tobin effect, so long as the degree to which the monetary friction binds is held constant. Formally, we observe from Lemma 3 that for fixed multiplier $\lambda_t^{\mathcal{M}}$, an increase in the nominal wealth share ϑ_t leads to a reduction in u_t.[19] The economics are again best understood in terms of wealth effects from nominal wealth. In analogy to a consumption-labor choice, agents with higher consumption c_t^i require a larger real compensation per unit of additional utilization effort u_t^i to accept the associated disutility cost. The real compensation, the analog to the wage in models with labor choice, is proportional to capital holdings k_t^i. As nominal wealth raises the aggregate consumption-capital capital ratio ($C_t/K_t = \rho q_t$), agents find it optimal to reduce effort by scaling back utilization u_t.

There can, however, be a countervailing effect on utilization from the role of money as a medium of exchange, as we have previously observed in Figures 1 and 3. An increase in nominal wealth for fixed money share $\vartheta_t^{\mathcal{M}}$ also makes media of exchange more abundant and relaxes the cash constraint (1). As a scarcity of media of exchange acts as a drag on economic activity, relaxing the constraint tends to increase both utilization and investment, mitigating and possibly even overturning the Tobin effect.

5 Monetary Policy (and its Fiscal Implications)

5.1 Interest Rate Policy

Interest rate policy sets targets for the nominal interest rates $i_t^{\mathcal{M}}$ and/or $i_t^{\mathcal{B}}$ and uses the available policy instruments to achieve these target rates. Instead of describing interest rate policy directly in terms of the pair $(i_t^{\mathcal{M}}, i_t^{\mathcal{B}})$, it is conceptually useful to parameterize the two dimensions of interest rate policy in terms of the overall level

[18] The previous explanation of the Tobin effect relies on a wealth effect from nominal liabilities in positive net supply. The Tobin effect does not apply to inside money or safe assets created by the financial sector, which are in zero net supply, see Merkel (2020).

[19] $\lambda_t^{\mathcal{M}}$ may remain fixed for an increase in ϑ_t if either the monetary friction does not bind or if $\vartheta_t^{\mathcal{M}}$ is reduced to prevent real balances from rising as well.

of interest rates and the spread between the two assets, that is in terms of the two variables,

$$i_t := \vartheta_t^{\mathcal{M}} i_t^{\mathcal{M}} + (1 - \vartheta_t^{\mathcal{M}}) i_t^{\mathcal{B}}, \qquad \Delta i_t := i_t^{\mathcal{B}} - i_t^{\mathcal{M}}.$$

The first variable, i_t, is the nominal rate of return on the total portfolio of nominal assets. We refer to this variable as the level of nominal interest rates. The second variable, Δi_t, is the spread between bonds and money, which equals the liquidity premium, compare equation (18): $\Delta i_t = \lambda_t^{\mathcal{M}} \nu_t$.

Within limits, these two dimensions of interest rate policy, adjusting the level i_t and adjust the spread Δi_t, can be implemented by adjusting the policy instruments $i_t^{\mathcal{M}}$ and $\vartheta_t^{\mathcal{M}}$.[20] In the following, we will discuss each dimension separately, assuming the other is kept unchanged. In reality, most monetary policy actions have effects on both dimensions simultaneously. This observation notwithstanding, the separation serves conceptual clarity.

Increasing Rate Level i_t Funded by Debt Expansion. We first consider adjustments to the level of interest rates, i_t. Changing the level of interest rates has fiscal implications via the government budget constraint (4), which we can rewrite equivalently as

$$\hat{\tau}_t + \mu_t^{\mathcal{MB}} \vartheta_t = i_t \vartheta_t.$$

Consider, for example, a rate hike. Increasing i_t increases the nominal interest rate burden on government liabilities (right-hand side), which can be funded either by an increase in taxes ($\hat{\tau}_t$) or by an increase in the growth rate of government liabilities ($\mu_t^{\mathcal{MB}}$).

Let us first discuss the latter possibility of increasing the debt growth rate. This form of interest rate policy turns out to be (super-)neutral:

Proposition 7 (Effects of Debt-funded Rate Level Policy)**.** *Consider a reference equilibrium* $\mathbf{e}$ *with taxes* $\hat{\tau}_t$ *and nominal interest rates* $i_t, \Delta i_t$. *Let* $\mathbf{e}'$ *be an alternative equilibrium with identical initial conditions, the same taxes* $\hat{\tau}_t$*, and the same nominal interest rate spread* Δi_t*, but with a different process* i_t' *for the level of nominal rates.*

1. Superneutrality*: The real allocations (all variables in Definition 1 other than* $\mathcal{MB}$, $\mathcal{P}$, $i^{\mathcal{M}}$*, and* $i^{\mathcal{B}}$*) are the same in* $\mathbf{e}$ *and* $\mathbf{e}'$*.*

[20] A central bank can set $i_t^{\mathcal{M}}$ by altering the interest rate paid on reserves, while it can use open market purchases or sales of bonds to set $\vartheta_t^{\mathcal{M}}$, as we discuss in Section 5.2.

2. Price level dynamics*: The initial price level is unaffected, $\mathcal{P}_0' = \mathcal{P}_0$, and the effect on inflation is* neo-Fisherian*, i.e., inflation increases one for one with nominal rates:*

$$\frac{d\mathcal{P}_t'}{\mathcal{P}_t'} = \frac{d\mathcal{P}_t}{\mathcal{P}_t} + (i_t' - i_t)dt.$$

3. Implementation*: To implement i_t', the government sets the reserve rate to $i_t^{\mathcal{M}\,\prime} = i_t^{\mathcal{M}} + i_t' - i_t$ while keeping $\vartheta_t^{\mathcal{M}\,\prime} = \vartheta_t^{\mathcal{M}}$ as in the reference equilibrium.*

This proposition is the flip side of the classical dichotomy propositions that we have discussed in Section 2.3 (Propositions 1 and 2). There, we have observed that we can add any predictable process to inflation dynamics and still obtain an equilibrium with the same real allocation. All we need to do is to add the same predictable process also to all nominal interest rates, so that real rates of return on nominal assets are unaffected. Because the interest rate changes are funded by growing government liabilities, there are also no fiscal side effects.

Intuitively, hiking the interest rate and issuing more government paper to fund it is like printing money and handing it out to existing money holders. Printing money dilutes existing holdings, but distributing the newly printed money exactly compensates for the dilution in the sense that each holder holds the same fraction of the total money stock as before. The only thing that has changed is the total nominal quantity of money in circulation. But because the unit of account does not matter with flexible prices, the real allocation is unaffected and nominal prices adjust one-for-one with the increase in the outstanding money stock. Relative to this analogy, the only difference in our model is that the government does not just print money to pay interest, which would raise $\vartheta_t^{\mathcal{M}}$ and lower Δi_t. Instead, it pays a fraction $\vartheta_t^{\mathcal{M}}$ of additional interest payments with newly printed money and the remaining fraction $1 - \vartheta_t^{\mathcal{M}}$ with newly printed bonds.

Increasing Rate Level i_t Funded by Fiscal Tightening. Let us now turn to the opposite extreme, that the nominal growth rate of government liabilities $\mu_t^{\mathcal{MB}}$ is kept unchanged, and a change in the interest burden is fully absorbed by changes in taxes. For concreteness, suppose the monetary authority hikes interest rates, $i_t' > i_t$ for all t.[21] To balance the budget, the government needs to raise the ratio of taxes to debt in all future

[21] We could relax this assumption and assume that interest rates are strictly raised only for some future dates and states, but this would make the precise mathematical statements in the following proposition more involved.

periods, $\hat{\tau}'_t/\vartheta'_t > \hat{\tau}_t/\vartheta_t$. Increased tax backing leads to an increase in portfolio demand for nominal assets by equation (24):

Proposition 8 (Effects of Tax-funded Rate Level Policy)**.** *Consider two equilibria* $\mathbf{e}$, $\mathbf{e}'$ *with identical initial conditions and* $\mu_t^{\mathcal{MB}\prime} = \mu_t^{\mathcal{MB}}$, $\Delta i'_t = \Delta i_t$, $\hat{\tau}_t, \hat{\tau}'_t \geq 0$, *and* $i'_t > i_t$ *for all* t.

1. Real effects of rate hike*: Taxes and the nominal wealth share are higher in* $\mathbf{e}'$ *than in* $\mathbf{e}$, *and capital utilization and investment are lower: for all* t,

$$\vartheta'_t > \vartheta_t, \qquad \hat{\tau}'_t > \hat{\tau}_t, \qquad u'_t < u_t, \qquad \iota'_t < \iota_t.$$

2. Initial price level*: For small* t, $\mathcal{P}'_t < \mathcal{P}_t$.

3. Implementation*: To implement* $\mathbf{e}'$, *in any state with* $\Delta i_t > 0$, *the money share of nominal liabilities must fall relative to* $\mathbf{e}$ *and the reserve rate rises by less than* $i'_t - i_t$*:*

$$\vartheta_t^{\mathcal{M}\prime} < \vartheta_t^{\mathcal{M}}, \qquad i_t^{\mathcal{M}\prime} - i_t^{\mathcal{M}} < i'_t - i_t.$$

   In a state with $\Delta i_t = 0$, *implementation requires* $i_t^{\mathcal{M}\prime} - i_t^{\mathcal{M}} = i'_t - i_t$.

In particular, the previous proposition implies that a rate hike funded by a fiscal tightening is both deflationary on impact, $\mathcal{P}'_0 < \mathcal{P}_0$, and leads to a contraction in current and future economic output, $Y'_t < Y_t$ for all $t \geq 0$. This contrasts with a debt-funded rate hike as in Proposition 7. For the effects of interest rate policy that involves adjusting the level of interest rates, the nature of the fiscal reaction is therefore crucial. Rate hikes have contractionary effects for output and the price level only if they are accompanied by a fiscal tightening. Symmetrically, rate reductions are only expansionary if they are accompanied by looser fiscal policy.

Beyond the prediction that the price level decreases for small t, the previous proposition does not make a clear prediction concerning price level dynamics for large t. This is because there are both inflationary and deflationary effects from the policy. The expected inflation rate can be backed out by a type of Fisher equation that relates expected inflation to the difference in the nominal rates on bonds and the real risk-free rate, corrected by an inflation risk premium (because bonds are generally not risk-free). Using the characterization of equilibrium from Section 2.2, we can determine the real risk-free rate, which allows us to derive an explicit representation for expected inflation:

Lemma 6. *In any equilibrium, the expected inflation rate is given by*

$$\pi_t := \frac{\mathbb{E}_t[d\mathcal{P}_t]}{\mathcal{P}_t dt} = i_t^{\mathcal{M}} + \Delta i_t - \left(1 - \frac{\Delta i_t}{\nu_t}\right) a_t u_t + \frac{\hat{\tau}_t}{1-\vartheta_t} + (1-\vartheta_t)\tilde{\sigma}_t^2 + \frac{(\sigma_t^{\vartheta})^2}{1-\vartheta_t} + \delta.$$

It is generally difficult to say anything about the how a change in policy affects the term $\frac{(\sigma_t^{\vartheta})^2}{1-\vartheta_t}$ without additional assumptions on aggregate shocks. For all other terms in the equation, we can use Proposition 8 to discuss how each of these terms is affected by a tax-funded rate hike. The first term, $i_t^{\mathcal{M}}$, is the only relevant one in the case of a debt-funded rate hike as in Proposition 7 and the reason for the neo-Fisherian prediction in that case. Here, Proposition 8 implies that $i_t^{\mathcal{M}}$ increases only one-for-one with the desired change in i_t if the monetary friction is not binding, $\Delta i_t = 0$. Otherwise, inflation pressures arising from this term tend to be lower than in the neo-Fisherian benchmark. The second term, Δi_t, is by construction unaffected by the rate level hike. The third and fourth term both increase, suggesting additional inflation pressures, whereas the $\tilde{\sigma}$-dependent term decreases, dampening inflation pressures.

In sum, if we disregard aggregate risk implications (the term $\frac{(\sigma_t^{\vartheta})^2}{1-\vartheta_t}$), then inflation increases by more than under the neo-Fisherian benchmark if the rate hike is tax-funded in the absence of binding monetary and financial frictions ($\Delta i_t = \tilde{\sigma}_t = 0$). This is because then the rate hike depresses output and growth (lower u_t and ι_t), so that the real interest rate falls. With (binding) frictions, this conclusion may not be true. Sufficiently positive Δi_t and $\tilde{\sigma}_t$ tend to dampen inflation pressures, so that the overall inflation impact of a rate hike could be less than one-for-one.[22]

Increasing Rate Spread Δi_t. Unlike a change in the level of interest rates, a change in the rate spread Δi_t that leaves the level i_t unaffected does not have direct fiscal implications from an altered interest burden for a given debt value (it may have indirect fiscal implications arising a change in the real value of debt). Let us therefore suppose that $\hat{\tau}_t$ is not adjusted together with Δi_t. In this case, the effects on the nominal wealth share ϑ_t depend on how the present value of medium of exchange services in equation (24) is affected. The impact on the latter is potentially ambiguous: to engineer a rise in $\Delta i_t = \lambda_t^{\mathcal{M}} \nu_t$, the government needs to reduce $\vartheta_t^{\mathcal{M}}$, but what matters for the total

[22]In principle, it is even possible that the net impact on inflation is negative if either friction is sufficiently important, although we have not found a plausible numerical example that delivers this. Specifically, for large Δi_t, implementation of an increase in i_t may, in fact, require a reduction in $i_t^{\mathcal{M}}$. Similarly, for large $\tilde{\sigma}_t$, the term $(1-\vartheta_t)\tilde{\sigma}_t^2$ in Lemma 6 could dominate.

value of medium of exchange services is the product of the two. Indeed, after using the equations for optimal utilization and the multiplier $\lambda_t^{\mathcal{M}}$ from Lemma 3, we can write the medium of exchange service flow at time t as

$$\lambda_t^{\mathcal{M}} \nu_t \vartheta_t^{\mathcal{M}} \vartheta_t = \nu_t \vartheta_t^{\mathcal{M}} \vartheta_t \left(1 - \rho \frac{\left(\nu_t \vartheta_t^{\mathcal{M}} \vartheta_t \right)^{\varphi}}{\left(a_t \left(1 - \vartheta_t \right) \right)^{1+\varphi}} \right),$$

which is inverted u-shaped in $\vartheta_t^{\mathcal{M}}$.

Depending on whether the value of these service flows decreases or increases, raising the current and future spread Δi_t may either increase or decrease ϑ. Let us suppose that we are in the former, "regular", case, in which a larger spread Δi_t raises the service flow at time t. This case is the only relevant one in the limit $\varphi \to \infty$, when the output adjustment in response to a shortage of media of exchange is muted. In this "regular" case, a monetary tightening in the form of an increase of the path of liquidity spreads Δi_t leads to similar conclusions as for a rate level hike accompanied by fiscal tightening:

Proposition 9 (Effects of Rate Spread Policy)**.** *Consider two equilibria* $\mathbf{e}$, $\mathbf{e}'$ *with identical initial conditions,* $\hat{\tau}_t' = \hat{\tau}_t$, $i_t' = i_t$, *and* $\Delta i_t' > \Delta i_t$ *for all t and suppose that* $\vartheta_t^{\mathcal{M}'} \Delta i_t' > \vartheta_t^{\mathcal{M}} \Delta i_t$ *for all t ("regular case").*

1. Real effects*: The nominal wealth share is higher in* $\mathbf{e}'$ *than in* $\mathbf{e}$*, and capital utilization and investment are lower: for all t,*

$$\vartheta_t' > \vartheta_t, \qquad u_t' < u_t, \qquad \iota_t' < \iota_t.$$

2. Initial price level*: For small t,* $\mathcal{P}_t' < \mathcal{P}_t$*.*

3. Implementation*: To implement* $\Delta i_t' > \Delta i_t$*, the money share of nominal liabilities must be lower in* $\mathbf{e}'$*, and to keep* i_t' *from rising above* i_t*, the reserve rate must be lower as well,*

$$\vartheta_t^{\mathcal{M}\prime} < \vartheta_t^{\mathcal{M}}, \qquad i_t^{\mathcal{M}\prime} < i_t^{\mathcal{M}}.$$

With respect to inflation dynamics for large t, there are again potentially ambiguous effects in analogy to the situation of a tax-funded rate hike, although most effects point to an increase in inflation. Specifically, Lemma 6 continues to hold. Under the assumptions of Proposition 9, it is easy to show that the combined impact on the first two terms in that lemma, $i_t^{\mathcal{B}} = i_t^{\mathcal{M}} + \Delta i_t$, must be positive. Similarly, also the next two terms in Lemma 6 increase. Aside from the aggregate risk term, which cannot be signed without additional assumptions, the only term that could possibly offset these inflation pressures is $(1 - \vartheta_t)\tilde{\sigma}_t^2$. Once again, this term rises because the monetary tightening raises the real value of safe assets, which improves risk sharing and increases the real interest rate. This means that there is an unambiguous conclusion for inflation, if idiosyncratic and aggregate risk are both small. Then, inflation rises also when the rate spread Δi_t is increased.

In total, we observe that a monetary contraction implemented by an increase in Δi_t has qualitatively very similar effects to a monetary contraction implemented by an increase in i_t funded with taxes. However, there are some noteworthy differences. On the one hand, a monetary contraction via a hike in the level of rates only has the desired effects when it triggers a fiscal contraction, whereas a monetary contraction via an increase in the rate spread does not require such fiscal support. On the other hand, a monetary contraction via an increase in the spread requires a larger output sacrifice than a comparable (fiscally supported) rate hike in the sense made precise in the next proposition:

Proposition 10. *Consider two policies, (1) a tax-funded i_t-hike as in Proposition 8 and (2) a Δi_t-increase without fiscal adjustment as in Proposition 9. Suppose that both policies lead to the same ϑ_t-process in equilibrium. Then output Y_t in the equilibrium under policy (1) is strictly larger than under policy (2) for all t.*

Finally, we note that adjusting the spread Δi_t may not have the desired effects because of the possibility that we are not in the "regular case". This happens if demand for money as a medium of exchange is very elastic to Δi_t, so that the rise in Δi_t lowers demand for nominal assets strongly.[23] In this case, the initial price level rises, $\mathcal{P}_0' > \mathcal{P}_0$ and the contractionary effects on capital utilization and investment are mitigated or even overturned.

Unpleasant Monetarist Arithmetic. In an influential paper, Sargent and Wallace (1981) point out that, even in an environment in which the price level is tightly linked to the

[23]This is particular relevant for small φ. Low φ means that it is not very costly for agents, in utility terms, to reduce capital utilization to economize on the need for media of exchange. The availability of this alternative adjustment margin raises the elasticity of money demand to the liquidity spread Δi_t.

money supply via a quantity equation, a monetary contraction without a change in fiscal policy lowers the price level at best temporarily and leads to larger inflation eventually. They call this observation the "unpleasant monetarist arithmetic" and trace it to the role played by seigniorage financing of government deficits: If the monetary tightening restricts the flow of seigniorage revenues that contribute to the government's budget initially and primary fiscal surpluses are unchanged, then debt must grow at a faster rate, so that, eventually, more seigniorage is required to balance the budget.

In the previous discussion, our observation has been more generally that a monetary tightening, interpreted either as an increase in i_t or an increase in Δi_t, tends to increase inflation in the long run, even if possibly supported by a fiscal tightening, at least if idiosyncratic risk is low or absent. But while the conclusion is similar, the seigniorage channel emphasized by Sargent and Wallace (1981) is only part of the story.

To understand the role of seigniorage from money creation clearly,[24] let us keep taxes $\hat{\tau}_t$ fixed and shut down idiosyncratic risk, $\tilde{\sigma}_t \equiv 0$. One measure of seigniorage is the interest advantage the government enjoys by issuing money instead of debt. The flow benefit from this advantage is $\Delta i_t \frac{\mathcal{M}_t}{\mathcal{P}_t} = \vartheta_t^{\mathcal{M}} \lambda_t^{\mathcal{M}} \nu_t \vartheta_t \cdot q_t K_t$, where the first factor is precisely our flow measure of medium-of-exchange service flows in equation (24). For fixed taxes and no idiosyncratic risk, we may read this equation as an intertemporal budget constraint for flow seigniorage. If, for whatever reason, the government seeks to lower seigniorage extracted from medium-of-exchange services over some time interval $[0, T]$, then either the initial value of government liabilities ϑ_0 must fall, which requires an upward adjustment in $\mathcal{P}_0$, or the government must eventually raise seigniorage after date T to keep the present value of medium-of-exchange service flows from falling. Sargent and Wallace (1981) exclude the first possibility in their (baseline) model,[25] so that a temporary reduction in seigniorage must necessarily be matched by an eventual increase to keep the initial value of money stable. This logic is quite general and works also in our setting.

How is the timing of seigniorage flows related to inflation dynamics? In Sargent and Wallace (1981), fairly immediately. They consider a setting with no output or growth effects from money, which can be mapped into ours by imposing exogenous constraints

[24] This type of seigniorage is the focus of Sargent and Wallace (1981). In our setting, one may think of safe asset creation as a second source of seigniorage (see, e.g., Brunnermeier et al., 2021, 2024b).

[25] In their model, a quantity equation ensures that $\mathcal{P}_t$ is strictly proportional to $\mathcal{M}_t$ at all times. Because they consider a policy that does not change $\mathcal{M}_0$, any adjustment in $\mathcal{P}_0$ is prevented, so that the initial *real* value of government liabilities is essentially exogenously given, not endogenous as in our model.

$\iota_t^i = \bar{\iota}$, $u_t^i = \bar{u}$ on agent choices. We provide a brief analysis of that model variant in Appendix B.3. In such a setting, and under the assumption that the cash constraint is always binding, there is a tight one-to-one relationship between medium-of-exchange service flows (or seigniorage) and inflation (the expected change in $1/\mathcal{P}_t$, to be precise):

$$\vartheta_t^{\mathcal{M}} \lambda_t^{\mathcal{M}} \nu_t \vartheta_t = \frac{\rho}{\nu_t} \left(\rho - \frac{\mathbb{E}_t[\mathrm{d}(1/\mathcal{P}_t)]}{1/\mathcal{P}_t \mathrm{d}t} - i_t^{\mathcal{M}} \right). \tag{26}$$

In Sargent and Wallace (1981), there is an additional assumption $i_t^{\mathcal{M}} = 0$ ("money = cash"), and then seigniorage is necessarily monotonically related to inflation. Seigniorage therefore plays indeed the pivotal role in their argument, which follows the logical chain

$$\begin{aligned} \text{lower inflation over } [0, T] \quad &\Rightarrow \quad \text{lower seigniorage over } [0, T] \\ &\Rightarrow \quad \text{larger seigniorage over } [T, \infty) \\ &\Rightarrow \quad \text{larger inflation over } [T, \infty). \end{aligned}$$

In our setting, in contrast, there is no such simple relationship between seigniorage and inflation. Even in equation (26), which requires exogenous u_t and ι_t and no idiosyncratic risk, the link between seigniorage and inflation can be broken by varying the reserve rate $i_t^{\mathcal{M}}$. More generally, when u_t and ι_t are endogenous and idiosyncratic risk is present, Lemma 6 describes inflation dynamics. As discussed previously, an increase in the spread Δi_t is plausibly still associated with an increase in inflation. But there are a number of additional channels that work through adjustments in output, growth, and safe asset demand. These are not tightly related to seigniorage from money creation.

Long-term Debt and Stepping on a Rake Effect. The discussion so far has assumed that government liabilities are floating rate or, equivalently, of infinitesimal duration. We now use our extended model with long-duration bonds from Section 2.4 to investigate how our conclusions for interest rate policy are affected by the maturity structure of government debt. Because of Proposition 3, our conclusions for the effect of interest rate policy on real allocations hold regardless of the maturity of government bonds, so long as i_t and Δi_t are still defined with reference to the nominal short rate $i_t^{\mathcal{B}}$. As mentioned in Section 2.4, the absence of real effects does not mean that the dynamics of the nominal price level do not change. Recall that

$$\mathcal{P}_t = \frac{\mathcal{MB}_t}{q_t^{\mathcal{MB}} K_t}.$$

For short-term debt, the effects of a policy change at $t = 0$ on the initial price level $\mathcal{P}_0$ only depend on the real effects in the denominator because $\mathcal{MB}_0$ is a backward-looking state that does not adjust to interest rate changes. At any future date, the numerator $\mathcal{MB}_t$ matters only to the extent that the growth rate of nominal liabilities is adjusted to clear the government budget constraint. For long-duration debt, in contrast, $\mathcal{MB}_t = \mathcal{M}_t + \mathcal{B}_t$ is the nominal value of government liabilities, and the $\mathcal{B}_t$-component of this value adjusts directly to the path of future nominal interest rates. To see this clearly, recall that $\mathcal{B}_t$ is monotonically related to the zero coupon bond prices $\mathcal{P}_t^{\mathcal{B}}(\Delta)$ (equation (22)). From equation (23), we observe that each $\mathcal{P}_t^{\mathcal{B}}(\Delta)$ is inversely related to the (expected) path of nominal short rates $i_t^{\mathcal{B}}$ for $t \in [0, \Delta]$. Hence, an increase in future nominal short rates directly depresses $\mathcal{B}_0$. Even in the situation of Proposition 7 (debt-funded rate policy), a rate hike then leads to a downward adjustment in the initial price level $\mathcal{P}_0$. Hence, the adjustment in long-term debt prices can overcome the neo-Fisherian prediction that (debt-funded) rate hikes are inflationary.[26]

For a mean-reverting interest rate hike, nominal bond prices must eventually return to their original values, so that, in the long run, the nominal value of government liabilities $\mathcal{MB}_t$ is the same as if all liabilities were of infinitesimal duration.[27] But this means that, eventually, the price level path $\mathcal{P}_t$ under long-duration debt must catch up with the one that would obtain under the same rate policy if all debt had infinitesimal duration. As a consequence, the initial price level drop after a rate hike is accompanied by a subsequent *inflation increase* that is even larger than in the neo-Fisherian benchmark of floating-rate debt. Following Sims (2011), we call this the "stepping on a rake effect" of interest rate policy.

[26] In a flexible price model like ours, only the initial price level $\mathcal{P}_0$ falls, but subsequent inflation still rises. Sticky prices, as in Li and Merkel (2024), would smooth out dynamics, such that the initial price level jump turns into a finite transition period with reduced inflation or even deflation.

[27] Recall, once more, that the real allocation is unaffected by the maturity structure, and this includes the real government budget constraint. Therefore, the government's finances are not affected differentially by a rate hike for different debt maturity.

5.2 Open Market Operations and Quantitative Easing

In this section, we discuss central bank asset purchases. These purchases serve the purpose of implementing changes in the money share of nominal assets, $\vartheta_t^{\mathcal{M}}$, and therefore in the liquidity spread Δi_t. Beyond effects on liquidity, asset purchases do not have real effects, in line with an observation first made in Wallace (1981). However, asset purchases may give the central bank a greater control over the evolution of the price level $\mathcal{P}_t$, even in the absence of any fiscal support.

Central Bank Purchases of Short-term Bonds. Let us first consider the baseline model with floating rate government bonds and suppose the central bank engages in an open market purchase of government bonds, funded by expanding the monetary base. To introduce central bank purchases in a rigorous way, let us replace the consolidated government budget constraint (3) from our model setup by two separate constraints,

$$\begin{aligned} \mathrm{d}\mathcal{B}_t &= (i_t^{\mathcal{B}}\mathcal{B}_t - \mathcal{P}_t\tau_t K_t)\mathrm{d}t - \mathrm{d}\mathcal{Q}_t, \\ \mathrm{d}\mathcal{M}_t &= i_t^{\mathcal{M}}\mathcal{M}_t\mathrm{d}t + \mathrm{d}\mathcal{Q}_t, \end{aligned}$$

where $\mathcal{B}_t$ denotes the quantity of nominal bonds held by private agents, exclusive of any debt that might be held by the central bank, and $\mathcal{Q}_t$ denotes the cumulative nominal value of bond purchases, i.e., $\mathrm{d}\mathcal{Q}_t$ is the nominal value of bonds purchased between t and $t+\mathrm{d}t$.[28]

The key observation is that controlling the purchase process $\mathcal{Q}_t$ is mathematically equivalent to controlling the money share in nominal liabilities,

$$\vartheta_t^{\mathcal{M}} = \frac{\mathcal{M}_t}{\mathcal{MB}_t}.$$

Indeed, $\mathrm{d}\mathcal{Q}_t$ cancels out in $\mathrm{d}\mathcal{MB}_t = \mathrm{d}\mathcal{M}_t + \mathrm{d}\mathcal{B}_t$, hence,

$$\mathrm{d}\vartheta_t^{\mathcal{M}} = \frac{\mathrm{d}\mathcal{M}_t}{\mathcal{MB}_t} - \vartheta_t^{\mathcal{M}}\frac{\mathrm{d}\mathcal{MB}_t}{\mathcal{MB}_t} = \vartheta_t^{\mathcal{M}}\left(i_t^{\mathcal{M}}\mathrm{d}t - \frac{\mathrm{d}\mathcal{MB}_t}{\mathcal{MB}_t}\right) + \frac{\mathrm{d}\mathcal{Q}_t}{\mathcal{MB}_t}.$$

By adjusting $\mathrm{d}\mathcal{Q}_t$ suitably, the central bank can attain any desired $\vartheta_t^{\mathcal{M}} \in [0,1]$.

[28] By not explicitly keeping track of the quantity of bonds held by the central bank, we abstract here from the possibility that sufficiently negative $\mathrm{d}\mathcal{Q}_t$ might eventually deplete the central banks' bond stock. We do so because, in this paper, we are not concerned with any strategic interaction or institutional frictions between the central bank and the treasury.

Corollary 1. *Open market purchases, dQ_t, are equivalent to adjusting the money share in nominal liabilities, $\vartheta_t^{\mathcal{M}}$.*

What is the effect of $\mathrm{d}Q_t > 0$ while keeping the reserve rate $i_t^{\mathcal{M}}$ and taxes $\hat{\tau}_t$ constant? By the previous equation, it raises $\vartheta_t^{\mathcal{M}}$, which by Lemma 3 reduces $\lambda_t^{\mathcal{M}}$ for given ϑ_t. This lowers both the rate spread $\Delta i_t = \lambda_t^{\mathcal{M}} \nu_t$ and the rate level $i_t = i_t^{\mathcal{M}} + (1 - \vartheta_t^{\mathcal{M}})\Delta i_t$, with the effects discussed in Section 5.1.

Central Bank Purchases of Long-term Bonds. Central bank purchases of long-term government bonds can be analyzed in full analogy in the context of the model extension presented in Section 2.4. In fact, the formal arguments that have lead us to conclude Corollary 1 remain valid almost word by word, so that the conclusion continues to hold: open market purchases of (long-term) bonds are equivalent to adjusting $\vartheta_t^{\mathcal{M}}$.

Nevertheless, there is one aspect about bond purchases, or, equivalently, changes in $\vartheta_t^{\mathcal{M}}$, in the presence of long-term bonds that is absent in our baseline model. Namely, raising $\vartheta_t^{\mathcal{M}}$ reduces the overall duration of nominal assets held by the private sector. While the duration of nominal assets is irrelevant for real allocations (Proposition 3), it affects the sensitivity of the price level to a given change in nominal interest rates in the stepping on a rake effect. If the fiscal authority has issued long-term bonds across the maturity spectrum, the central bank can exploit this feature to recalibrate the price level response to future interest rate policy, possibly independently from the choice of $\vartheta_t^{\mathcal{M}}$. By purchasing and selling different bonds across the maturity spectrum, the central bank can alter the average duration of bonds held by the private sector and thereby prepare how the next interest rate change will affect the nominal price level.[29]

For example, purchasing long-term bonds in exchange for short-term bonds (or money) shortens the average duration of government liabilities. This mitigates the stepping on a rake effect, making the price level less sensitive to future rate changes.

Purchases of Private Assets and Wallace Neutrality. The previous observation that long-term bond purchases are not fundamentally different from short-term bond purchases can be generalized substantially. We now present a result that even arbitrary asset purchases are neutral if they have no implications for fiscal policy (changes in $\hat{\tau}_t$), idiosyncratic risk exposures, and liquidity (changes in Δi_t). Under these assumptions

[29] Li and Merkel (2024) and Alexandrov and Brunnermeier (2025) show how nominal asset duration modifies the effects of interest rate policy in a related model with sticky prices, where classical dichotomy does not hold, so that different effects on the price level translate into different effects on real variables.

all sources of the value of money are unaffected from the perspective of our key equation (24), so that there should be indeed no real effects from the asset purchase. In credit to Wallace (1981), this type of neutrality is called *Wallace neutrality*.[30]

Formally, let us suppose that there is a generic asset in zero net supply with nominal return[31]

$$\mathrm{d}i_t^{\mathcal{A}} = \mathbb{E}_t[\mathrm{d}i_t^{\mathcal{A}}] + \sigma^{\mathcal{A}}(X_t)\mathrm{d}Z_t$$

that does not load on idiosyncratic shocks. Both the government and private agents can trade in the market for this generic asset. Denote by $\mathcal{A}_t \geq 0$ the nominal value of the asset purchased by the government. The consolidated nominal government budget constraint is then

$$\mathrm{d}\mathcal{MB}_t = \left(i_t^{\mathcal{M}}\mathcal{M}_t + i_t^{\mathcal{B}}\mathcal{B}_t - \mathcal{P}_t\tau_t K_t\right)\mathrm{d}t - \underbrace{\left(\mathcal{A}_t\mathrm{d}i_t^{\mathcal{A}} - \mathrm{d}\mathcal{A}_t\right)}_{\text{cash flows from asset portfolio}}.$$

Private agents $i \in \mathbb{I}$ have a portfolio choice between capital, money, bonds, and the generic asset.

We present formal details of this model extension and an equilibrium definition in Appendix B.2. Our key result is that the real allocation is independent of the $\mathcal{A}_t$-path, provided that taxes $\hat{\tau}_t$ and the money share in *net* nominal government liabilities, $\vartheta_t^{\mathcal{M}} := \mathcal{M}_t/\mathcal{G}_t$, where $\mathcal{G}_t := \mathcal{M}_t + \mathcal{B}_t - \mathcal{A}_t$, are kept fixed when changing $\mathcal{A}_t$. Specifically, our key equation (24) continues to hold if we define $\vartheta_t := \frac{\mathcal{G}_t/\mathcal{P}_t}{q_t K_t}$ as the share of private-sector net wealth that is due to nominal government liabilities net of assets $\mathcal{A}_t$ sold to the government.

Asset purchases do not affect the fundamental sources of value of nominal assets in that equation. Tax backing and medium-of-exchange services are unaffected by assumption on fixed $\hat{\tau}_t$ and $\vartheta_t^{\mathcal{M}}$. Safe asset services are unaffected because the generic asset does not load on idiosyncratic shocks, so introducing this asset does not improve risk sharing in dimensions in which markets are incomplete.

[30] We only claim neutrality for the real allocation, but not for the price level path, whereas in Wallace (1981), also the price level path is unaffected by asset purchases. This is because, in that paper, lump-sum taxes are assumed to adjust to sterilize the effects on the price level, whereas such taxes are absent here.

[31] Our formal arguments would also work if the volatility loading of the real instead of the nominal return process was taken exogenously given. We have chosen the formulation here as it leads to slightly simpler notation and formal arguments that are more symmetric to maturity structure irrelevance, Proposition 3.

Nevertheless, just like for long-term bond purchases, asset purchases can affect the price level path by altering the exposure of nominal net government liabilities $\mathcal{G}_t$ to aggregate shocks. For example, if $\mathrm{d}i_t^{\mathcal{A}}$ loads negatively on the first component of $\mathrm{d}Z_t$, then a larger $\mathcal{A}_t$ means that $\mathcal{G}_t$ has a larger (positive) loading on that component of $\mathrm{d}Z_t$, so the evolution of the real value $\mathcal{G}_t/\mathcal{P}_t$ can only be independent of the $\mathcal{A}_t$-path, if also $\mathcal{P}_t$ has a larger loading on that component of $\mathrm{d}Z_t$ to offset the effect on $\mathcal{G}_t/\mathcal{P}_t$. If the central bank has access to a sufficiently rich menu of assets, it can effectively use asset purchases to control $\sigma_t^{\mathcal{P}}$, the unanticipated component of inflation. Together with debt-funded rate policy as in Proposition 7, the central bank may then be able to perfectly control the time evolution of $\mathcal{P}_t$ even without any fiscal support.[32]

5.3 Optimal Policy

We now turn to normative considerations. Our model allows for a clean analysis of optimal policy because any two allocations that feature the same initial wealth distribution can be Pareto ranked. In addition, just two variables are sufficient statistics for the welfare impact of any policy, the nominal wealth share ϑ_t and the capital utilization rate u_t. Formally, the following proposition holds.

Proposition 11 (Welfare Representation and Constrained Efficient Allocation)**.** *For any social welfare function that is strictly increasing in individual expected utilities $\{V_0^i\}_{i\in\mathbb{I}}$, and for any control variables that do not affect the initial wealth distribution $\{\eta_0^i\}_{i\in\mathbb{I}}$, maximizing the welfare function is equivalent to maximizing the objective*

$$\mathbb{E}\left[\int_0^\infty e^{-\rho t}\left(\mathcal{W}_{\vartheta,t}(\vartheta_t)+\mathcal{W}_{u,t}(u_t)\right)dt\right]$$

with respect to $\{u_t,\vartheta_t\}_{t=0}^\infty$, where

$$\mathcal{W}_{\vartheta,t}(\vartheta) = -\frac{1}{1-\vartheta}-\log\left(1-\vartheta\right)-\frac{\left(1-\vartheta\right)^2\tilde{\sigma}_t^2}{2\rho},$$

$$\mathcal{W}_{u,t}(u) = \frac{a_t u}{\rho}-\frac{u^{1+\varphi}}{1+\varphi}.$$

[32]The precise statement is that asset purchases and nominal rate policy can jointly control $\mathcal{P}_t$ for any given tax and money share processes $\hat{\tau}_t$ and $\vartheta_t^{\mathcal{M}}$, provided the central bank can choose both positive and negative asset positions (so possibly $\mathcal{A}_t < 0$) and it has access to d_Z distinct assets whose nominal volatility loadings are linearly independent at all states.

A constrained planner that can control portfolios and capital utilization directly but is unable to provide direct idiosyncratic risk insurance optimally chooses a $\tilde{\sigma}_t$-dependent nominal wealth share $\vartheta_t = \vartheta^{e*}(\tilde{\sigma}_t)$ and an a_t-dependent utilization rate $u_t = u^{e*}(a_t)$, where ϑ^{e*} and u^{e*} are both strictly increasing functions.

Constrained Efficient vs. Equilibrium Allocations. Motivated by the previous proposition, we ask how the nominal wealth share and utilization in any given competitive equilibrium relate to their constrained efficient levels, $\vartheta_t^{e*}, u_t^{e*}$. We start with utilization. It turns out that capital is always underutilized in a competitive equilibrium:

Lemma 7. *Equilibrium utilization satisfies the relationship*

$$u_t = \left[(1-\lambda_t^{\mathcal{M}})(1-\vartheta_t)\right]^{\frac{1}{\varphi}} u_t^{e*} < u_t^{e*}.$$

The reason for underutilization is as follows. By Proposition 11, the welfare effects of utilization are additively separated from any welfare benefits from nominal assets, and therefore optimal utilization should only depend on productivity a_t. It does so in a frictionless world without incomplete markets, monetary frictions, and distortions from capital taxes because then the first welfare theorem holds. Yet, in the equilibrium of our model, utilization is lowered compared to this frictionless benchmark for two reasons. First, whenever the monetary friction is binding ($\lambda_t^{\mathcal{M}} > 0$), agents optimally reduce utilization to economize on the need to hold money. Second, even in the absence of the monetary friction, the wealth effect from nominal assets discussed in Section 4.3 crowds out utilization effort relative to a situation without nominal assets, $\vartheta_t = 0$.

Let us next turn to the relationship between the equilibrium and constrained efficient levels of the nominal wealth share ϑ_t. Here, the situation is not as clear-cut as in Lemma 7. ϑ_t can be either too low or too high. We illustrate this with a specific example, a steady state economy as in Section 4.1.

Figure 4 compares the steady-state value for ϑ with the constrained efficient level ϑ^{e*} as a function of idiosyncratic risk $\tilde{\sigma}$. We observe that the red solid line, which depicts the equilibrium value for ϑ is above ϑ^{e*} (blue solid line) for both low and high values of idiosyncratic risk $\tilde{\sigma}$ but below ϑ^{e*} for intermediate values. That ϑ is too high for very low and very large idiosyncratic risk turns out to be a general feature in the steady state model, so long as taxes are nonnegative. But for sufficiently large taxes or sufficiently severe monetary friction, the intermediate region of too low ϑ may disappear as depicted in the dashed gray line in Figure 4, which plots the steady state for the same parameters as for the red solid line, except that $\hat{\tau}$ has been increased substantially.

That the equilibrium ϑ-value generally differs from its constrained efficient counterpart, even in the absence of tax distortions and when the monetary friction does not bind, can be traced back to pecuniary externalities due to incomplete markets. We refer the reader to a more in-depth discussion of these externalities to Brunnermeier et al. (2021, Section 5.4), which analyzes a version of our model without monetary frictions.

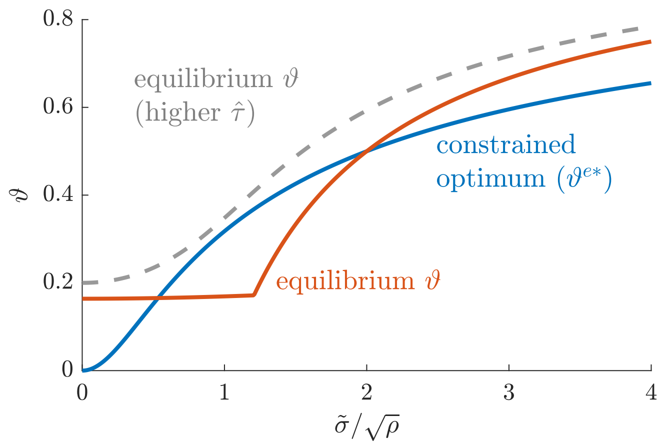

Figure 4: Constrained efficient ϑ-value (ϑ^{e*}) versus steady-state value in the competitive equilibrium as a function of $\tilde{\sigma}/\sqrt{\rho}$.

Problem of a Ramsey Planner. After having established the constrained efficient benchmark and its relationship to attainable equilibrium allocations, let us turn to optimal policy when the government has only access to the limited set of instruments $i_t^{\mathcal{M}}$, $\vartheta_t^{\mathcal{M}}$, and $\hat{\tau}_t$. By Lemma 7, it is generally impossible to implement $u_t = u_t^{e*}$, and because the gap between u_t and u_t^{e*} depends on ϑ_t, it is then generally also no longer optimal to target $\vartheta_t = \vartheta_t^{e*}$. In the following, we consider the problem of a Ramsey planner that chooses stochastic processes for $i_t^{\mathcal{M}}$, $\vartheta_t^{\mathcal{M}} \in [0,1]$, and $\hat{\tau}_t$ to maximize the welfare objective stated in Proposition 11.

For conceptual clarity, let us discuss the choice of each of the three policy instruments in isolation, holding the others fixed in a suitable sense to be clarified in the course of the discussion. We note that choosing $i_t^{\mathcal{M}}$ is equivalent to choosing the level of nominal interest rates i_t (compare Proposition 7), choosing $\vartheta_t^{\mathcal{M}}$ is closely related to choosing the interest rate spread Δi_t, and that, by equation (24), the path of ϑ_t can be directly controlled with the tax instrument $\hat{\tau}_t$, so that we can also consider ϑ_t instead of

$\hat{\tau}_t$ as the third instrument and leave the tax process required to implement it implicit.[33]

Optimal Level of Nominal Rates. Let us start with the choice of i_t, the level of nominal interest rates. As we have observed in Proposition 7, changing the level of nominal rates while holding both taxes and the interest rate spread constant is superneutral in the sense that it has no effects on the real allocation. We therefore conclude:

Corollary 2. *The choice of the level of nominal interest rates i_t is irrelevant for welfare.*

Optimal Money Share: Modified Friedman Rule. We now turn to the optimal choice of $\vartheta_t^{\mathcal{M}}$ holding the ϑ_t-process fixed. Recall that, for fixed ϑ_t, there is a (strictly decreasing) one-to-one relationship between $\vartheta_t^{\mathcal{M}}$ and the interest rate spread Δi_t, so that we can interpret this aspect of the policy choice as the choice of the spread Δi_t. Because ϑ_t is held fixed, changing Δi_t is relevant for welfare only insofar as it changes the equilibrium utilization process u_t. By Lemma 7, raising u_t is always beneficial in any equilibrium, so the optimal Δi_t-choice given ϑ_t is the one that maximizes u_t. Because u_t is strictly decreasing in $\lambda_t^{\mathcal{M}} = \frac{\Delta i_t}{\nu_t}$, this means the planner seeks to make Δi_t as small as possible:

Corollary 3 (Modified Friedman Rule)**.** *For any given ϑ_t-process, it is welfare-maximizing to set the interest rate spread Δi_t to the smallest feasible level in each period. This can be achieved by funding the government with money only, $\vartheta_t^{\mathcal{M}} = 1$.*

Intuitively, so long as providing additional money is not socially costly, a positive private opportunity cost of holding money, $\Delta i_t > 0$, is wasteful as it induces agents to economize on money holdings and avoid socially beneficial transactions, in this model by lowering production $u_t a_t K_t$. Based on this reasoning, the so-called Friedman rule, proposed in Friedman (1969), holds that optimal policy should set the spread Δi_t to zero. However, in the present setting, this may not be feasible for a given target process for ϑ_t. The optimal policy instead calls for lowering Δi_t so long as this remains consistent with the target for ϑ_t. It is always possible to lower Δi_t further if $\Delta i_t > 0$ and $\vartheta_t^{\mathcal{M}} < 1$ by swapping bonds for money. But this process cannot be taken further once money is the only government liability, $\vartheta_t^{\mathcal{M}} = 1$. Due to the shared logic with the original Friedman rule, we call this the "modified Friedman rule".

[33]By considering the equation in differential form and solving for $\hat{\tau}_t$, we observe that it is always possible to back out a tax process required to implement the desired target for ϑ_t.

Optimal ϑ. We turn to the final choice of the Ramsey planner, the choice of ϑ_t or, equivalently, of capital taxes. By the previous analysis, there is no need to study interactions between ϑ_t and other policy instruments than taxes if we set $\vartheta_t^{\mathcal{M}} = 1$ in line with the modified Friedman rule. The previous analysis also shows that the problem of choosing ϑ_t is separated across time (and states), so that we can split the dynamic problem of the planner into a continuum of (uncoupled) static problems, which are much simpler to analyze.

Specifically, for a fixed date t (and a fixed shock history), consider the choice of ϑ_t at this particular date in order to maximize the time-t flow in the welfare objective of Proposition 11,

$$\vartheta \mapsto \mathcal{W}_{\vartheta,t}(\vartheta) + \mathcal{W}_{u,t}(u_t(\vartheta)), \tag{27}$$

recognizing that for fixed $\vartheta_t^{\mathcal{M}} = 1$ and exogenous state X_t, u_t is a function of ϑ_t, compare Lemma 3. We discuss the three possible cases in Lemma 3 separately.

First, suppose that $u_t(\vartheta) = u^0(\vartheta; X_t)$ locally around the optimal choice of the planner. This is equivalent to the assumption that the cash constraint (11) has positive slack at date t. We denote the optimal ϑ-choice under this assumption by $\underline{\vartheta}_t^*$. $\underline{\vartheta}_t^*$ is characterized by taking first-order condition in the objective function (27) after substituting in $u_t(\vartheta) = u^0(\vartheta, X_t)$:

$$0 = \mathcal{W}'_{\vartheta,t}(\underline{\vartheta}_t^*) - \frac{1}{\varphi}\left(\frac{a_t}{\rho}\right)^{1+1/\varphi} \frac{\underline{\vartheta}_t^*}{(1-\underline{\vartheta}_t^*)^{1-1/\varphi}}. \tag{28}$$

Assuming $\varphi \geq 1$, this equation has a unique solution $\underline{\vartheta}_t^* \leq \vartheta_t^{e*}$.

Second, suppose that $u_t(\vartheta) = u^c(\vartheta, 1; X_t)$ locally around the optimal choice of the planner. This means that the cash constraint (11) is binding with a strictly positive multiplier $\lambda_t^{\mathcal{M}} > 0$. We denote the optimal ϑ-choice under this assumption by $\overline{\vartheta}_t^*$. It solves the first-order condition

$$0 = \mathcal{W}'_{\vartheta,t}(\overline{\vartheta}_t^*) + \left(\frac{a_t}{\rho} - \left(\frac{\nu_t}{a_t}\frac{\overline{\vartheta}_t^*}{1-\overline{\vartheta}_t^*}\right)^{\varphi}\right)^{+} \frac{\nu_t}{a_t}\frac{1}{(1-\overline{\vartheta}_t^*)^2}. \tag{29}$$

We show in the appendix that also this equation has a unique solution. Because the second term is nonnegative, the solution satisfies $\overline{\vartheta}_t^* \geq \vartheta_t^{e*}$.

There is a third case. Namely, the optimal choice might be right at the boundary between the previous two cases. In this case, there is a kink in the $u_t(\vartheta)$-function, so that neither of the two conditions (28) or (29) has to hold at the optimum. The location of the kink is the value $\hat{\vartheta}_t$ that solves $u^0(\hat{\vartheta}_t; X_t) = u^c(\hat{\vartheta}_t, 1; X_t)$, or more explicitly:

$$\rho(\nu_t\hat{\vartheta}_t)^{\varphi} = (a_t(1-\hat{\vartheta}_t))^{1+\varphi}, \tag{30}$$

which clearly has a unique solution $\hat{\vartheta}_t \in (0,1)$.

Furthermore, if $\vartheta_t > \hat{\vartheta}_t$, then the constraint has positive slack locally around ϑ_t, and if $\vartheta_t < \hat{\vartheta}_t$, then the constraint is binding with positive multiplier locally around ϑ_t. Therefore, $\hat{\vartheta}_t$ is indeed optimal whenever it falls inside the interval $[\underline{\vartheta}_t^*, \overline{\vartheta}_t^*]$ because then none of the other two cases is possible. Otherwise, one of the other two cases applies. Specifically, the boundary of the interval closer to $\hat{\vartheta}_t$ must be optimal. Figure 5 provides a graphical illustration of this idea. In sum, we arrive at the following proposition.

Proposition 12 (Optimal ϑ)**.** *Suppose that $\varphi \geq 1$. Then equations (28), (29), and (30) all have unique solutions $\underline{\vartheta}_t^*$, $\overline{\vartheta}_t^*$, $\hat{\vartheta}_t$, respectively, and there is a unique optimal ϑ-process ϑ^* chosen by the Ramsey planner, which is given by*

$$\vartheta_t^* = \min\{\max\{\hat{\vartheta}_t, \underline{\vartheta}_t^*\}, \overline{\vartheta}_t^*\} > 0.$$

Furthermore,

1. *if $\vartheta_t^* = \underline{\vartheta}_t^* > \hat{\vartheta}_t$, then $\vartheta_t^* \leq \vartheta_t^{e*}$ and the Friedman rule is optimal;*
2. *if $\vartheta_t^* = \hat{\vartheta}_t$, ϑ_t^* can be on either side of ϑ_t^{e*} and the Friedman rule is optimal;*
3. *if $\vartheta_t^* = \overline{\vartheta}_t^* < \hat{\vartheta}_t$, then $\vartheta_t^* \geq \vartheta_t^{e*}$ and the Friedman rule is not optimal.*

We remark that in case 3., the Friedman rule is not optimal because providing additional money balances would require an increase in ϑ_t, which has a positive social cost on the margin. Specifically, because $\vartheta_t^* \geq \vartheta_t^{e*}$, increasing ϑ_t moves ϑ_t away from the level ϑ_t^{e*} that is optimally balancing the growth-risk sharing trade-off encoded by $\mathcal{W}_{\vartheta,t}$. In principle, this welfare loss could be compensated by the positive impact on resource utilization u_t, which happens in case 2. if $\vartheta_t^* \geq \vartheta_t^{e*}$ but not in case 3.

Non-optimality of Modified Friedman Rule and Financial Repression. The modified Friedman rule here calls for a government that funds itself by issuing only money

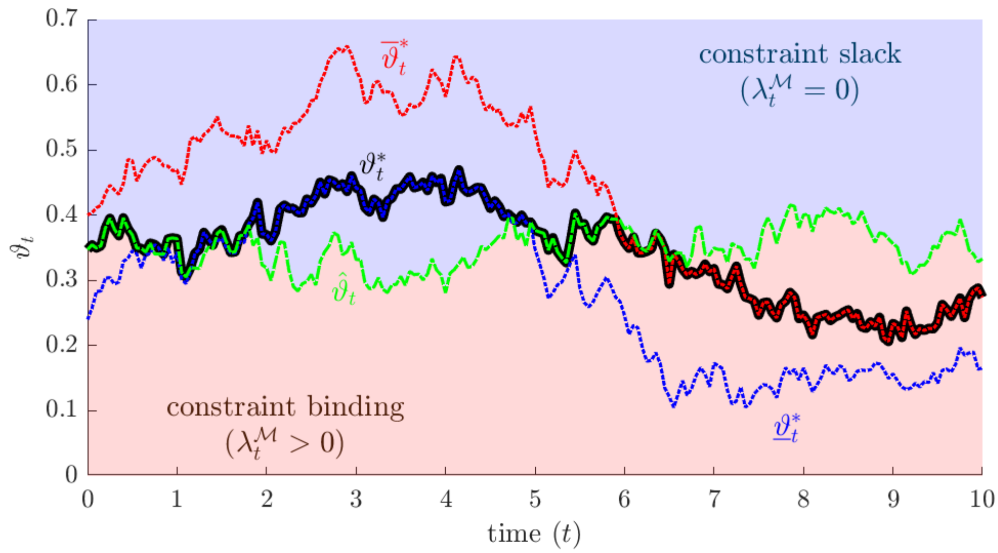

Figure 5: Optimal ϑ for a simulated path. The black solid line denotes the optimal policy ϑ_t^*. The green dash-dotted line denotes $\hat{\vartheta}_t$, which divides the figure vertically into two areas: for $\vartheta_t > \hat{\vartheta}_t$, the cash constraint is slack (blue area), for $\vartheta_t < \hat{\vartheta}_t$, it binds with positive multiplier (red area). The blue dotted line depicts $\underline{\vartheta}_t^*$, the optimal policy conditional on a slack constraint. This is optimal whenever it falls inside the blue area. The red dotted line depicts $\overline{\vartheta}_t^*$, the optimal policy conditional on a binding constraint. Again, this is optimal whenever it falls inside the red area.

but no bonds, except perhaps in states when $\Delta i_t = 0$, as then the two assets are perfect substitutes on the margin. This contrasts with the real-world observation that, typically, there is a positive liquidity premium $\Delta i_t > 0$ yet the government issues substantial quantities of nominal bonds. A slight modification of our model can rationalize such a policy. Specifically, it may be optimal to deviate even from the modified Friedman rule and choose $\vartheta_t^{\mathcal{M}} < 1$ if we impose constraints on taxes or add other sources of tax distortions. If, for $\vartheta_t^{\mathcal{M}} \approx 1$ and $\Delta i_t > 0$, the medium-of-exchange service flows in equation (24) are strictly decreasing in $\vartheta_t^{\mathcal{M}}$, i.e., if we are in the "regular" case for Δi-policy, then lowering $\vartheta_t^{\mathcal{M}}$ permits the government to achieve the same target path for ϑ_t with lower taxes. This can be interpreted as either seigniorage funding of government debt or a form of financial repression. In our model, this is not optimal because taxes have no welfare costs beyond their effects on the nominal wealth share ϑ_t. However, if there was an upper bound on taxes or if there were additional tax distortions, then it would be possible that optimal policy deviates even from the modified Friedman rule and uses seigniorage funding to keep taxes low.

6 Conclusion

This paper outlines a tractable incomplete markets model that captures and unifies many key results of monetary economics in a world with safe assets, in the form of government debt and money in positive supply. As in Tobin (1965), it stresses the importance of portfolio choice next to the investment and the consumption-saving choice. Risk and risk premia are central to the analysis. The paper discusses, within a unified framework, classic concepts such as the Tobin effect, Sargent–Wallace's unpleasant monetarist arithmetic, Wallace's neutrality of open market operations, Sims' stepping on a rake effect, and the Friedman rule.

The paper also highlights several promising directions for further research. First, the decomposition of the value of nominal assets can be related to the question of price level determination (Brunnermeier and Merkel, 2025). Second, the model could be extended to incorporate a banking sector that issues inside money, while outside money takes the form of central bank reserves (e.g., Brunnermeier and Sannikov, 2016a; Merkel, 2020). Third, introducing various forms of price rigidities can help build a bridge to New Keynesian models à la HANK, as in Li and Merkel (2024). With price stickiness and a banking system in place, one can also analyze the interaction of different monetary policy instruments, including distinct interest rates on required and excess reserves, as well as unconventional monetary policy measures (Alexandrov and Brunnermeier, 2025). Finally, deriving the optimal size and composition of the central bank's balance sheet remains an important task for future work.

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Appendix

A Derivations and Proofs in Baseline Model

Notation Convention. For any positive scalar Ito process x_t driven by aggregate shocks $\mathrm{d}Z_t$ and possibly the idiosyncratic shock $\mathrm{d}\tilde{Z}^i_t$ for a single $i \in \mathbb{I}$, we denote by μ^x_t, σ^x_t, $\tilde{\sigma}^x_t$ the *geometric* drift, aggregate volatility loading, and idiosyncratic volatility loading, that is we write

$$\mathrm{d}x_t = x_t\mu^x_t\mathrm{d}t + x_t\sigma^x_t\mathrm{d}Z_t + x_t\tilde{\sigma}^x_t\mathrm{d}\tilde{Z}^i_t.$$

Standard results guarantee that μ^x_t, σ^x_t, $\tilde{\sigma}^x_t$ are (almost surely) uniquely determined by the process x_t.

If a variable has a superscript, say x^A_t, we regularly write $\mu^{x,A}_t$, $\sigma^{x,A}_t$, $\tilde{\sigma}^{x,A}_t$ in place of $\mu^{x^A}_t$, $\sigma^{x^A}_t$, $\tilde{\sigma}^{x^A}_t$ to avoid double superscripts.

We remind the reader that Z_t is a d_Z-dimensional Brownian motion with possibly $d_Z > 1$, so aggregate volatility loadings σ^x_t are $1 \times d_Z$-vectors. Throughout, we interpret the product of two aggregate volatility loadings as an inner product of d_Z-dimensional row vectors, i.e., we write $\sigma^x_t\sigma^y_t$ instead of $\sigma^x_t(\sigma^y_t)^T$, omitting the transposition of the second factor to a column vector.

A.1 Model Solution

Proof of Lemma 1. We solve dynamic decision problems using the stochastic maximum principle. The Hamiltonian of household i is

$$\begin{aligned}H^i_t =& e^{-\rho t}\left(\log c^i_t - \frac{(u^i_t)^{1+\varphi}}{1+\varphi}\right) \\ &+ \xi^i_t\left(-c^i_t + n^i_t\left(\theta^i_t\frac{\mathbb{E}_t\left[\mathrm{d}r^{\mathcal{MB}}_t(\theta^{\mathcal{M},i}_t)\right]}{\mathrm{d}t} + (1-\theta^i_t)\frac{\mathbb{E}_t\left[\mathrm{d}r^{K,i}_t(\iota^i_t, u^i_t)\right]}{\mathrm{d}t}\right)\right) \\ &- \varsigma^i_t\xi^i_t n^i_t\left(\theta^i_t\sigma^{q,\mathcal{MB}}_t + (1-\theta^i_t)\sigma^{q,K}\right) - \tilde{\varsigma}^i_t\xi^i_t n^i_t(1-\theta^i_t)\tilde{\sigma}_t\end{aligned}$$

where

$$\mathrm{d}r_t^{\mathcal{MB}}(\theta^{\mathcal{M}}) := \theta^{\mathcal{M}}\mathrm{d}r_t^{\mathcal{M}} + (1-\theta^{\mathcal{M}})\mathrm{d}r_t^{\mathcal{B}}$$

and where ξ_t^i denotes the costate associated with household net worth n_t^i and $\varsigma_t^i := -\sigma_t^{\xi,i}$, $\tilde{\varsigma}_t^i := -\tilde{\sigma}_t^{\xi,i}$ are the negatives of the geometric volatility loadings of ξ_t^i on the aggregate and idiosyncratic shock, respectively.

According to the stochastic maximum principle, the optimal choices must maximize the Hamiltonian H_t^i. We determine c_t^i, ι_t^i, u_t^i, $\theta_t^{\mathcal{M},i}$, and θ_t^i that maximize H_t^i subject to the cash constraint (11). Denote by $\lambda_t^{\mathcal{M},i}\xi_t^i n_t^i$ the Lagrange multiplier on the constraint (11). The first-order conditions (FOCs) are:

$$c_t^i: \qquad \xi_t^i = e^{-\rho t}\frac{1}{c_t^i},$$

$$\iota_t^i: \qquad \left.\frac{\partial(\mathbb{E}_t[\mathrm{d}r_t^{K,i}(\iota,u_t^i)]/\mathrm{d}t)}{\partial\iota}\right|_{\iota=\iota_t^i} = 0$$

$$u_t^i: \qquad \frac{e^{-\rho t}}{(1-\theta_t^i)\xi_t^i n_t^i}(u_t^i)^{\varphi} = \left.\frac{\partial(\mathbb{E}_t[\mathrm{d}r_t^{K,i}(\iota_t^i,u)]/\mathrm{d}t)}{\partial\iota}\right|_{u=u_t^i} - \lambda_t^{\mathcal{M},i}\frac{a_t}{q_t^K}$$

$$\theta_t^{\mathcal{M},i}: \qquad \left.\frac{\partial(\mathbb{E}_t[\mathrm{d}r_t^{\mathcal{MB}}(\theta^{\mathcal{M}})]/\mathrm{d}t)}{\partial\theta^{\mathcal{M}}}\right|_{\theta^{\mathcal{M}}=\theta_t^{\mathcal{M},i}} = -\lambda_t^{\mathcal{M},i}\nu_t$$

$$\theta_t^i: \qquad \frac{\mathbb{E}_t\left[\mathrm{d}r_t^{K,i}(\iota_t^i,u_t^i)\right]}{\mathrm{d}t} - \frac{\mathbb{E}_t\left[\mathrm{d}r_t^{\mathcal{MB},i}(\theta_t^{\mathcal{M},i})\right]}{\mathrm{d}t} = \varsigma_t^i(\sigma_t^{q,K}-\sigma_t^{q,\mathcal{MB}}) + \tilde{\varsigma}_t^i\tilde{\sigma}_t + \lambda_t^{\mathcal{M},i}\left(\nu_t\theta_t^{\mathcal{M},i} + \frac{a_t u_t^i}{q_t^K}\right)$$

We now derive the equations stated in Lemma 1:

- Equation (15):
  Following Brunnermeier et al. (2024b), we guess that $\xi_t^i = e^{-\rho t}/(\rho n_t^i)$. This guess can be verified using a standard verification argument based on the costate equation, which we omit here. Substituting this guess into the FOC for c_t^i and rearranging yields equation (15).

- Equation (16):
  Recall from equation (8) that

$$\frac{\mathbb{E}_t[dr_t^{K,i}(u,\iota)]}{\mathrm{d}t} = \frac{ua_t-\iota}{q_t^K} + \frac{\breve{\mu}_t^{\mathcal{MB}}q_t^{\mathcal{MB}}}{q_t^K} + \iota - \delta + \mu_t^{q,K}.$$

Taking the derivative of this expression with respect to ι and plugging it into the

FOC for ι_t^i yields equation (16).

We remark that this condition holds, of course, only at an interior maximum. Due to the linearity of the Hamiltonian in ι_t^i, the agent chooses $\iota_t^i = -\infty$ if $q_t^K < 1$ and $\iota_t^i = \infty$ if $q_t^K > 1$, cases that cannot occur in an equilibrium.

- Equation (17):

  Taking the derivative of the previously stated equation for the expected return on capital with respect to u, using once again the guess $\xi_t^i = e^{-\rho t}/(\rho n_t^i)$ and plugging both into the FOC for u_t^i, implies

$$\frac{e^{-\rho t}}{(1-\theta_t^i)e^{-\rho t}/(\rho n_t^i)n_t^i}(u_t^i)^{\varphi} = \frac{a_t}{q_t^K} - \lambda_t^{\mathcal{M},i}\frac{a_t}{q_t^K}.$$

  After canceling terms and rearranging slightly, we obtain equation (17).

- Equation (18):

  By definition of $\mathrm{d}r_t^{\mathcal{MB}}(\theta^{\mathcal{M}})$,

$$\frac{\partial(\mathbb{E}_t[\mathrm{d}r_t^{\mathcal{MB}}(\theta^{\mathcal{M}})]/\mathrm{d}t)}{\partial\theta^{\mathcal{M}}} = \frac{\mathbb{E}_t[\mathrm{d}r_t^{\mathcal{M}} - \mathrm{d}r_t^{\mathcal{B}}]}{\mathrm{d}t} = i_t^{\mathcal{M}} - i_t^{\mathcal{B}},$$

  where the last equality follows from the return expressions (9) and (10). Substituting this into the FOC for $\theta_t^{\mathcal{M},i}$ yields equation (18).

- Equation (19):

  Our guess for ξ_t^i implies, in particular,

$$\varsigma_t^i = \sigma_t^{n,i}, \qquad \tilde{\varsigma}_t^i = (1-\theta_t^i)\tilde{\sigma}_t.$$

  In addition, the $\lambda_t^{\mathcal{M},i}$-term in the FOC for θ_t^i either vanishes or the constraint (11) is binding, in which case

$$\frac{a_t u_t^i}{q_t^K} = \nu_t \theta_t^{\mathcal{M},i}\frac{\theta_t^i}{1-\theta_t^i}.$$

  In either case, we therefore obtain

$$\lambda_t^{\mathcal{M},i}\left(\nu_t\theta_t^{\mathcal{M},i} + \frac{a_t u_t^i}{q_t^K}\right) = \frac{\theta_t^{\mathcal{M},i}}{1-\theta_t^i}\lambda_t^{\mathcal{M},i}\nu_t.$$

Plugging these observations into the FOC for θ_t^i yields

$$\frac{\mathbb{E}_t\left[\mathrm{d}r_t^{K,i} - \mathrm{d}r_t^{\mathcal{MB},i}\right]}{\mathrm{d}t} = \sigma_t^{n,i}\left(\sigma_t^{q,K} - \sigma_t^{q,\mathcal{MB}}\right) + (1-\theta_t^i)\tilde{\sigma}_t^2 + \frac{\theta_t^{\mathcal{M},i}}{1-\theta_t^i}\lambda_t^{\mathcal{M},i}\nu_t$$
$$= \frac{\mathbb{E}_t[\mathrm{d}\langle n_t^i, q_t^K\rangle / q_t^K - \mathrm{d}\langle n_t^i, q_t^{\mathcal{MB}}\rangle / q_t^{\mathcal{MB}}]}{n_t^i \mathrm{d}t} + (1-\theta_t^i)\tilde{\sigma}_t^2 + \frac{\theta_t^{\mathcal{M},i}}{1-\theta_t^i}\lambda_t^{\mathcal{M},i}\nu_t,$$

where $\mathrm{d}\langle n_t^i, q_t^X\rangle = n_t^i q_t^X \sigma_t^{n,i}\sigma_t^{q,K}\mathrm{d}t$ is the quadratic covariation of n_t^i and q_t^X ($X \in \{\mathcal{MB}, K\}$). This is equation (19) in the main text.

Finally, the complementary slackness condition is a standard requirement on Lagrange multipliers on inequality constraints. □

Proof of Lemma 2. The two equations preceding the statement of the lemma in the main text represent linear equations for ι_t and q_t for any given fixed ϑ_t and u_t. Solving these equations results in the first and third equation stated in the lemma. The second equation is a restatement of $C_t = \rho q_t K_t$ derived in the main text.

□

Proof of Lemma 3. Under our assumption of a symmetric equilibrium, all agents choose the same portfolios, so that $\theta_t^i = \vartheta_t$ by market clearing for government liabilities, equation (13). We can therefore simplify the optimal utilization choice, equation (17), as follows:

$$\rho u_t^{\varphi} = (1-\lambda_t^{\mathcal{M}})\frac{a_t}{q_t}.$$

Plugging in q_t from Lemma 2 and solving for u_t yields:

$$u_t = \left((1-\lambda_t^{\mathcal{M}})\frac{(1-\vartheta_t)\,a_t}{\rho}\right)^{1/\varphi} \leq \left(\frac{(1-\vartheta_t)\,a_t}{\rho}\right)^{1/\varphi} = u^0(\vartheta_t; X_t)$$

and this holds with equality if and only if $\lambda_t^{\mathcal{M}} = 0$. Furthermore, the cash constraint (11), combined with symmetry and market clearing, $\theta_t^i = \vartheta_t$ and $\theta_t^{\mathcal{M},i} = \vartheta_t^{\mathcal{M}}$, can be written as an inequality for u_t:

$$u_t \leq \frac{\nu_t}{a_t}\frac{\vartheta_t^{\mathcal{M}}\vartheta_t}{1-\vartheta_t} = u^c(\vartheta_t, \vartheta_t^{\mathcal{M}}; X_t).$$

By the complementary slackness condition, at least one of the two conditions needs to hold with equality, which proves the stated equation for u_t. The equation for $\lambda_t^{\mathcal{M}}$ follows directly from $u_t = \left((1-\lambda_t^{\mathcal{M}})\frac{(1-\vartheta_t)a_t}{\rho}\right)^{1/\varphi}$.

□

Proof of Lemma 4. Consider the portfolio choice condition for θ_t^i, equation (19), and impose market clearing. First, using the return expressions (8), (9), and (10), the left-hand side is of the choice condition is

$$\begin{aligned}\frac{\mathbb{E}_t\left[\mathrm{d}r_t^{K,i} - \mathrm{d}r_t^{\mathcal{MB}}\right]}{\mathrm{d}t} &= \frac{u_t a_t - \iota_t}{q_t^K} + \frac{\breve{\mu}_t^{\mathcal{MB}} q_t^{\mathcal{MB}}}{q_t^K} + \breve{\mu}_t^{\mathcal{MB}} + \frac{\mathbb{E}_t[\mathrm{d}q_t^K]}{q_t^K \mathrm{d}t} - \frac{\mathbb{E}_t[\mathrm{d}q_t^{\mathcal{MB}}]}{q_t^{\mathcal{MB}}\mathrm{d}t} \\ &= \frac{\rho}{1-\vartheta_t} + \frac{\breve{\mu}_t^{\mathcal{MB}}}{1-\vartheta_t} + \frac{\mathbb{E}_t[\mathrm{d}q_t^K]}{q_t^K \mathrm{d}t} - \frac{\mathbb{E}_t[\mathrm{d}q_t^{\mathcal{MB}}]}{q_t^{\mathcal{MB}}\mathrm{d}t},\end{aligned} \tag{31}$$

where the second line follows by imposing goods market clearing and using the definition of ϑ_t.

Next, consider the aggregate risk premium term on the right-hand side of the portfolio choice condition. Note that the wealth share $\eta_t^i := n_t^i/(q_t K_t)$ only loads on the idiosyncratic shocks $\mathrm{d}\tilde{Z}_t^i$ and that K_t does not load on any shock. Therefore, for any process ϑ_t^X that only loads on aggregate shocks and $q_t^X := \vartheta_t^X q_t$, we obtain

$$\frac{\mathrm{d}\langle n_t^i, q_t^X\rangle}{n_t^i q_t^X} = -\frac{\mathrm{d}\langle 1/n_t^i, q_t^X\rangle}{q_t^X/n_t^i} = -\frac{\mathrm{d}\langle 1/q_t, q_t^X\rangle}{\vartheta_t^X} = -\frac{d\vartheta_t^X}{\vartheta_t^X} + \frac{\mathrm{d}q_t^X}{q_t^X} + \frac{\mathrm{d}(1/q_t)}{1/q_t}$$

Applying this to the aggregate risk premium term in the portfolio condition yields

$$\begin{aligned}\frac{\mathbb{E}_t[\mathrm{d}\langle n_t^i, q_t^K\rangle/q_t^K - \mathrm{d}\langle n_t^i, q_t^{\mathcal{MB}}\rangle/q_t^{\mathcal{MB}}]}{n_t^i \mathrm{d}t} &= -\frac{\mathbb{E}_t[\mathrm{d}(1-\vartheta_t)]}{(1-\vartheta_t)\mathrm{d}t} + \frac{\mathbb{E}_t[\mathrm{d}q_t^K]}{q_t^K\mathrm{d}t} + \frac{\mathbb{E}_t[\mathrm{d}\vartheta_t]}{\vartheta_t \mathrm{d}t} - \frac{\mathbb{E}_t[\mathrm{d}q_t^{\mathcal{MB}}]}{q_t^{\mathcal{MB}}\mathrm{d}t} \\ &= \frac{1}{1-\vartheta_t}\frac{\mathbb{E}_t[\mathrm{d}\vartheta_t]}{\vartheta_t \mathrm{d}t} + \frac{\mathbb{E}_t[\mathrm{d}q_t^K]}{q_t^K\mathrm{d}t} - \frac{\mathbb{E}_t[\mathrm{d}q_t^{\mathcal{MB}}]}{q_t^{\mathcal{MB}}\mathrm{d}t}.\end{aligned}$$

Finally, we can eliminate i-dependent variables in the remaining terms on the right-hand side of equation (19) by using $\theta_t^i = \vartheta_t$, $\theta_t^{\mathcal{M},i} = \vartheta_t^{\mathcal{M}}$, which follows from asset market clearing (equations (13) and (14), respectively) and the fact that all agents chose the same portfolios.

Combining these facts and plugging them into the portfolio choice condition yields

$$\frac{\rho + \breve{\mu}_t^{\mathcal{MB}}}{1-\vartheta_t} + \frac{\mathbb{E}_t[\mathrm{d}q_t^K]}{q_t^K \mathrm{d}t} - \frac{\mathbb{E}_t[\mathrm{d}q_t^{\mathcal{MB}}]}{q_t^{\mathcal{MB}} \mathrm{d}t}$$
$$= \frac{1}{1-\vartheta_t}\frac{\mathbb{E}_t[\mathrm{d}\vartheta_t]}{\vartheta_t \mathrm{d}t} + \frac{\mathbb{E}_t[\mathrm{d}q_t^K]}{q_t^K \mathrm{d}t} - \frac{\mathbb{E}_t[\mathrm{d}q_t^{\mathcal{MB}}]}{q_t^{\mathcal{MB}} \mathrm{d}t} + (1-\vartheta_t)\tilde{\sigma}_t^2 + \frac{\vartheta_t^{\mathcal{M}}}{1-\vartheta_t}\lambda_t^{\mathcal{M}}\nu_t$$

Canceling terms and solving for $\mathbb{E}_t[d\vartheta_t]$ implies

$$\mathbb{E}_t[d\vartheta_t] = \left(\rho + \breve{\mu}_t^{\mathcal{MB}} - (1-\vartheta_t)^2\tilde{\sigma}_t^2 - \vartheta_t^{\mathcal{M}}\lambda_t^{\mathcal{M}}\nu_t\right)\vartheta_t \mathrm{d}t.$$

□

A.2 Classical Dichotomy Propositions

Proof of Proposition 1. Follows directly from the more general Proposition 2. □

Proof of Proposition 2. Let the original equilibrium be given by processes

$$K_t, \mathcal{MB}_t, \mathcal{P}_t, q_t^{\mathcal{M}}, q_t^{\mathcal{MB}}, q_t^K, \breve{\mu}_t^{\mathcal{MB}}, \tau_t, i_t^{\mathcal{M}}, i_t^{\mathcal{B}}, c_t^i, u_t^i, \theta_t^i, \theta_t^{\mathcal{M},i}, n_t^i$$

for $t \geq 0$ and $i \in \mathbb{I}$. For each variable x in this list except for $x \in \{\mathcal{MB}, \mathcal{P}, i^{\mathcal{M}}, i^{\mathcal{B}}\}$ define

$$x' := x$$

and for the remaining variables, except for $\mathcal{P}'$, which is already defined, define

$$\mathcal{MB}_t' := \exp\left(\int_0^t \Delta\pi_{t'}\mathrm{d}t'\right)\mathcal{MB}_t, \qquad i_t^{\mathcal{M}\prime} := i_t^{\mathcal{M}} + \Delta\pi_t, \qquad i_t^{\mathcal{B}\prime} := i_t^{\mathcal{B}} + \Delta\pi_t,$$

where

$$\Delta\pi_t := \frac{\mathbb{E}_t[\mathrm{d}\mathcal{P}_t']}{\mathcal{P}_t'\mathrm{d}t} - \frac{\mathbb{E}_t[\mathrm{d}\mathcal{P}_t]}{\mathcal{P}_t\mathrm{d}t}.$$

We show that the so-defined list of processes

$$K_t', \mathcal{MB}_t', \mathcal{P}_t', q_t^{\mathcal{M}\prime}, q_t^{\mathcal{MB}\prime}, q_t^{K\prime}, \breve{\mu}_t^{\mathcal{MB}\prime}, \tau_t', i_t^{\mathcal{M}\prime}, i_t^{\mathcal{B}\prime}, c_t^{\prime i}, u_t^{\prime i}, \theta_t^{\prime i}, \theta_t^{\mathcal{M}\prime,i}, n_t^{\prime i}$$

satisfies again all properties of a competitive equilibrium according to Definition 1 by checking the conditions in that definition. We first note that $K_0 = K_0'$, $\mathcal{MB}_0 = \mathcal{MB}_0'$, so the initial conditions are the same as for the original equilibrium. Next, in conditions 2., 5., and 6. of Definition 1, only variables x enter for which $x' = x$, so these conditions hold because the original list of processes was assumed to constitute an equilibrium.

In condition 1., variables with $x' \neq x$ only enter via the return expressions on nominal assets in the dt-terms of equations (9) and (10). But in these terms, the adjustments cancel out:

$$i_t^{\mathcal{M}\prime} - \mu_t^{\mathcal{MB}\prime} = i_t^{\mathcal{M}} + \Delta\pi_t - \left(\mu_t^{\mathcal{MB}} + \Delta\pi_t\right) = i_t^{\mathcal{M}} - \mu_t^{\mathcal{MB}}$$

and

$$i_t^{\mathcal{B}\prime} - \mu_t^{\mathcal{MB}\prime} = i_t^{\mathcal{B}} + \Delta\pi_t - \left(\mu_t^{\mathcal{MB}} + \Delta\pi_t\right) = i_t^{\mathcal{B}} - \mu_t^{\mathcal{MB}}.$$

Consequently, also condition 1. continues to hold.

In condition 3., the validity of the government flow budget constraint (4) for all t follows from condition 3. for the original equilibrium because this equation only contains variables x for which $x' = x$. The remaining part of condition 3. concerns the evolution of the nominal liability stock $\mathcal{MB}_t'$, which satisfies by definition

$$\begin{aligned}
\mathrm{d}\mathcal{MB}_t' &= \exp\left(\int_0^t \Delta\pi_{t'}\mathrm{d}t'\right)\mathrm{d}\mathcal{MB}_t + \Delta\pi_t \exp\left(\int_0^t \Delta\pi_{t'}\mathrm{d}t'\right)\mathcal{MB}_t\mathrm{d}t \\
&= \left(\breve{\mu}_t^{\mathcal{MB}} + i_t^{\mathcal{B}} + \frac{q_t^{\mathcal{M}}}{q_t^{\mathcal{MB}}}(i_t^{\mathcal{M}} - i_t^{\mathcal{B}}) + \Delta\pi_t\right)\mathcal{MB}_t'\mathrm{d}t \\
&= \left(\breve{\mu}_t^{\mathcal{MB}} + i_t^{\mathcal{B}\prime} + \frac{q_t^{\mathcal{M}}}{q_t^{\mathcal{MB}}}(i_t^{\mathcal{M}\prime} - i_t^{\mathcal{B}\prime})\right)\mathcal{MB}_t'\mathrm{d}t,
\end{aligned}$$

which is the required evolution according to Definition 1.

This leaves condition 4. By Ito's lemma,

$$\frac{\mathcal{P}_t'}{\mathcal{P}_t} = \frac{e^{\log \mathcal{P}_t'}}{e^{\log \mathcal{P}_t}} = \frac{\exp\left(\int_0^t \left(\frac{\mathrm{d}\mathcal{P}_{t'}'}{\mathcal{P}_{t'}'} - \frac{1}{2}\frac{\mathrm{d}\langle\mathcal{P}_{t'}'\rangle}{(\mathcal{P}_{t'}')^2}\right)\right)\mathcal{P}_0'}{\exp\left(\int_0^t \left(\frac{\mathrm{d}\mathcal{P}_{t'}}{\mathcal{P}_{t'}} - \frac{1}{2}\frac{\mathrm{d}\langle\mathcal{P}_{t'}\rangle}{(\mathcal{P}_{t'})^2}\right)\right)\mathcal{P}_0}$$

$$= \underbrace{\exp\left(\int_0^t \left(\frac{\mathbb{E}_t[\mathrm{d}\mathcal{P}'_{t'}]}{\mathcal{P}'_{t'}} - \frac{\mathbb{E}_t[\mathrm{d}\mathcal{P}_{t'}]}{\mathcal{P}_{t'}}\right)\right)}_{=\exp\left(\int_0^t \Delta\pi_{t'}\mathrm{d}t'\right)} \underbrace{\frac{\exp\left(\int_0^t \left(\frac{\mathrm{d}\mathcal{P}'_{t'} - \mathbb{E}_t[\mathrm{d}\mathcal{P}'_{t'}]}{\mathcal{P}'_{t'}} - \frac{1}{2}\frac{\mathrm{d}\langle\mathcal{P}'_{t'}\rangle}{(\mathcal{P}'_{t'})^2}\right)\right)\mathcal{P}'_0}{\exp\left(\int_0^t \left(\frac{\mathrm{d}\mathcal{P}_{t'} - \mathbb{E}_t[\mathrm{d}\mathcal{P}_{t'}]}{\mathcal{P}_{t'}} - \frac{1}{2}\frac{\mathrm{d}\langle\mathcal{P}_{t'}\rangle}{(\mathcal{P}_{t'})^2}\right)\right)\mathcal{P}_0}}_{=1},$$

where the second factor equals 1 because of the initial condition $\mathcal{P}'_0 = \mathcal{P}_0$ and because of the relative surprises in $\mathcal{P}_t$ and $\mathcal{P}'_t$ are assumed to be the same.[34] Hence,

$$\frac{\mathcal{MB}'_t}{\mathcal{P}'_t} = \frac{\exp\left(\int_0^t \Delta\pi_{t'}\mathrm{d}t'\right)\mathcal{MB}_t}{\exp\left(\int_0^t \Delta\pi_{t'}\mathrm{d}t'\right)\mathcal{P}_t} = \frac{\mathcal{MB}_t}{\mathcal{P}_t} = q_t^{\mathcal{MB}}K_t = q_t^{\mathcal{MB}\prime}K'_t,$$

which is condition 4. □

A.3 Key Valuation Equation and Dynamic Trading Perspective

We first prove Proposition 4 and then discuss the dynamic trading perspective, including the proof of Proposition 5.

Omitted Details in Proof of Proposition 4. The idea of the proof has been stated in the main text. Here we fill in the formal details. Applying Ito's product rule to $e^{-\rho t}\vartheta_t$ yields

$$\mathrm{d}(e^{-\rho t}\vartheta_t) = -\rho e^{-\rho t}\vartheta_t \mathrm{d}t + e^{-\rho t}\mathrm{d}\vartheta_t.$$

Combining this with equation (20) from Lemma 4 yields

$$\begin{aligned}\mathrm{d}(e^{-\rho t}\vartheta_t) &= e^{-\rho t}\left(\breve{\mu}_t^{\mathcal{MB}} - (1-\vartheta_t)^2\tilde{\sigma}_t^2 - \vartheta_t^{\mathcal{M}}\lambda_t^{\mathcal{M}}\nu_t\right)\vartheta_t\mathrm{d}t + e^{-\rho t}\sigma_{\vartheta,t}\mathrm{d}Z_t \\ &= -e^{-\rho t}\left(\hat{\tau}_t + (1-\vartheta_t)^2\tilde{\sigma}_t^2\vartheta_t + \vartheta_t^{\mathcal{M}}\lambda_t^{\mathcal{M}}\nu_t\vartheta_t\right)\mathrm{d}t + e^{-\rho t}\sigma_{\vartheta,t}\mathrm{d}Z_t\end{aligned}$$

where $\sigma_{\vartheta,t}$ denotes the $(\mathrm{d}Z_t)$-volatility loading of ϑ_t and the second line uses the gov-

[34] The equality of inflation surprises implies immediately that the stochastic components of the integrals in the numerator and denominator are equal. But equality of relative surprise components also implies $\frac{\mathrm{d}\langle\mathcal{P}'_t\rangle}{\mathcal{P}_t'^2} = \frac{\mathrm{d}\langle\mathcal{P}_t\rangle}{\mathcal{P}_t^2}$, which is sufficient to conclude that also the integrals with respect to the quadratic variation terms are equal.

ernment budget constraint (4) and the definition of $\hat{\tau}_t$. Next, write this stochastic differential equation in integral form from t to T:

$$e^{-\rho T}\vartheta_T - e^{-\rho t}\vartheta_t = \int_t^T \mathrm{d}\left(e^{-\rho t'}\vartheta_{t'}\right)$$
$$= -\int_t^T e^{-\rho t'}\left(\hat{\tau}_{t'} + (1-\vartheta_{t'})^2\tilde{\sigma}_{t'}^2\vartheta_{t'} + \vartheta_{t'}^{\mathcal{M}}\lambda_{t'}^{\mathcal{M}}\nu_{t'}\vartheta_{t'}\right)\mathrm{d}t' + \int_t^T e^{-\rho t'}\sigma_{\vartheta,t'}\mathrm{d}Z_{t'}.$$

The last term is a martingale and vanishes once we take time-t expectations $\mathbb{E}_t[\cdot]$ on both sides. We recover equation (24) stated in the proposition after taking the limit $T \to \infty$ and rearranging the remaining terms. Note that $\vartheta_T \in [0,1)$ is bounded, so that $e^{-\rho T}\vartheta_T \to 0$ as $T \to \infty$. □

We now turn to the claims made in Section 3.2 concerning the dynamic trading perspective. Let us first define the equilibrium share of nominal assets held by agent i:

$$\eta_t^{\mathcal{MB},i} := \frac{m_t^i + b_t^i}{\int (m_t^{i'} + b_t^{i'})\mathrm{d}i'} = \frac{m_t^i + b_t^i}{\mathcal{MB}_t} = \frac{\theta_t^i n_t^i}{q_t^{\mathcal{MB}} K_t}. \tag{32}$$

To define the variables $v_t^{\text{gov. CF},i}$, $v_t^{\text{trading CF},i}$, $v_t^{\text{money dividend},i}$ introduced in the main text rigorously in our continuous-time environment, we take the formal limit of the (purely suggestive) discrete-time expressions stated in the main text as $\mathrm{d}t \to 0$:[35]

$$\xi_t^i v_t^{\text{gov. CF},i} := \mathbb{E}_t\left[\int_t^\infty \xi_{t'}^i \eta_{t'}^{\mathcal{MB},i}\tau_{t'}K_{t'}\mathrm{d}t'\right], \tag{33}$$

$$\xi_t^i v_t^{\text{trading CF},i} := \mathbb{E}_t\left[-\int_t^\infty \xi_{t'}^i q_{t'}^{\mathcal{MB}} K_{t'}\mathrm{d}\eta_{t'}^{\mathcal{MB},i} - \int_t^\infty \mathrm{d}\left\langle \xi_{t'}^i q_{t'}^{\mathcal{MB}} K_{t'}, \eta_{t'}^{\mathcal{MB},i}\right\rangle\right], \tag{34}$$

$$\xi_t^i v_t^{\text{money dividend},i} := \mathbb{E}_t\left[\int_t^\infty \xi_{t'}^i \eta_{t'}^{\mathcal{MB},i}\vartheta_{t'}^{\mathcal{M}}\lambda_{t'}^{\mathcal{M}}\nu_{t'}q_{t'}^{\mathcal{MB}}K_{t'}\mathrm{d}t'\right]. \tag{35}$$

We furthermore define

$$v_t^i := v_t^{\text{gov. CF},i} + v_t^{\text{trading CF},i} + v_t^{\text{money dividend},i}.$$

[35]The first and third definitions are relatively straightforward. For the second, note that the Riemann sum approximation for a (Ito) stochastic integral requires the integrand to be evaluated at the left end of each interval, while the terms in the discrete sum other than $(\eta_{t+j\mathrm{d}t}^{\mathcal{MB},i} - \eta_{t+(j+1)\mathrm{d}t}^{\mathcal{MB},i})$ are evaluated at the right endpoint $t+(j+1)\mathrm{d}t$. Correcting the timing results in an additional term that converges to an integral with respect to the quadratic covariation of $\xi_t^i q_t^{\mathcal{MB}} K_t$ and $\eta_t^{\mathcal{MB},i}$.

Given these definitions, we first prove Proposition 5:

Proof of Proposition 5. We begin by deriving some preliminary identities that will prove useful in the course of the argument. First, note that by equation (32),

$$\eta_t^{\mathcal{MB},i} = \frac{\vartheta_t n_t^i}{\vartheta_t q_t K_t} = \frac{n_t^i}{q_t K_t} = \eta_t^i,$$

where η_t^i is the wealth share of agent i at time t. Furthermore,[36]

$$\xi_t^i = e^{-\rho t}\frac{c_0^i}{c_t^i} = e^{-\rho t}\frac{\rho n_0^i}{\rho n_t^i} = \underbrace{e^{-\rho t}\frac{q_0 K_0}{q_t K_t}}_{=:\xi_t^{**}}\frac{\eta_0^i}{\eta_t^i}.$$

Combining the previous two equations, we obtain for all $t \leq t'$

$$\frac{\xi_{t'}^i}{\xi_t^i}\eta_{t'}^{\mathcal{MB},i} = \eta_t^i \frac{\xi_{t'}^{**}}{\xi_t^{**}}. \tag{36}$$

In addition, by definition of ξ_t^{**}, we also have for all $t \leq t'$

$$\frac{1}{q_t K_t}\frac{\xi_{t'}^{**}}{\xi_t^{**}} = e^{-\rho(t'-t)}\frac{1}{q_{t'} K_{t'}}. \tag{37}$$

Substituting equation (36) into equation (33) yields

$$v_t^{\text{gov. CF},i} = \eta_t^i \mathbb{E}_t\left[\int_t^\infty \frac{\xi_{t'}^{**}}{\xi_t^{**}}\tau_{t'} K_{t'} \mathrm{d}t'\right]. \tag{38}$$

Dividing both sides by $q_t K_t$, and using equation (37) and the fact that $\int \eta_t^i \mathrm{d}i = 1$ implies

$$\frac{\int v_t^{\text{gov. CF},i}\mathrm{d}i}{q_t K_t} = \mathbb{E}_t\left[\int_t^\infty e^{-\rho(t'-t)}\frac{\tau_{t'} K_{t'}}{q_{t'} K_{t'}}\mathrm{d}t'\right] = \mathbb{E}_t\left[\int_t^\infty e^{-\rho(t'-t)}\hat{\tau}_{t'}\mathrm{d}t'\right],$$

which is the first equation in Proposition 5.

An analogous argument can be applied to derive the third equation in Proposition 5.

[36]The notation ξ_t^{**} is taken from Brunnermeier et al. (2024b) because this object is mathematically identical to the weighted average SDF that appears in that paper.

Substituting equation (36) into equation (35) yields

$$v_t^{\text{money dividend},i} = \eta_t^i \mathbb{E}_t \left[\int_t^\infty \frac{\xi_{t'}^{**}}{\xi_t^{**}} \vartheta_{t'}^{\mathcal{M}} \lambda_{t'}^{\mathcal{M}} \nu_{t'} q_{t'}^{\mathcal{MB}} K_{t'} \mathrm{d}t' \right]. \tag{39}$$

Once again, dividing by $q_t K_t$, using equation (37), and integrating over $i \in \mathbb{I}$ implies

$$\frac{\int v_t^{\text{money dividend},i} \mathrm{d}i}{q_t K_t} = \mathbb{E}_t \left[\int_t^\infty e^{-\rho(t'-t)} \frac{\vartheta_{t'}^{\mathcal{M}} \lambda_{t'}^{\mathcal{M}} \nu_{t'} q_{t'}^{\mathcal{MB}} K_{t'}}{q_{t'} K_{t'}} \mathrm{d}t' \right] = \mathbb{E}_t \left[\int_t^\infty e^{-\rho(t'-t)} \vartheta_{t'}^{\mathcal{M}} \lambda_{t'}^{\mathcal{M}} \nu_{t'} \vartheta_{t'} \mathrm{d}t' \right],$$

which is the third equation in the proposition.

The proof of the second equation in Proposition 5 is slightly more involved because equation (34) does not only contain $\mathrm{d}t'$-integrals. Note that $\mathrm{d}\eta_t^i = \eta_t^i (1 - \vartheta_t) \tilde{\sigma}_t \mathrm{d}\tilde{Z}_t^i$, i.e., η_t^i is a martingale that only loads on the idiosyncratic shock $\mathrm{d}\tilde{Z}_t^i$. This implies that the expectation of the first integral in equation (34) vanishes. In addition, the quadratic covariation in the second integral in equation (34) can be written as

$$\begin{aligned}
\mathrm{d} \left\langle \xi_t^i q_t^{\mathcal{MB}} K_t, \eta_t^{\mathcal{MB},i} \right\rangle &= \mathrm{d} \left\langle \eta_0^i \xi_t^{**} q_t^{\mathcal{MB}} K_t \frac{1}{\eta_t^i}, \eta_t^i \right\rangle = \eta_0^i \xi_t^{**} q_t^{\mathcal{MB}} K_t \mathrm{d} \left\langle \frac{1}{\eta_t^i}, \eta_t^i \right\rangle \\
&= -\eta_0^i \xi_t^{**} q_t^{\mathcal{MB}} K_t (1 - \vartheta_t)^2 \tilde{\sigma}_t^2 \mathrm{d}t,
\end{aligned}$$

where the first equality uses equation (36) and the third uses the fact that $\eta_0^i \xi_t^{**} q_t^{\mathcal{MB}} K_t$ does not load on the $\mathrm{d}\tilde{Z}_t^i$ shock whereas η_t^i only loads on that shock. Using equation (36) once more, we can therefore also write the trading value defined in equation (34) in terms of the expectation of a $\mathrm{d}t'$-integral:

$$v_t^{\text{trading CF},i} = \eta_t^i \mathbb{E}_t \left[\int_t^\infty \frac{\xi_{t'}^{**}}{\xi_t^{**}} (1 - \vartheta_{t'})^2 \tilde{\sigma}_{t'}^2 q_{t'}^{\mathcal{MB}} K_{t'} \mathrm{d}t' \right]. \tag{40}$$

From here on, the remaining steps are identical as for the previous two equations: dividing by $q_t K_t$, using equation (37), and integrating over $i \in \mathbb{I}$ implies

$$\begin{aligned}
\frac{\int v_t^{\text{trading CF},i} \mathrm{d}i}{q_t K_t} &= \mathbb{E}_t \left[\int_t^\infty e^{-\rho(t'-t)} \frac{(1 - \vartheta_{t'})^2 \tilde{\sigma}_{t'}^2 q_{t'}^{\mathcal{MB}} K_{t'}}{q_{t'} K_{t'}} \mathrm{d}t' \right] \\
&= \mathbb{E}_t \left[\int_t^\infty e^{-\rho(t'-t)} (1 - \vartheta_{t'})^2 \tilde{\sigma}_{t'}^2 \vartheta_{t'} \mathrm{d}t' \right].
\end{aligned}$$

□

Next, we verify the remaining claims we made in Section 3.2. Specifically, we show the following:

1. Equation (25) holds, i.e.,
$$q_t^{\mathcal{MB}} K_t = \int v_t^i \mathrm{d}i.$$
2. v_t^i is indeed the value of agent i's nominal asset portfolio, i.e., for each $i \in \mathbb{I}$ and all $t \geq 0$,
$$v_t^i = \theta_t^i n_t^i.$$

For a proof of 1., let us combine equations (38), (39), and (40) derived in the course of the previous proof:

$$v_t^i = \eta_t^i \underbrace{\mathbb{E}_t \left[\int_t^\infty \frac{\xi_{t'}^{**}}{\xi_t^{**}} \left(\tau_{t'} K_{t'} + (1 - \vartheta_{t'})^2 \tilde{\sigma}_{t'}^2 q_{t'}^{\mathcal{MB}} K_{t'} + \vartheta_{t'}^{\mathcal{M}} \lambda_{t'}^{\mathcal{M}} \nu_{t'} q_{t'}^{\mathcal{MB}} K_{t'} \right) \mathrm{d}t' \right]}_{=: \bar{v}_t}.$$

Using $\int \eta_t^i \mathrm{d}i = 1$, we conclude $\bar{v}_t = \int v_t^i \mathrm{d}i$ and then with Propositions 4 and 5

$$\frac{\bar{v}_t}{q_t K_t} = \vartheta_t \Rightarrow \bar{v}_t = \vartheta_t q_t K_t = q_t^{\mathcal{MB}} K_t,$$

which implies 1.

For a proof of 2., we use that $\eta_t^{\mathcal{MB},i} = \eta_t^i$, so

$$\theta_t^i n_t^i = \eta_t^{\mathcal{MB},i} q_t^{\mathcal{MB}} K_t = \eta_t^i q_t^{\mathcal{MB}} K_t = \eta_t^i \bar{v}_t = v_t^i,$$

where the last two equalities use the two previously derived equations.

A.4 Steady State Results

Proof of Proposition 6. In any steady-state equilibrium, monetary or otherwise, $\mathrm{d}\vartheta_t = 0$. Combining this with equation (20), we obtain the equation

$$0 = \left(\rho + \breve{\mu}^{\mathcal{MB}} - (1 - \vartheta)^2 \tilde{\sigma}^2 - \vartheta^{\mathcal{M}} \lambda^{\mathcal{M}} \nu \right) \vartheta.$$

In a monetary steady state, $\vartheta > 0$, so this equation is satisfied if and only if

$$\rho + \breve{\mu}^{\mathcal{MB}} - (1-\vartheta)^2\tilde{\sigma}^2 - \vartheta^{\mathcal{M}}\lambda^{\mathcal{M}}\nu = 0.$$

Because $0 \leq \lambda^{\mathcal{M}}, 1-\vartheta \leq 1$, the existence of a monetary steady state implies, together with the previous equation, the inequalities stated in the proposition in weak form. Due to $\vartheta > 0$, the first inequality must, in fact, hold strictly, and due to $\vartheta < 1$ (this has to hold in any equilibrium due to $1 = q_t^K = (1-\vartheta_t)q_t$), also the second inequality must be strict.

Let us now assume that the inequality conditions in the proposition are satisfied. We construct an explicit steady-state solution and show that it is unique. By Lemma 3, in any steady state

$$\vartheta^{\mathcal{M}}\lambda^{\mathcal{M}}\nu = \left(\vartheta^{\mathcal{M}}\nu - \rho\vartheta^{\varphi}\left(\frac{\nu}{a}\frac{\vartheta^{\mathcal{M}}}{1-\vartheta}\right)^{1+\varphi}\right)^{+}$$

and plugging this into the previous equation yields

$$\rho + \breve{\mu}^{\mathcal{MB}} = (1-\vartheta)^2\tilde{\sigma}^2 + \left(\vartheta^{\mathcal{M}}\nu - \rho\vartheta^{\varphi}\left(\frac{\nu}{a}\frac{\vartheta^{\mathcal{M}}}{1-\vartheta}\right)^{1+\varphi}\right)^{+} = 0.$$

For $\vartheta \in (0,1)$, the right-hand side of this equation is strictly decreasing in ϑ, it approaches $\tilde{\sigma}^2 + \vartheta^{\mathcal{M}}\nu > \rho + \breve{\mu}^{\mathcal{MB}}$ as $\vartheta \to 0$ and $0 < \rho + \breve{\mu}^{\mathcal{MB}}$ as $\vartheta \to 1$. Therefore, there is a unique solution ϑ to this equation. By construction, this solution satisfies equation (20) if $\lambda^{\mathcal{M}}$ and u are defined as in Lemma 3. This means, ϑ is the only possible (monetary) steady state solution. With the help of ϑ and u, we can use Lemma 2 and the relationships $q^K = (1-\vartheta)q$, $q^{\mathcal{MB}} = \vartheta q$, $q^{\mathcal{M}} = \vartheta^{\mathcal{M}}q^{\mathcal{MB}}$ to define all aggregates in Definition 1 other than K_t, $\mathcal{MB}_t$, $\mathcal{P}_t$, $i^{\mathcal{M}}$, $i^{\mathcal{B}}$. We can pick $i^{\mathcal{M}}$ arbitrary, define $i^{\mathcal{B}} := \lambda^{\mathcal{M}}\nu$ and define the paths K_t, $\mathcal{MB}_t$, $\mathcal{P}_t$ (which are not necessarily constant in steady state) to satisfy properties 2., 3., and 4. of Definition 1, respectively.

It is then straightforward to verify that these aggregate paths satisfy properties 2.–6. of Definition 1 (this is mostly by construction or, in the face of market clearing, follows from straightforward calculations). The final step is to construct individual choices, $\iota^i = \iota$, $u^i = u$, $\theta^i = \vartheta$, $\theta^{\mathcal{M},i} = \vartheta^{\mathcal{M}}$, $c_t^i = \rho n_t^i$ and n_t^i as implied by the net worth evolution (7) (and these choices) and show that these indeed solve the agent problem. It is sufficient to verify the conditions in Lemma 1. It is obvious that all conditions hold here by construction, except perhaps for equation (19), so let us verify this one explicitly.

Due to the absence of aggregate risk, the right-hand side is simply

$$(1-\vartheta)\tilde{\sigma}^2 + \frac{\vartheta^{\mathcal{M}}}{1-\vartheta}\lambda^{\mathcal{M}}\nu = \frac{1}{1-\vartheta}\left((1-\vartheta)^2\tilde{\sigma}^2 + \vartheta^{\mathcal{M}}\lambda^{\mathcal{M}}\nu\right),$$

and the expression in parentheses is exactly the right-hand side of the equation we have used to define ϑ. The left-hand side of equation (19) is (using $\mathrm{d}q_t^K = \mathrm{d}q_t^{\mathcal{MB}} = 0$ in a steady state)

$$\begin{aligned}\frac{\mathbb{E}_t\left[\mathrm{d}r_t^{K,i} - \mathrm{d}r_t^{\mathcal{MB},i}\right]}{\mathrm{d}t} &= ua - \iota + \breve{\mu}^{\mathcal{MB}}q^{\mathcal{MB}} + \iota - \delta - \left(\vartheta^{\mathcal{M}}i^{\mathcal{M}} + (1-\vartheta^{\mathcal{M}})i^{\mathcal{B}} - \mu^{\mathcal{MB}} + \iota - \delta\right) \\ &= ua - \left(au - \frac{\rho}{1-\vartheta}\right) + \breve{\mu}^{\mathcal{MB}}\frac{\vartheta}{1-\vartheta} + \breve{\mu}^{\mathcal{MB}} \\ &= \frac{1}{1-\vartheta}\left(\rho + \breve{\mu}^{\mathcal{MB}}\right),\end{aligned}$$

and the last expression in parentheses is exactly the left-hand side of the equation we have used to define ϑ. Hence, the validity of that equation also implies condition (19) for agent choices.

This concludes the proof that there is indeed a steady-state equilibrium and ϑ is unique. Finally, the two equations for ϑ^0 and ϑ^c in the proposition are just special cases of the equation for ϑ stated above depending on whether the $(\cdot)^+$-term is positive of zero. From this observation it is also clear that $\vartheta = \max\{\vartheta^0, \vartheta^c\}$.

□

A.5 Interest Rate Policy Results

Proof of Proposition 7. The claims in 2. about the evolution of the price level fully determine a candidate process for $\mathcal{P}'$ in the new equilibrium $\mathbf{e}'$. This process satisfies the assumptions of Proposition 2. Indeed, $\mathcal{P}_0' = \mathcal{P}_0$ by assumption and

$$\frac{\mathrm{d}\mathcal{P}_t' - \mathbb{E}_t[\mathrm{d}\mathcal{P}_t']}{\mathcal{P}_t'} = \left(\frac{\mathrm{d}\mathcal{P}_t}{\mathcal{P}_t} + (i_t' - i_t)\mathrm{d}t\right) - \left(\frac{\mathbb{E}_t[\mathrm{d}\mathcal{P}_t]}{\mathcal{P}_t} + (i_t' - i_t)\mathrm{d}t\right) = \frac{\mathrm{d}\mathcal{P}_t - \mathbb{E}_t[\mathrm{d}\mathcal{P}_t]}{\mathcal{P}_t}$$

by definition of $\mathcal{P}_t'$. Applying Proposition 2, we can conclude that there is an equilibrium $\mathbf{e}'$ with price level process $\mathcal{P}_t'$ and identical real allocation as in $\mathbf{e}$. Hence, for this specific $\mathbf{e}'$, the assertions in 1. and 2. of Proposition 7 hold.

Furthermore, an identical real allocation implies, in particular, that

$$\vartheta_t^{\mathcal{M}\,\prime} = \frac{q_t^{\mathcal{M}\,\prime}}{q_t^{\mathcal{MB}\prime}} = \frac{q_t^{\mathcal{M}}}{q_t^{\mathcal{MB}}} = \vartheta_t^{\mathcal{M}}.$$

Proposition 2 also tells us that $i_t^{\mathcal{M}\,\prime} = i_t^{\mathcal{M}} + i_t' - i_t$ and $i_t^{\mathcal{B}\,\prime} = i_t^{\mathcal{B}} + i_t' - i_t$, which implies, together with $\vartheta_t^{\mathcal{M}\,\prime} = \vartheta_t^{\mathcal{M}}$ that indeed $\vartheta_t^{\mathcal{M}\,\prime} i_t^{\mathcal{M}\,\prime} + (1 - \vartheta_t^{\mathcal{M}\,\prime}) i_t^{\mathcal{B}\,\prime} = i_t'$, so in equilibrium $\mathbf{e}'$, the target for the level of nominal rates is indeed attained, and $\vartheta_t^{\mathcal{M}\,\prime}$ and $i_t^{\mathcal{M}\,\prime}$ are as asserted in part 3. of Proposition 7.

The previous argument shows that there is one equilibrium $\mathbf{e}'$ with the asserted properties. Because the equilibrium given $\hat{\tau}_t$, i_t, Δi_t is unique (see Section 3.3 and Brunnermeier and Merkel (2025, Proposition 8)), this is sufficient to establish the proposition. □

Proof of Proposition 8. We first show that $\vartheta_t' > \vartheta_t$. To do so, consider the valuation BSDE from Lemma 4, written in the form

$$\mathbb{E}_t[d\vartheta_t] = \left(\rho + \breve{\mu}_t^{\mathcal{MB}} - (1 - \vartheta_t)^2 \tilde{\sigma}_t^2 - \hat{\vartheta}^{\mathcal{M}}(\vartheta_t, \Delta i_t; X_t)\Delta i_t\right) \vartheta_t \mathrm{d}t,$$

where $\hat{\vartheta}^{\mathcal{M}}$ is defined by

$$\hat{\vartheta}^{\mathcal{M}}(\vartheta, \Delta i; X) := \left(\frac{a(X)}{\nu(X)}\right)^{1+1/\varphi} \left(\frac{\nu(X) - \Delta i}{\rho}\right)^{1/\varphi} \frac{(1-\vartheta)^{1+1/\varphi}}{\vartheta}.$$

The equations stated in Lemma 3 imply that $\vartheta_t^{\mathcal{M}} = \hat{\vartheta}^{\mathcal{M}}(\vartheta_t, \Delta i_t; X_t)$ in all states in which $\Delta i_t > 0$. While this does not need to be true in states in which $\Delta i_t = 0$, then Δi_t-term in the valuation BSDE vanishes and so it does not matter that we replace $\vartheta_t^{\mathcal{M}}$ with $\hat{\vartheta}^{\mathcal{M}}(\vartheta_t, \Delta i_t; X_t)$. Therefore, the BSDE just stated must hold for ϑ_t (from the $\mathbf{e}$-

equilibrium). ϑ'_t solves the same equation but with $\breve{\mu}_t^{\mathcal{MB}}$ replaced with $\breve{\mu}_t^{\mathcal{MB}\prime}$. Because $\breve{\mu}_t^{\mathcal{MB}\prime} = \breve{\mu}_t^{\mathcal{MB}} - i'_t + i_t < \breve{\mu}_t^{\mathcal{MB}}$ in all states, $\vartheta_t = \vartheta'_t > 0$ implies $\mathbb{E}_t[\mathrm{d}\vartheta_t] > \mathbb{E}_t[\mathrm{d}\vartheta'_t]$ in any state. By the comparison theorem for BSDEs, we can conclude $\vartheta'_t > \vartheta_t$.

It follows then immediately that

$$\hat{\tau}'_t = -\breve{\mu}_t^{\mathcal{MB}\prime}\vartheta'_t > -\breve{\mu}_t^{\mathcal{MB}}\vartheta_t = \hat{\tau}_t,$$

where the middle inequality follows from $\vartheta'_t > \vartheta_t > 0$ and $\breve{\mu}_t^{\mathcal{MB}\prime} < \breve{\mu}_t^{\mathcal{MB}} \leq 0$ (the latter being a consequence of nonnegative taxes).

For the inequality of utilization rates, note that Δi_t is the same across the two equilibria and, therefore so is $\lambda_t^{\mathcal{M}}$. By Lemma 3, for given $\lambda_t^{\mathcal{M}}$, utilization is inversely related to ϑ_t:

$$u_t = \left((1 - \lambda_t^{\mathcal{M}}) \frac{(1 - \vartheta_t) a_t}{\rho} \right)^{1/\varphi}.$$

Hence, $\vartheta'_t > \vartheta_t$ implies $u'_t < u_t$. Lemma 2 then also implies $\iota'_t < \iota_t$. This completes the proof of the first part.

For the second part, $\vartheta'_0 > \vartheta_0$ implies

$$\mathcal{P}'_0 = \frac{\mathcal{MB}'_0}{q_0^{\mathcal{B}\prime} K'_0} = \frac{\mathcal{MB}_0}{\vartheta'_0 q'_0 K_0} = \frac{1 - \vartheta'_0}{\vartheta'_0} \frac{\mathcal{MB}_0}{K_0} < \frac{1 - \vartheta_0}{\vartheta_0} \frac{\mathcal{MB}_0}{K_0} = \mathcal{P}_0,$$

where the second equality follows from the assumption of identical initial conditions and the inequality follows from $\vartheta'_0 > \vartheta_0$ by part 1. Because $\mathcal{P}$, $\mathcal{P}'$ are continuous processes, the inequality extends to all sufficiently small t.

For the third part of the proposition, let us first consider a state in which $\Delta i_t = \Delta i'_t > 0$. Then the cash constraint (11) is binding in both $\mathbf{e}$ and $\mathbf{e}'$, which makes $\vartheta_t^{\mathcal{M}} = \hat{\vartheta}^{\mathcal{M}}(\vartheta_t, \Delta i_t; X_t)$ a necessary equilibrium requirement, where $\hat{\vartheta}^{\mathcal{M}}$ is as defined above. Because $\vartheta'_t > \vartheta_t$ and $\hat{\vartheta}^{\mathcal{M}}$ is decreasing in ϑ_t, it must be that

$$\vartheta_t^{\mathcal{M}\prime} = \hat{\vartheta}^{\mathcal{M}}(\vartheta'_t, \Delta i_t; X_t) < \hat{\vartheta}^{\mathcal{M}}(\vartheta_t, \Delta i_t; X_t) = \vartheta_t^{\mathcal{M}}.$$

This also implies for the reserve rate

$$i_t^{\mathcal{M}\prime} = i'_t - (1 - \vartheta_t^{\mathcal{M}\prime})\Delta i'_t = i_t - (1 - \vartheta_t^{\mathcal{M}\prime})\Delta i_t + i'_t - i_t$$

$$< i_t - (1-\vartheta_t^{\mathcal{M}})\Delta i_t + i_t' - i_t = i_t^{\mathcal{M}} + i_t' - i_t.$$

The previous two inequalities are the two inequalities asserted in part 3. of the proposition for a state in which $\Delta i_t > 0$.

For the remaining case, suppose that $\Delta i_t = \Delta i_t' = 0$. In this case, $i_t^{\mathcal{M}} = i_t$ and $i_t^{\mathcal{M}\prime} = i_t'$, which immediately implies the asserted equality. □

Proof of Lemma 6. Standard martingale asset pricing arguments imply that for any agent $i \in \mathbb{I}$ marginal in the bond market (all i, in equilibrium)

$$\mathbb{E}_t[\mathrm{d}r_t^{\mathcal{B}}] = -\frac{\mathbb{E}_t[\mathrm{d}\xi_t^i] + \mathrm{d}\langle \xi_t^i, r_t^{\mathcal{B}}\rangle}{\xi_t^i} = (r_t^i + \varsigma_t^i \sigma_t^{1/\mathcal{P}})\mathrm{d}t,$$

where $\xi_t^i = e^{-\rho t}\frac{c_0^i}{c_t^i}$ is the SDF process of agent i, $r_t^i := -\frac{\mathbb{E}_t[\mathrm{d}\xi_t^i]}{\xi_t^i \mathrm{d}t}$ is the shadow real risk-free rate, and ς_t^i is the negative of the geometric aggregate risk loading of ξ_t^i (the "price of risk") as in the proof of Lemma 1.

Substituting in $\mathbb{E}_t[\mathrm{d}r_t^{\mathcal{B}}]/\mathrm{d}t = i_t^{\mathcal{B}} + \mu_t^{1/\mathcal{P}}$ from equation (10), using $\mu_t^{1/\mathcal{P}} = -\mu_t^{\mathcal{P}} + (\sigma_t^{1/\mathcal{P}})^2 = -\pi_t + (\sigma_t^{1/\mathcal{P}})^2$, and solving for expected inflation yields

$$\pi_t = i_t^{\mathcal{B}} - r_t^i + \sigma_t^{1/\mathcal{P}}\left(\sigma_t^{1/\mathcal{P}} - \varsigma_t^i\right).$$

The asserted equation follows from the following three equations, which we proof separately:

$$i_t^{\mathcal{B}} = i_t^{\mathcal{M}} + \Delta i_t \tag{41}$$

$$r_t^i = \left(1 - \frac{\Delta i_t}{\nu_t}\right) a_t u_t - \frac{\hat{\tau}_t}{1-\vartheta_t} - (1-\vartheta_t)\tilde{\sigma}_t^2 - \delta \tag{42}$$

$$\sigma_t^{1/\mathcal{P}}\left(\sigma_t^{1/\mathcal{P}} - \varsigma_t^i\right) = \frac{(\sigma_t^{\vartheta})^2}{1-\vartheta_t} \tag{43}$$

- Equation (41):
  This follows directly from the definition of $\Delta i_t = i_t^{\mathcal{B}} - i_t^{\mathcal{M}}$.

- Equation (42):
  The SDF can be written in the form $\xi_t^i = \frac{c_0^i}{\rho} e^{-\rho t}\frac{1}{K_t}\frac{1}{q_t}\frac{1}{\eta_t^i}$. Applying Ito's lemma and using that $\frac{c_0^i}{\rho}$ is a constant, $e^{-\rho t}\frac{1}{K_t}$ is locally deterministic, $\frac{1}{q_t} = 1 - \vartheta_t$ loads only

on the aggregate shock $\mathrm{d}Z_t$, and $\frac{1}{\eta_t^i}$ loads only on the idiosyncratic shock $\mathrm{d}\tilde{Z}_t^i$, we obtain

$$\begin{aligned}\frac{\mathrm{d}\xi_t^i}{\xi_t^i} &= -\rho\mathrm{d}t - \frac{\mathrm{d}K_t}{K_t} - \frac{\mathrm{d}\vartheta_t}{1-\vartheta_t} - \frac{\mathrm{d}\eta_t^i}{\eta_t^i} + \frac{\mathrm{d}\langle\eta_t^i\rangle}{(\eta_t^i)^2} \\ &= -\left(\rho + \iota_t - \delta - (1-\vartheta_t)^2\tilde{\sigma}_t^2\right)\mathrm{d}t - \frac{\mathrm{d}\vartheta_t}{1-\vartheta_t} - (1-\vartheta_t)\tilde{\sigma}_t\mathrm{d}\tilde{Z}_t^i.\end{aligned}$$

Therefore,

$$r_t = -\frac{\mathbb{E}_t[\mathrm{d}\xi_t]}{\xi_t\mathrm{d}t} = \rho + \iota_t - \delta - (1-\vartheta_t)^2\tilde{\sigma}_t^2 + \frac{\mathbb{E}_t[\mathrm{d}\vartheta_t]}{(1-\vartheta_t)\mathrm{d}t}.$$

Substituting in ι_t from Lemma 2 and $\mathbb{E}_t[\mathrm{d}\vartheta_t]$ from Lemma 4 (equation (20)) yields

$$\begin{aligned}r_t &= \rho + \frac{(1-\vartheta_t)a_t u_t - \rho}{1-\vartheta_t} - \delta - (1-\vartheta_t)^2\tilde{\sigma}_t^2 + \frac{\left(\rho + \breve{\mu}_t^{\mathcal{MB}} - (1-\vartheta_t)^2\tilde{\sigma}_t^2 - \vartheta_t^{\mathcal{M}}\lambda_t^{\mathcal{M}}\nu_t\right)\vartheta_t}{1-\vartheta_t} \\ &= a_t u_t - \underbrace{\frac{\vartheta_t^{\mathcal{M}}\lambda_t^{\mathcal{M}}\nu_t\vartheta_t}{1-\vartheta_t}}_{=\lambda_t^{\mathcal{M}}a_t u_t \text{ by Lemma } 3} + \underbrace{\frac{\breve{\mu}_t^{\mathcal{MB}}\vartheta_t}{1-\vartheta_t}}_{=-\frac{\hat{\tau}_t}{1-\vartheta_t} \text{ by } (4)} - \underbrace{\left[(1-\vartheta_t)\vartheta_t\tilde{\sigma}_t^2 + (1-\vartheta_t)^2\tilde{\sigma}_t^2\right]}_{=(1-\vartheta_t)\tilde{\sigma}_t^2} - \delta.\end{aligned}$$

- Equation (43):
  $1/\mathcal{P}_t = \frac{q_t^{\mathcal{MB}}K_t}{\mathcal{MB}_t} = \vartheta_t q_t \frac{K_t}{\mathcal{MB}_t}$. By Ito's lemma, and because $\frac{K_t}{\mathcal{MB}_t}$ is locally deterministic,

  $$\sigma_t^{1/\mathcal{P}} = \sigma_t^{\vartheta} + \sigma_t^{q}.$$

  Furthermore, $\varsigma_t^i = \sigma_t^q$ (by Ito's lemma and multiplicative decomposition of ξ_t^i stated above), so

  $$\sigma_t^{1/\mathcal{P}}\left(\sigma_t^{1/\mathcal{P}} - \varsigma_t^i\right) = \left(\sigma_t^{\vartheta} + \sigma_t^{q}\right)\sigma_t^{\vartheta}.$$

  Finally, $q_t = \frac{1}{1-\vartheta_t}$ (Lemma 2), so $\sigma_t^q = \frac{\vartheta_t\sigma_t^{\vartheta}}{1-\vartheta_t}$. Combining this with the previous equation yields equation (43).

□

Proof of Proposition 9. We use a similar argument as in the proof of Proposition 8 to show that $\vartheta_t' > \vartheta_t$. We can write the valuation BSDE from Lemma 4 in the form

$$\mathbb{E}_t[d\vartheta_t] = \underbrace{\left(\left(\rho - (1-\vartheta_t)^2\tilde{\sigma}_t^2 - \vartheta_t^{\mathcal{M}}\Delta i_t\right)\vartheta_t \mathrm{d}t - \hat{\tau}_t\right)}_{=:f(\vartheta_t, \vartheta_t^{\mathcal{M}}\Delta i_t, X_t)}\mathrm{d}t.$$

Furthermore, define

$$g_t(\vartheta) := f(\vartheta, \vartheta_t^{\mathcal{M}}\Delta i_t, X_t), \qquad g'(\vartheta) := f(\vartheta, \vartheta_t^{\mathcal{M}\prime}\Delta i_t', X_t).$$

f is strictly decreasing in its second argument for any positive first argument ϑ. Because, by assumption, $\vartheta_t^{\mathcal{M}\prime}\Delta i_t' > \vartheta_t^{\mathcal{M}}\Delta i_t$ in all states, we obtain $g_t'(\vartheta) < g_t(\vartheta)$ in all states and for all $\vartheta \in (0,1)$. By using the BSDE comparison theorem to compare the BSDEs $\mathbb{E}_t[\mathrm{d}\vartheta_t] = g_t(\vartheta_t)\mathrm{d}t$ and $\mathbb{E}_t[\mathrm{d}\vartheta_t'] = g_t(\vartheta_t')\mathrm{d}t$, we can conclude that the solution to the former, namely ϑ_t must be strictly below the solution to the latter, ϑ_t'.

Having established $\vartheta_t' > \vartheta_t$, the inequalities for the utilization and investment rate follow easily. For the utilization rate, by Lemma 3 and due to $\lambda_t^{\mathcal{M}\prime} > \lambda_t^{\mathcal{M}}$,

$$u_t' = \left((1-\lambda_t^{\mathcal{M}\prime})\frac{(1-\vartheta_t')a_t}{\rho}\right)^{1/\varphi} < \left((1-\lambda_t^{\mathcal{M}})\frac{(1-\vartheta_t)a_t}{\rho}\right)^{1/\varphi} = u_t.$$

$\iota_t' < \iota_t$ follows then from Lemma 2. This concludes the proof of the first part.

The proof of part 2. is identical to the proof of part 2. of Proposition 8 and therefore omitted here.

Finally, regarding implementation, the cash constraint (11) must be binding in equilibrium $\mathbf{e}'$ throughout (because $\Delta i_t' > \Delta i_t \geq 0$), so $u_t' = u^c(\vartheta_t', \vartheta_t^{\mathcal{M}\prime}; X_t)$ for u^c as defined in Lemma 3. The binding constraint and the previously derived inequalities imply

$$\vartheta_t^{\mathcal{M}\prime} = \frac{a_t u_t'}{\nu_t}\frac{1-\vartheta_t'}{\vartheta_t'} < \frac{a_t u_t}{\nu_t}\frac{1-\vartheta_t}{\vartheta_t} \leq \frac{a_t u^c(\vartheta_t, \vartheta_t^{\mathcal{M}}; X_t)}{\nu_t}\frac{1-\vartheta_t}{\vartheta_t} = \vartheta_t^{\mathcal{M}}.$$

Because $i_t' = i_t$, we then obtain for the reserve rates

$$i_t^{\mathcal{M}\prime} = i_t' - (1-\vartheta_t^{\mathcal{M}\prime})\Delta i_t' < i_t - (1-\vartheta_t^{\mathcal{M}})\Delta i_t = i_t^{\mathcal{M}}.$$

□

Proof of Proposition 10. Equilibrium utilization is given by (compare Lemma 3)

$$u_t = \left((1 - \lambda_t^{\mathcal{M}}) \frac{(1 - \vartheta_t) a_t}{\rho} \right)^{1/\varphi}.$$

By assumption, both policies lead to the same ϑ_t, but policy (2) features a larger interest rate spread Δi_t and therefore a larger liquidity Lagrange multiplier $\lambda_t^{\mathcal{M}}$ than policy (1). Therefore, u_t under policy (1) must be strictly larger than under policy (2) throughout.

Next, the equilibrium investment rate ι_t is related to ϑ_t and u_t by (compare Lemma 2)

$$\iota_t = \frac{(1 - \vartheta_t) a_t u_t - \rho}{1 - \vartheta_t},$$

which is strictly increasing in u_t. Hence, also the investment rate ι_t is larger under policy (1) than under policy (2), which implies a higher capital growth rate $\mathrm{d}K_t / (K_t \mathrm{d}t)$ under policy (1).

Total output at time t is

$$Y_t = a_t u_t K_t.$$

Because we have already established that both utilization u_t and capital K_t are larger under policy (1), output must be larger as well. □

A.6 Optimal Policy Results

Proof of Proposition 11. We first compute the expected utility V_0^i of an individual agent. To do so, let us first justify the following mathematical result: for any Ito process X_t satisfying $\mathrm{d}X_t / X_t = \mu_t^X \mathrm{d}t + \sigma_t^X \mathrm{d}Z_t + \tilde{\sigma}_t^X \mathrm{d}\tilde{Z}_t^i$,

$$\mathbb{E}_0 \left[\int_0^\infty e^{-\rho t} \log X_t \mathrm{d}t \right] = \frac{1}{\rho} \log X_0 + \frac{1}{\rho} \mathbb{E}_0 \left[\int_0^\infty e^{-\rho t} \left(\mu_t^X - \frac{1}{2} \left(\sigma_t^X \right)^2 - \frac{1}{2} (\tilde{\sigma}_t^X)^2 \right) \mathrm{d}t \right].$$

To show this, apply Ito's lemma to $\log X_t$, which implies

$$\mathrm{d} \log X_t = \left(\mu_t^X - \frac{1}{2} (\sigma_t^X)^2 - \frac{1}{2} (\tilde{\sigma}_t^X)^2 \right) \mathrm{d}t + \sigma_t^X \mathrm{d}Z_t + \tilde{\sigma}_t^X \mathrm{d}\tilde{Z}_t^i.$$

Writing this in integral form and taking expectations,

$$\mathbb{E}_0[\log X_t] = \log X_0 + \int_0^t \mathbb{E}_0 \left(\mu_s^X - \frac{1}{2}(\sigma_s^X)^2 - \frac{1}{2}(\tilde{\sigma}_s^X)^2 \right) \mathrm{d}s + \underbrace{\mathbb{E}_0 \left[\int_0^t \sigma_s^X \mathrm{d}Z_s \right] + \mathbb{E}_0 \left[\int_0^t \tilde{\sigma}_s^X \mathrm{d}\tilde{Z}_s \right]}_{=0}.$$

Hence,

$$\begin{aligned}
\mathbb{E}_0 \left[\int_0^\infty e^{-\rho t} \log X_t \mathrm{d}t \right] &= \int_0^\infty e^{-\rho s} \mathbb{E}_0 \left[\log X_t \right] \mathrm{d}s \\
&= \int_0^\infty e^{-\rho s} \log X_0 \mathrm{d}s + \int_0^\infty e^{-\rho s} \int_0^s \mathbb{E}_0 \left(\mu_t^X - \frac{1}{2}(\sigma_t^X)^2 - \frac{1}{2}(\tilde{\sigma}_t^X)^2 \right) \mathrm{d}t \mathrm{d}s \\
&= \frac{1}{\rho} \log X_0 + \int_0^\infty \int_t^\infty e^{-\rho s} \mathrm{d}s \mathbb{E}_0 \left(\mu_t^X - \frac{1}{2}(\sigma_t^X)^2 - \frac{1}{2}(\tilde{\sigma}_t^X)^2 \right) \mathrm{d}t \\
&= \frac{1}{\rho} \log X_0 + \frac{1}{\rho} \mathbb{E}_0 \left(\int_0^\infty e^{-\rho t} \left(\mu_t^X - \frac{1}{2}(\sigma_t^X)^2 - \frac{1}{2}(\tilde{\sigma}_t^X)^2 \right) \mathrm{d}t \right).
\end{aligned}$$

We now use $\log c_t^i = \log (a_t u_t - \iota_t) + \log K_t + \log \eta_t^i$, where $\eta_t^i = n_t^i / (q_t K_t)$ is the wealth share of household i. Applying the previous result for both $X_t = K_t$ and $X_t = \eta_t^i$ and using that K_t has zero risk terms whereas η_t^i has zero drift and aggregate risk, we obtain for individual expected utility

$$\begin{aligned}
V_0^i &= \mathbb{E}_0 \left[\int_0^\infty e^{-\rho t} \left(\log c_t^i - \frac{u_t^{1+\varphi}}{1+\varphi} \right) \mathrm{d}t \right] \\
&= \frac{\log K_0 + \log \eta_0^i}{\rho} + \mathbb{E}_0 \left[\int_0^\infty e^{-\rho t} \left(\log (a_t u_t - \iota_t) + \frac{\mu_t^K}{\rho} - \frac{(\tilde{\sigma}_t^{\eta,i})^2}{2\rho} - \frac{u_t^{1+\varphi}}{1+\varphi} \right) \mathrm{d}t \right].
\end{aligned}$$

Next, plug in $a_t u_t - \iota_t = \rho q_t = \rho \frac{1}{1-\vartheta_t}$, $\mu_t^K = \iota_t - \delta = \frac{(1-\vartheta_t) a_t u_t - \rho}{1-\vartheta_t} - \delta$, $\tilde{\sigma}_t^{\eta,i} = (1-\vartheta_t)\tilde{\sigma}_t$ and rearrange,

$$\begin{aligned}
V_0^i = &\frac{\log K_0 + \log \eta_0^i + \log \rho}{\rho} - \frac{\delta}{\rho^2} \\
&+ \mathbb{E}_0 \left[\int_0^\infty e^{-\rho t} \left(\log \left(\frac{1}{1-\vartheta_t} \right) + \frac{1}{\rho} \frac{(1-\vartheta_t) a_t u_t - \rho}{1-\vartheta_t} - \frac{(1-\vartheta_t)^2 \tilde{\sigma}_t^2}{2\rho} - \frac{u_t^{1+\varphi}}{1+\varphi} \right) \mathrm{d}t \right]
\end{aligned}$$

$$
\begin{aligned}
&= \frac{\log K_0 + \log \eta_0^i + \log \rho}{\rho} - \frac{\delta}{\rho^2} \\
&\quad + \mathbb{E}_0 \left[\int_0^\infty e^{-\rho t} \left(\log \left(\frac{1}{1-\vartheta_t} \right) - \frac{1}{1-\vartheta_t} - \frac{(1-\vartheta_t)^2 \tilde{\sigma}_t^2}{2\rho} \right) \mathrm{d}t \right] \\
&\quad + \mathbb{E}_0 \left[\int_0^\infty e^{-\rho t} \left(\frac{a_t u_t}{\rho} - \frac{u_t^{1+\varphi}}{1+\varphi} \right) \mathrm{d}t \right] \\
&= \underbrace{\frac{\log K_0 + \log \eta_0^i + \log \rho}{\rho} - \frac{\delta}{\rho^2}}_{=: V_0^{(0),i}} + \mathbb{E}_0 \left[\int_0^\infty e^{-\rho t} \left(\mathcal{W}_{\vartheta,t}(\vartheta_t) + \mathcal{W}_{u,t}(u_t) \right) \mathrm{d}t \right].
\end{aligned}
$$

The term $V_0^{(0),i}$ captures initial conditions which are independent of control variables by the assumption of no redistribution of initial wealth. This term is therefore irrelevant to welfare assessments for such policy control variables. The remaining term, $V_0^i - V_0^{(0),i}$ is identical for all households and corresponds to the objective stated in the proposition. Hence, so long as the social welfare function is strictly increasing in individual welfare, maximizing the welfare function is equivalent to maximizing the objective stated in the proposition.

Now consider a constrained planner who can directly control u_t and ϑ_t but has to respect the competitive equilibrium relationships implicit in the functions $\mathcal{W}_{\vartheta,t}$ and $\mathcal{W}_{u,t}$. This planner finds it optimal to choose u_t and ϑ_t each period in a way that maximizes $\mathcal{W}_{u,t}(u_t)$ and $\mathcal{W}_{\vartheta,t}(\vartheta_t)$, respectively.

Maximizing $\mathcal{W}_u(u_t)$ (which is strictly concave for $\varphi > 0$) leads to the first-order condition:

$$
0 = \mathcal{W}_u'(u_t) = \frac{a_t}{\rho} - u_t^{\varphi} \Leftrightarrow u_t = \left(\frac{a_t}{\rho} \right)^{1/\varphi} =: u^{e*}(a_t).
$$

Evidently, u^{e*} is a strictly increasing function.

Maximizing $\mathcal{W}_{\vartheta,t}(\vartheta_t)$ (which is strictly quasiconcave for $\vartheta_t \in [0,1)$) leads to the first-order condition

$$
\begin{aligned}
& 0 = \mathcal{W}_{\vartheta,t}'(\vartheta_t) = \frac{1}{1-\vartheta_t} - \frac{1}{(1-\vartheta_t)^2} + \frac{(1-\vartheta_t)\tilde{\sigma}_t^2}{\rho} \\
\Leftrightarrow \quad & 0 = (1-\vartheta_t)^3 \, \tilde{\sigma}_t^2 + \rho \, (1-\vartheta_t) - \rho
\end{aligned}
$$

The right-hand side of the last equation is a third-order polynomial in $1 - \vartheta_t$, which means the equation has, in principle, up to three real solutions. However, because all coefficients on terms of positive order are positive and the coefficient on the zero order term is negative, well-known results on polynomial roots imply that there is a unique positive solution $1-\vartheta_t^{e*}>0$. It is easy to see that this solution must also satisfy $\vartheta_t^{e*}>0$. Finally, observe that ϑ_t^{e*} is a function of $\tilde{\sigma}_t$ only, $\vartheta_t^{e*}=\vartheta^*(\tilde{\sigma}_t)$, and a straightforward application of the implicit function theorem yields $\vartheta^{e*\prime}>0$. □

Proof of Lemma 7. By Lemma 3, the utilization rate in the competitive equilibrium is

$$u_t=\left(\left(1-\lambda_t^{\mathcal{M}}\right)\frac{(1-\vartheta_t)a_t}{\rho}\right)^{1/\varphi}$$

and comparing this with the expression for u_t^{e*} derived in the proof of Proposition 11 yields

$$u_t=\left[\left(1-\lambda_t^{\mathcal{M}}\right)(1-\vartheta_t)\right]^{1/\varphi}u_t^{e*}.$$

The expression in square brackets is smaller than 1 unless $\vartheta_t=\lambda_t^{\mathcal{M}}=0$. Due to Lemma 3, $\vartheta_t=\lambda_t^{\mathcal{M}}=0$ is not possible in any equilibrium: if $\vartheta_t=0$, then $u_t\le u^c(0,\vartheta_t^{\mathcal{M}},X_t)=0$ and hence $\lambda_t^{\mathcal{M}}=1$. □

Proof of Proposition 12. We first prove the claim that the three equations have unique solutions:

- Equation (28):
  First, an explicit calculation shows that $\mathcal{W}'_{\vartheta,t}$ is a strictly decreasing function:

$$\mathcal{W}'_{\vartheta,t}(\vartheta)=\frac{1}{1-\vartheta}-\frac{1}{(1-\vartheta)^2}+\frac{(1-\vartheta)\tilde{\sigma}_t^2}{\rho}=-\frac{\vartheta}{(1-\vartheta)^2}+\frac{(1-\vartheta)\tilde{\sigma}_t^2}{\rho}.$$

  This implies that the first term on the right-hand side of equation (28) is strictly decreasing in $\underline{\vartheta}_t^*$. For $\varphi\ge 1$, also the second term is strictly decreasing in $\underline{\vartheta}_t^*$. Therefore, there can be at most one solution to the equation. Furthermore, the above representation for $\mathcal{W}'_{\vartheta,t}$ shows that $\mathcal{W}'_{\vartheta,t}(\underline{\vartheta}_t^*)\to\frac{\tilde{\sigma}_t^2}{\rho}\ge 0$ as $\underline{\vartheta}_t^*\to 0$ and $\mathcal{W}'_{\vartheta,t}(\underline{\vartheta}_t^*)\to-\infty<0$ as $\underline{\vartheta}_t^*\to 1$. Similarly, the second term on the right-hand side of equation (28) is nonpositive and vanishes for $\underline{\vartheta}_t^*=0$. By continuity, there must be at least one solution $\underline{\vartheta}_t^*\in[0,\vartheta_t^{e*}]\subset[0,1)$.

- Equation (29):
  Using the explicit expression for $\mathcal{W}'_{\vartheta,t}(\vartheta)$ computed above, we can write the right-hand side of equation (29) in the form

$$\underbrace{\frac{(1-\overline{\vartheta}_t^*)\tilde{\sigma}_t^2}{\rho}}_{\text{decreasing in } \overline{\vartheta}_t^*} + \underbrace{\frac{1}{(1-\overline{\vartheta}_t^*)^2}}_{\text{positive and increasing in } \overline{\vartheta}_t^*} \underbrace{\left(\left(\frac{\nu_t}{\rho} - \left(\frac{\nu_t}{a_t}\right)^{1+\varphi}\left(\frac{\overline{\vartheta}_t^*}{1-\overline{\vartheta}_t^*}\right)^{\varphi}\right)^+ - \overline{\vartheta}_t^*\right)}_{\text{decreasing in } \overline{\vartheta}_t^*}.$$

  While this expression is not globally decreasing in $\overline{\vartheta}_t^*$, it is so whenever the third term is negative. As this is the only relevant region in which there can be a solution, it follows that there is at most one solution. For existence, note that the right-hand side of the equation is nonnegative for $\overline{\vartheta}_t^* \leq \vartheta_t^{e*}$ and it approaches $-\infty$ as $\overline{\vartheta}_t^* \to 1$, so by continuity, there must be a solution $\overline{\vartheta}_t^* \in [\vartheta_t^{e*}, 1)$.

- Equation (30):
  The left-hand side of this equation is continuous and strictly increasing in $\hat{\vartheta}_t$ and varies from 0 to a strictly positive number as $\hat{\vartheta}_t$ varies from 0 to 1. The right-hand side is continuous and strictly decreasing and varies from a strictly positive number to 0 as $\hat{\vartheta}_t$ varies from 0 to 1. Therefore, there must be a unique solution in $(0, 1)$.

For the claims about optimality of ϑ_t^*, note that by Proposition 11, Corollary 3, and Lemma 3, $\vartheta_t^* \in [0, 1)$ must maximize the function

$$\vartheta \mapsto \mathcal{W}_{\vartheta,t}(\vartheta) + \mathcal{W}_{u,t}\left(\min\{u^0(\vartheta; X_t), u^c(\vartheta, 1; X_t)\}\right)$$

state by state. Because this objective function is continuous, a maximizer ϑ_t^* must exist in the compact interval $[0, 1]$ but we can exclude the boundary case $\vartheta_t^* = 1$ from the outset because $\mathcal{W}_{\vartheta,t}(\vartheta) \to -\infty$ as $\vartheta \to 1$.

Furthermore, this objective function is also continuously differentiable, except possibly at $\vartheta = \hat{\vartheta}_t$ where both arguments of the minimum operator are equal. Let us therefore distinguish three cases, for any given state and date:

1. $\vartheta_t^* \in [0, \hat{\vartheta}_t)$:
   Then $\min\{u^0(\vartheta; X_t), u^c(\vartheta, 1; X_t)\} = u^c(\vartheta, 1; X_t)$ for $\vartheta \approx \vartheta_t^*$, and therefore $\vartheta_t^* \geq 0$

must satisfy the first-order condition

$$0 \geq \mathcal{W}'_{\vartheta,t}(\vartheta^*_t) + \mathcal{W}'_{u,t}\left(u^c(\vartheta, 1; X_t)\right) u^{c\prime}(\vartheta^*_t, 1; X_t)$$

and with equality if $\vartheta^*_t > 0$. The right-hand side of this condition is the same as the one in equation (29) (so long as $\vartheta < \hat{\vartheta}$, so that the argument of the $(\cdot)^+$-term in (29) is always positive). Observe that this expression is positive for $\vartheta = 0$, so $\vartheta^*_t = 0$ is, in fact, impossible and it must be that the first-order condition holds with equality. But then by uniqueness of the solution to equation (29), $\vartheta^*_t = \overline{\vartheta}^*_t$. Because $\hat{\vartheta}_t > \vartheta^*_t = \overline{\vartheta}^*_t$, it is then also the case that

$$\vartheta^*_t = \min\{\max\{\hat{\vartheta}_t, \underline{\vartheta}^*_t\}, \overline{\vartheta}^*_t\}.$$

2. $\vartheta^*_t \in (\hat{\vartheta}_t, 1)$:
Then $\min\{u^0(\vartheta; X_t), u^c(\vartheta, 1; X_t)\} = u^0(\vartheta; X_t)$ for $\vartheta \approx \vartheta^*_t$, and therefore ϑ^*_t must satisfy the first-order condition

$$0 = \mathcal{W}'_{\vartheta,t}(\vartheta^*_t) + \mathcal{W}'_{u,t}\left(u^0(\vartheta; X_t)\right) u^{0\prime}(\vartheta^*_t; X_t).$$

Comparing this with equation (28), we conclude $\vartheta^*_t = \underline{\vartheta}^*_t$. Because $\hat{\vartheta}_t < \vartheta^*_t = \underline{\vartheta}^*_t$ and $\underline{\vartheta}^*_t \leq \vartheta^{e*}_t \leq \overline{\vartheta}^*_t$, it is then also the case that

$$\vartheta^*_t = \min\{\max\{\hat{\vartheta}_t, \underline{\vartheta}^*_t\}, \overline{\vartheta}^*_t\}.$$

3. $\vartheta^*_t = \hat{\vartheta}_t$:
For any $\vartheta < \vartheta^*_t$, the constraint is binding and so the objective is locally the same as in case 1. As this objective is maximized at $\vartheta = \overline{\vartheta}^*_t$, it must be that $\overline{\vartheta}^*_t \geq \hat{\vartheta}_t$. Otherwise, the objective would be strictly larger at $\overline{\vartheta}^*_t$ than at ϑ^*_t, in contradiction to the assumption that ϑ^*_t is optimal.

   Similarly, for any $\vartheta > \vartheta^*_t$, the constraint is binding and so the objective is locally the same as in case 2. As this objective is maximized at $\vartheta = \underline{\vartheta}^*_t$, it must be that $\underline{\vartheta}^*_t \leq \hat{\vartheta}_t$. Otherwise, the objective would be strictly larger at $\underline{\vartheta}^*_t$ than at ϑ^*_t, in contradiction to the assumption that ϑ^*_t is optimal.

Taken together, the previous two arguments imply

$$\underline{\vartheta}_t^* \leq \hat{\vartheta}_t \leq \overline{\vartheta}_t^*,$$

hence

$$\vartheta_t^* = \hat{\vartheta}_t = \min\{\max\{\hat{\vartheta}_t, \underline{\vartheta}_t^*\}, \overline{\vartheta}_t^*\}.$$

Because in all cases we end up with the same formula for ϑ_t^*, the optimal ϑ-choice must be unique. In addition, case 1. proves the claim $\vartheta_t^* > 0$.

Let us briefly comment on the additional statements in the proposition, which directly follow from the previous arguments in the proof:

1. If $\vartheta_t^* = \underline{\vartheta}_t^* > \hat{\vartheta}_t$, we are in case 2. above. As observed there,

   $$u_t^* = \min\{u^0(\vartheta_t^*; X_t), u^c(\vartheta_t^*, 1; X_t)\} = u^0(\vartheta_t^*; X_t),$$

   so $\lambda_t^{\mathcal{M}} = 0$ by Lemma 3, so the Friedman rule is optimal.

   $\vartheta_t^* = \underline{\vartheta}_t^* \leq \vartheta_t^{e*}$ has already been shown above.

2. If $\vartheta_t^* = \hat{\vartheta}_t$, we are in case 3. above. Also there

   $$u_t^* = \min\{u^0(\vartheta_t^*; X_t), u^c(\vartheta_t^*, 1; X_t)\} = u^0(\vartheta_t^*; X_t),$$

   so $\lambda_t^{\mathcal{M}} = 0$ by Lemma 3, so the Friedman rule is optimal.

   For the claim that $\vartheta_t^* = \hat{\vartheta}_t$ can be on either side of ϑ_t^{e*}, note that the latter only depends on $\tilde{\sigma}_t$, whereas the former depends on a_t and ν_t. By inspecting equation (30), it is clear that, for any given $\tilde{\sigma}_t$, a_t, and $\varepsilon > 0$, it is always possible to choose ν_t such that $\hat{\vartheta}_t = \vartheta_t^{e*} \pm \varepsilon$. By making ε small enough and a_t large enough, we can also ensure that $\hat{\vartheta}_t \in [\underline{\vartheta}_t^*, \overline{\vartheta}_t^*]$, so that $\vartheta_t^* = \hat{\vartheta}_t$. There exist therefore parameters that generate either of the sub-cases $\vartheta_t^* = \underline{\vartheta}_t^* < \vartheta_t^{e*}$ and $\vartheta_t^* = \underline{\vartheta}_t^* > \vartheta_t^{e*}$.

3. If $\vartheta_t^* = \overline{\vartheta}_t^* < \hat{\vartheta}_t$, we are in case 1. above. Then

   $$u_t^* = \min\{u^0(\vartheta_t^*; X_t), u^c(\vartheta_t^*, 1; X_t)\} = u^c(\vartheta_t^*; X_t) < u^0(\vartheta_t^*; X_t),$$

   so $\lambda_t^{\mathcal{M}} > 0$ by Lemma 3, which implies a positive liquidity premium $\Delta i_t = \lambda_t^{\mathcal{M}} \nu_t > 0$. The Friedman rule is not optimal.

$\vartheta_t^* = \overline{\vartheta}_t^* \geq \vartheta_t^{e*}$ has already been shown above.

□

B Model Variants and Extensions

B.1 Model with Long-term Bonds

B.1.1 Setup

We briefly restate the setup and definition of notation outlined in the main text. The government bond stock consists of zero coupon bonds of arbitrary duration $\Delta \geq 0$. We assume that the government uses the whole continuum of available bonds in a way that the promised payment at any given date is a dt-flow.[37] $\mathcal{X}_t(\Delta)$ denotes the face value of zero coupon bonds outstanding at time t with time to maturity Δ. The nominal price of such a bond is

$$\mathcal{P}_t^{\mathcal{B}}(\Delta) = \mathbb{E}_t\left[\frac{\xi_{t+\Delta}/\xi_t}{\mathcal{P}_{t+\Delta}/\mathcal{P}_t}\right],$$

where ξ_t is any real SDF that prices aggregate claims (this is a restatement of equation (21) from the main text). The total nominal value of outstanding government bonds at time t is

$$\mathcal{B}_t = \int_0^\infty \mathcal{P}_t^{\mathcal{B}}(\Delta)\mathcal{X}_t(\Delta)\mathrm{d}\Delta$$

and the flow budget constraint of the government is

$$\underbrace{\mathrm{d}\mathcal{B}_t - \int_0^\infty \mathcal{X}_t(\Delta)\left(\mathrm{d}\mathcal{P}_t^{\mathcal{B}}(\Delta) - \mathcal{P}_t^{\mathcal{B}\prime}(\Delta)\mathrm{d}t\right)\mathrm{d}\Delta}_{\text{new debt issuance}} + \mathrm{d}\mathcal{M}_t = i_t^{\mathcal{M}}\mathcal{M}_t\mathrm{d}t - \mathcal{P}_t\tau_t K_t\mathrm{d}t. \tag{44}$$

This equation replaces equation (3) from the baseline model. Like in the baseline model, we denote by $\mathcal{MB}_t := \mathcal{M}_t + \mathcal{B}_t$ the *nominal market value* of government liabilities, and use the notation $q_t^{\mathcal{MB}}K_t := \frac{\mathcal{MB}_t}{\mathcal{P}_t}$ for their real value.

The remaining model setup is as in Section 2.1 with the difference that the return on money and bonds are not given by equations (9) and (10). Compared to equation (9),

[37] Everything could be generalized beyond this case but this requires heavier notation as then $\mathcal{X}_t(\Delta)$ would really be a measure over $[0,\infty)$ instead of a density of promised repayments.

the return on money differs only slightly:

$$\mathrm{d}r_t^{\mathcal{M}} = i_t^{\mathcal{M}}\mathrm{d}t + \frac{\mathrm{d}(1/\mathcal{P}_t)}{1/\mathcal{P}_t} = i_t^{\mathcal{M}}\mathrm{d}t + \frac{\mathrm{d}(q_t^{\mathcal{MB}}K_t/\mathcal{MB}_t)}{q_t^{\mathcal{MB}}K_t/\mathcal{MB}_t}. \tag{45}$$

Here, we cannot remove the $\mathcal{MB}_t$-portion on the right-hand side from the stochastic differential term and replace it with a drift as in equation (9) because $\mathcal{MB}_t$ may have nonzero volatility loading due to price variation in its $\mathcal{B}_t$-component. The return on bonds is even more involved and now given by

$$\mathrm{d}r_t^{\mathcal{B}} = \frac{\int_0^\infty \mathcal{X}_t(\Delta)\left(\frac{\mathrm{d}(\mathcal{P}_t^{\mathcal{B}}(\Delta)/\mathcal{P}_t)}{1/\mathcal{P}_t} - \mathcal{P}_t^{\mathcal{B}\prime}(\Delta)\mathrm{d}t\right)\mathrm{d}\Delta}{\int_0^\infty \mathcal{X}_t(\Delta)\mathcal{P}_t^{\mathcal{B}}(\Delta)\mathrm{d}\Delta}. \tag{46}$$

We note that this is a real rate of return because of the presence of the inverse price level $1/\mathcal{P}_t$ in the term $\frac{\mathrm{d}(\mathcal{P}_t^{\mathcal{B}}(\Delta)/\mathcal{P}_t)}{1/\mathcal{P}_t}$. If $\mathcal{P}_t$ is locally deterministic, the presence of $1/\mathcal{P}_t$ is equivalent to subtracting the inflation rate from the nominal return. More generally, it adds covariance terms between changes in nominal bond prices and the price level.

The decision problem of agent $i \in \mathbb{I}$ is to choose consumption c_t^i, investment ι_t^i, capital utilization u_t^i, and portfolio weights $\theta_t^i, \theta_t^{\mathcal{M},i}$ to maximize utility V_0^i subject to the net worth evolution (7), the return expressions (8), (45), and (46), the cash constraint (11), and a solvency constraint $n_t^i \geq 0$. Note that even the second line in equation (8) for the return on capital remains valid here because we have defined the notation $\breve{\mu}_t^{\mathcal{MB}}$ as the negative of the ratio of taxes to government liabilities, i.e., $\breve{\mu}_t^{\mathcal{MB}} = -\frac{\tau_t}{q_t^{\mathcal{MB}}}$.

The market clearing conditions remain the same as in the baseline model, i.e., equation (12) for the goods market, equation (14) for the market of government liabilities, and equation (14) for the money market.[38]

In analogy to Definition 1, we define an equilibrium in the extended model as follows:

Definition 2. *Given an exogenous process X_t, functions as in (5), initial stocks of capital, money, and government bonds, K_0, $\mathcal{M}_0$, $\{\mathcal{X}_0(\Delta)\}_{\Delta\geq 0}$, and an initial cross-sectional wealth*

[38] By pricing individual bonds according to equation (21), we ensure that agents are indifferent between all bond maturities and do not need to impose bond market clearing for each individual Δ separately. Equivalently, but at the expense of more notation, we could have not imposed these equations and instead added a continuum of separate portfolio weights $\theta_t^{\mathcal{B}}(\Delta)$ for each Δ-bond in the household problem as well as a continuum of market clearing conditions for them. We would then recover the pricing equation (21) in equilibrium.

distribution $\{\eta_0^i\}_{i\in\mathbb{I}}$ *satisfying* $\int_{\mathbb{I}} \eta_0^i di = 1$, *a* competitive equilibrium *consists of aggregate (Ito) stochastic processes* K_t, $\mathcal{MB}_t$, $\mathcal{P}_t$, $q_t^{\mathcal{M}}$, $q_t^{\mathcal{MB}}$, q_t^K, $\breve{\mu}_t^{\mathcal{MB}}$, $i_t^{\mathcal{M}}$, $\{\mathcal{X}_t(\Delta)\}_{\Delta\geq 0}$, $\{\mathcal{P}_t^{\mathcal{B}}(\Delta)\}_{\Delta\geq 0}$ *adapted to* $\mathcal{F}_t$ *and, for each* $i \in \mathbb{I}$, *individual (Ito) stochastic processes* c_t^i, ι_t^i, u_t^i, θ_t^i, $\theta_t^{\mathcal{M},i}$, n_t^i *adapted to* $\tilde{\mathcal{F}}_t^i$, *such that*

1. *For each agent* $i \in \mathbb{I}$, c_t^i, ι_t^i, u_t^i, θ_t^i, $\theta_t^{\mathcal{M},i}$ *solve i's optimization problem for initial wealth* $n_0^i = \eta_0^i(q_0^K + q_0^{\mathcal{MB}})K_0$ *and* n_t^i *is as implied by the net worth evolution* (7).

2. K_t *satisfies the aggregate capital evolution*

$$dK_t = \left(\int_{\mathbb{I}} \iota_t^i k_t^i di - \delta K_t\right) dt$$

   with the given initial condition K_0.

3. $\mathcal{MB}_t$ *satisfies the evolution*

$$d\mathcal{MB}_t = \left(\breve{\mu}_t^{\mathcal{MB}} + \frac{q_t^{\mathcal{M}}}{q_t^{\mathcal{MB}}} i_t^{\mathcal{M}} + \frac{\mathcal{X}_t(0)}{\mathcal{MB}_t}\right) \mathcal{MB}_t dt + \int_0^\infty \mathcal{X}_t(\Delta)(d\mathcal{P}_t^{\mathcal{B}}(\Delta) - \mathcal{P}_t^{\mathcal{B}\prime}(\Delta)dt)d\Delta$$

   with the initial condition

$$\mathcal{MB}_0 = \mathcal{M}_0 + \int_0^\infty \mathcal{P}_0^{\mathcal{B}}(\Delta)\mathcal{X}_0(\Delta)d\Delta.$$

4. $q_t^{\mathcal{MB}} K_t$ *is the real value of government liabilities: for all* t,

$$q_t^{\mathcal{MB}} K_t = \frac{\mathcal{MB}_t}{\mathcal{P}_t}.$$

5. *Bond prices* $\mathcal{P}_t^{\mathcal{B}}$ *satisfy*[39]

$$\mathcal{P}_t^{\mathcal{B}}(\Delta) = \mathbb{E}_t\left[e^{-\rho\Delta}\frac{\mathcal{P}_t c_t^i}{\mathcal{P}_{t+\Delta} c_{t+\Delta}^i}\right]$$

   for all $t, \Delta \geq 0$ *and some* $i \in \mathbb{I}$.

6. *All asset values and nominal bond quantities are nonnegative,* $q_t^{\mathcal{M}}, q_t^{\mathcal{B}} := q_t^{\mathcal{MB}} - q_t^{\mathcal{M}}, q_t^K \geq 0$ *for all* t *and* $\mathcal{X}_t(\Delta) \geq 0$ *for all* t *and all* Δ.

[39]This is essentially equation (21) with $\xi_t = e^{-\rho t}/c_t^i$. All agents agree in any equilibrium on the valuation of aggregate cash flows, so the identity i of the agent pricing the bond does not matter.

7. *All markets clear: for all t, equations (12), (13), and (14) hold.*

B.1.2 Auxiliary Results

In this section, we prove several technical auxiliary results that will be helpful in proving Lemma 5 and Proposition 3 stated in the main text.

Useful Lemmas.

Lemma 8. *In a Definition 2 equilibrium (long-term bond model),*

$$dr_t^{\mathcal{B}} = \frac{\int_0^\infty \mathcal{X}_t(\Delta)\left(d\mathcal{P}_t^{\mathcal{B}}(\Delta) - \mathcal{P}_t^{\mathcal{B}\prime}(\Delta)dt\right)d\Delta}{\mathcal{B}_t} + \frac{d(1/\mathcal{P}_t)}{1/\mathcal{P}_t} + \frac{d\langle \mathcal{MB}_t, 1/\mathcal{P}_t\rangle}{\mathcal{B}_t/\mathcal{P}_t}.$$

Proof. Because of $\mathcal{B}_t = \int_0^\infty \mathcal{X}_t(\Delta)\mathcal{P}_t^{\mathcal{B}}(\Delta)\mathrm{d}\Delta$, the asserted equation follows immediately from equation (46) if we can show that

$$\int_0^\infty \mathcal{X}_t(\Delta)\frac{\mathrm{d}(\mathcal{P}_t^{\mathcal{B}}(\Delta)/\mathcal{P}_t)}{1/\mathcal{P}_t}\mathrm{d}\Delta = \int_0^\infty \mathcal{X}_t(\Delta)\mathrm{d}\mathcal{P}_t^{\mathcal{B}}(\Delta)\mathrm{d}\Delta + \mathcal{B}_t\frac{\mathrm{d}(1/\mathcal{P}_t)}{1/\mathcal{P}_t} + \frac{\mathrm{d}\langle \mathcal{MB}_t, 1/\mathcal{P}_t\rangle}{1/\mathcal{P}_t} \tag{47}$$

For the proof of equation (47), start with Ito's product rule applies to $\mathcal{P}_t^{\mathcal{B}}(\Delta)/\mathcal{P}_t$:

$$\frac{\mathrm{d}(\mathcal{P}_t^{\mathcal{B}}(\Delta)/\mathcal{P}_t)}{1/\mathcal{P}_t} = \mathrm{d}\mathcal{P}_t^{\mathcal{B}}(\Delta) + \mathcal{P}_t^{\mathcal{B}}(\Delta)\frac{\mathrm{d}(1/\mathcal{P}_t)}{1/\mathcal{P}_t} + \frac{\mathrm{d}\langle \mathcal{P}_t^{\mathcal{B}}(\Delta), 1/\mathcal{P}_t\rangle}{1/\mathcal{P}_t}.$$

Integrating over all Δ yields

$$\begin{aligned}
&\int_0^\infty \mathcal{X}_t(\Delta)\frac{\mathrm{d}(\mathcal{P}_t^{\mathcal{B}}(\Delta)/\mathcal{P}_t)}{1/\mathcal{P}_t}\mathrm{d}\Delta \\
&\quad = \int_0^\infty \left(\mathcal{X}_t(\Delta)\mathrm{d}\mathcal{P}_t^{\mathcal{B}}(\Delta) + \mathcal{X}_t(\Delta)\mathcal{P}_t^{\mathcal{B}}(\Delta)\frac{\mathrm{d}(1/\mathcal{P}_t)}{1/\mathcal{P}_t} + \mathcal{X}_t(\Delta)\frac{\mathrm{d}\langle \mathcal{P}_t^{\mathcal{B}}(\Delta), 1/\mathcal{P}_t\rangle}{1/\mathcal{P}_t}\right)\mathrm{d}\Delta \\
&\quad = \int_0^\infty \mathcal{X}_t(\Delta)\mathrm{d}\mathcal{P}_t^{\mathcal{B}}(\Delta)\mathrm{d}\Delta + \mathcal{B}_t\frac{\mathrm{d}(1/\mathcal{P}_t)}{1/\mathcal{P}_t} + \int_0^\infty \mathcal{X}_t(\Delta)\frac{\mathrm{d}\langle \mathcal{P}_t^{\mathcal{B}}(\Delta), 1/\mathcal{P}_t\rangle}{1/\mathcal{P}_t}\mathrm{d}\Delta.
\end{aligned}$$

To complete the proof of equation (47), we therefore have to show that

$$\int_0^\infty \mathcal{X}_t(\Delta)\mathrm{d}\langle \mathcal{P}_t^{\mathcal{B}}(\Delta), 1/\mathcal{P}_t\rangle\mathrm{d}\Delta = \mathrm{d}\langle \mathcal{MB}_t, 1/\mathcal{P}_t\rangle.$$

This can be seen as follows. Define A_t by

$$\mathrm{d}A_t := \mathrm{d}\mathcal{MB}_t - \int_0^\infty \mathcal{X}_t(\Delta)\mathrm{d}\mathcal{P}_t^{\mathcal{B}}(\Delta)\mathrm{d}\Delta$$

for arbitrary initial condition A_0. By the government budget constraint (44), A_t is a finite variation process, so

$$0 = \langle A_t, 1/\mathcal{P}_t\rangle = \langle \mathcal{MB}_t, 1/\mathcal{P}_t\rangle - \int_0^\infty \mathcal{X}_t(\Delta)\mathrm{d}\left\langle \mathcal{P}_t^{\mathcal{B}}(\Delta), 1/\mathcal{P}_t\right\rangle \mathrm{d}\Delta.$$

□

Lemma 9. *If bond prices $\{\mathcal{P}_t^{\mathcal{B}}(\Delta)\}_{\Delta\geq 0}$ satisfy property 5. in Definition 2 for agent $i \in \mathbb{I}$ and $i_t^{\mathcal{B}} = -\mathcal{P}_t^{\mathcal{B}\prime}(0)$, then*

$$d\mathcal{P}_t^{\mathcal{B}}(\Delta) - \mathcal{P}_t^{\mathcal{B}\prime}(\Delta)dt = \left(i_t^{\mathcal{B}} + \sigma_t^{\mathcal{P}c^i}\sigma_t^{\mathcal{P}^{\mathcal{B}}(\Delta)}\right)\mathcal{P}_t^{\mathcal{B}}(\Delta)dt + \mathcal{P}_t^{\mathcal{B}}(\Delta)\sigma_t^{\mathcal{P}^{\mathcal{B}}(\Delta)}dZ_t.$$

Proof. For fixed $T \geq t$, define

$$\tilde{\mathcal{P}}_t^{\mathcal{B}}(T) := \mathcal{P}_t^{\mathcal{B}}(T-t).$$

Taking the time differential in the definition of $\tilde{P}_t^{\mathcal{B}}(T)$ yields for $\Delta := T - t$

$$\mathrm{d}\tilde{\mathcal{P}}_t^{\mathcal{B}}(T) = \mathrm{d}\mathcal{P}_t^{\mathcal{B}}(\Delta) - \mathcal{P}_t^{\mathcal{B}\prime}(\Delta)\mathrm{d}t. \tag{48}$$

We first determine $\mathrm{d}\tilde{\mathcal{P}}_t^{\mathcal{B}}(T)$.

By assumption, the bond pricing equation holds for agent i, i.e.,

$$\mathcal{P}_t^{\mathcal{B}}(\Delta) = \mathbb{E}_t\left[e^{-\rho\Delta}\frac{\mathcal{P}_t c_t^i}{\mathcal{P}_{t+\Delta}c_{t+\Delta}^i}\right]$$

for all $\Delta \geq 0$. With $T := t + \Delta$, we can write this equivalently as

$$\tilde{\mathcal{P}}_t^{\mathcal{B}}(T) = \mathbb{E}_t\left[e^{-\rho(T-t)}\frac{\mathcal{P}_t c_t^i}{\mathcal{P}_T c_T^i}\right] = e^{\rho t}\mathcal{P}_t c_t^i \cdot \underbrace{\mathbb{E}_t\left[e^{-\rho T}\frac{1}{\mathcal{P}_T c_T^i}\right]}_{=:M_t(T)}.$$

Applying Ito to both sides of this equation and comparing volatility loadings yields

$$\sigma_t^{\tilde{\mathcal{P}}^{\mathcal{B}}(T)} = \sigma_t^{\mathcal{P}c^i} + \sigma_t^{M(T)} \Rightarrow \sigma_t^{M(T)} = \sigma_t^{\tilde{\mathcal{P}}^{\mathcal{B}}(T)} - \sigma_t^{\mathcal{P}c^i}.$$

Next, compare the drift terms and note that, for any fixed T, $M_t(T)$ is a martingale. Therefore,

$$\begin{aligned}
\mu_t^{\tilde{\mathcal{P}}^{\mathcal{B}}(T)} &= \rho + \mu_t^{\mathcal{P}c^i} + \sigma_t^{\mathcal{P}c^i}\sigma_t^{M(T)} \\
&= \rho + \mu_t^{\mathcal{P}c^i} + \sigma_t^{\mathcal{P}c^i}\left(\sigma_t^{\tilde{\mathcal{P}}^{\mathcal{B}}(T)} - \sigma_t^{\mathcal{P}c^i}\right) \\
&= \rho + \mu_t^{\mathcal{P}c^i} - \left(\sigma_t^{\mathcal{P}c^i}\right)^2 + \sigma_t^{\mathcal{P}c^i}\sigma_t^{\tilde{\mathcal{P}}^{\mathcal{B}}(T)} \\
&= \rho + \mu_t^{\mathcal{P}c^i} - \left(\sigma_t^{\mathcal{P}c^i}\right)^2 + \sigma_t^{\mathcal{P}c^i}\sigma_t^{\mathcal{P}^{\mathcal{B}}(T-t)},
\end{aligned}$$

where the last line follows from equation (48) above. Hence, using again equation (48),

$$\mathrm{d}\mathcal{P}_t^{\mathcal{B}}(\Delta) - \mathcal{P}_t^{\mathcal{B}\prime}(\Delta)\mathrm{d}t = \left(\rho + \mu_t^{\mathcal{P}c^i} - \left(\sigma_t^{\mathcal{P}c^i}\right)^2 + \sigma_t^{\mathcal{P}c^i}\sigma_t^{\mathcal{P}^{\mathcal{B}}(\Delta)}\right)\mathcal{P}_t^{\mathcal{B}}(\Delta)\mathrm{d}t + \mathcal{P}_t^{\mathcal{B}}(\Delta)\sigma_t^{\mathcal{P}^{\mathcal{B}}(\Delta)}\mathrm{d}Z_t.$$

This holds for all Δ, including $\Delta = 0$. But because of $\mathcal{P}_t^{\mathcal{B}}(0) = 1$ and $\mathcal{P}_t^{\mathcal{B}\prime}(0) = -i_t^{\mathcal{B}}$, in this specific case the equation simplifies to

$$i_t^{\mathcal{B}}\mathrm{d}t = \left(\rho + \mu_t^{\mathcal{P}c^i} - \left(\sigma_t^{\mathcal{P}c^i}\right)^2\right)\mathrm{d}t.$$

Substituting this into the previous equation implies, for all Δ,

$$\mathrm{d}\mathcal{P}_t^{\mathcal{B}}(\Delta) - \mathcal{P}_t^{\mathcal{B}\prime}(\Delta)\mathrm{d}t = \left(i_t^{\mathcal{B}} + \sigma_t^{\mathcal{P}c^i}\sigma_t^{\mathcal{P}^{\mathcal{B}}(\Delta)}\right)\mathcal{P}_t^{\mathcal{B}}(\Delta)\mathrm{d}t + \mathcal{P}_t^{\mathcal{B}}(\Delta)\sigma_t^{\mathcal{P}^{\mathcal{B}}(\Delta)}\mathrm{d}Z_t.$$

□

Lemma 10. *Suppose that bond prices* $\{\mathcal{P}_t^{\mathcal{B}}(\Delta)\}_{\Delta\geq 0}$ *satisfy property 5. in Definition 2 for agent* $i \in \mathbb{I}$, $i_t^{\mathcal{B}} = -\mathcal{P}_t^{\mathcal{B}\prime}(0)$, *and the government budget constraint (44) holds. Then*

$$dr_t^{\mathcal{B}} = \left(i_t^{\mathcal{B}} + \frac{\sigma_t^{c,i}\sigma_{\mathcal{MB},t}}{\mathcal{B}_t}\right)dt + \frac{\sigma_{\mathcal{MB},t}}{\mathcal{B}_t}dZ_t + \frac{d(1/\mathcal{P}_t)}{1/\mathcal{P}_t},$$

where $\sigma_{\mathcal{MB},t}$ *is the arithmetic volatility loading of* $\mathcal{MB}_t$.

Proof. By definition of $\sigma_{\mathcal{MB},t}$, $\frac{\mathrm{d}\langle \mathcal{MB}_t, 1/\mathcal{P}_t\rangle}{\mathcal{B}_t/\mathcal{P}_t} = -\frac{\sigma_t^{\mathcal{P}}\sigma_{\mathcal{MB},t}}{\mathcal{B}_t}\mathrm{d}t$. Substituting this into the equation from Lemma 8 yields

$$\mathrm{d}r_t^{\mathcal{B}} = \frac{\int_0^\infty \mathcal{X}_t(\Delta)\left(\mathrm{d}\mathcal{P}_t^{\mathcal{B}}(\Delta) - \mathcal{P}_t^{\mathcal{B}\prime}(\Delta)\mathrm{d}t\right)\mathrm{d}\Delta - \sigma_t^{\mathcal{P}}\sigma_{\mathcal{MB},t}\mathrm{d}t}{\mathcal{B}_t} + \frac{\mathrm{d}(1/\mathcal{P}_t)}{1/\mathcal{P}_t}.$$

The asserted equation follows if we can show that

$$\int_0^\infty \mathcal{X}_t(\Delta)\left(\mathrm{d}\mathcal{P}_t^{\mathcal{B}}(\Delta) - \mathcal{P}_t^{\mathcal{B}\prime}(\Delta)\mathrm{d}t\right)\mathrm{d}\Delta - \sigma_t^{\mathcal{P}}\sigma_{\mathcal{MB},t}\mathrm{d}t = \left(i_t^{\mathcal{B}}\mathcal{B}_t + \sigma_t^{c,i}\sigma_{\mathcal{MB},t}\right)\mathrm{d}t + \sigma_{\mathcal{MB},t}\mathrm{d}Z_t. \tag{49}$$

For a proof of equation (49), start from the equation in Lemma 9 and take the $\mathcal{X}_t(\Delta)$-weighted integral over all $\Delta \in [0,\infty)$:

$$\begin{aligned}
\mathrm{d}\mathcal{A}_t &:= \int_0^\infty \mathcal{X}_t(\Delta)(\mathrm{d}\mathcal{P}_t^{\mathcal{B}}(\Delta) - \mathcal{P}_t^{\mathcal{B}\prime}(\Delta)\mathrm{d}t)\mathrm{d}\Delta \\
&= \left(i_t^{\mathcal{B}}\underbrace{\int_0^\infty \mathcal{X}_t(\Delta)\mathcal{P}_t^{\mathcal{B}}(\Delta)\mathrm{d}\Delta}_{=\mathcal{B}_t} + \sigma_t^{\mathcal{P}c^i}\underbrace{\int_0^\infty \mathcal{X}_t(\Delta)\mathcal{P}_t^{\mathcal{B}}(\Delta)\sigma_t^{\mathcal{P}^{\mathcal{B}}(\Delta)}\mathrm{d}\Delta}_{=\sigma_{\mathcal{A},t}}\right)\mathrm{d}t \\
&\quad + \underbrace{\int_0^\infty \mathcal{X}_t(\Delta)\mathcal{P}_t^{\mathcal{B}}(\Delta)\sigma_t^{\mathcal{P}^{\mathcal{B}}(\Delta)}\mathrm{d}\Delta}_{=:\sigma_{\mathcal{A},t}}\mathrm{d}Z_t.
\end{aligned}$$

By the government budget constraint,

$$\mathrm{d}\mathcal{A}_t = \mathrm{d}\mathcal{MB}_t + \mathrm{d}t\text{-terms}$$

and therefore $\sigma_{\mathcal{A},t} = \sigma_{\mathcal{MB},t}$. Hence,

$$\begin{aligned}
\mathrm{d}\mathcal{A}_t &= \left(i_t^{\mathcal{B}}\mathcal{B}_t + \sigma_t^{\mathcal{P}c^i}\sigma_{\mathcal{MB},t}\right)\mathrm{d}t + \sigma_{\mathcal{MB},t}\mathrm{d}Z_t \\
&= \left(i_t^{\mathcal{B}}\mathcal{B}_t + \sigma_t^{c,i}\sigma_{\mathcal{MB},t}\right)\mathrm{d}t + \sigma_{\mathcal{MB},t}\mathrm{d}Z_t + \sigma_t^{\mathcal{P}}\sigma_{\mathcal{MB},t}\mathrm{d}t,
\end{aligned}$$

which implies equation (49).

□

Lemma 11. *In a Definition 2 equilibrium (long-term bond model), if agent $i \in \mathbb{I}$ chooses optimal consumption $c_t^i = \rho n_t^i$ and if bond prices $\{\mathcal{P}_t^{\mathcal{B}}(\Delta)\}_{\Delta \geq 0}$ satisfy property 5. in Definition 2 for that agent i, then equation (18) holds for that agent i.*

Proof. The proof uses a stochastic maximum principle approach as in the proof of Lemma 1. The Hamiltonian is

$$H_t^i = \cdots + \xi_t^i n_t^i \theta_t^i \left(\theta_t^{\mathcal{M},i} \frac{\mathbb{E}_t \left[\mathrm{d}r_t^{\mathcal{M}} \right]}{\mathrm{d}t} + (1 - \theta_t^{\mathcal{M},i}) \frac{\mathbb{E}_t \left[\mathrm{d}r_t^{\mathcal{B}} \right]}{\mathrm{d}t} \right) \\ - \varsigma_t^i \xi_t^i n_t^i \theta_t^i \left(\theta_t^{\mathcal{M},i} \left(-\sigma_t^{\mathcal{P}} \right) + (1 - \theta_t^{\mathcal{M},i}) \left(-\sigma_t^{\mathcal{P}} + \frac{\sigma_{\mathcal{MB},t}}{\mathcal{B}_t} \right) \right),$$

where "$\cdots$" captures portions that are not dependent on $\theta_t^{\mathcal{M},i}$ (these are identical to the ones in the proof of Lemma 1, but this is not important here). Here, we have used equation (45) to recover the risk loading of the return on money and Lemma 10 to recover the risk loading of the return on bonds. Lemma 10 is applicable because all assumptions of that lemma are satisfied in a Definition 2 equilibrium.

Again, $\theta_t^{\mathcal{M},i}$ maximizes H_t^i subject to the constraint (11), and we denote by $\lambda_t^{\mathcal{M},i} \xi_t^i n_t^i$ the Lagrange multiplier on that constraint, just like in Lemma 1. The first-order condition is

$$\frac{\mathbb{E}_t \left[\mathrm{d}r_t^{\mathcal{B}} - \mathrm{d}r_t^{\mathcal{M}} \right]}{\mathrm{d}t} = \varsigma_t^i \frac{\sigma_{\mathcal{MB},t}}{\mathcal{B}_t} + \lambda_t^{\mathcal{M},i} \nu_t.$$

By Lemma 10, and the fact that the formula holds for this agent i,

$$\frac{\mathbb{E}_t \left[\mathrm{d}r_t^{\mathcal{B}} - \mathrm{d}r_t^{\mathcal{M}} \right]}{\mathrm{d}t} = i_t^{\mathcal{B}} + \frac{\sigma_t^{c,i} \sigma_{\mathcal{MB},t}}{\mathcal{B}_t} - i_t^{\mathcal{M}}.$$

Now, in full analogy to the proof of Lemma 1, $\varsigma_t^i = \sigma_t^{n,i} = \sigma_t^{c,i}$ (where the last equation follows from $c_t^i = \rho n_t^i$, which holds by assumption for agent i). Combining these equations yields

$$i_t^{\mathcal{B}} - i_t^{\mathcal{M}} = \lambda_t^{\mathcal{M},i} \nu_t,$$

as claimed.

□

Real Equilibrium in Baseline Model. Our ultimate goal in proving Proposition 3 is to construct a mapping between equilibria of the long-term bond model according to Definition 2 and equilibria of the baseline model according to Definition 2 that preserve the real allocation and nominal rates. To prepare this construction, we first define and characterize real allocations in the baseline model by introducing the following auxiliary construct of a "real equilibrium".

Definition 3 (Real Equilibrium)**.** *Given an exogenous process X_t, functions as in* (5)*, an initial stock of capital, K_0, and an initial cross-sectional wealth distribution $\{\eta_0^i\}_{i\in\mathbb{I}}$ satisfying $\int_{\mathbb{I}} \eta_0^i di = 1$, a* real equilibrium *consists of aggregate (Ito) stochastic processes K_t, $q_t^{\mathcal{M}}$, $q_t^{\mathcal{MB}}$, q_t^K, $\breve{\mu}_t^{\mathcal{MB}}$ adapted to $\mathcal{F}_t$ and, for each $i \in \mathbb{I}$, individual (Ito) stochastic processes c_t^i, ι_t^i, u_t^i, θ_t^i, $\theta_t^{\mathcal{M},i}$, n_t^i adapted to $\tilde{\mathcal{F}}_t^i$, with the property that there are processes $\mathcal{MB}_t$, $\mathcal{P}_t$, $i_t^{\mathcal{M}}$, and $i_t^{\mathcal{B}}$, adapted to $\mathcal{F}_t$, such that the collection*

$$\{K_t, \mathcal{MB}_t, \mathcal{P}_t, q_t^{\mathcal{M}}, q_t^{\mathcal{MB}}, q_t^K, \breve{\mu}_t^{\mathcal{MB}}, i_t^{\mathcal{M}}, i_t^{\mathcal{B}}, \{c_t^i, \iota_t^i, u_t^i, \theta_t^i, \theta_t^{\mathcal{M},i}, n_t^i\}_{i\in\mathbb{I}}\}_{t\geq 0}$$

is a competitive equilibrium according to Definition 1 for initial conditions K_0, $\mathcal{MB}_0$, and $\{\eta_0^i\}_{i\in\mathbb{I}}$.

A real equilibrium is called symmetric*, if c_t^i/n_t^i, ι_t^i, u_t^i, θ_t^i, and $\theta_t^{\mathcal{M},i}$ do not depend on i.*

We seek to characterize symmetric real equilibria in a way that makes sense both in the baseline and in the long-term bond model. To do so, let us consider the following auxiliary decision problem for agent $i \in \mathbb{I}$. The net worth evolution is

$$\mathrm{d}n_t^i = -c_t^i \mathrm{d}t + n_t^i \left(\theta_t^i \mathrm{d}r_t^{\mathcal{MB}} + (1-\theta_t^i)\mathrm{d}r_t^{K,i}(\iota_t^i, u_t^i)\right), \tag{50}$$

in place of equation (7), where the return on nominal assets is given by

$$\mathrm{d}r_t^{\mathcal{MB}} = -\breve{\mu}_t^{\mathcal{MB}} \mathrm{d}t + \frac{\mathrm{d}(q_t^{\mathcal{MB}} K_t)}{q_t^{\mathcal{MB}} K_t}. \tag{51}$$

Furthermore, there is no choice of the money portfolio weight $\theta_t^{\mathcal{M},i}$, which is fixed instead at an exogenous value $\vartheta_t^{\mathcal{M}} \in [0,1]$. The cash constraint (11) is therefore replaced with

$$\frac{a_t u_t^i}{q_t^K}(1-\theta_t^i) \leq \nu_t \vartheta_t^{\mathcal{M}} \theta_t^i. \tag{52}$$

In sum, the auxiliary decision problem given a process $\vartheta_t^{\mathcal{M}}$ is to choose consumption c_t^i, investment ι_t^i, capital utilization u_t^i, and a portfolio weight θ_t^i to maximize V_0^i subject to the net worth evolution (50), the return expressions (8) and (51), the cash constraint (52), and a solvency constraint $n_t^i \geq 0$.

We note that the constraints in the auxiliary decision problem only contain exogenous processes and aggregate endogenous variables that are part of a real equilibrium.[40]

Lemma 12 (Characterization of Real Equilibria)**.** *Let*

$$\mathbf{e}_r = \{K_t, q_t^{\mathcal{M}}, q_t^{\mathcal{MB}}, q_t^K, \breve{\mu}_t^{\mathcal{MB}}, \{c_t^i, \iota_t^i, u_t^i, \theta_t^i, \theta_t^{\mathcal{M},i}, n_t^i\}_{i\in\mathbb{I}}\}_{t\geq 0}$$

be a collection of Ito processes adapted to $\mathcal{F}_t$ or $\mathcal{F}_t^i$ (for i-dependent variables).

1. *$\mathbf{e}_r$ is a symmetric real equilibrium, if and only if the following are true:*
   (a) *for all $i \in \mathbb{I}$, c_t^i, ι_t^i, u_t^i, θ_t^i solve i's auxiliary optimization problem for exogenous $\vartheta_t^{\mathcal{M}} := q_t^{\mathcal{M}}/q_t^{\mathcal{MB}}$ and initial wealth $n_0^i = \eta_0^i(q_0^K + q_0^{\mathcal{MB}})K_0$, and n_t^i is as implied by the net worth evolution (7);*
   (b) *for all $i \in \mathbb{I}$ and t, c_t^i/n_t^i, ι_t^i, u_t^i, θ_t^i, and $\theta_t^{\mathcal{M},i}$ do not depend on i and $\theta_t^{\mathcal{M},i} = \vartheta_t^{\mathcal{M}}$;*
   (c) *properties 2., 5., and 6. of Definition 1 hold.*
2. *If the properties in 1. are true and $i_t^{\mathcal{M}}$ is an arbitrary Ito process, then $\mathbf{e}_r$ can be extended to a symmetric Definition 1 equilibrium $\mathbf{e}$ with nominal money rate process $i_t^{\mathcal{M}}$.*

Proof. We first show the "only if" direction in the first claim. Because a real equilibrium can be extended to a competitive equilibrium according to Definition 1 by adding nominal variables, any property of Definition 1 that only involves real variables must hold for a real equilibrium. This applies, in particular, for properties 2., 5., and 6. of Definition 1. The characterization of equilibrium in the main text shows further that, in any (symmetric) competitive equilibrium, c_t^i/n_t^i, ι_t^i, u_t^i, θ_t^i, and $\theta_t^{\mathcal{M},i}$ do not depend on i and that $\theta_t^{\mathcal{M},i} = \vartheta_t^{\mathcal{M}} = q_t^{\mathcal{M}}/q_t^{\mathcal{MB}}$ by money market clearing (equation (14)). Hence, we only need to show the first property, that the auxiliary decision problem is solved. To do so, let us note that, conditional on the optimal equilibrium choice $\theta_t^{\mathcal{M},i} = \vartheta_t^{\mathcal{M}}$, the

[40]This is not true for the original decision problem of an agent in the model because the returns $dr_t^{\mathcal{M}}$ and $dr_t^{\mathcal{B}}$ on money and bonds individually depend directly on nominal variables. This is the main reason why we have removed the $\theta_t^{\mathcal{M},i}$-choice and instead taken $\vartheta_t^{\mathcal{M}}$ exogenously given.

cash constraint (11) in the original agent problem is equivalent to the constraint (52) in the auxiliary problem and the net worth evolution (7) in the original problem is also equivalent to the one in the auxiliary problem, equation (50), provided we can show that, in equilibrium, $\mathrm{d}r_t^{\mathcal{MB}}$ takes the form (51) from the auxiliary problem. Joint optimality of all choices, as in the original problem, implies optimality of a subset of these choices, conditional on all other choices being held fixed at the optimum, so that we can conclude that also the auxiliary problem must be solved.

It is therefore left to show that, indeed, $\mathrm{d}r_t^{\mathcal{MB}}$ takes the form (51) in any Definition 1 equilibrium. By combining equations (9) and (10), we obtain

$$\begin{aligned}\mathrm{d}r_t^{\mathcal{MB}} &= \vartheta_t^{\mathcal{M}}\mathrm{d}r_t^{\mathcal{M}} + (1-\vartheta_t^{\mathcal{M}})\mathrm{d}r_t^{\mathcal{B}} \\ &= \underbrace{\left(\vartheta_t^{\mathcal{M}} i_t^{\mathcal{M}} + (1-\vartheta_t) i_t^{\mathcal{B}} - \mu_t^{\mathcal{MB}}\right)}_{=-\check{\mu}_t^{\mathcal{MB}} \text{ by (4)}} \mathrm{d}t + \frac{\mathrm{d}(q_t^{\mathcal{MB}} K_t)}{q_t^{\mathcal{MB}} K_t},\end{aligned}$$

i.e., equation (51) holds. This concludes the proof of the "only if" direction.

Let us next prove the "if" direction in the first claim. We construct an explicit extension by nominal variables $i_t^{\mathcal{M}}$, $i_t^{\mathcal{B}}$, $\mathcal{MB}_t$, $\mathcal{P}_t$ to obtain a competitive equilibrium according to Definition 1 for an arbitrary given $i_t^{\mathcal{M}}$ (Ito) process adapted to $\mathcal{F}_t$. We next construct $i_t^{\mathcal{B}}$ such that c_t^i, ι_t^i, u_t^i, θ_t^i, $\theta_t^{\mathcal{M},i}$ solve the original decision problem of agent $i \in \mathbb{I}$ from property 1. in Definition 1. To do so, pick a fixed $i \in \mathbb{I}$ and note that, by assumption (a), c_t^i, ι_t^i, u_t^i, θ_t^i solve the auxiliary decision problem. As this decision problem is just the original decision problem with the $\theta_t^{\mathcal{M},i}$-choice held fixed at $\vartheta_t^{\mathcal{M}}$, the first-order conditions (15), (16), (17), (19), and the complementary slackness condition for $\lambda_t^{\mathcal{M},i}$ hold here, too (the derivation is the same as in the proof of Lemma 1). In particular, equation (17) allows us to recover $\lambda_t^{\mathcal{M},i}$ from variables in $\mathbf{e}_r$. By assumption (b), all i-dependent variables in that equation other than $\lambda_t^{\mathcal{M},i}$ are in fact independent of i, so that also $\lambda_t^{\mathcal{M},i} = \lambda_t^{\mathcal{M}}$ cannot depend on the identity of the agent. Given such a i-independent value for $\lambda_t^{\mathcal{M}}$, there is then precisely one way to choose $i_t^{\mathcal{B}}$ to make it consistent with the equation (18), which is the $\theta_t^{\mathcal{M},i}$-FOC in Lemma 1, for all agents $i \in \mathbb{I}$, namely

$$i_t^{\mathcal{B}} := i_t^{\mathcal{M}} + \lambda_t^{\mathcal{M}} \nu_t.$$

With this definition, the choice $\theta_t^{\mathcal{M},i} = \vartheta_t^{\mathcal{M}}$ for all $i \in \mathbb{I}$ is optimal, so that not just the auxiliary decision problem but also the original decision problem is solved for all i. This implies that our equilibrium candidate satisfies property 1. of Definition 1 irrespective of our choice for the remaining variables $\mathcal{MB}_t$ and $\mathcal{P}_t$. By assumption (c), it also satisfies properties 2., 5., and 6.

We now construct the remaining variables $\mathcal{MB}_t$ and $\mathcal{P}_t$ so that the remaining properties 3. and 4. are automatically satisfied. For $\mathcal{MB}_t$, we pick an arbitrary initial condition $\mathcal{MB}_0 > 0$ and note that, given this initial condition and the already constructed processes, the equation in property 3. of Definition 1 is a linear forward stochastic differential equation that has a (unique) solution process $\mathcal{MB}_t$.

Finally, we define $\mathcal{P}_t := \frac{\mathcal{MB}_t}{q_t^{\mathcal{MB}} K_t}$, which ensures that also property 4. is satisfied. This concludes the construction of the extension of $\mathbf{e}_r$ to a Definition 1 equilibrium. Because the specific extension constructed here is always symmetric and works for an arbitrarily given nominal money rate process, we have also established the second claim in the lemma. □

It turns out, the construction in part 2. of the previous lemma is also unique, provided we fix the initial condition $\mathcal{MB}_0$ for nominal assets:

Lemma 13. *For any pair $(\mathbf{e}_r, i^{\mathcal{M}})$ consisting of a symmetric real equilibrium according to Definition 3 and an (Ito) stochastic process $i_t^{\mathcal{M}}$ adapted to $\mathcal{F}_t$ and any initial condition $\mathcal{MB}_0$ for nominal assets, there is a unique symmetric equilibrium $\mathbf{e}$ according to Definition 1 with real initial conditions as in $\mathbf{e}_r$ and nominal asset initial condition $\mathcal{MB}_0$ such that (i) all variables in $\mathbf{e}_r$ are the same as their counterparts in $\mathbf{e}$ and (ii) $i^{\mathcal{M}}$ equals the $i^{\mathcal{M}}$-process in $\mathbf{e}$.*

Proof. Existence of $\mathbf{e}$ follows from Lemma 12. For uniqueness, we only need to prove uniqueness of the three processes $i_t^{\mathcal{B}}$, $\mathcal{MB}_t$, and $\mathcal{P}_t$, as all other processes in $\mathbf{e}$ are already uniquely defined by the given pair $(\mathbf{e}_r, i^{\mathcal{M}})$ and conditions (i) and (ii). Uniqueness of $i_t^{\mathcal{B}}$ follows from the fact that, by Lemmas 1 and 3, the spread $i_t^{\mathcal{B}} - i_t^{\mathcal{M}}$ is uniquely determined by the real allocation. Uniqueness of $\mathcal{MB}_t$ follows then from property 3. of the equilibrium definition 1. Finally, uniqueness of $\mathcal{P}_t$ follows from property 4. of the equilibrium definition. □

Real Equilibrium in Long-term Bond Model. We conclude this section of auxiliary results by showing that also equilibria of the long-term bond model can be mapped into real equilibria as defined previously:

Lemma 14. *Let*

$$\mathbf{e} = \{K_t, \mathcal{MB}_t, \mathcal{P}_t, q_t^{\mathcal{M}}, q_t^{\mathcal{MB}}, q_t^K, \breve{\mu}_t^{\mathcal{MB}}, i_t^{\mathcal{M}}, \{\mathcal{X}_t(\Delta), \mathcal{P}_t^{\mathcal{B}}(\Delta)\}_{\Delta \geq 0}, \{c_t^i, \iota_t^i, u_t^i, \theta_t^i, \theta_t^{\mathcal{M},i}, n_t^i\}_{i \in \mathbb{I}}\}_{t \geq 0}$$

be a symmetric Definition 2 equilibrium (competitive equilibrium in the long-term bond model). Then

$$\mathbf{e}_r := \{K_t, q_t^{\mathcal{M}}, q_t^{\mathcal{MB}}, q_t^K, \breve{\mu}_t^{\mathcal{MB}}, \{c_t^i, \iota_t^i, u_t^i, \theta_t^i, \theta_t^{\mathcal{M},i}, n_t^i\}_{i \in \mathbb{I}}\}_{t \geq 0}$$

is a symmetric real equilibrium.

Proof. We show that $\mathbf{e}_r$ satisfies the three properties in Lemma 12. We start with the easy part: properties 2., 6., and 7. in Definition 2 are formally identical to properties 2., 5., and 6. in Definition 1, and they only depend on variables contained in $\mathbf{e}_r$. Hence, $\mathbf{e}_r$ satisfies property (c) in Lemma 12. Property (b) is also satisfied because, by assumption, $\mathbf{e}$ is a symmetric equilibrium.

The remaining property (a) follows from arguments analogous to the "only if" direction in the proof of Lemma 12: Also the agent decision problem in the long-term bond model collapses to the auxiliary decision problem if we hold $\theta_t^{\mathcal{M},i} = \vartheta_t^{\mathcal{M}}$ fixed instead of letting it be chosen, provided $\mathrm{d}r_t^{\mathcal{MB}}$ again takes the form (51) in any Definition 2 equilibrium. This follows from Lemma 5 stated in the main text, which we prove in the next section (and whose proof does not make use of this lemma). □

B.1.3 Proofs of Results Stated in Main Text

Proof of Lemma 5. By equation (45) and Lemma 8,

$$\begin{aligned}
\mathrm{d}r_t^{\mathcal{MB}} &= \vartheta_t^{\mathcal{M}} \mathrm{d}r_t^{\mathcal{M}} + \left(1 - \vartheta_t^{\mathcal{M}}\right) \mathrm{d}r_t^{\mathcal{B}} \\
&= \frac{\mathcal{M}_t}{\mathcal{MB}_t} \left(i_t^{\mathcal{M}} \mathrm{d}t + \frac{\mathrm{d}(1/\mathcal{P}_t)}{1/\mathcal{P}_t} \right) \\
&\quad + \frac{\mathcal{B}_t}{\mathcal{MB}_t} \left(\frac{\int_0^\infty \mathcal{X}_t(\Delta) \left(\mathrm{d}\mathcal{P}_t^{\mathcal{B}}(\Delta) - \mathcal{P}_t^{\mathcal{B}\prime}(\Delta) \mathrm{d}t \right) \mathrm{d}\Delta}{\mathcal{B}_t} + \frac{\mathrm{d}\langle \mathcal{MB}_t, 1/\mathcal{P}_t \rangle}{\mathcal{B}_t/\mathcal{P}_t} + \frac{\mathrm{d}(1/\mathcal{P}_t)}{1/\mathcal{P}_t} \right) \\
&= \frac{i_t^{\mathcal{M}} \mathcal{M}_t + \int_0^\infty \mathcal{X}_t(\Delta) \left(\mathrm{d}\mathcal{P}_t^{\mathcal{B}}(\Delta) - \mathcal{P}_t^{\mathcal{B}\prime}(\Delta) \mathrm{d}t \right) \mathrm{d}\Delta}{\mathcal{MB}_t} + \frac{\mathrm{d}\langle \mathcal{MB}_t, 1/\mathcal{P}_t \rangle}{\mathcal{MB}_t/\mathcal{P}_t} + \frac{\mathrm{d}(1/\mathcal{P}_t)}{1/\mathcal{P}_t}.
\end{aligned}$$

Using the government budget constraint (44) and the product rule, we obtain

$$\begin{aligned}\mathrm{d}r_t^{\mathcal{MB}} &= \frac{\mathrm{d}\mathcal{MB}_t + \mathcal{P}_t \tau_t K_t \mathrm{d}t}{\mathcal{MB}_t} + \frac{\mathrm{d}\langle \mathcal{MB}_t, 1/\mathcal{P}_t \rangle}{\mathcal{MB}_t/\mathcal{P}_t} + \frac{\mathrm{d}(1/\mathcal{P}_t)}{1/\mathcal{P}_t} \\ &= \underbrace{\frac{\mathcal{P}_t \tau_t K_t}{\mathcal{MB}_t}}_{=-\breve{\mu}_t^{\mathcal{MB}}} \mathrm{d}t + \underbrace{\frac{\mathrm{d}(\mathcal{MB}_t/\mathcal{P}_t)}{\mathcal{MB}_t/\mathcal{P}_t}}_{=\frac{\mathrm{d}(q_t^{\mathcal{MB}} K_t)}{q_t^{\mathcal{MB}} K_t}}.\end{aligned}$$

□

Proof of Proposition 3. We first define two mappings. Using Lemma 14, we define a mapping $\Phi(\mathbf{e}) := (\mathbf{e}_r, i^{\mathcal{M}})$ that maps (symmetric) equilibria $\mathbf{e}$ in the long-term bond model into pairs of (symmetric) real equilibria and nominal money rate processes. Using Lemma 13, we further define a mapping $\Psi(\mathbf{e}_r, i^{\mathcal{M}}) := \mathbf{e}$ that maps any pair of a (symmetric) real equilibrium and a nominal money rate process to the unique equilibrium $\mathbf{e}$ in the baseline model that shares the same real allocation and $i^{\mathcal{M}}$-process. We use these two mappings to prove the proposition:

Let $\mathbf{e}^{LTB}$ be a Definition 2 equilibrium (long-term bond model). By composing the previously defined mappings, we can define a Definition 1 equilibria (baseline model)

$$\mathbf{e} := \Psi(\Phi(\mathbf{e}^{LTB}))$$

that features the same real allocation and the same nominal money rate process as $\mathbf{e}^{LTB}$. In addition, because equation (18) holds in both the baseline model (by Lemma 1) and in the long-term bond model (by Lemma 11[41]), the nominal rate spread $\Delta i_t = i_t^{\mathcal{B}} - i_t^{\mathcal{M}}$ must be the same in both $\mathbf{e}$ and $\mathbf{e}^{LTB}$ if the real allocation is the same.[42] Because also $i_t^{\mathcal{M}}$ is the same, so must be $i_t^{\mathcal{B}}$.

This proves the existence part of the proposition. For uniqueness, note that the required equality of the real allocation and nominal rates (plus the implicit equality of initial conditions), implies in particular that $\mathbf{e}$ must be a Definition 1 equilibrium that

[41]Note that because the real allocations are the same, the additional requirement $c_t^i = \rho n_t^i$ follows from the fact that this holds in the baseline model

[42]Note that the multiplier $\lambda_t^{\mathcal{M},i}$, which appears in equation (18), is also the multiplier on the same constraint in the auxiliary decision problem that must be solved in any real equilibrium according to Definition 3. Therefore, if the real equilibrium is the same, this multiplier must take on the same value in both $\mathbf{e}$ and $\mathbf{e}^{LTB}$.

extends the pair $(\mathbf{e}_r, i^{\mathcal{M}}) := \Phi(\mathbf{e}^{LTB})$ in the sense of Lemma 13. By that lemma, this extension is unique and given by $\Psi(\mathbf{e}_r, i^{\mathcal{M}})$.

Finally, for Lemmas 1, 2, 3, and 4, note that they hold for the equilibrium $\mathbf{e}$ and they do not involve the processes $\mathcal{MB}_t$ or $\mathcal{P}_t$, which are the only two processes in $\mathbf{e}$ that are not identical also in $\mathbf{e}^{LTB}$. Therefore, all equations in these lemmas must hold for this specific long-term bond equilibrium $\mathbf{e}^{LTB}$ as well. Because the construction carried out here works for any $\mathbf{e}^{LTB}$, we conclude that the lemmas must hold for the long-term bond model in general.

□

Proof of equation (23). Let $T := t + \Delta$ and define as in the proof of Lemma 9

$$\tilde{\mathcal{P}}^{\mathcal{B}}_t(T) := \mathcal{P}^{\mathcal{B}}_t(\Delta) = \mathcal{P}^{\mathcal{B}}_t(T - t).$$

Clearly, $\mathrm{d}\tilde{\mathcal{P}}^{\mathcal{B}}_t(T) = \mathrm{d}\mathcal{P}^{\mathcal{B}}_t(\Delta) - \mathcal{P}^{\mathcal{B}\prime}_t(\Delta)\mathrm{d}t$, and by Lemma 9, for all $t \leq T$

$$\mathrm{d}\tilde{\mathcal{P}}^{\mathcal{B}}_t(T) = \left(i^{\mathcal{B}}_t + \sigma^{\mathcal{P}c^i}_t \sigma^{\tilde{\mathcal{P}}^{\mathcal{B}}(T)}_t\right) \tilde{\mathcal{P}}^{\mathcal{B}}_t(T)\mathrm{d}t + \sigma^{\tilde{\mathcal{P}}^{\mathcal{B}}(T)}_t \tilde{\mathcal{P}}^{\mathcal{B}}_t(T)\mathrm{d}Z_t.$$

This is a linear BSDE for $\tilde{\mathcal{P}}^{\mathcal{B}}_t(T)$. By standard results it has a unique solution for any fixed terminal condition. Here, the bond pricing equation (23) implies the terminal condition

$$\tilde{\mathcal{P}}^{\mathcal{B}}_T(T) = 1.$$

We prove equation (23) by showing that the guess

$$\tilde{\mathcal{P}}^{\mathcal{B}}_t(T) := \mathbb{E}_t\Bigg[\underbrace{\frac{\Theta_T}{\Theta_t}\exp\left(-\int_t^T i^{\mathcal{B}}_{t'}\mathrm{d}t'\right)}_{=:A_T/A_t}\Bigg]$$

indeed solve the above BSDE for this terminal condition. By uniqueness of the BSDE solution, the bond price must then necessarily take this form.

Verifying the terminal condition is trivial because $A_T/A_T = 1$. We show that the

BSDE holds for $\tilde{\mathcal{P}}^{\mathcal{B}}_t(T)$ and all $t \leq T$. By the product rule,

$$\begin{aligned}
\mathrm{d}\tilde{\mathcal{P}}^{\mathcal{B}}_t(T) &= \frac{\mathrm{d}(A_t\tilde{\mathcal{P}}^{\mathcal{B}}_t(T)) - \tilde{\mathcal{P}}^{\mathcal{B}}_t(T)\mathrm{d}A_t - \mathrm{d}\langle A_t, \tilde{\mathcal{P}}^{\mathcal{B}}_t(T)\rangle}{A_t} \\
&= \frac{\mathrm{d}(A_t\tilde{\mathcal{P}}^{\mathcal{B}}_t(T))}{A_t} - \frac{\tilde{\mathcal{P}}^{\mathcal{B}}_t(T)(-i^{\mathcal{B}}_t A_t\mathrm{d}t + (\sigma^{\vartheta}_t - \sigma^{\mathcal{MB}}_t)A_t\mathrm{d}Z_t) + (\sigma^{\vartheta}_t - \sigma^{\mathcal{MB}}_t)A_t\sigma^{\tilde{\mathcal{P}}^{\mathcal{B}}(T)}_t\tilde{\mathcal{P}}^{\mathcal{B}}_t(T)\mathrm{d}t}{A_t} \\
&= \left(i^{\mathcal{B}}_t + (\sigma^{\mathcal{MB}}_t - \sigma^{\vartheta}_t)\sigma^{\tilde{\mathcal{P}}^{\mathcal{B}}(T)}_t\right)\tilde{\mathcal{P}}^{\mathcal{B}}_t(T)\mathrm{d}t + \frac{\mathrm{d}(A_t\tilde{\mathcal{P}}^{\mathcal{B}}_t(T))}{A_t} - \tilde{\mathcal{P}}^{\mathcal{B}}_t(T)(\sigma^{\vartheta}_t - \sigma^{\mathcal{MB}}_t)\mathrm{d}Z_t,
\end{aligned}$$

where the second line uses the definition of A_t. Now, by definition of $\tilde{\mathcal{P}}^{\mathcal{B}}_t(T)$, the product $A_t\tilde{\mathcal{P}}^{\mathcal{B}}_t(T)$ is a martingale, so the term $\frac{\mathrm{d}(A_t\tilde{\mathcal{P}}^{\mathcal{B}}_t(T))}{A_t}$ has zero drift. Therefore, $\tilde{\mathcal{P}}^{\mathcal{B}}_t(T)$ satisfies the BSDE

$$\mathrm{d}\tilde{\mathcal{P}}^{\mathcal{B}}_t(T) = \left(i^{\mathcal{B}}_t + (\sigma^{\mathcal{MB}}_t - \sigma^{\vartheta}_t)\sigma^{\tilde{\mathcal{P}}^{\mathcal{B}}(T)}_t\right)\tilde{\mathcal{P}}^{\mathcal{B}}_t(T)\mathrm{d}t + \sigma^{\tilde{\mathcal{P}}^{\mathcal{B}}(T)}_t\tilde{\mathcal{P}}^{\mathcal{B}}_t(T)\mathrm{d}Z_t.$$

To complete the proof, we only need to show $\sigma^{\mathcal{P}c^i}_t = \sigma^{\mathcal{MB}}_t - \sigma^{\vartheta}_t$ for some agent i (and we show that this holds for all i). For arbitrary $i \in \mathbb{I}$, consumption is $c^i_t = \rho n^i_t = \rho\eta^i_t K_t \cdot q_t$. Note that the factor $\rho\eta^i_t K_t$ has a zero loading on aggregate shocks $\mathrm{d}Z_t$, so that necessarily $\sigma^{c,i}_t = \sigma^q_t$. Hence, by the product rule (recall $\mathcal{P}_t = \mathcal{MB}_t/(q^{\mathcal{MB}}_t K_t)$

$$\sigma^{\mathcal{P}c^i}_t = \sigma^{\mathcal{P}}_t + \sigma^{c,i}_t = \sigma^{\mathcal{MB}}_t - \sigma^{q,\mathcal{MB}}_t + \sigma^q_t = \sigma^{\mathcal{MB}}_t - \sigma^{\vartheta}_t.$$

□

B.2 Model with Purchases of Privately Issued Assets

Setup. The setup is the same as in the baseline model, except that the government budget constraint (3) is replaced by the equation stated in the main text,

$$\mathrm{d}\mathcal{MB}_t = \left(i^{\mathcal{M}}_t\mathcal{M}_t + i^{\mathcal{B}}_t\mathcal{B}_t - \mathcal{P}_t\tau_t K_t\right)\mathrm{d}t - \left(\mathcal{A}_t\mathrm{d}i^{\mathcal{A}}_t - \mathrm{d}\mathcal{A}_t\right),$$

and agents $i \in \mathbb{I}$ face a portfolio choice that includes the $\mathcal{A}$-asset. Specifically, the net worth evolution of agent $i \in \mathbb{I}$ is

$$\mathrm{d}n_t^i = -c_t^i \mathrm{d}t + n_t^i \left(\theta_t^i \left(\theta_t^{\mathcal{M},i} \mathrm{d}r_t^{\mathcal{M}} + \theta_t^{\mathcal{A},i} \mathrm{d}r_t^{\mathcal{A}} + (1 - \theta_t^{\mathcal{M},i} - \theta_t^{\mathcal{A},i}) \mathrm{d}r_t^{\mathcal{B}} \right) + (1 - \theta_t^i) \mathrm{d}r_t^{K,i}(\iota_t^i, u_t^i) \right), \tag{53}$$

where $\theta_t^{\mathcal{A},i}$ denotes the share of the non-capital portfolio invested into the $\mathcal{A}$-asset (which is usually negative because the government purchases this asset) and $\mathrm{d}r_t^{\mathcal{A}}$ is the *real* return on that asset, i.e.,

$$\begin{aligned} \mathrm{d}r_t^{\mathcal{A}} &= \mathrm{d}\imath_t^{\mathcal{A}} + \frac{\mathrm{d}(1/\mathcal{P}_t)}{1/\mathcal{P}_t} + \frac{\mathrm{d}\langle \imath_t^{\mathcal{A}}, 1/\mathcal{P}_t \rangle}{1/\mathcal{P}_t} \\ &= \left(i_t^{\mathcal{A}} - \sigma^{\mathcal{A}}(X_t)\sigma_t^{\mathcal{P}} \right) \mathrm{d}t + \sigma^{\mathcal{A}}(X_t)\mathrm{d}Z_t + \frac{\mathrm{d}(1/\mathcal{P}_t)}{1/\mathcal{P}_t}. \end{aligned} \tag{54}$$

where we define $i_t^{\mathcal{A}} := \frac{\mathbb{E}_t[\mathrm{d}\imath_t^{\mathcal{A}}]}{\mathrm{d}t}$ for the expected nominal return on the asset. The market clearing condition for the asset is

$$\frac{\mathcal{A}_t}{\mathcal{P}_t} + \int \theta_t^{\mathcal{A},i} n_t^i \mathrm{d}i = 0. \tag{55}$$

We further define the following notation. The nominal value of *net* government liabilities is denoted by

$$\mathcal{G}_t := \mathcal{MB}_t - \mathcal{A}_t$$

and its real value by

$$q_t^{\mathcal{G}} K_t := \frac{\mathcal{G}_t}{\mathcal{P}_t}.$$

We let

$$\breve{\mu}_t^{\mathcal{G}} := -\frac{\tau_t}{q_t^{\mathcal{G}}}$$

denote the negative of the ratio of tax revenues to net government liabilities. In the equilibrium definition below, $\mathcal{G}_t$, $q_t^{\mathcal{G}}$, and $\breve{\mu}_t^{\mathcal{G}}$ take the roles of $\mathcal{MB}_t$, $q_t^{\mathcal{MB}}$, and $\breve{\mu}_t^{\mathcal{MB}}$, respectively, from the baseline model.

The decision problem of agent $i \in \mathbb{I}$ is to choose consumption c_t^i, investment ι_t^i, capital utilization u_t^i, and portfolio weights $\theta_t^i, \theta_t^{\mathcal{M},i}, \theta_t^{\mathcal{A},i}$ to maximize utility V_0^i subject to the net worth evolution (53), the return expressions (8), (9), (10), and (54), the cash constraint (11), and a solvency constraint $n_t^i \geq 0$, with the following qualifications:

equation (8) for the return on capital is to be interpreted with $\breve{\mu}_t^{\mathcal{G}}$ in place of $\breve{\mu}^{\mathcal{MB}}$ and $q_t^{\mathcal{G}}$ in place of $q_t^{\mathcal{MB}}$ and only the first equalities in equations (9) and (10) for the returns on money and bonds are relevant constraints while the second equalities are to be ignored. With these qualifications, variables referring to the aggregate $\mathcal{MB}$ no longer appear in the return expressions faced by the agent (and are therefore not required in the equilibrium definition either).

Definition 4. *Given an exogenous process X_t, functions as in (5) and a function $\sigma^{\mathcal{A}}$, initial stocks of capital and nominal assets, K_0, $\mathcal{G}_0$, and an initial cross-sectional wealth distribution $\{\eta_0^i\}_{i\in\mathbb{I}}$ satisfying $\int_{\mathbb{I}}\eta_0^i di = 1$, a* competitive equilibrium *consists of aggregate (Ito) stochastic processes K_t, $\mathcal{G}_t$, $\mathcal{A}_t$ $\mathcal{P}_t$, $q_t^{\mathcal{M}}$, $q_t^{\mathcal{G}}$, q_t^K, $\breve{\mu}_t^{\mathcal{G}}$, $i_t^{\mathcal{M}}$, $i_t^{\mathcal{B}}$, $i_t^{\mathcal{A}}$ adapted to $\mathcal{F}_t$ and, for each $i \in \mathbb{I}$, individual (Ito) stochastic processes c_t^i, ι_t^i, u_t^i, θ_t^i, $\theta_t^{\mathcal{M},i}$, $\theta_t^{\mathcal{A},i}$, n_t^i adapted to $\tilde{\mathcal{F}}_t^i$, such that*

1. *For each agent $i \in \mathbb{I}$, c_t^i, ι_t^i, u_t^i, θ_t^i, $\theta_t^{\mathcal{M},i}$, $\theta_t^{\mathcal{A},i}$ solve i's optimization problem for initial wealth $n_0^i = \eta_0^i(q_0^K + q_0^{\mathcal{MB}})K_0$ and n_t^i is as implied by the net worth evolution (53).*

2. *K_t satisfies the aggregate capital evolution*

$$dK_t = \left(\int_{\mathbb{I}} \iota_t^i k_t^i di - \delta K_t\right) dt$$

   with the given initial condition K_0.

3. *$\mathcal{G}_t$ satisfies the evolution*

$$d\mathcal{G}_t = \left(\breve{\mu}_t^{\mathcal{G}} + i_t^{\mathcal{B}} + \frac{q_t^{\mathcal{M}}}{q_t^{\mathcal{G}}}(i_t^{\mathcal{M}} - i_t^{\mathcal{B}})\right)\mathcal{G}_t dt - \mathcal{A}_t\left((i_t^{\mathcal{A}} - i_t^{\mathcal{B}})dt + \sigma^{\mathcal{A}}(X_t)dZ_t\right)$$

   with the given initial condition $\mathcal{G}_0$.

4. *$q_t^{\mathcal{G}} K_t$ is the real value of net government liabilities: for all t,*

$$q_t^{\mathcal{G}} K_t = \frac{\mathcal{G}_t}{\mathcal{P}_t}.$$

5. *All asset values are nonnegative, $q_t^{\mathcal{M}}, q_t^{\mathcal{G}} - q_t^{\mathcal{M}}, q_t^K \geq 0$ and $\mathcal{G}_t, \mathcal{A}_t \geq 0$ for all t.*

6. *All markets clear: for all t, equations (12), (13), (14), and (55) hold.*

Irrelevance of $\mathcal{A}_t$ for Real Allocation. We now state and prove an irrelevance proposition that is in analogy to Proposition 3 in the long-term bond model. The key is, once again, that the return on all nominal assets can be written in a form analogous to Lemma 5:

Lemma 15. *In any symmetric Definition 4 equilibrium, the return on the portfolio of net government liabilities is*

$$dr_t^{\mathcal{G}} = -\breve{\mu}_t^{\mathcal{G}} dt + \frac{d(q_t^{\mathcal{G}} K_t)}{q_t^{\mathcal{G}} K_t},$$

where $\breve{\mu}_t^{\mathcal{G}} := -\frac{\tau_t}{q_t^{\mathcal{G}}}$.

Proof. In equilibrium, the return on the portfolio of net government liabilities is (we can suppress i-superscripts by symmetry)

$$\begin{aligned}
\mathrm{d}r_t^{\mathcal{G}} &= \theta_t^{\mathcal{M}} \mathrm{d}r_t^{\mathcal{M}} + \theta_t^{\mathcal{A}} \mathrm{d}r_t^{\mathcal{A}} + (1 - \theta_t^{\mathcal{M}} - \theta_t^{\mathcal{A}}) \mathrm{d}r_t^{\mathcal{B}} \\
&= \frac{\mathcal{M}_t}{\mathcal{G}_t} \mathrm{d}r_t^{\mathcal{M}} + \frac{\mathcal{A}_t}{\mathcal{G}_t} \mathrm{d}r_t^{\mathcal{A}} + \frac{\mathcal{B}_t}{\mathcal{G}_t} \mathrm{d}r_t^{\mathcal{B}} \\
&= \frac{\mathcal{M}_t i_t^{\mathcal{M}} + \mathcal{B}_t i_t^{\mathcal{B}} - \mathcal{A}_t \mathrm{d}i_t^{\mathcal{A}}}{\mathcal{G}_t} + \frac{\mathrm{d}(1/\mathcal{P}_t)}{1/\mathcal{P}_t} - \frac{\mathcal{A}_t}{\mathcal{G}_t} \frac{\mathrm{d}\langle i_t^{\mathcal{A}}, 1/\mathcal{P}_t \rangle}{1/\mathcal{P}_t} \\
&= \frac{\mathcal{P}_t \tau_t K_t \mathrm{d}t}{\mathcal{G}_t} + \frac{\mathrm{d}\mathcal{G}_t}{\mathcal{G}_t} + \frac{\mathrm{d}(1/\mathcal{P}_t)}{1/\mathcal{P}_t} - \frac{\mathcal{A}_t}{\mathcal{G}_t} \frac{\mathrm{d}\langle i_t^{\mathcal{A}}, 1/\mathcal{P}_t \rangle}{1/\mathcal{P}_t},
\end{aligned}$$

where the last line follows from the government budget constraint. Using Ito's product rule, we can write further

$$\frac{\mathrm{d}\mathcal{G}_t}{\mathcal{G}_t} + \frac{\mathrm{d}(1/\mathcal{P}_t)}{1/\mathcal{P}_t} = \frac{\mathrm{d}(\mathcal{G}_t/\mathcal{P}_t)}{\mathcal{G}_t/\mathcal{P}_t} - \frac{\mathrm{d}\langle \mathcal{G}_t, 1/\mathcal{P}_t \rangle}{\mathcal{G}_t/\mathcal{P}_t}$$

and therefore

$$\begin{aligned}
\mathrm{d}r_t^{\mathcal{G}} &= \frac{\mathcal{P}_t \tau_t K_t \mathrm{d}t}{\mathcal{G}_t} + \frac{\mathrm{d}(\mathcal{G}_t/\mathcal{P}_t)}{\mathcal{G}_t/\mathcal{P}_t} - \frac{\overbrace{\mathrm{d}\langle \mathcal{G}_t, 1/\mathcal{P}_t \rangle + \mathcal{A}_t \mathrm{d}\langle i_t^{\mathcal{A}}, 1/\mathcal{P}_t \rangle}^{=0}}{\mathcal{G}_t/\mathcal{P}_t} \\
&= -\breve{\mu}_t^{\mathcal{G}} \mathrm{d}t + \frac{\mathrm{d}(q_t^{\mathcal{G}} K_t)}{q_t^{\mathcal{G}} K_t},
\end{aligned}$$

where the last term in the first line vanishes because $\mathrm{d}\mathcal{G}_t - \mathcal{A}_t \mathrm{d}i_t^{\mathcal{A}}$ has only $\mathrm{d}t$-terms by the government budget constraint and the second line uses the definition $\mu^{\mathcal{G}} := -\frac{\tau_t}{q_t^{\mathcal{G}}}$.

□

Proposition 13 (Irrelevance of $\mathcal{A}_t$)**.** *Let*

$$\{K_t, \mathcal{G}_t, \mathcal{A}_t, \mathcal{P}_t, q_t^{\mathcal{M}}, q_t^{\mathcal{G}}, q_t^K, \breve{\mu}_t^{\mathcal{G}}, i_t^{\mathcal{M}}, i_t^{\mathcal{B}}, i_t^{\mathcal{A}}, \{c_t^i, \iota_t^i, u_t^i, \theta_t^i, \theta_t^{\mathcal{M},i}, \theta_t^{\mathcal{A},i}, n_t^i\}_{i\in\mathbb{I}}\}$$

be a symmetric equilibrium according to Definition 4 and suppose that $\vartheta_t^{\mathcal{M}} = q_t^{\mathcal{M}}/q_t^{\mathcal{G}} \leq 1$. Then, for any arbitrary nominal initial condition $\mathcal{MB}_0 > 0$, there are processes $\hat{\mathcal{MB}}_t$, $\hat{\mathcal{P}}_t$ such that

$$\begin{aligned}&\{K_t, \mathcal{MB}_t, \mathcal{P}_t, q_t^{\mathcal{M}}, q_t^{\mathcal{MB}}, q_t^K, \breve{\mu}_t^{\mathcal{MB}}, i_t^{\mathcal{M}}, i_t^{\mathcal{B}}, \{c_t^i, \iota_t^i, u_t^i, \theta_t^i, \theta_t^{\mathcal{M},i}, n_t^i\}_{i\in\mathbb{I}}\}\\ &:= \{K_t, \hat{\mathcal{MB}}_t, \hat{\mathcal{P}}_t, q_t^{\mathcal{M}}, q_t^{\mathcal{G}}, q_t^K, \breve{\mu}_t^{\mathcal{G}}, i_t^{\mathcal{M}}, i_t^{\mathcal{B}}, \{c_t^i, \iota_t^i, u_t^i, \theta_t^i, \theta_t^{\mathcal{M},i}, n_t^i\}_{i\in\mathbb{I}}\}\end{aligned}$$

is an equilibrium according to Definition 1.

Sketch of Proof. The proof follows the same logic as the proof of the irrelevance Proposition 3 in the long-term bond model and makes use of auxiliary results proven in Appendix B.1.

Specifically, one shows, using Lemma 12, that

$$\mathbf{e}_r := \{K_t, q_t^{\mathcal{M}}, q_t^{\mathcal{MB}}, q_t^K, \breve{\mu}_t^{\mathcal{MB}}, \{c_t^i, \iota_t^i, u_t^i, \theta_t^i, \theta_t^{\mathcal{M},i}, n_t^i\}_{i\in\mathbb{I}}\}$$

is a symmetric real equilibrium (compare Definition 3). Then by Lemma 13, for any initial condition $\mathcal{MB}_0$, there is a unique extension of $\mathbf{e}_r$ to a Definition 1 equilibrium $\mathbf{e}$ that features the nominal money rate process $i_t^{\mathcal{M}}$. Because the $\theta_t^{\mathcal{M},i}$-choice has not changed in this model relative to the baseline model, one shows as in the proof of Lemma 1 that equation (18) continues to hold, so that also the nominal bond rate $i_t^{\mathcal{B}}$ must be the same as in the originally given (Definition 4) equilibrium. Choosing $\hat{\mathcal{MB}}_t$ and $\hat{\mathcal{P}}_t$ as in the extension $\mathbf{e}$ implies the claim in the proposition.

For the remaining argument that $\mathbf{e}_r$ is indeed a real equilibrium, one verifies properties (a)–(c) in Lemma 12. The argument is completely analogous to the proof of Lemma 14, which establishes a similar claim in the long-term bond model. The details are therefore not restated here. Key is, once again, that the return on nominal assets takes the form (51) required in the auxiliary decision problem. This follows here by Lemma 15 established above (and the definitions $\breve{\mu}_t^{\mathcal{MB}} = \breve{\mu}_t^{\mathcal{G}}$, $q_t^{\mathcal{MB}} = q_t^{\mathcal{G}}$). □

Proposition 13 implies that, for understanding the real allocation, and possibly the interaction between nominal interest rates and the real allocation, the asset portfolio $\mathcal{A}_t$ held by the central bank is irrelevant.

Asset Purchases in Terms of dQ_t-Processes. We can link the previous result closer to our formal presentation of asset purchases in the main text by separating the consolidated government budget constraint and introducing explicit $\mathrm{d}\mathcal{Q}_t$-processes for the purchase behavior of the central bank. Here, it makes sense to split the budget constraint as follows:

$$
\begin{aligned}
\mathrm{d}\mathcal{B}_t &= (i_t^{\mathcal{B}}\mathcal{B}_t - \mathcal{P}_t\tau_t K_t)\mathrm{d}t - \mathrm{d}\mathcal{Q}_t^{\mathcal{B}}, && \text{(fiscal budget)}\\
\mathrm{d}\mathcal{M}_t &= i_t^{\mathcal{M}}\mathcal{M}_t\mathrm{d}t + \mathrm{d}\mathcal{Q}_t^{\mathcal{B}} + \mathrm{d}\mathcal{Q}_t^{\mathcal{A}}, && \text{(monetary budget)}\\
\mathrm{d}\mathcal{A}_t &= \mathcal{A}_t\mathrm{d}i_t^{\mathcal{A}} + \mathrm{d}\mathcal{Q}_t^{\mathcal{A}}. && \text{(asset portfolio)}
\end{aligned}
$$

The process $\mathrm{d}\mathcal{Q}_t^{\mathcal{B}}$ describes bond purchases, as before, whereas $\mathrm{d}\mathcal{Q}_t^{\mathcal{A}}$ captures $\mathcal{A}$-asset purchases.

Because both $\mathrm{d}\mathcal{Q}_t^{\mathcal{A}}$ and $\mathrm{d}\mathcal{Q}_t^{\mathcal{B}}$ cancel out when we consolidate the three equations into a single equation for net government liabilities $\mathcal{G}_t = \mathcal{MB}_t - \mathcal{A}_t$, the exact same formal argument that led us to Corollary 1 in the baseline model still applies, with the understanding that the corollary applies to bond purchases (i.e., $\mathrm{d}\mathcal{Q}_t$ there means $\mathrm{d}\mathcal{Q}_t^{\mathcal{B}}$ here). Given an arbitrary $\mathrm{d}\mathcal{Q}_t^{\mathcal{A}}$-process, variation in $\mathrm{d}\mathcal{Q}_t^{\mathcal{B}}$ can therefore still be used to implement any desired target path for $\vartheta_t^{\mathcal{M}} \in [0,1]$.[43]

Let us now consider variation in $\mathrm{d}\mathcal{Q}_t^{\mathcal{A}}$. Holding fixed processes for $\hat{\tau}_t$ and $\vartheta_t^{\mathcal{M}}$, Proposition 13 tells us that this may alter the paths of nominal net government liabilities $\mathcal{G}_t$ and the price level $\mathcal{P}_t$, but not affect the ratio $\mathcal{G}_t/\mathcal{P}_t$ or any other real variable. In this sense, asset purchases $\mathrm{d}\mathcal{Q}_t^{\mathcal{A}}$ are neutral.[44]

More specifically, we may use the proposition to map both equilibria, the one with the original $\mathrm{d}\mathcal{Q}_t^{\mathcal{A}}$-process and the one with an altered $\mathrm{d}\mathcal{Q}_t^{\mathcal{A}}$-process into the baseline

[43] Here we need to assume that the $\mathrm{d}\mathcal{Q}_t^{\mathcal{A}}$-process ensures $\mathcal{A}_t \geq 0$, as we have done in the setup. In fact, $\mathcal{A}_t > 0$ may permit the central bank to implement $\vartheta_t^{\mathcal{M}} > 1$ because the private assets serve as additional backing for money balances, and the exact upper bound on $\vartheta_t^{\mathcal{M}}$ does interact with the asset purchase process $\mathrm{d}\mathcal{Q}_t^{\mathcal{A}}$. Once we restrict $\vartheta_t^{\mathcal{M}} \in [0,1]$, as we do here, we do not need to worry about this additional complication.

[44] Note that the previous discussion of bond purchases implies that it is a legitimate exercise to vary $\mathrm{d}\mathcal{Q}_t^{\mathcal{A}}$ independently of $\vartheta_t^{\mathcal{M}}$. Ex post, we can always find a bond purchase process $\mathrm{d}\mathcal{Q}_t^{\mathcal{B}}$ to implement the desired $\vartheta_t^{\mathcal{M}}$-path. Of course, the exact required $\mathrm{d}\mathcal{Q}_t^{\mathcal{B}}$-path for implementation will in general depend on the $\mathrm{d}\mathcal{Q}_t^{\mathcal{A}}$-path and the return on the $\mathcal{A}$-asset.

model in a way that keeps the real allocation and nominal rates unchanged. Because Proposition 4 applies in the baseline model and because, by assumption, the endogenous processes $\hat{\tau}_t$ and $\vartheta_t^{\mathcal{M}}$ are the same in both scenarios, the implied ϑ_t-process must be the same.[45] Then Lemmas 2 and 3 and variable definitions imply that also u_t, ι_t, q_t, q_t^K, $q_t^{\mathcal{MB}}$, $q_t^{\mathcal{M}}$ must be the same in both equilibria. This establishes that all real aggregates are the same across the two equilibria. It is then straightforward to conclude from the optimal decision rules in Lemma 1 (and the symmetry assumption) that also all i-dependent variables must be the same.

B.3 Simple Model with Fixed Output and no Safe Assets

In this appendix, we briefly outline a variant of our model with exogenous output that we have referred to in our discussion of Sargent and Wallace (1981)'s unpleasant arithmetic and prove equation (26).

Setup. The setup of this variant is the same as in our baseline model, except that we shut down idiosyncratic risk, $\tilde{\sigma}_t \equiv 0$, and variation in productivity, $a_t \equiv a$, and we change the decision problems of private agents $i \in \mathbb{I}$ by imposing two additional constraints:

$$\iota_t^i = \delta, \tag{56}$$

$$u_t^i = 1. \tag{57}$$

Constraint (56) removes the investment choice from agents' choice set and ensures that aggregate capital K_t is constant. We may normalize $K_t \equiv 1$ w.l.o.g. Constraint (57) also removes the utilization choice from agents' choice set, so that output $u_t a_t K_t = a$ becomes constant.

An equilibrium is defined precisely as in Definition 1, except that in property 1., we now require that i-dependent variables solve the modified decision problem with the additional constraints (56) and (57).

Clearly, an equilibrium can only exist if we assume $a > \delta$ at all times, as otherwise the goods market could not clear. W.l.o.g., we can then even assume $\delta = \iota = 0$ (by just replacing a with $a - \delta$).

[45] Strictly speaking, this also requires a uniqueness result. The uniqueness result summarized in Section 3.3 and formally stated in the companion paper Brunnermeier and Merkel (2025) is sufficient here.

Solution. By following the solution steps in Appendix A.1 and imposing the additional constraints (56) and (57) throughout, the reader readily verifies that Lemmas 1–4 continue to hold with the following modifications:

- Lemma 1: The choice conditions (16) and (17) relating to the optimal investment and utilization choice, respectively, are no longer relevant. Everything else in the lemma continues to hold unchanged.
- Lemma 2: The representation for q_t is now

$$q_t = \frac{a}{\rho}$$

  and the last equation for ι_t is no longer relevant (instead $\iota_t = \delta = 0$).
- Lemma 3: This lemma no longer applies. There is no relationship between the liquidity premium and utilization anymore, because the latter is fixed by the constraint.
- Lemma 4: This lemma applies unchanged (except that here $\tilde{\sigma}_t \equiv 0$) because its proof does not rely on any of the results from the previous lemmas that are no longer valid here.

In addition to these observations, we can compute the spread Δi_t more explicitly. While the expression for $\lambda_t^{\mathcal{M}}$ from Lemma 3 does not carry over to this variant of the model, we can back out Δi_t from equation (20) if we derive an alternative equation for ϑ_t. Specifically,

$$\vartheta_t = \frac{q_t^{\mathcal{MB}}}{q_t} = \frac{\rho}{a}\frac{\mathcal{MB}_t}{\mathcal{P}_t}.$$

Applying Ito's lemma to this equation (and using that $\mathcal{MB}_t$ is a FV process) yields

$$\frac{\mathrm{d}\vartheta_t}{\vartheta_t} = \left(\mu_t^{\mathcal{MB}} + \mu_t^{1/\mathcal{P}}\right)\mathrm{d}t + \sigma_t^{1/\mathcal{P}}\mathrm{d}Z_t.$$

Combining this with equation (20) implies

$$\begin{aligned}\mu_t^{\mathcal{MB}} + \mu_t^{1/\mathcal{P}} &= \rho + \breve{\mu}_t^{\mathcal{MB}} - \vartheta_t^{\mathcal{M}}\lambda_t^{\mathcal{M}}\nu_t \\ &= \rho + \mu_t^{\mathcal{MB}} - i_t - \vartheta_t^{\mathcal{M}}\Delta i_t \\ &= \rho + \mu_t^{\mathcal{MB}} - i_t^{\mathcal{M}} - \Delta i_t,\end{aligned}$$

and solving for Δi_t yields

$$\Delta i_t = \rho - \mu_t^{1/\mathcal{P}} - i_t^{\mathcal{M}}. \tag{58}$$

This derivation is only valid in states in which $\vartheta_t > 0$, but this is sufficient for our purposes.

Equation (26). In equilibrium, the cash constraint (11) takes the form

$$\frac{a}{q_t} \leq \nu_t \vartheta_t^{\mathcal{M}} \vartheta_t,$$

and plugging in $q_t = a/\rho$ permits us to rewrite this as follows:

$$\vartheta_t^{\mathcal{M}} \vartheta_t \geq \frac{\rho}{\nu_t}.$$

If the previous equation is binding, then multiplying the left-hand side of equation (58) by $\vartheta_t^{\mathcal{M}} \vartheta_t$ and the right-hand side by $\frac{\rho}{\nu_t} = \vartheta_t^{\mathcal{M}} \vartheta_t$ yields immediately equation (26).

If the constraint is slack, then $\Delta i_t = 0$. Hence, equation (58), which holds in all states, simplifies to "$0 = 0$" and continues to hold even if we multiply the two sides by different numbers. Once again, multiplying the left-hand side by $\vartheta_t^{\mathcal{M}} \vartheta_t$ and the right-hand side by $\frac{\rho}{\nu_t}$ yields equation (26).