Safe Assets
Online Appendix

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December 20, 2023

B Further Derivations, Proofs, and Model Extensions

B.1 Uniqueness of Stationary Monetary Equilibria

BSDE (10) is a fixed-point condition for the key equilibrium process ϑ. In this appendix, we show that the BSDE is well-behaved on the domain $(0, 1)$ and represents a contraction in a suitable sense to be made precise. The contraction property implies that the equation has at most one nondegenerate stationary solution on this domain.

Let us first consider the finite-horizon version of the BSDE (10) for a fixed terminal condition ϑ_T. In integral form, this BSDE can be written as

$$\forall t \in [0, T]: \; \vartheta_t = \mathbb{E}_t \left[\vartheta_T + \int_t^T \left((1 - \vartheta_s)^2 \bar{\chi}^2 \tilde{\sigma}_s^2 - \rho - \breve{\mu}_s^{\mathcal{B}} \right) \vartheta_s ds \right]. \tag{38}$$

Standard results from BSDE theory imply that, under suitable conditions on $\breve{\mu}^{\mathcal{B}}$ and $\tilde{\sigma}$ (boundedness is sufficient), there is a unique solution to the BSDE for any bounded terminal condition (see, e.g., Pham (2009, Theorem 6.2.2)).

The following auxiliary lemma shows that solutions to the finite-horizon BSDE satisfy a type of monotonicity property with respect to the terminal condition. It also implies that, if the terminal condition is in $(0, 1]$, then so is the full solution.

Lemma 2. *Let ϑ_t solve the BSDE (38) with terminal condition ϑ_T taking values in $(0, 1]$. If ϑ'_t is another solution with terminal condition $\vartheta'_T < \vartheta_T$, then $\vartheta'_t < \vartheta_t$ for all $t \in [0, T]$. If $\rho + \breve{\mu}^{\mathcal{B}}_t > 0$ for all $t \in [0, T]$, then $\vartheta_t \in (0, 1)$ for all $t < T$.*

Proof. First, observe that whenever $\vartheta_T \geq \vartheta'_T$, the comparison principle for BSDEs implies that $\vartheta_t \geq \vartheta'_t$ (see, for example, Pham (2009, Theorem 6.2.2)). Furthermore, $\vartheta_t > \vartheta'_t$ if $\vartheta_T > \vartheta'_T$ with positive probability.

Second, let us compare the solution ϑ_t of BSDE (38), which we write in integrated form as

$$-d\vartheta_t = \underbrace{\left((1 - \vartheta_t)^2 \breve{\chi}^2 \tilde{\sigma}_t^2 - \rho - \breve{\mu}^{\mathcal{B}}_t\right) \vartheta_t}_{=f^1_t(\vartheta_t)} dt - \nu_t dZ_t$$

with the solution $\bar{\vartheta}_t = 1$ of BSDE

$$-d\bar{\vartheta}_t = \underbrace{0}_{=f^2_t(\bar{\vartheta}_t)} dt - \nu_t dZ_t$$

with terminal condition $\bar{\vartheta}_T = 1$.

Since terminal conditions satisfy $\bar{\vartheta}_T \geq \vartheta_T$, and generators satisfy

$$0 = f^2_t(\bar{\vartheta}_t) > f^1_t(\bar{\vartheta}_t) = -\left(\rho + \breve{\mu}^{\mathcal{B}}_t\right),$$

the comparison principle implies that $\vartheta_t < \bar{\vartheta}_t = 1$ for all $t \in [0, T)$. □

One could now attempt to solve the infinite-horizon BSDE (10) by starting at some terminal guess ϑ_T of the finite-horizon BSDE and considering longer and longer time horizons ($T \to \infty$). It is, however, a priori unclear whether this procedure converges and, if so, whether the limit is independent of the assumed terminal guess.

The following technical lemma is key in establishing that this strategy succeeds (under certain conditions).

Lemma 3. *Suppose $\breve{\mu}^{\mathcal{B}}_t + \rho > 0$ for all t. Then the finite-horizon BSDE (38) is a contraction on logarithmic scale:*

Consider any two distinct terminal conditions ϑ_T and ϑ'_T with values in $(0,1)$. Let ϑ_t and ϑ'_t be the corresponding solutions. Then for all $t < T$, ϑ_t and ϑ'_t have values in $(0,1)$ and satisfy[1]

$$\log \vartheta_t - \log \vartheta'_t \in \Big(\operatorname{ess\,inf} \left(\log \vartheta_T - \log \vartheta'_T \right), \operatorname{ess\,sup} \left(\log \vartheta_T - \log \vartheta'_T \right) \Big). \tag{39}$$

Proof. The statement of the lemma is equivalent to

$$\frac{\vartheta_t}{\vartheta'_t} \in \left(\operatorname{ess\,inf} \frac{\vartheta_T}{\vartheta'_T}, \operatorname{ess\,inf} \frac{\vartheta_T}{\vartheta'_T} \right).$$

Let us prove that

$$\frac{\vartheta_t}{\vartheta'_t} > x := \operatorname{ess\,inf} \frac{\vartheta_T}{\vartheta'_T}$$

as the other bound is symmetric. Without loss of generality, let us assume that $\vartheta_T \le \vartheta'_T$, because replacing ϑ_T with $\min(\vartheta_T, \vartheta'_T)$ only weakly lowers ϑ_t by Lemma 2 and makes the bound harder to prove.

Equation (38) implies that ϑ_t satisfies

$$\vartheta_t = \mathbb{E}_t \left[\exp \left(\int_t^T \left((1-\vartheta_s)^2 \bar{\chi}^2 \tilde{\sigma}_s^2 - \rho - \breve{\mu}_s^{\mathcal{B}} \right) ds \right) \vartheta_T \right] \tag{40}$$

and an analogous expression holds for ϑ'_t. Since $\vartheta_s \le \vartheta'_s < 1$ for all $s \in [t, T]$ we have $(1-\vartheta_s)^2 \le (1-\vartheta'_s)^2$ and so

$$\begin{aligned} \vartheta_t &\ge \mathbb{E}_t \left[\exp \left(\int_t^T \left((1-\vartheta'_s)^2 \bar{\chi}^2 \tilde{\sigma}_s^2 - \rho - \breve{\mu}_s^{\mathcal{B}} \right) ds \right) \vartheta_T \right] \\ &\ge \mathbb{E}_t \left[\exp \left(\int_t^T \left((1-\vartheta'_s)^2 \bar{\chi}^2 \tilde{\sigma}_s^2 - \rho - \breve{\mu}_s^{\mathcal{B}} \right) ds \right) x \vartheta'_T \right] = x \vartheta'_t, \end{aligned}$$

Hence, $\vartheta_t / \vartheta'_t \ge x$. If ϑ_T and ϑ'_T are distinct, then the inequality must be strict. □

Lemma 3 has important implications for the infinite-horizon BSDE (10) if the economy is *stationary* (compare Definition 2).

[1]Here ess inf and ess sup denote the essential infimum and essential supremum, respectively, and are taken over all outcomes of the underlying probability space.

Proposition 8. *Suppose the exogenous processes are stationary and $\rho + \breve{\mu}^{\mathcal{B}}(X) > 0$ for all $X \in \mathbb{X}$. Then, equation (10) has at most one stationary nondegenerate (i.e. not identically 0) solution.*

If this solution exists, $\vartheta_t = \vartheta^*(X_t)$, then

- *for all $X \in \mathbb{X}$, $\vartheta^*(X) > 0$;*
- *for any function $\vartheta' : \mathbb{X} \to (0,1)$, the solution to the finite-horizon equation (38) with terminal condition $\vartheta_T = \vartheta'(X_T)$ converges to $\vartheta_t = \vartheta^*(X_t)$ as $T \to \infty$.*

If equation (10) has no stationary nondegenerate solution, then for any terminal condition $\vartheta_T = \vartheta'(X_T)$, the solution to the finite-horizon equation converges to $\vartheta_t = 0$ as $T \to \infty$.

We note that Proposition 8 encompasses all statements in Proposition 2 stated in the main text. Proving the former therefore also implies the latter. Before we present the proof, we first establish another small technical lemma.

Lemma 4. *Suppose that economy is stationary and let $\vartheta_t = \vartheta(t, X_t)$ solve (38) with terminal condition $\vartheta_T = \vartheta(T, X_T)$, with values in $(0,1]$. If $\vartheta(t,X) > \vartheta(T,X)$ for all X and $t < T$, then $\vartheta(t,X)$ increases as t declines. If $\vartheta(t,X) < \vartheta(T,X)$, then $\vartheta(t,X)$ declines as t declines.*

Proof. The two statements are symmetric, so let us prove the first one. We would like to show that $\vartheta(t-s,X) > \vartheta(t,X)$. These are solutions to (38) with time horizon $T-t$ and terminal conditions $\vartheta(T-s,X) > \vartheta(T,X)$. By Lemma 2, $\vartheta(t-s,X) > \vartheta(t,X)$. □

Proof of Proposition 8. First, let us show that any stationary nondegenerate solution $\vartheta(X)$ must be strictly positive. If $\vartheta(X') = 0$ for some $X' \in \mathbb{X}$, then $\vartheta_t = 0$ when $X_t = X$, hence equation (40) can hold only if $\vartheta_T = 0$ almost surely for all future T. Since state process X_t is ergodic, it follows that $\vartheta(X) = 0$ almost surely, but then (40) implies that $\vartheta(X) = 0$ for all X. Therefore, ϑ cannot degenerate to 0 at any single point.

Let us prove that there is at most one stationary nondegenerate solution. Suppose $\vartheta_1(X)$ and $\vartheta_2(X)$ are two distinct solutions, with $\vartheta_1(X) < \vartheta_2(X)$ for some $X \in \mathbb{X}$. Then $x := \inf_{X \in \mathbb{X}} \frac{\vartheta_1(X)}{\vartheta_2(X)} < 1$, and by the compactness of the domain $\mathbb{X}$, the infimum is attained at some point $\underline{X}$ (as ϑ_1, ϑ_2 are assumed to be continuous).

Now, suppose $X_t = \underline{X}$ and consider solutions ϑ and ϑ' of equation (38) with terminal conditions $\vartheta_T = \vartheta_1(X_T)$ and $\vartheta'_T = \vartheta_2(X_T)$. Then, by uniqueness of solutions to the

BSDE (38), we have $\vartheta_t = \vartheta_1(\underline{X})$ and $\vartheta'_t = \vartheta_2(\underline{X})$. But then

$$\frac{\vartheta_t}{\vartheta'_t} = \frac{\vartheta_1(\underline{X})}{\vartheta_2(\underline{X})} \leq \inf \frac{\vartheta_T}{\vartheta'_T},$$

a contradiction to Lemma 3.

Now, suppose (10) has no stationary nondegenerate solution. Consider the solution to equation (38) with terminal condition $\vartheta_T = 1$. Then by Lemma 2, $\vartheta_{T-s} < 1$ for all $s > 0$, and by Lemma 4, ϑ_t declines for each X as the horizon T increases. Hence, ϑ_t must converge to some function $\vartheta^*(X)$. By continuity $\vartheta^*(X)$ is a solution to (10), and because there are no stationary nondegenerate solutions, the limit must be $\vartheta^*(X) = 0$. Now, if ϑ'_t a solution with a different terminal condition $\vartheta'_T < 1$, then $\vartheta'_t < \vartheta_t$ by the comparison principle (Lemma 2), hence ϑ'_t must also converge to 0.

Finally, suppose (10) does have a stationary nondegenerate solution. Then the solution ϑ from the terminal condition $\vartheta_T = 1$ is likewise declining as we go backwards in time and converges to a solution. Since ϑ stays above the stationary nondegenerate solution ϑ^* by Lemma 2, it must converge to ϑ^*. Likewise, the solution ϑ' from the terminal condition $\vartheta'_T = \epsilon\hat{\vartheta}$ increases as we go backwards in time (by Lemmas 4 and 3), and converges to ϑ^*. By the comparison principle, any other solution ϑ''_t with terminal condition $\vartheta''_T(X) \in [\epsilon\hat{\vartheta}(X), 1]$ will also be squeezed between ϑ'_t and ϑ_t, hence will converge to $\hat{\vartheta}$. □

Proposition 8 implies that to solve (10), we do not need a good guess of the terminal condition. Any nonzero guess will converge to a stationary solution and, if it exists, the nondegenerate one.

We remark that when the standard solution is nondegenerate, then equation (10) does have many other nonstationary solutions (i.e. the uniqueness result applies only to the stationary solution). However, Proposition 8 implies that all nonstationary solutions converge to 0 in the distant future.

B.2 Proof of Proposition 3

Note that, in general, because $\xi_t^i = e^{-\rho t}/(\rho n_t^i)$ and $dr_t^{n,i}$ has the same risk loadings as dn_t^i/n_t^i (compare equation (4)),

$$\mathrm{Cov}_t\left(\frac{d\xi_t^i}{\xi_t^i}, dr_t^{n,i}\right) = \mathrm{Cov}_t\left(\frac{d(1/n_t^i)}{1/n_t^i}, \frac{dn_t^i}{n_t^i}\right) = -\left(\left(\sigma_t^{n,i}\right)^2 + \left(\tilde{\sigma}_t^{n,i}\right)^2\right)$$

Under the assumptions of the proposition, ϑ_t and thus prices q_t^K, q_t^B do not load on the aggregate shock dZ_t, $\sigma_t^{\vartheta} = \sigma_t^{q,B} = \sigma_t^{q,K} = 0$. In particular, $\sigma_t^{n,i} = 0$ (compare equation (30) and recall that prices of risk and net worth loadings coincide). Thus

$$\mathrm{Cov}_t\left(\frac{d\xi_t^i}{\xi_t^i}, dr_t^{n,i}\right) = \mathrm{Cov}_t\left(\frac{d(1/n_t^i)}{1/n_t^i}, \frac{dn_t^i}{n_t^i}\right) = -\left(\tilde{\sigma}_t^{n,i}\right)^2 = -(1-\vartheta)^2\bar{\chi}^2\tilde{\sigma}^2 < 0,$$

where the last equation follows from equation (30) and market clearing for $\theta_t^{K,i}$.

In contrast, $\sigma_t^{q,B} = 0$ implies that $dr_t^{\mathcal{B}}$ is locally deterministic (does not load on Brownian shocks), so that

$$\mathrm{Cov}_t\left(\frac{d\xi_t^i}{\xi_t^i}, dr_t^{\mathcal{B}}\right) = 0.$$

Comparing the two covariances reveals that the former is always strictly smaller. Thus the bond is a safe asset for agent i at all times t.

B.3 Representative Agent Formulation

In this appendix we present additional details on the representative agent formulation summarized in Section 3. In Part B.3.1, we outline the setup of the hypothetical representative agent tree economy that generates the same asset prices and allocations as our incomplete markets economy and discuss substantive economic takeaways. Additional technical derivation details, including the omitted steps in the arguments that lead to Proposition 6 in the main text, can be found in Part B.3.2.

B.3.1 The Representative Agent Economy

We present a Lucas (1978)-type asset pricing economy that generates the same allocation as in the competitive equilibrium of our incomplete markets economy. In this economy, we interpret aggregate capital and aggregate bonds as two "trees" and we show that equation (13) is precisely the valuation equation for the "bond tree" from the perspective of the representative agent. The dynamic trading perspective is therefore equivalent to the perspective of a hypothetical representative agent.

As stated in the main text, we consider a representative agent whose preferences are represented by a weighted welfare function $\mathcal{W}_0 = \int \lambda^i V_0^i di$. We denote by $\eta_t^i := c_t^i / C_t$ the consumption share of agent i and assume that $d\eta_t^i = \tilde{\sigma}_t^{\eta} d\tilde{Z}_t^i$ with volatility process $\tilde{\sigma}_t^{\eta}$ specified below in equation (42). As shown in the main text, utility $\mathcal{W}_0$ satisfies equation (17), which expresses utility in terms of aggregate consumption C_t and consumption shares η_t^i. We show below (Part B.3.2) that utility can also be represented in the form (this is equation (18) in the main text)

$$\mathcal{W}_0 = w_0 + \mathbb{E}\left[\int_0^{\infty} e^{-\rho t}\left(\log C_t - \frac{1}{2\rho}\left(\tilde{\sigma}_t^{\eta}\right)^2\right) dt\right] \tag{41}$$

with some constant w_0. This equation eliminates the direct dependence on i and gives us the alternative interpretation that two "goods" enter the representative agent's utility function, the aggregate consumption good and a "volatility good" (which generates disutility).[2]

We assume that the representative agent has access to two assets, capital K_t, which produces a certain bundle of the aggregate consumption good and volatility $\tilde{\sigma}_t^{\eta}$, and "derivatives" X_t, which mimic the cash flows to individuals i generated by bond trades in our incomplete markets model and thereby reduce volatility. Capital grows at rate $g_t := \Phi(\iota_t) - \delta$ over time and generates consumption goods at rate $\left((1-\tau_t)a_t - \iota_t\right) K_t dt$. For the purpose of this representative agent economy, g_t, τ_t, a_t, ι_t are exogenous processes. But, of course, we choose for them the stochastic processes implied by the competitive equilibrium of our incomplete markets model.[3] The same remark holds

[2]The representative agent's objective is akin to a money in utility (MIU) model. Holding the derivative asset introduced below reduces volatility $\tilde{\sigma}_t^{\eta}$ in a similar way as holding money in a MIU model generates utility services.

[3]We could also endogenize the real investment decision by letting the representative agent choose ι_t.

for other lower-case variables q_t^B, q_t^K, $\breve{\mu}_t^{\mathcal{B}}$ used below. The face value X_t of derivatives evolves according to

$$dX_t/X_t = \left(g_t + \mu_t^{q,B}\right)dt + \sigma_t^{q,B}dZ_t,$$

where $\mu_t^{q,B}$, $\sigma_t^{q,B}$ are the drift and volatility processes of q_t^B implied by the competitive equilibrium of the incomplete markets model. Derivatives generate a cash flow $-\breve{\mu}_t^{\mathcal{B}}X_t$ and reduce fluctuations in consumption shares η_t^i. Specifically, the volatility loading $\tilde{\sigma}_t^{\eta}$ satisfies the equation

$$\left(q_t^K K_t + X_t\right)\tilde{\sigma}_t^{\eta} = q_t^K K_t \bar{\chi}\tilde{\sigma}_t, \tag{42}$$

where q_t^K is the capital price process from the incomplete markets economy. We can interpret the product $X_t\tilde{\sigma}_t^{\eta}$ as a measure of the aggregate gross trading cash flows from bond trades in response to idiosyncratic shocks in the incomplete markets economy.[4]

Let Q_t^K be the capital price that the representative agent faces, P_t^X the price per unit (face value) of derivatives, and let $N_t := Q_t^K K_t + P_t^X X_t$ be the representative agent's total net worth. The budget constraint of the representative agent is

$$dN_t = -C_t dt + Q_t^K K_t dr_t^K + P_t^X X_t dr_t^X \tag{43}$$

with return processes

$$\begin{aligned} dr_t^K &= \left(\frac{(1-\tau_t)a_t - \iota_t}{Q_t^K} + \mu_t^{Q,K} + g_t\right)dt + \sigma_t^{Q,K}dZ_t, \\ dr_t^X &= \left(\mu_t^{P,X} + g_t - \breve{\mu}_t^{\mathcal{B}} + \sigma_t^{q,B}\sigma_t^{P,X}\right)dt + \left(\sigma_t^{q,B} + \sigma_t^{P,X}\right)dZ_t. \end{aligned}$$

The representative agent chooses $C_t, \tilde{\sigma}_t^{\eta}, K_t, X_t$ to maximize utility $\mathcal{W}_0$ subject to the budget constraint (43) and the risk constraint (42) taking the prices Q_t^K, P_t^X and the return processes as given. The representative agent model is closed by time-zero supplies of capital (K_0) and derivatives (X_0). We impose the additional relationship $X_0 = q_0^B K_0$, where q_0^B is the initial value of q_t^B in the incomplete markets model. While this supply restriction for X_0 may appear ad hoc, it can be micro-founded in an environment with information frictions in which idiosyncratic shocks are private information and

The representative agent would choose precisely the rate ι_t we are taking here as exogenous.

[4] $q_t^K k_t^i \bar{\chi}\tilde{\sigma}_t$ is sensitivity of an agent i's capital wealth to shocks $d\tilde{Z}_t^i$ before portfolio rebalancing and $q_t^K k_t^i \tilde{\sigma}_t^{\eta}$ is the shock sensitivity after rebalancing. The difference, $q_t^K k_t^i\left(\bar{\chi}\tilde{\sigma}_t - \tilde{\sigma}_t^{\eta}\right)$ measures trading cash flows per unit of $d\tilde{Z}_t^i$ and aggregating over all agents yields $X_t\tilde{\sigma}_t^{\eta}$.

agents have access to hidden trade and savings.[5] In such an environment, incentive compatibility requires that any insurance transfer to an agent must be precisely offset by a reduction in the present value of that agent's future consumption. Otherwise, the agent would have incentives to misreport the size of the shock and secretly trade capital. Incentive compatibility thus limits the amount of insurance that can be provided, i.e. the quantity X of derivatives.

We show below that the competitive equilibrium of this representative agent economy features prices $Q_t^K = q_t^K$ and $P_t^X = 1$ (and thus $P_t^X X_t = q_t^B K_t$), so that asset prices are the same as in the incomplete markets economy.[6] Also, as we have already stated in the main text, the representative agent's SDF process satisfies $\Xi_t = \xi_t^{**}$ (compare Proposition 6).

The valuation equation for derivatives from the perspective of the representative agent is

$$P_0^X X_0 = \mathbb{E}\left[\int_0^\infty \Xi_t \cdot \left(-\breve{\mu}_t^{\mathcal{B}} X_t\right) dt\right] + \mathbb{E}\left[\int_0^\infty \Xi_t \cdot (1-\vartheta_t)^2 \,\bar{\chi}^2 \tilde{\sigma}_t^2 X_t dt\right]. \tag{44}$$

Here, the first term represents the discounted present value of cash flows $-\breve{\mu}_t^{\mathcal{B}} X_t$ and the second term represents the discounted volatility reduction service flows that derivatives provide by lowering $\tilde{\sigma}^\eta$ in the utility function (18). As derivatives in the representative agent economy play the same role as bonds in the incomplete markets economy, we can make the identification $X_t = q_t^B K_t$ and $-\breve{\mu}_t^{\mathcal{B}} X_t = s_t K_t$. With these replacements (and $P_0^X = 1$), equation (44) becomes equation (13), the debt valuation equation from the dynamic trading perspective.

B.3.2 Additional Derivation Details and Proofs

Missing Step in Proof of Proposition 6: Ξ is Independent of Welfare Weights. For CRRA utility with parameter γ, we have

$$\Xi_t = e^{-\rho t} \frac{\int \lambda^i \eta_t^i u'(\eta_t^i C_t) di}{\int \lambda^i \eta_0^i u'(\eta_0^i C_0) di} = e^{-\rho t} \frac{\int \lambda^i (\eta_0^i)^{1-\gamma} (\eta_t^i/\eta_0^i)^{1-\gamma} di}{\int \lambda^i (\eta_0^i)^{1-\gamma} di} \frac{C_t^{-\gamma}}{C_0^{-\gamma}}.$$

[5] Details on this micro-foundation can be found in Brunnermeier et al. (2020). This information environment has also been employed by Di Tella (2020) in a closely related model.

[6] Also aggregate consumption C_t and the consumption shares η_t^i are as in the incomplete markets economy. The representative agent economy therefore leads to the same allocation.

Furthermore, η_t^i/η_0^i is given by

$$(\eta_t^i/\eta_0^i)^{1-\gamma} = \exp\left((1-\gamma)\int_0^t \tilde{\sigma}_\tau^\eta d\tilde{Z}_\tau^i - \frac{1-\gamma}{2}\int (\tilde{\sigma}_\tau^\eta)^2 d\tau\right)$$

and the distribution of this object conditional on aggregate information (i.e. information in Z) does not depend on i. In particular, there is a random variable X_t such that

$$X_t = \mathbb{E}[(\eta_t^i/\eta_0^i)^{1-\gamma} \mid Z_\tau : \tau \leq t]$$

for all i. Because Ξ_t is adapted to the filtration generated by the aggregate Brownian motion Z,

$$\begin{aligned}
\Xi_t = \mathbb{E}[\Xi_t \mid Z_\tau : \tau \leq t] &= e^{-\rho t}\frac{\mathbb{E}[\int \lambda^i(\eta_0^i)^{1-\gamma}(\eta_t^i/\eta_0^i)^{1-\gamma} di \mid Z_\tau : \tau \leq t]}{\int \lambda^i(\eta_0^i)^{1-\gamma} di}\frac{C_t^{-\gamma}}{C_0^{-\gamma}} \\
&= e^{-\rho t}\frac{\int \lambda^i(\eta_0^i)^{1-\gamma}\mathbb{E}[(\eta_t^i/\eta_0^i)^{1-\gamma} \mid Z_\tau : \tau \leq t] di}{\int \lambda^i(\eta_0^i)^{1-\gamma} di}\frac{C_t^{-\gamma}}{C_0^{-\gamma}} = e^{-\rho t}\underbrace{\frac{\int \lambda^i(\eta_0^i)^{1-\gamma} di}{\int \lambda^i(\eta_0^i)^{1-\gamma} di}}_{=1} X_t\frac{C_t^{-\gamma}}{C_0^{-\gamma}}.
\end{aligned}$$

Hence, Ξ_t does not depend on the choice of the weights λ^i.

Derivation of Utility Representation (18). By Ito's formula,

$$\log \eta_t^i = \log \eta_0^i - \frac{1}{2}\int_0^t \left(\tilde{\sigma}_s^\eta\right)^2 ds + \int_0^t \tilde{\sigma}_s^\eta d\tilde{Z}_s^i$$

and thus

$$\begin{aligned}
\int_0^\infty e^{-\rho t}\int \lambda^i \mathbb{E}\left[\log \eta_t^i\right] didt &= \int_0^\infty e^{-\rho t}\int \lambda^i \log \eta_0^i didt - \frac{1}{2}\int_0^\infty e^{-\rho t}\int \lambda^i \int_0^t \left(\tilde{\sigma}_s^\eta\right)^2 dsdidt \\
&= \frac{1}{\rho}\int \lambda^i \log \eta_0^i di - \frac{1}{2}\int \lambda^i di \int_0^\infty e^{-\rho t}\int_0^t \left(\tilde{\sigma}_s^\eta\right)^2 dsdt \\
&= \frac{1}{\rho}\int \lambda^i \log \eta_0^i di - \frac{1}{2\rho}\int_0^\infty e^{-\rho t}\left(\tilde{\sigma}_t^\eta\right)^2 dt,
\end{aligned}$$

where the last line uses that $\int \lambda^i di = 1$. Substituting this into equation (17) (with interchanged order of integration where necessary) implies

$$\mathcal{W}_0 = \frac{1}{\rho} \int \lambda^i \log \eta_0^i di + \mathbb{E}\left[\int_0^\infty e^{-\rho t} \left(\log C_t - \frac{1}{2\rho}\left(\tilde{\sigma}_t^\eta\right)^2\right) dt\right]$$

With the definition $w_0 := \frac{1}{\rho} \int \lambda^i \log \eta_0^i di$, this is precisely equation (18).

Competitive Equilibrium in Representative Agent Economy. As this is a representative agent economy, we can fully characterize the allocation by determining goods and asset supplies. The problem of the representative agent only needs to be considered to determine asset prices.

The assumed growth rate process for capital K_t is the same as in the equilibrium of the incomplete markets model, so that K_t must follow precisely the same process as in that equilibrium if we start from the same initial K_0 (which we can assume w.l.o.g. as this only scales the overall size of the economy). Because $dX_t/X_t = d(q_t^B K_t)/(q_t^B K_t)$ and $X_0 = q_0^B X_0$ by the condition on initial supply, we then also have $X_t = q_t^B K_t$ for all t. Total consumption goods produced by the two "trees" in period t are

$$\begin{aligned} C_t &= \left((1-\tau_t) a_t - \iota_t\right) K_t - \breve{\mu}_t^{\mathcal{B}} X_t \\ &= \left(a_t - \iota_t + \tau_t a_t - \breve{\mu}_t^{\mathcal{B}} q_t^B\right) K_t \\ &= \left(a_t - \mathfrak{g} - \iota_t\right) K_t, \end{aligned}$$

where the last line follows from the government budget constraint (2) (in the incomplete markets model). The aggregate consumption goods supply is thus the same as the (endogenous) aggregate consumption process in the incomplete markets economy.

We now turn to the remaining "good", volatility reduction. Total volatility "supply" is determined by equation (42),

$$\tilde{\sigma}_t^\eta = \frac{q_t^K K_t}{q_t^K K_t + X_t} \bar{\chi} \tilde{\sigma}_t = \frac{q_t^K K_t}{q_t^K K_t + q_t^B K_t} \bar{\chi} \tilde{\sigma}_t = \left(1 - \vartheta_t\right) \bar{\chi} \tilde{\sigma}_t.$$

This is also the same as the (endogenous) volatility of consumption shares η_t^i in the incomplete markets economy. The representative agent economy therefore generates the same allocation as the equilibrium in our incomplete markets model.

We now turn to asset prices. As this is the decision problem of a consumer with logarithmic utility, the optimal consumption rule is $C_t = \rho N_t$, exactly as for the agents in our incomplete markets economy.[7] This fact can be derived using the stochastic maximum principle in precisely the same way as in Appendix A.1, so that we skip the details here. Using the definition $N_t = Q_t^K K_t + P_t^X X_t$ and the supplies $X_t = q_t^B K_t$, $C_t = (q_t^B + q_t^K)K_t$ derived previously, we obtain

$$(q_t^B + q_t^K)K_t = \frac{C_t}{\rho} = Q_t^K K_t + P_t^X X_t = (Q_t^K + P_t^X q_t^B)K_t.$$

Therefore, if we can show $P_t^X = 1$, $Q_t^K = q_t^K$ is automatically implied. $P_t^X = 1$, in turn, follows from equation (44) and the remarks following it in the main text. Consequently, we only need to derive equation (44) to complete the equilibrium characterization.

Valuation Formula (44) for "Derivatives". We can use standard asset pricing logic. From the perspective of the representative agent, this is an entirely standard complete markets economy with two consumption goods. The price of a single unit of an asset measured in time-zero consumption units must thus equal the sum of the present discounted value of its future marginal consumption flow dividends and the present discounted future consumption value of its marginal volatility flow dividends, both discounted with the SDF Ξ_t, the marginal rate of substitution between consumption at time t and consumption at time 0.

The consumption flow term is straightforward. One unit of derivatives at time 0 turns into X_t/X_0 units of derivatives at time t and each of them produces a consumption flow $-\breve{\mu}_t^{\mathcal{B}} dt$. The present discounted value of these future consumption flows is therefore

$$\mathbb{E}\left[\int_0^\infty \Xi_t \left(-\breve{\mu}_t^{\mathcal{B}} \frac{X_t}{X_0}\right) dt\right].$$

For the volatility flow term, note that the "marginal volatility product of derivatives" at time t is

$$\frac{\partial \tilde{\sigma}_t^\eta}{\partial X_t} = -\frac{q_t^K K_t}{\left(q_t^K K_t + X_t\right)^2} \bar{\chi}\tilde{\sigma}_t = -\frac{\tilde{\sigma}_t^\eta}{\left(q_t^K + q_t^B\right) K_t} = -\frac{\tilde{\sigma}_t^\eta}{N_t}$$

[7] In the utility function here, there is also a second term ($\tilde{\sigma}_t^\eta$). But because it is additively separated, it does not affect the optimal consumption rule.

and the marginal rate of substitution between time-t consumption and time-t volatility is

$$\frac{\partial\left(\log C_t - \frac{1}{2\rho}\left(\tilde{\sigma}_t^{\eta}\right)^2\right)/\partial\tilde{\sigma}_t^{\eta}}{\partial\left(\log C_t - \frac{1}{2\rho}\left(\tilde{\sigma}_t^{\eta}\right)^2\right)/\partial C_t} = \frac{-\tilde{\sigma}_t^{\eta}/\rho}{1/C_t} = -\frac{C_t}{\rho}\tilde{\sigma}_t^{\eta}.$$

The consumption value of the marginal volatility reduction of X_t/X_0 derivatives at time t is therefore

$$\frac{\partial\left(\log C_t - \frac{1}{2\rho}\left(\tilde{\sigma}_t^{\eta}\right)^2\right)/\partial\tilde{\sigma}_t^{\eta}}{\partial\left(\log C_t - \frac{1}{2\rho}\left(\tilde{\sigma}_t^{\eta}\right)^2\right)/\partial C_t} \cdot \frac{\partial\tilde{\sigma}_t^{\eta}}{\partial X_t} \cdot \frac{X_t}{X_0} = \frac{C_t}{\rho N_t}\left(\tilde{\sigma}_t^{\eta}\right)^2 \frac{X_t}{X_0} = \left(\tilde{\sigma}_t^{\eta}\right)^2 \frac{X_t}{X_0},$$

here the last equation follows from $C_t = \rho N_t$. Consequently, the discounted value of volatility flows generates by one unit of derivatives is

$$\mathbb{E}\left[\int_0^\infty \Xi_t \left(\tilde{\sigma}_t^{\eta}\right)^2 \frac{X_t}{X_0} ds\right].$$

Combining the two present values and using $\tilde{\sigma}_t^{\eta} = (1-\vartheta_t)\bar{\chi}\tilde{\sigma}_t$ (derived previously) yields

$$P_0^X = \mathbb{E}\left[\int_0^\infty \Xi_t \left(-\breve{\mu}_t^{\mathcal{B}}\frac{X_t}{X_0}\right) dt\right] + \mathbb{E}\left[\int_0^\infty \Xi_t \left(1-\vartheta_t\right)^2 \bar{\chi}^2 \tilde{\sigma}_t^2 \frac{X_t}{X_0} ds\right].$$

After multiplying both sides by X_0, we obtain equation (44).

B.4 Model Solution with Stochastic Differential Utility

The model setup is identical to the one described in Section 2, except that logarithmic preferences are replaced with the utility recursion

$$V_t^i = \mathbb{E}_t\left[\int_t^\infty f(c_s^i, V_s^i) ds\right],$$

where the aggregator f is defined by

$$f(c,V) = (1-\gamma)\rho V\left(\log(c) - \frac{1}{1-\gamma}\log\left((1-\gamma)V\right)\right)$$

We can solve this augmented model as we have solved the baseline model in Section 2.2 (compare also Appendix A.1). The Hamiltonian of the household problem is precisely as stated in Appendix A.1, except that the very first term $e^{-\rho t} \log c_t^i$ must be replaced with $f(c^i, V_t(n^i))$.[8]

We use again a standard guess for the value function to eliminate the costate variable from the Hamiltonian. The guess here is $V_t(n^i) = v_t \frac{(n^i)^{1-\gamma}}{1-\gamma}$, where v_t is, again, a variable that does not depend on individual net worth. The relationship between the value function and the costate requires $\xi_t^i = V_t'(n_t^i) = v_t (n_t^i)^{-\gamma}$.[9] We write μ_t^v and σ_t^v for the (geometric) drift and aggregate volatility of v_t. Note that v_t does not load on the idiosyncratic Brownian because it merely depends on aggregate conditions.

The model solution procedure follows the same steps as for the baseline model. Here, we merely highlight the differences that occur on the way.

The first difference is that the first-order condition for optimal consumption is not immediately equation (23), but instead of the more complicated form

$$v_t (n_t^i)^{-\gamma} = \partial_c f(c_t, V_t) = (1-\gamma)\rho \frac{V_t}{c_t}.$$

However, once the value function $V_t = v_t \frac{(n_t^i)^{1-\gamma}}{1-\gamma}$ is plugged in, the condition reduces again to the familiar form of equation (23).

The second difference is in the characterization of the costate volatility loadings ς_t^i and $\tilde{\varsigma}_t^i$. Because the costate is now $\xi_t^i = v_t (n_t^i)^{-\gamma}$, Ito's lemma implies

$$\varsigma_t^i = \gamma \sigma_t^{n,i} - \sigma_t^v, \qquad \tilde{\varsigma}_t^i = \gamma \tilde{\sigma}_t^{n,i}. \tag{45}$$

The net worth volatilities $\sigma_t^{n,i}$ and $\tilde{\sigma}_t^{n,i}$ take the same form as before such that we simply need to replace the final equation (30) with the slightly more complicated form

$$\varsigma_t^i = \gamma \left(\sigma_t^{q,B} - \theta_t^{K,i} \frac{\sigma_t^{\vartheta}}{1-\vartheta_t} \right) - \sigma_t^v, \qquad \tilde{\varsigma}_t^i = \gamma \theta_t^{K,i} \bar{\chi} \tilde{\sigma}_t.$$

[8]On a small technical note, the resulting Hamiltonian here is a "current value Hamiltonian" whereas the one used in Appendix A.1 is a "present value Hamiltonian". The costate must thus be discounted differently here. Otherwise, this does not affect the solution procedure.

[9]It is here that the difference between "present value" and "current value" matters. For this reason, there is no time discounting term (such as $e^{-\rho t}$) in this equation, unlike in Appendix A.1.

The third difference is that the modified expressions for ς_t^i and $\tilde{\xi}_t^i$ affect the derivation and final result of equation (10). Following the same steps as in Appendix A.1, we obtain the slightly modified equation

$$\mathbb{E}_t\left[d\vartheta_t\right] = \left(\rho + \breve{\mu}_t^{\mathcal{B}} - \left(\sigma_t^v - (\gamma-1)\,\sigma_t^{\bar{q}}\right)\sigma_t^{\vartheta} - \gamma\left(1-\vartheta_t\right)^2\bar{\chi}^2\tilde{\sigma}_t^2\right)\vartheta_t dt,$$

where $\sigma_t^{\bar{q}}$ is the volatility of $\bar{q}_t := q_t^B + q_t^K$.

The fourth and final difference is that we now also have to characterize the process v_t as it affects the BSDE for ϑ_t through the term σ_t^v.[10] To characterize v_t, we start from the costate equation (a necessary optimality condition by the stochastic maximum principle), which is here given by

$$\begin{aligned}
\mathbb{E}_t[d\xi_t^i] &= -\left(\partial_V f(c_t^i, V_t^i)\xi_t^i + \frac{\partial H_t^i}{\partial n_t^i}\right)dt \\
&= -\left((1-\gamma)\rho\log(c_t^i/n_t^i) - \rho\log v_t - \rho + \mu_t^{n,i} + \frac{c_t^i}{n_t^i} - \varsigma_t^i\sigma_t^{n,i} - \tilde{\varsigma}_t^i\tilde{\sigma}_t^{n,i}\right)\xi_t^i dt \\
&= -\left((1-\gamma)\rho\log\rho - \rho\log v_t + \mu_t^{n,i} - \left(\gamma\sigma_t^{n,i} - \sigma_t^v\right)\sigma_t^{n,i} - \gamma\left(\tilde{\sigma}_t^{n,i}\right)^2\right)\xi_t^i dt,
\end{aligned} \tag{46}$$

where the last line uses $c_t^i/n_t^i = \rho$ and the price of risk formulas (45). We also know $\xi_t^i = v_t(n_t^i)^{-\gamma}$ and applying Ito's lemma to this equation yields for the drift term

$$\mathbb{E}_t[d\xi_t^i] = \left(\mu_t^v - \gamma\mu_t^{n,i} + \frac{\gamma(\gamma+1)}{2}\left(\left(\sigma_t^{n,i}\right)^2 + \left(\tilde{\sigma}_t^{n,i}\right)^2\right) - \gamma\sigma_t^v\sigma_t^{n,i}\right)\xi_t^i dt \tag{47}$$

Combining equations (46) and (47) and solving for μ_t^v yields

$$\begin{aligned}
\mu_t^v &= \gamma\mu_t^{n,i} - \frac{\gamma(\gamma+1)}{2}\left(\left(\sigma_t^{n,i}\right)^2 + \left(\tilde{\sigma}_t^{n,i}\right)^2\right) + \gamma\sigma_t^v\sigma_t^{n,i} \\
&\quad - \left((1-\gamma)\rho\log\rho - \rho\log v_t^i + \mu_t^{n,i} - \left(\gamma\sigma_t^{n,i} - \sigma_t^v\right)\sigma_t^{n,i} - \gamma\left(\tilde{\sigma}_t^{n,i}\right)^2\right) \\
&= \rho\log v_t + (\gamma-1)\left(\rho\log\rho + \mu_t^{n,i} - \frac{\gamma}{2}\left(\left(\sigma_t^{n,i}\right)^2 + \left(\tilde{\sigma}_t^{n,i}\right)^2\right) + \sigma_t^v\sigma_t^{n,i}\right)
\end{aligned}$$

[10]There is no need to solve for v_t in the baseline model because there it enters the value function additively and thus only impacts total utility but not optimal choices.

$$= \rho \log v_t + (\gamma - 1)\left(\rho \log \rho + \mu_t^{\bar{q}} + \Phi(\iota_t) - \delta - \frac{\gamma}{2}\left(\left(\sigma_t^{\bar{q}}\right)^2 + (1-\vartheta)^2 \bar{\chi}^2 \tilde{\sigma}_t^2\right) + \sigma_t^v \sigma_t^{\bar{q}}\right),$$

where in the last line we use that individual net worth has the same drift and aggregate volatility as aggregate net worth $\bar{q}_t K_t$, while its idiosyncratic volatility is $\tilde{\sigma}_t^{n,i}$, as determined previously. The previous equation for μ_t^v leads to a second BSDE

$$\mathbb{E}_t[dv_t] = \mu_t^v v_t dt$$

that has to be solved numerically jointly with the BSDE for ϑ_t stated previously.

Numerical Model Solution. We solve the model numerically using a finite difference method. This is a standard approach employed in the literature to solve models of this type. Here, we only briefly outline the procedure. A more comprehensive description of the method can be found, e.g., in Brunnermeier et al. (2020), Chapter 3 (specifically Sections 3.2.6 and 3.2.7).

For our numerical solution, we impose the functional relationships $\vartheta_t = \vartheta(t, \tilde{\sigma}_t)$, $v_t = v(t, \tilde{\sigma}_t)$ and use the known forward equation for the state variable $\tilde{\sigma}_t$ to transform the two BSDEs into partial differential equations in time t and the state $\tilde{\sigma}_t$. We choose suitable terminal guesses for the functions ϑ and v[11] at a finite terminal time T and solve the two PDEs backward in time using a finite difference method. We choose T sufficiently large such that an increase in T no longer changes the solutions at $t = 0$, $\vartheta(0, \cdot)$ and $v(0, \cdot)$, noticeably. These solution functions $\vartheta(0, \cdot)$ and $v(0, \cdot)$ represent our numerical approximation to the stationary (Markov) equilibrium functions $\tilde{\sigma} \mapsto \vartheta(\tilde{\sigma}), v(\tilde{\sigma})$.[12]

B.5 Model Extension with Privately Issued Safe Assets

In this appendix, we present the formal details for the model extension with privately issued safe assets. We restrict attention to the baseline model from Section 2 with logarithmic preferences.

[11] Specifically, we use the functions implied by the steady state equilibrium with $\tilde{\sigma}_t = \tilde{\sigma}^0$ forever.

[12] Note that our results in Appendix B.1 imply that this solution procedure always selects the unique nondegenerate stationary solution the BSDE for ϑ.

Setup and Model Solution. Each agent i issues nominally risk-free bonds ("i-bonds") of total real value $B_t(i) \geq 0$ and holds a real quantity $b_t^i(j) \geq 0$ of j-bonds issued by other agents $j \neq i$. The clearing conditions at all times t and for all varieties j are

$$B_t(j) = \int b_t^i(j) di.$$

We denote by i_t^p the nominal interest a household has to pay in equilibrium on its privately issued debt[13] and by $B_t^p := \int B_t(j)dj$ the aggregate quantity of privately issued bonds outstanding. Because privately issued debt is nominally risk-free, its return is

$$dr_t^b = \left(i_t^p - i_t\right) dt + dr_t^{\mathcal{B}},$$

where, as before, $dr_t^{\mathcal{B}}$ is the return on government bonds (compare equation (6)). By no arbitrage, in equilibrium $i_t^p = i_t$. Thus, the yields on privately issued bonds and government bonds are identical.

We can solve household i's problem as in the baseline model. Denote by $\theta_t^{B,i} := -B_t(i)/n_t^i \leq 0$ the negative of bond issuance as a share of net worth and by $\theta_t^{b,i}(j) := b_t^i(j)/n_t^i \geq 0$ holdings of j-bonds as a fraction of net worth. Relative to the baseline model, the household has the additional choice variables $\theta_t^{B,i}$ and $(\theta_t^{b,i}(j))_{j\in[0,1]}$ subject to the nonnegativity constraints. However, the Hamiltonian of the household's problem does not change relative to Appendix A.1: due to $dr_t^b = dr_t^{\mathcal{B}}$, choices of $\theta_t^{B,i}$ and $(\theta_t^{b,i}(j))_{j\in[0,1]}$ do not affect either the expected return or the risk characteristics of the household's portfolio, such that the additional terms in the Hamiltonian cancel out.

We can draw two immediate conclusions from the previous observation. First, because the Hamiltonian remains unaffected, the model solution steps outlined in Appendix A.1 remain valid in this extended model. Consequently, all equilibria with private bond issuance must feature the same real allocation and the same prices of government bond (q_t^B) and capital (q_t^K) as in the baseline model. Second, all households are indifferent between any choice of private bond issuance and holdings of bonds issued by other agents as long as these holdings do not interfere with the optimal plans for capital holdings ($\theta_t^{K,i}$), outside equity issuance ($\theta_t^{E,i}$), and diversified equity holdings ($\theta_t^{\bar{E},i}$).

[13] Theoretically, i_t^p could depend on the issuing household j. However, as all privately issued bonds are required to be nominally risk-free, it is obvious that they all have to pay the same nominal rate in equilibrium.

There are thus many different equilibria that all feature the same consumption allocation and valuation of government bonds, equity, and capital, but differ with regard to the quantities $B_t(j)$ of private bonds in circulation.

A Simple Example. To illustrate how privately issued bonds can serve as safe assets in precisely the same way as government bonds, we consider an example in which all agents trade private and government bonds in equal proportions.[14] Specifically, we make the following choices: (a) the aggregate real value of privately issued bonds is proportional to the value of government bonds, $B_t^p \propto q_t^B K_t$, (b) the total bonds issued by each agent j is proportional to the agent's net worth share, $B_t(j) = \eta_t^j B_t^p$, and (c) all agents hold a portfolio of j-bonds for $j \neq i$ and government bonds according to market capitalization weights.

We now discuss the debt valuation equations verbally referenced in the main text. We defer a derivation of the following equations to the end of this appendix.

For each agent i, the value of the long position $b_t^i(j)$ in j-bonds must equal the present value of future cash inflows from the portfolio of j-bonds, either due to payments made by agent j or due to trading of j-bonds. This insight leads to an equation in full analogy to equations (14) and (16) for government bonds that we have derived in the context of the dynamic trading perspective:

$$b_0^i(j) = \mathbb{E}\left[\int_0^\infty \xi_t^i x_t b_t^i(j) dt\right] + \mathbb{E}\left[\int_0^\infty \xi_t^i \left(1 - \vartheta_t\right)^2 \bar{\chi}^2 \tilde{\sigma}_t^2 b_t^i(j) dt\right]. \tag{48}$$

Here, x_t denotes the expected net payouts made by agent j to all holders of j-bonds per real unit of j-bonds outstanding. Total expected net payouts $x_t B_t(j)$ made by agent j are the private debt counterparts of primary surpluses $s_t K_t$, which represent the net payouts made by the government to public debt holders.

Equation (48) emphasizes that the valuation of j-bonds for agent i depends on a cash flow component resulting from payouts made by agent j and a service flow component resulting from the fact that i trades j-bonds with agents other than j. When aggregating these equations for all $i \neq j$, we obtain a debt valuation equation from the dynamic

[14] While valuation equations for individual bond types depend on what we assume about trading of individual bonds (which is indeterminate due to indifference), none of the economic conclusions from the example crucially depend on this choice.

trading perspective for the aggregate long position in j-bonds:[15]

$$B_0(j) = \mathbb{E}\left[\int_0^\infty \xi_t^{**} x_t B_t(j) dt\right] + \mathbb{E}\left[\int_0^\infty \xi_t^{**} \left(1 - \vartheta_t\right)^2 \bar{\chi}^2 \tilde{\sigma}_t^2 B_t(j) dt\right]. \tag{49}$$

The key takeaway is that this equation looks precisely like equation (13) for government bonds. In particular, the service flow component is identical.

Equation (49) emphasizes the similarity between government bonds and privately issued bonds for their holders. However, private bond issuance also comes with a short position in the bond for the issuer j. In the same spirit as before, we can value that short position by determining the present value of all net payouts that j makes to holders of j-bonds,

$$-B_0(j) = \mathbb{E}\left[\int_0^\infty \xi_t^j \left(-x_t\right) B_t(j) dt\right] + \mathbb{E}\left[\int_0^\infty \xi_t^j \left(-\left(1 - \vartheta_t\right)^2 \bar{\chi}^2 \tilde{\sigma}_t^2\right) B_t(j) dt\right]. \tag{50}$$

This equation illustrates that issuing bonds according to the specified issuance strategy effectively exposes the agent to negative service flows. Because $B_t(j) = \eta_t^j B_t^p$ is proportional to η_t^j, cash flows from debt issuance and repayments are systematically correlated with marginal utility in a way that increases the riskiness of j's portfolio.

Once we integrate equations (49) and (50) over all bond types j, the integrated service flow terms on the right-hand side become identical in absolute value but have opposite sign. In other words, in the aggregate the positive service flows derived from privately issued bonds by their holders exactly cancel with the negative service flows generated for their issuers. Private safe asset creation does not generate additional net service flows for the economy.

Derivation of Equations (48), (49), and (50). In precisely the same way as in Appendix A.3, we can derive equations in analogy to equation (14) for the portfolios of j-bonds held by agents i and j:

$$b_0^i(j) = -\mathbb{E}_0\left[\int_0^\infty \xi_t^i b_t^i(j) \left(\mu_t^{\Delta,i}(j) - \varsigma_t \sigma_t^{\Delta,i}(j) - \tilde{\varsigma}_t \tilde{\sigma}_t^{\Delta,i,i}(j)\right)\right], \tag{51}$$

[15]Relative to equation (48), the following equation also interchanges integrals and uses $b_t^i(j) = \eta_t^i B_t(j)$.

$$-B_0^j = -\mathbb{E}_0\left[\int_0^\infty \xi_t^i(-B_t^j)\left(\mu_t^{\Delta,j}(j) - \varsigma_t\sigma_t^{\Delta,j}(j) - \tilde{\varsigma}_t\tilde{\sigma}_t^{\Delta,j}(j)\right)\right]. \tag{52}$$

Here, $d\Delta^{b,i}(j)_t$ and $d\Delta_t^{B,j}$ are the trading processes for j-bonds of agents i and j, respectively:

$$\begin{aligned} d\Delta_t^{b,i}(j) &= \mu_t^{\Delta,i}(j)dt + \sigma_t^{\Delta,i}(j)dZ_t + \tilde{\sigma}_t^{\Delta,i,i}(j)d\tilde{Z}_t^i + \tilde{\sigma}_t^{\Delta,i,j}(j)d\tilde{Z}_t^j, \\ d\Delta_t^{B,j} &= \mu_t^{\Delta,j}(j)dt + \sigma_t^{\Delta,j}(j)dZ_t + \tilde{\sigma}_t^{\Delta,j}(j)d\tilde{Z}_t^i. \end{aligned}$$

As in Section 3 and Appendix A.3, $b_t^i(j)d\Delta^{b,i}(j)_t$ represents the real value of new j-bonds purchased by agent i at time t (net of payouts made by agent j on existing bonds). Similarly, but with opposite sign due to the short position, $-B_t^j d\Delta_t^{B,j}$ represents the real value of new j-bonds (re-)purchased by agent j. In other words, $-d\Delta_t^{B,j}$ corresponds to the payouts that the issuer j makes to bond holders.

To derive equations (48) and (50), we have to characterize the trading processes. In full analogy to Appendix A.3, these processes must satisfy

$$d\Delta_t^{b,i}(j) = \frac{db_t^i(j)}{b_t^i(j)} - dr_t^b, \tag{53}$$

$$d\Delta_t^{B,j} = \frac{dB_t^j}{B_t^j} - dr_t^b. \tag{54}$$

We first characterize the second process. By definition, $\mu_t^{\Delta,j}(j) = -x_t$ corresponds to the negative of the expected net payouts made by agent j to holders of j-bonds per real unit of bonds outstanding. To determine the volatility loadings of the trading process, we use $B_t^j = \eta_t^j B_t^p \propto \eta_t^j q_t^B K_t$, so that

$$\frac{dB_t^j}{B_t^j} = \frac{d\eta_t^j}{\eta_t^j} + \frac{d(q_t^B K_t)}{q_t^B K_t}.$$

The volatility loadings of $dr_t^b = dr_t^{\mathcal{B}}$ coincide with the ones of $d(q_t^B K_t)/(q_t^B K_t)$, compare equation (6). Thus,

$$d\Delta_t^{B,j} = \text{drift terms} + \tilde{\sigma}_t^{\eta} d\tilde{Z}_t^j.$$

In total, we get

$$\mu_t^{\Delta,j}(j) = -x_t, \qquad \sigma_t^{\Delta,j}(j) = 0, \qquad \tilde{\sigma}_t^{\Delta,j}(j) = \tilde{\sigma}_t^{\eta}.$$

Substituting this into equation (52) and using $\tilde{\zeta}_t = \tilde{\sigma}_t^{\eta} = \bar{\chi}(1-\vartheta_t)\tilde{\sigma}_t$ implies equation (50).

The previous discussion also implies (using equation (54))

$$dr_t^b = \frac{dB_t(j)}{B_t(j)} - d\Delta_t^{B,j} = x_t dt + \frac{d\left(q_t^B K_t\right)}{q_t^B K_t}$$

and substituting this into equation (53) and using $b_t^i(j) = \eta_t^i B_t^j = \eta_t^i \eta_t^j B_t^p$ implies

$$\begin{aligned}
d\Delta_t^{b,i}(j) &= \frac{d\eta_t^i}{\eta_t^i} + \frac{d\eta_t^j}{\eta_t^j} + \frac{dB_t^p}{B_t^p} - \left(x_t dt + \frac{d\left(q_t^B K_t\right)}{q_t^B K_t} \right) \\
&= \tilde{\sigma}_t^{\eta} d\tilde{Z}_t^i + \tilde{\sigma}_t^{\eta} d\tilde{Z}_t^j + \frac{d\left(q_t^B K_t\right)}{q_t^B K_t} - x_t dt - \frac{d\left(q_t^B K_t\right)}{q_t^B K_t} \\
&= -x_t dt + \tilde{\sigma}_t^{\eta} d\tilde{Z}_t^i + \tilde{\sigma}_t^{\eta} d\tilde{Z}_t^j.
\end{aligned}$$

In other words,

$$\mu_t^{\Delta,i}(j) = -x_t, \qquad \sigma_t^{\Delta,i}(j) = 0, \qquad \tilde{\sigma}_t^{\Delta,i,i}(j) = \tilde{\sigma}_t^{\Delta,i,j}(j) = \tilde{\sigma}_t^{\eta}.$$

Substituting these equations into equation (51) implies equation (48).

It is left to derive equation (49). This equation easily follows from the previously derived equation (48) by integrating over all holders i:

$$\begin{aligned}
B_0(j) &= \int b_t^i(j) di \\
&= \int \left(\mathbb{E}\left[\int_0^{\infty} \xi_t^i x_t b_t^i(j) dt \right] + \mathbb{E}\left[\int_0^{\infty} \xi_t^i \left(1-\vartheta_t\right)^2 \bar{\chi}^2 \tilde{\sigma}_t^2 b_t^i(j) dt \right] \right) di \\
&= \mathbb{E}\left[\int_0^{\infty} \int \xi_t^i x_t \eta_t^i B_t(j) di dt \right] + \mathbb{E}\left[\int_0^{\infty} \int \xi_t^i \left(1-\vartheta_t\right)^2 \bar{\chi}^2 \tilde{\sigma}_t^2 \eta_t^i B_t(j) di dt \right] \\
&= \mathbb{E}\left[\int_0^{\infty} \int \xi_t^i \eta_t^i di \cdot x_t B_t(j) dt \right] + \mathbb{E}\left[\int_0^{\infty} \int \xi_t^i \eta_t^i di \cdot \left(1-\vartheta_t\right)^2 \bar{\chi}^2 \tilde{\sigma}_t^2 B_t(j) dt \right]
\end{aligned}$$

$$= \mathbb{E}\left[\int_0^\infty \xi_t^{**} x_t B_t(j)dt\right] + \mathbb{E}\left[\int_0^\infty \xi_t^{**}\left(1-\vartheta_t\right)^2 \bar{\chi}^2 \tilde{\sigma}_t^2 B_t(j)dt\right].$$

B.6 Model Extension with Convenience Yields

.In this appendix, we present the model extension with bonds in the utility function to generate a convenience yield and derive the two debt valuation equations stated in Section 6

Setup and Equilibrium Characterization. To keep equations as simple as possible, we only consider the case of logarithmic consumption preferences and introduce separable logarithmic bond utility as in Di Tella (2020). Each agent i maximizes

$$\mathbb{E}\left[\int_0^\infty e^{-\rho t}\left((1-v)\log c_t^i + v\log b_t^i\right)dt\right],$$

where

$$b_t^i = (1-\theta_t^{K,i} - \theta_t^{E,i} - \theta_t^{\bar{E},i})n_t^i$$

are real government bond holdings of the agent as in Section 3. v measures the utility share derived from bond holdings. For $v = 0$, the model collapses to the baseline model. As in the main text, but unlike in Appendix B.5, we assume here that the gross holdings or privately issued nominal debt are zero, so that all bonds are government bonds. So long as privately issued bonds do not provide utility, this assumption is without loss of generality.

However, as in Appendix B.5, we use the notation i_t^p to denote the (shadow) nominal short rate on such privately issued bonds. As these bonds do not enter utility, the spread $\Delta i_t := i_t^p - i_t$ can be positive in this augmented model. It captures the convenience yield on government bonds.

The augmented model has almost the same equilibrium solution as our baseline model. ι, q^B, and q^K are given by the equations

$$\iota_t = \frac{(1-\vartheta_t)\left(a_t - \mathfrak{g}\right) - (1-v)\rho}{1-\vartheta_t + \phi\left(1-v\right)\rho}, \tag{55}$$

$$q_t^B = \vartheta_t \frac{1+\phi\left(a_t - \mathfrak{g}\right)}{1-\vartheta_t + \phi\left(1-v\right)\rho}, \tag{56}$$

$$q_t^K = (1-\vartheta_t)\,\frac{1+\phi\,(a_t-\mathfrak{g})}{1-\vartheta_t+\phi\,(1-v)\,\rho} \tag{57}$$

as a function of the bond wealth share ϑ_t. The latter is determined by the dynamic equation

$$\mathbb{E}_t\left[d\vartheta_t\right] = \left(\rho+\breve{\mu}_t^{\mathcal{B}}-\Delta i_t-(1-\vartheta_t)^2\,\bar{\chi}^2\tilde{\sigma}_t^2\right)\vartheta_t dt, \tag{58}$$

where $\Delta i_t = v\rho/\vartheta_t$ is the equilibrium convenience yield on government bonds. This equation differs from equation (10) only by the presence of the convenience yield term Δi_t, which raises the equilibrium level of ϑ_t.

We present a proof of equations (55)–(57) and (58) at the end of this appendix.

Debt Valuation Equations (Proposition 7). We next sketch the derivations of the two debt valuation equations stated in the main text. The derivation steps are in complete analogy to the ones presented in Section 3 for the baseline model.

The valuation from the buy and hold perspective starts again from the government flow budget constraint (2) and follows precisely the same steps as in Section 3 up to the derivation of equation (31) stated in the main text and restated here for convenience:

$$\xi_0^i\frac{\mathcal{B}_0}{\mathcal{P}_0} = \mathbb{E}\left[\int_0^T \xi_t^i s_t K_t dt\right] - \mathbb{E}\left[\int_0^T \mathcal{B}_t\left(d\left(\xi_t^i/\mathcal{P}_t\right)+i_t\xi_t^i/\mathcal{P}_t dt\right)\right] + \mathbb{E}\left[\xi_T^i\frac{\mathcal{B}_T}{\mathcal{P}_T}\right].$$

From here on, the derivation departs slightly. Because the nominal SDF $\xi_t^i/\mathcal{P}_t$ in this model does *not* price nominal government debt but nominal private debt, it decays on average at rate $i_t^p = i_t + \Delta i_t$, and the second term does not vanish. Instead, we obtain

$$-\mathbb{E}\left[\int_0^T \mathcal{B}_t\left(d\left(\xi_t^i/\mathcal{P}_t\right)+i_t\xi_t^i/\mathcal{P}_t dt\right)\right] = \mathbb{E}\left[\int_0^T \xi_t^i\Delta i_t\frac{\mathcal{B}_t}{\mathcal{P}_t}dt\right],$$

which is the present value of convenience yield service flows derived from government debt between $t=0$ and $t=T$. From here on, the derivation is again analogous to the one in Section 3. Once we replace ξ_t^i with $\bar{\xi}_t$ and take the limit $T\to\infty$, we arrive at the equation stated in Section 6.

The valuation from the dynamic trading perspective proceeds precisely as in Section 3. The only difference is that the derivation no longer results in the intermediate equation (14) but in the slightly modified version

$$b_0^i = -\mathbb{E}\left[\int_0^\infty \xi_t^i b_t^i \left(-\Delta i_t + \mu_t^{\Delta,i} - \varsigma_t \sigma_t^{\Delta,i} - \tilde{\varsigma}_t^i \tilde{\sigma}_t^{\Delta,i}\right) dt\right],$$

where the term $-\Delta i_t$ is new. After replacing equation (14) with this variant and otherwise following the steps outlined in Section 3, we obtain the valuation equation from the dynamic trading perspective stated in Section 6.

To understand where the additional term $-\Delta i_t$ comes from, note that also the derivation steps for equation (14) in Appendix A.3 remain unchanged except for one detail: in that appendix, we have used in equation (33) that

$$\mu_t^{r^{\mathcal{B}}} - r_t^f - \varsigma_t \sigma_t^{r^{\mathcal{B}}} = 0$$

by standard asset pricing logic. That argument is valid if the SDF ξ_t^i prices the government bond, so that the expected return $\mu_t^{r^{\mathcal{B}}}$ equals the risk-adjusted required return $r_t^f + \varsigma_t \sigma_t^{r^{\mathcal{B}}}$. Due to the presence of utility services from government bonds, this is not true anymore in the augmented model. The expected return on a privately issued bond $\mu_t^{r^{\mathcal{B}}} + \Delta i_t$ still equals the required return, but the expected return on the government bond is lower by Δi_t. Consequently, we must use the modified relationship

$$\mu_t^{r^{\mathcal{B}}} - r_t^f - \varsigma_t \sigma_t^{r^{\mathcal{B}}} = -\Delta i_t$$

in equation (33). This explains the additional term $-\Delta i_t$ above.

Model Solution Details. The model solution follows the same steps as in Appendix A.1. The difference here is that the term $\log c_t^i$ in the Hamiltonian must be replaced with

$$(1-v)\log c_t^i + v\log\left(1 - \theta_t^{K,i} - \theta_t^{E,i} - \theta_t^{\bar{E},i}\right) + v\log n_t^i.$$

We only discuss how this affects the solution without repeating all steps from Appendix A.1 explicitly.

With the same conjecture for the value function (and thus for ξ_t^i) as in Appendix A.1, the first-order condition for optimal consumption becomes

$$c_t^i = (1-v)\,\rho n_t^i$$

while the first-order condition for the optimal investment choice remains unaffected. Following the aggregation steps in Appendix A.1, we obtain again equations (7), (8), and (9) for ι_t, q_t^B, and q_t^K from the maintext with the difference that ρ in these equations, which represents the consumption-wealth ratio, must be replaced with $(1-v)\,\rho$. With this replacement, these equations take the form equations (55), (56), and (57).

The first-order conditions for the portfolio shares $\theta_t^{K,i}$, $\theta_t^{E,i}$, and $\theta_t^{\bar{E},i}$ are the same as in Appendix A.1 except that there is an additional term[16]

$$\frac{\rho v}{1-\theta_t^{K,i}-\theta_t^{E,i}-\theta_t^{\bar{E},i}} = \rho v \frac{n_t^i}{b_t^i} = \Delta i_t$$

on the right-hand side of all three conditions that is due to the marginal utility of bond holdings:

$$\begin{aligned}
\frac{\mathbb{E}_t\left[dr_t^{K,i}(\iota_t^i)\right]}{dt} - \frac{\mathbb{E}_t\left[dr_t^{\mathcal{B}}\right]}{dt} &= -\varsigma_t^i \frac{\sigma_t^{\vartheta}}{1-\vartheta_t} + \tilde{\varsigma}_t^i \tilde{\sigma}_t - \lambda_t^i\,(1-\bar{\chi}) + \Delta i_t, \\
\frac{\mathbb{E}_t\left[dr_t^{E,i}\right]}{dt} - \frac{\mathbb{E}_t\left[dr_t^{\mathcal{B}}\right]}{dt} &= -\varsigma_t^i \frac{\sigma_t^{\vartheta}}{1-\vartheta_t} + \tilde{\varsigma}_t^i \tilde{\sigma}_t - \lambda_t^i + \Delta i_t, \\
\frac{\mathbb{E}_t\left[d\bar{r}_t^{E}\right]}{dt} - \frac{\mathbb{E}_t\left[dr_t^{\mathcal{B}}\right]}{dt} &= -\varsigma_t^i \frac{\sigma_t^{\vartheta}}{1-\vartheta_t} + \Delta i_t.
\end{aligned}$$

From here, we can follow the same steps as in Appendix A.1, which yield again equation (28), but a modified version of equation (29):

$$\frac{a_t-\mathfrak{g}-\iota_t}{q_t^K} - \frac{\mu_t^{\vartheta}-\breve{\mu}_t^{\mathcal{B}}}{1-\vartheta_t} - \frac{\left(\sigma_t^{q,B}-\sigma_t^{\vartheta}\right)\sigma_t^{\vartheta}}{1-\vartheta_t} = -\varsigma_t^i \frac{\sigma_t^{\vartheta}}{1-\vartheta_t} + \tilde{\varsigma}_t^i \bar{\chi} \tilde{\sigma}_t + \Delta i_t.$$

Replacing equation (29) with the previous one but following otherwise the steps in Appendix A.1 yields for μ_t^{ϑ}

$$\mu_t^{\vartheta} = (1-v)\rho + \breve{\mu}_t^{\mathcal{B}} - (1-\vartheta_t)^2\bar{\chi}^2\tilde{\sigma}_t^2 - (1-\vartheta_t)\Delta i_t.$$

[16]The last equality follows from the fact that a hypothetical zero net supply nominal bond not entering the utility function but with otherwise identical risk profile would only have this term in the first-order condition for its excess return.

To bring this into the form (58), note that $\Delta i_t = \rho v \frac{n_t^i}{b_t^i} = \frac{\rho v}{\vartheta_t}$ in equilibrium and hence

$$(1-\vartheta_t)\Delta i_t + v\rho = \frac{\rho v}{\vartheta_t} - \rho v + v\rho = \frac{v\rho}{\vartheta_t} = \Delta i_t.$$

The previous equation therefore simplifies to

$$\mu_t^{\vartheta} = \rho + \breve{\mu}_t^{\mathcal{B}} - (1-\vartheta)^2 \bar{\chi}^2 \tilde{\sigma}_t^2 - \Delta i_t.$$

Multiplying both sides by ϑ_t yields equation (58).

B.7 A Model with Two Types

In this appendix we present a model variant with two types of agents that have heterogeneous access to the different asset markets in our economy and therefore heterogeneous idiosyncratic and possibly aggregate risk exposures. We derive theoretical results that link the predictions of the two-type model to the predictions of the one-type model presented in the main text.

Setup. The model is the same as the baseline model in the main text, except for the following modification. At each time, each agent i is either an expert ("e") or a household ("h"). Experts can manage capital directly and therefore face precisely the same portfolio (and real investment) choice as all agents in our baseline model. Households, in turn, are restricted to only hold financial assets (equity and bonds). All agents have identical preferences regardless of type. We allow for both logarithmic preferences as in Section 2 ($\gamma = 1$) and more general stochastic differential utility preferences with risk aversion $\gamma > 0$ as considered in Section 4.

Agents receive idiosyncratic (Poisson) type switching shocks. Experts become households with arrival rate $\lambda^e > 0$ and households become experts with arrival rate $\lambda^h > 0$.[17]

Let e_t^i be an indicator that is 1 if agent i is an expert at time t and zero otherwise. In

[17]Without type switching experts would eventually dominate the economy because they earn higher expected returns on average. In the stationary distribution, the model would then reduce to the one-type model studied in the main text.

what follows, we define the shares

$$\eta_t^e := \int \eta_t^i e_t^i di, \qquad \eta_t^h := \int \eta_t^i (1 - e_t^i) di$$

of total wealth that is owned by experts (η_t^e) and households (η_t^h), respectively. One of these variables is a sufficient summary of the cross-sectional wealth distribution for the purposes of solving for the aggregate dynamics and asset prices in this model (the other variable can be backed out form $\eta_t^e + \eta_t^h = 1$). When solving the model, we therefore include η_t^e as an additional state variable.

Sketch of the Model Solution. The model can be solved along the same lines as our baseline model. We briefly sketch the solution procedure here and provide more details on the steps that are new relative to the baseline model.

First, everything that is said in Section 2.2 before Lemma 1 as well as that lemma itself remains valid in the two-type model without any modification. As a consequence, the dynamics of asset prices, aggregate consumption, and aggregate investment are fully determined by the dynamics of the endogenous process ϑ_t and the exogenous process a_t.

Second, the optimal portfolio choice conditions (25), (26), and (27) remain unchanged for those agents i that are experts at time t. For households, instead, the first two conditions do not apply, as households do not hold capital and issue outside equity. Nevertheless, equation (27) remains valid also for households. Therefore, for experts the exact same steps as in Appendix A.1 lead once again to equation (29) stated there. This equation depends on the agent index i only through the prices of risk ς_t^i and $\tilde{\varsigma}_t^i$. We argue next that these prices of risk are actually not i-dependent. Specifically, because equation (27) holds for all agents regardless of type, $\varsigma_t^i = \varsigma_t$ is the same for all i.[18] Furthermore, using $\varsigma_t^i = \varsigma_t$ and $\lambda_t^i = \tilde{\varsigma}_t^i \tilde{\sigma}_t$ (compare Appendix A.1) in equation (25) for any agent i that is an expert implies that $\tilde{\varsigma}_t^i$ is identical for all experts. We call this common value $\tilde{\varsigma}_t^e$ from now on. Consequently, the combined portfolio choice condition

[18] This conclusion rests on the implicit assumption $\sigma_t^\vartheta \neq 0$. However, it is easy to verify ex post that $\sigma_t^\vartheta = 0$ if and only if there is no consumption-relevant aggregate risk. But in this case, trivially $\varsigma_t^i = 0 =: \varsigma_t$ for all agents.

equation (29) can be written here as

$$\frac{a_t - \mathfrak{g}_t - \iota_t}{q_t^K} - \frac{\mu_t^{\vartheta} - \breve{\mu}_t^{\mathcal{B}}}{1 - \vartheta_t} - \frac{\left(\sigma_t^{q,B} - \sigma_t^{\vartheta}\right) \sigma_t^{\vartheta}}{1 - \vartheta_t} = -\varsigma_t \frac{\sigma_t^{\vartheta}}{1 - \vartheta_t} + \tilde{\varsigma}_t^e \bar{\chi} \tilde{\sigma}_t. \tag{59}$$

Third, the factor v_t in the costate ξ_t^i for agent i now becomes type-specific, v_t^e if $e_t^i = 1$ and v_t^h if $e_t^i = 0$. Hence, equations (45) for the prices of risk remain valid. However, in the first equation, σ_t^v has to be interpreted as $\sigma_t^{v,e}$ for experts and and as $\sigma_t^{v,h}$ for households. In this model, it makes sense to determine the aggregate net worth risk loadings $\sigma_t^{n,i}$ slightly differently to before. Specifically, $\eta_t^i = n_t^i / N_t$ implies that (by Ito's lemma) $\sigma_t^{n,i} = \sigma_t^N + \sigma_t^{\eta,i}$. Using $N_t = q_t^B / \vartheta_t K_t$, we have furthermore $\sigma_t^N = \sigma_t^{q,B} - \sigma_t^{\vartheta}$. We can therefore write for the price of aggregate risk

$$\varsigma_t = \gamma \left(\sigma_t^{q,B} - \sigma_t^{\vartheta}\right) + \gamma \sigma_t^{\eta,e} - \sigma_t^{v,e} = \gamma \left(\sigma_t^{q,B} - \sigma_t^{\vartheta}\right) + \gamma \sigma_t^{\eta,h} - \sigma_t^{v,h}.$$

The price of idiosyncratic risk is type-specific and given by

$$\tilde{\varsigma}_t^e = \gamma \frac{1 - \vartheta_t}{\eta_t^e} \bar{\chi} \tilde{\sigma}_t, \qquad \tilde{\varsigma}_t^h = 0.$$

Fourth, while the steps in Appendix A.1 that lead to Proposition 1 remain exactly the same, the ultimate dynamic equation for ϑ_t is different from equation (10) because the prices of risk are different. Plugging the prices of risk for experts into the combined portfolio choice equation (59) and otherwise following the same steps as in Appendix A.1 yields the equation

$$\mathbb{E}_t \left[d\vartheta_t\right] = \left(\rho + \breve{\mu}_t^{\mathcal{B}} - \left(\eta_t^e \sigma_t^{v,e} + \eta_t^h \sigma_t^{v,h} - (\gamma - 1)\, \sigma_t^{\bar{q}}\right) \sigma_t^{\vartheta} - \gamma \frac{(1 - \vartheta_t)^2 \bar{\chi}^2 \tilde{\sigma}_t^2}{\eta_t^e}\right) \vartheta_t dt, \tag{60}$$

where, as in Appendix B.4, $\sigma_t^{\bar{q}}$ denotes the volatility of $\bar{q}_t := q_t^B + q_t^K$.

Fifth, an additional law of motion for the endogenous state variable η_t^e needs to be determined. This is relatively straightforward for the volatility $\sigma_t^{\eta,e}$. Using the two expressions for ς_t, from experts' and households' perspective, and the fact that $\eta_t^e \sigma_t^{\eta,e} +$

$\eta_t^h \sigma_t^{\eta,h} = 0$ (by construction), we obtain

$$\sigma_t^{\eta,e} = \frac{(1-\eta_t^e)(\sigma_t^{v,e} - \sigma_t^{v,h})}{\gamma}.$$

In particular, in the special case of log utility, $\sigma_t^{v,e} = \sigma_t^{v,h} = 0$ and, hence, also $\sigma_t^{\eta,e} = 0$, so that the wealth share η_t^e evolves locally deterministically.

The drift of η_t^e can be computed using some straightforward but tedious algebra that is omitted here in the interest of space.[19] The final result is

$$\mu_t^{\eta,e} = \left(-\sigma_t^{v,e} + \gamma\sigma_t^{\eta,e} + (\gamma-1)\,\sigma_t^{\bar{q}}\right)\sigma_t^{\eta,e} + \frac{1-\eta_t^e}{\eta_t^e}\gamma\frac{\bar{\chi}^2\,(1-\vartheta_t)^2}{\eta_t^e}\tilde{\sigma}_t^2 + \frac{\lambda^h\left(1-\eta_t^e\right) - \lambda^e\eta_t^e}{\eta_t^e}. \tag{61}$$

Sixth and finally, there are now two BSDEs for the value function factors $\mathbb{E}_t[dv_t^e]$ and $\mathbb{E}_t[dv_t^h]$. These can be derived in precisely the same way as in Appendix B.4, except that we have to account for type switching. The two counterparts of the costate equation, equation (46), are entirely analogous:

$$\mathbb{E}_t[d\xi_t^{e,i}] = -\left((1-\gamma)\rho\log\rho - \rho\log v_t^e + \mu_t^{n,e,i} - \left(\gamma\sigma_t^{n,e,i} - \sigma_t^{v,e}\right)\sigma_t^{n,e,i} - \gamma\left(\tilde{\sigma}_t^{n,e,i}\right)^2\right)\xi_t^{e,i}dt,$$
$$\mathbb{E}_t[d\xi_t^{h,i}] = -\left((1-\gamma)\rho\log\rho - \rho\log v_t^h + \mu_t^{n,h,i} - \left(\gamma\sigma_t^{n,h,i} - \sigma_t^{v,h}\right)\sigma_t^{n,h,i}\right)\xi_t^{h,i}dt.$$

The counterparts of equation (47) change slightly because, when applying Ito's lemma to $\xi_t^e = v_t^e(n_t^i)^{-\gamma}$ and $\xi_t^h = v_t^h(n_t^i)^{-\gamma}$, additional jump terms appear:

$$\mathbb{E}_t[d\xi_t^{e,i}] = \left(\mu_t^{v,e} - \gamma\mu_t^{n,e,i} + \frac{\gamma(\gamma+1)}{2}\left(\left(\sigma_t^{n,e,i}\right)^2 + \left(\tilde{\sigma}_t^{n,e,i}\right)^2\right) - \gamma\sigma_t^{v,e}\sigma_t^{n,e,i} + \lambda^e\frac{\xi_t^{h,i} - \xi_t^{e,i}}{\xi_t^{e,i}}\right)\xi_t^{e,i}dt,$$
$$\mathbb{E}_t[d\xi_t^{h,i}] = \left(\mu_t^{v,h} - \gamma\mu_t^{n,h,i} + \frac{\gamma(\gamma+1)}{2}\left(\sigma_t^{n,h,i}\right)^2 - \gamma\sigma_t^{v,h}\sigma_t^{n,h,i} + \lambda^h\frac{\xi_t^{e,i} - \xi_t^{h,i}}{\xi_t^{h,i}}\right)\xi_t^{h,i}dt.$$

Combining the two sets of equations and solving for $\mu_t^{v,e}$ and $\mu_t^{v,h}$, respectively, yields

$$\mu_t^{v,e} = \rho\log v_t^e + (\gamma-1)\left(\rho\log\rho + \mu_t^{\bar{q}} + \Phi(\iota_t) - \delta + \mu_t^{\eta,e} + \frac{\lambda^e\eta_t^e - \lambda^h\eta_t^h}{\eta_t^e}\right)$$

[19] Essentially, one applies Ito's lemma to $\eta_t^e = N_t^e/N_t$, where $N_t^e := \int n_t^i e_t^i di$ and its evolution as an Ito process can be determined from the evolution of n_t^i for all i that are experts.

$$+ (\gamma - 1)\left(-\frac{\gamma}{2}\left(\left(\sigma_t^{\bar{q}} + \sigma_t^{\eta,e}\right)^2 + \frac{(1-\vartheta)^2\,\bar{\chi}^2}{(\eta_t^e)^2}\tilde{\sigma}_t^2\right) + \sigma_t^{v,e}\left(\sigma_t^{\bar{q}} + \sigma_t^{\eta,e}\right)\right) + \lambda^e \frac{v_t^h - v_t^e}{v_t^e},$$

$$\mu_t^{v,h} = \rho \log v_t^h + (\gamma - 1)\left(\rho \log \rho + \mu_t^{\bar{q}} + \Phi(\iota_t) - \delta + \mu_t^{\eta,h} + \frac{\lambda^h \eta_t^h - \lambda^e \eta_t^e}{\eta_t^h}\right)$$

$$+ (\gamma - 1)\left(-\frac{\gamma}{2}\left(\sigma_t^{\bar{q}} + \sigma_t^{\eta,h}\right)^2 + \sigma_t^{v,h}\left(\sigma_t^{\bar{q}} + \sigma_t^{\eta,h}\right)\right) + \lambda^h \frac{v_t^e - v_t^h}{v_t^h}.$$

Relationship with Baseline Model. We next provide two theoretical results that highlight relationships between the dynamics of the two-type model and the dynamics of our baseline model with just one type. These results emphasize that the statistic of the cross-sectional distribution of idiosyncratic risk exposures that matters most for our model's predictions is $\int \eta_t^i (\tilde{\sigma}_t^{n,i})^2 di$, i.e. the wealth-weighted cross-sectional mean of the idiosyncratic net worth variance. We conjecture that a similar conclusion would also hold in more general n-type models. To improve the reading flow, we first summarize here the results and present the proofs at the end of this appendix.

In what follows, we always make the following assumptions and use the following notation:

Let K_0 be an initial condition for the capital stock and a_t, $\breve{\mu}_t^{\mathcal{B}}$, $\tilde{\sigma}_t^1$, $\tilde{\sigma}_t^2$ be exogenous processes, such that both $(a_t, \breve{\mu}_t^{\mathcal{B}}, \tilde{\sigma}_t^1)$ and $(a_t, \breve{\mu}_t^{\mathcal{B}}, \tilde{\sigma}_t^2)$ are functions of some finite-dimensional Markov process. Suppose that stationary monetary equilibria exist both for the one-type model with exogenous processes $(a_t, \breve{\mu}_t^{\mathcal{B}}, \tilde{\sigma}_t^1)$ and for the two-type model with exogenous processes $(a_t, \breve{\mu}_t^{\mathcal{B}}, \tilde{\sigma}_t^2)$ based on the same parameters for ρ, $\mathfrak{g}$, and ϕ (but not necessarily for other model parameters).[20] *For any model variable x, denote by x_t^j the stochastic process for x in the equilibrium for the j-type model ($j \in \{1, 2\}$).*

Before establishing the main results, we remark that most interesting predictions of the two models only depend on the stochastic processes for the exogenous variables a_t and $\breve{\mu}_t^{\mathcal{B}}$ and the endogenous variable ϑ_t in equilibrium.

Lemma 5. *If $\vartheta_t^1 = \vartheta_t^2$ for all t (almost surely), then also the following equations hold for all t*

[20] By Proposition 1 and the results in Appendix B.1, these equilibria are then also unique. These results hold analogously also for the two-type model.

(almost surely):[21]

$$\begin{aligned} K_t^1 &= K_t^2, & q_t^{B,1} &= q_t^{B,2}, & q_t^{K,1} &= q_t^{K,2}, & \iota_t^1 &= \iota_t^2, \\ C_t^1 &= C_t^2, & \tau_t^1 &= \tau_t^2, & \mathbb{E}_t[dr_t^{\mathcal{B},1}] &= \mathbb{E}_t[dr_t^{\mathcal{B},2}], & \mathbb{E}_t[dr_t^{K,1}] &= \mathbb{E}_t[dr_t^{K,2}] \end{aligned}$$

The previous lemma establishes that if two equilibria feature the same endogenous process ϑ_t, then they make exactly the same predictions for a large range of variables. We next provide sufficient conditions for $\vartheta_t^1 = \vartheta_t^2$.

We start with the special case of log utility ($\gamma = 1$) as then several equations simplify. First, note that for log utility $v_t^e = v_t^h = 1$, so that the decision-relevant portion of agents' value functions is independent of the agent type. As a consequence, we also obtain $\sigma_t^{\eta,e} = 0$, agents find it optimal to fully share aggregate risk. Equation (60) then simplifies to

$$\begin{aligned} \mathbb{E}_t\left[d\vartheta_t\right] &= \left(\rho + \breve{\mu}_t^{\mathcal{B}} - \frac{(1-\vartheta_t)^2 \bar{\chi}^2 \tilde{\sigma}_t^2}{\eta_t^e}\right)\vartheta_t dt \\ &= \left(\rho + \breve{\mu}_t^{\mathcal{B}} - \left(\eta_t^e \left(\tilde{\sigma}_t^{n,e}\right)^2 + \eta_t^h \left(\tilde{\sigma}_t^{n,h}\right)^2\right)\right)\vartheta_t dt, \end{aligned} \tag{62}$$

where it has been used that $\tilde{\sigma}_t^{n,e} = \frac{1-\vartheta_t}{\eta_t}\bar{\chi}\tilde{\sigma}_t$ and $\tilde{\sigma}_t^{n,h} = 0$. Similarly, the corresponding equation in the baseline model with just one type is[22]

$$\mathbb{E}_t\left[d\vartheta_t\right] = \left(\rho + \breve{\mu}_t^{\mathcal{B}} - \left(\tilde{\sigma}_t^{n}\right)^2\right)\vartheta_t dt. \tag{63}$$

Note that, beyond the log utility case, equations (62) (for the two-type model) and (63) (for the one-type model) continue to hold more generally if we shut down aggregate shocks (by setting the dZ_t-loading of the exogenous processes to zero).

In either case, log utility or no aggregate shocks, equations (62) and (63) are identical if (and only if) the stochastic process for $\eta_t^e \left(\tilde{\sigma}_t^{n,e}\right)^2 + \eta_t^h \left(\tilde{\sigma}_t^{n,h}\right)^2$ in the two-type model is the same as the stochastic process for $\left(\tilde{\sigma}_t^{n}\right)^2$ in the one-type model. If this is the case,

[21] For the last equality, note that in both models, $dr_t^{K,i}$ does not depend on i except for the identity of the idiosyncratic shock $d\tilde{Z}_t^i$ which plays no role for the expectation (and it only makes sense if i is an expert in the two-type model). Therefore, the equation is written without i-superscripts

[22] Compare equation (10) in Proposition 1. Alternatively, simply set $\eta_t^e = 1$, $\eta_t^h = 0$ in the previous equation – the two-type model effectively collapses to the baseline model if η_t^e is held fixed at 1.

then the two models imply the same dynamics for ϑ_t and, hence, the same dynamics for aggregates and asset prices. This reasoning leads to the following proposition.

Proposition 9. *Let either $\gamma = 1$ (log utility) or assume that there are no aggregate shocks. Suppose that the following condition hold (for all t almost surely)*

$$\left(\tilde{\sigma}_t^{n,1}\right)^2 = \eta_t^e \left(\tilde{\sigma}_t^{n,e,2}\right)^2 + \eta_t^h \left(\tilde{\sigma}_t^{n,h,2}\right)^2. \tag{64}$$

Then the equilibrium dynamics for all macro aggregates and the expectations and aggregate volatility loadings of all asset returns are identical in both equilibria.

We now turn to the general case that there are aggregate shocks and, potentially, $\gamma \neq 1$. In this case, similar derivations as before show that equation (62) for the two-type model takes the form

$$\mathbb{E}_t\left[d\vartheta_t\right] = \left(\rho + \breve{\mu}_t^{\mathcal{B}} - \left(\eta_t^e \sigma_t^{v,e} + \eta_t^h \sigma_t^{v,h} - (\gamma - 1)\,\sigma_t^{\bar{q}}\right)\sigma_t^{\vartheta} - \gamma\left(\eta_t^e \left(\tilde{\sigma}_t^{n,e}\right)^2 + \eta_t^h \left(\tilde{\sigma}_t^{n,h}\right)^2\right)\right)\vartheta_t dt \tag{65}$$

and equation (63) for the one-type model takesthe form

$$\mathbb{E}_t\left[d\vartheta_t\right] = \left(\rho + \breve{\mu}_t^{\mathcal{B}} - \left(\sigma_t^{v} - (\gamma - 1)\,\sigma_t^{\bar{q}}\right)\sigma_t^{\vartheta} - \gamma\left(\tilde{\sigma}_t^{n}\right)^2\right)\vartheta_t dt. \tag{66}$$

In this general case, condition (64) is no longer sufficient to make the two equations identical. In addition, we would need the extra condition $\sigma_t^v = \eta_t^e \sigma_t^{v,e} + \eta_t^h \sigma_t^{v,h}$, which is unlikely to be satisfied in general as an inspection of the BSDEs for v_t in the one-type model and for v_t^e, v_t^h in the two-type model reveals. This is because, for $\gamma \neq 1$ and aggregate shocks, hedging demands induce different types to take on heterogeneous aggregate risk exposures, which generates additional dynamics that are absent from the more stylized one-type model. However, these additional dynamics disappear in the limit case that type switching is infinitely fast as then the value functions of households and experts align. Then, condition (64) is again sufficient for the two models to generate identical predictions:

Proposition 10. *The conclusion of Proposition 9 remains valid even for $\gamma \neq 1$ and with aggregate shocks, if the equilibrium in the two-type model is understood to be the limit as $\lambda^e, \lambda^h \to \infty$ with the ratio $\lambda^h/\lambda^e \in (0, \infty)$ held constant.*

Proofs.

Proof of Lemma 5. Lemma 1 holds for both models. We observe immediately from that lemma that if ϑ_t and a_t are the same in two equilibria, then so are q_t^B, q_t^K, and ι_t, provided the equilibria correspond to models with identical parameters ρ, $\mathfrak{g}$, and ϕ, as we have assumed here. If ι_t is the same across the two equilibria, then so is dK_t/K_t by the law of motion of aggregate capital (compare Definition 1 and note that an identical equation also holds in the two-type model). This completes the proof of the first four equations. We now discuss the remaining four equations.

For equality of aggregate consumption C_t, note that $a_t^1 = a_t^2$, $K_t^1 = K_t^2$, and $\iota_t^1 = \iota_t^2$ imply together with goods market clearing (equation (3))

$$C_t^1 = (a_t^1 - \mathfrak{g} - \iota_t^1)K_t^1 = (a_t^2 - \mathfrak{g} - \iota_t^2)K_t^2 = C_t^2.$$

For equality of taxes, we use similarly the government budget constraint and $a_t^1 = a_t^2$, $\breve{\mu}_t^{\mathcal{B},1} = \breve{\mu}_t^{\mathcal{B},2}$, and $q_t^{B,1} = q_t^{B,2}$:

$$\tau_t^1 = \frac{\mathfrak{g} - \breve{\mu}_t^{\mathcal{B},1} q_t^{B,1}}{a_t^1} = \frac{\mathfrak{g} - \breve{\mu}_t^{\mathcal{B},2} q_t^{B,2}}{a_t^2} = \tau_t^2.$$

For equality of the expected return on bonds consider the equation for the return $dr_t^{\mathcal{B}}$ stated in the first part of Appendix A.1. This equations holds for both the one-type and the two-type model. Observe that the drift of $dr_t^{\mathcal{B}}$ only depends on ι_t, $\mu_t^{q,B}$, and $\breve{\mu}_t^{\mathcal{B}}$, which are identical for $j = 1$ and $j = 2$. Hence, $\mathbb{E}_t[dr_t^{\mathcal{B},1}] = \mathbb{E}_t[dr_t^{\mathcal{B},2}]$.

An analogous argument holds for the last equality. Also the final expression for $dr_t^{K,i}(\iota_t^i)$ in Appendix A.1 holds for both models. This expression depends on a_t, ι_t, q_t^B, q_t^K, $\mu_t^{q,K}$, all of which have been shown to be identical in both equilibria. $\square$

Proof of Proposition 9. Comparing equations (62) and (63), it is apparent that if $\vartheta_t^1 = \vartheta_t^2$ and condition (64) holds, then the right-hand sides of both equations are identical state by state. Uniqueness of the solution (compare Appendix B.1) then implies that, indeed, $\vartheta_t^1 = \vartheta_t^2$ is the only possibility.

From Lemma 5 we can then immediately conclude that K_t, q_t^B, q_t^K, ι_t, C_t, τ_t, $\mathbb{E}_t[dr_t^{\mathcal{B}}]$, and $\mathbb{E}_t[dr_t^K]$ must be identical across the two models. All macro aggregates can be written as functions of the first six variables (and possibly the exogenous processes a_t

and $\breve{\mu}_t^{\mathcal{B}}$, which are the same for both models), so that, indeed, the dynamics of all macro aggregates must be identical in both equilibria.

In addition, if q_t^B and q_t^K are identical, then so are $\sigma_t^{q,B}$ and $\sigma_t^{q,K}$ by Ito's lemma. Hence, the aggregate volatility loadings on all returns $dr_t^{\mathcal{B}}$, dr_t^K, dr_t^E, $d\bar{r}_t^E$ must be identical.

Because Lemma 5 already implies that the expected returns on capital and bonds are identical across the two equilibria, it is only left to show that also $\mathbb{E}_t[dr_t^E] = \mathbb{E}_t[d\bar{r}_t^E]$ is the same in both equilibria. Because of equation (27), which holds in both models, and $\vartheta_t^1 = \vartheta_t^2 \Rightarrow \sigma_t^{\vartheta,1} = \sigma_t^{\vartheta,2}$, the desired equality holds if and only if $\varsigma_t^1 = \varsigma_t^2$. This is trivially satisfied if there are no aggregate shocks so that we can from now on assume that $\gamma = 1$. Then, $\varsigma_t^1 = \varsigma_t^2$ follows from the following considerations:

- In the one-type model, the aggregate price of risk is given by (compare Appendix A.1)

$$\varsigma_t^1 = \sigma_t^n = \sigma_t^{\bar{q}}.$$

- In the two-type model, the aggregate price of risk is given by

$$\varsigma_t^2 = \sigma_t^{\bar{q}} + \sigma_t^{\eta,e} = \sigma_t^{\bar{q}},$$

  because $\sigma_t^{\eta,e} = 0$ in the log utility case.

□

Proof of Proposition 10. We first establish some properties of the limit economy in the two-type model. Let $\eta^* := \frac{\lambda^h/\lambda^e}{1+\lambda^h/\lambda^e}$. By the assumptions that $\lambda^h/\lambda^e \in (0,\infty)$ is held constant, we know that $\eta^* \in (0,1)$ is constant along any limit sequence. Also $\frac{\lambda^e}{1-\eta^*} = \frac{\lambda^h}{\eta^*}$ by definition of η^*.

Consider now the last term in the drift of η_t, equation (61):

$$\begin{aligned}
\frac{\lambda^h\left(1-\eta_t^e\right)-\lambda^e\eta_t^e}{\eta_t^e} &= \frac{1}{\eta_t^e}\left(\frac{\lambda^h}{\eta^*}\eta^*\left(1-\eta_t^e\right)-\frac{\lambda^e}{1-\eta^*}\left(1-\eta^*\right)\eta_t^e\right) \\
&= \frac{\lambda^h}{\eta^*}\frac{1}{\eta_t^e}\left(\eta^*-\eta_t^e\right).
\end{aligned}$$

This term is positive for $\eta_t^e < \eta^*$ and negative for $\eta_t^e > \eta^*$ and as $\lambda^h \to \infty$, it becomes arbitrarily large in absolute value. In contrast, all other terms in equation (61) remain bounded for η_t in a (sufficiently small) neighborhood of η^*. Hence, the last term dominates in the limit and ensures that $\eta_t = \eta^*$ at all times.

Next, consider the equations for $\mu_t^{v,e}$ and $\mu_t^{v,h}$ for the two-type model stated above. We only discuss the equation for $\mu_t^{v,e}$ but note that everything said here applies symmetrically to $\mu_t^{v,h}$. The type switching intensities λ^e and λ^h appear in two places. First, in the first line, there is a term

$$\frac{\lambda^e \eta_t^e - \lambda^h \eta_t^h}{\eta_t^e} = \frac{\lambda^h}{\eta^*} \frac{\eta_t^e - \eta^*}{\eta_t^e},$$

which is, up to the sign, exactly the same term as the last term in the drift of η_t. Because the drift is finite (in fact, zero) in the limit equilibrium, this term must also vanish in the limit $\lambda^h \to \infty$. Second, the last term in the expression for $\mu_t^{v,e}$ also depends on switching intensities,

$$\lambda^e \frac{v_t^h - v_t^e}{v_t^e},$$

and, in the limit $\lambda^e \to \infty$, this term becomes arbitrarily large unless $v_t^e = v_t^h$. Because the term is positive if $v_t^h > v_t^e$, negative if $v_t^h < v_t^e$, and this equation describes a backward equation for v_t^e, it must indeed be the case that $v_t^e = v_t^h$ in the limit.

Furthermore, once we impose $v_t^e = v_t^h =: v_t$ and $\eta_t = \eta^*$, use that either of these two equations implies $\sigma_t^{\eta,e} = 0$, and plugs these equations into the equation for either $\mu_t^{v,h}$ or $\mu_t^{v,e}$ (in the limit as $\lambda^e, \lambda^h \to \infty$), we obtain an equation that is identical to the equation for μ_t^v in the one-type model stated at the end of Appendix B.4.

We use the previous considerations to conclude that if condition (64) is satisfied, then $v_t^1 = v_t^2$ and $\vartheta_t^1 = \vartheta_t^2$ (where, v_t^2 is the common value for $v_t^h = v_t^e$ in the two-type model in the limit economy). First, if the condition is satisfied and these two equations hold, then equations (65) and (66) have identical right-hand sides state by state, so indeed the ϑ-solutions must satisfy $\vartheta_t^1 = \vartheta_t^2$. Similarly, as just observed, then $\mu_t^{v,1}$ and $\mu_t^{v,2}$ must be identical state by state, such that the value function solutions must satisfy $v_t^1 = v_t^2$. While this logical appears somewhat circular, the previous observations are indeed sufficient to establish that under condition (64), there is a solution such that

$v_t^1 = v_t^2$ and $\vartheta_t^1 = \vartheta_t^2$.[23] Uniqueness of the non-degenerate stationary solution then also implies that this is the only possibility.

Having established that $\vartheta_t^1 = \vartheta_t^2$, arguments in full analogy to the proofs of Lemma 5 and Proposition 9 show that the conclusion of Proposition 9 remains valid.

□

C Calibration and Robustness

C.1 Calibration Details

C.1.1 Data Sources and Definitions

The data series for the CIV factor (Herskovic et al., 2016) have been retrieved from Bernard Herskovic's website (https://bernardherskovic.com/data/). That series (column "CIV") represents an annualized return variance measure of the common idiosyncratic volatility in stock returns.

All other data used in this paper have been retrieved from the FRED database maintained by the Federal Reserve Bank of St. Louis (https://fred.stlouisfed.org/). We briefly describe next how we map model quantities into FRED data series.

For the macro aggregates Y, C, I, and G, we use quarterly data from 1970Q1 to 2019Q4. Output is defined as $Y = C + I + G$ (in particular, exclusive of net exports) while we define the three series C, I, and G as follows:

- In line with the business cycle literature, we exclude consumption of durable goods from our consumption measure. To compute C, we start from real personal consumption expenditures (FRED code PCECC96) and subtract real expenditures for durable goods. We identify the latter by multiplying total real consumption expenditures by the ratio of nominal expenditures for durable goods (PCDG) and nominal total consumption expenditures (PCEC).

[23] Simply take the solution for $j = 1$ as given and conjecture that processes defined by $\vartheta_t^2 := \vartheta_t^1, v_t^2 := v_t^1$ represent a valid solution to the equations in the two-type model. The previous logic verifies that this is indeed the case.

- We define investment I as the sum of two components: (1) real gross private domestic investment (GPDIC1) net of the change in private inventories (CBIC1) and (2) real consumption expenditures for durable goods (measured as described previously). We include durables in investment as we have removed them from consumption but they nevertheless represent an important part of overall private expenditures.[24]

- We government spending G as real government consumption expenditures and gross investment (GCEC1).

The ratios of primary surpluses and government debt to GDP, S/Y and $q^B K/Y$, respectively, are measured from nominal data. We use again quarterly data series from 1970Q1 to 2019Q4. We define the nominal primary surplus as current receipts (FGRECPT) minus current expenditures (FGEXPND) but add back current interest expenditures (A091RC1Q027SBEA) of the federal government. We define nominal debt as the market value of marketable treasury debt (MVMTD027MNFRBDAL). We compute the ratios S/Y and $q^B K/Y$ by dividing both nominal primary surpluses and nominal debt by nominal GDP (GDP).[25]

Data on the capital stock to compute the capital-output ratio is based on the Penn World Tables (Feenstra et al., 2015) and only available annually. We again choose the time period from 1970 to 2019. The capital-output ratio $q^K K/Y$ is defined as capital stock at constant national prices (RKNANPUSA666NRUG) divided by real GDP at constant national prices (RGDPNAUSA666NRUG), both for the US.

For returns on bonds and equity, we use monthly data from February 1971 to December 2019.[26] We first construct monthly log returns from these data sources as follows:[27]

- We measure the return on government debt using data on the market yield on treasury securities at 5-year constant maturity (DGS5). We chose the 5-year ma-

[24]Excluding durables altogether from our measures of economic activity does not substantially change our computed data moments: it lowers the volatility of output somewhat but otherwise only marginally affects results.

[25]Unlike for our time series of macro aggregates, we do not correct the GDP measure for components not in the model. Doing so would have only a minor impact on the resulting numbers. Not doing so is also consistent with how we compute the capital-output ratio below.

[26]February 1971 is the first month at which all of the required series are available on FRED.

[27]To be precise, the following definitions are for nominal returns while the returns in the model are real. However, for the purpose of computing return differentials, as we do, this distinction is irrelevant.

turity as this approximately reflects the average duration of federal debt. We convert the yield data into (holding period) returns using the well-known formula

$$r_{t+1}^{T} = Ty_t^T - (T-1)y_{t+1}^{T-1}$$

that relates the log holding period return r_{t+1}^{T} over the period from t to $t+1$ of a bond with time to maturity of T at date t to the log yield y_t^T of a T-period bond at t and the log y_{t+1}^{T-1} of a $T-1$-period bond at $t+1$. To operationalize this formula, we approximate the unknown 59-month yield y_{t+1}^{T-1} with the observed 60-month yield y_{t+1}^{T}. This procedure generates a series $\hat{r}_t^{\mathcal{B}}$ of monthly log returns for government bonds.

- As a proxy for the total equity market, we take the Wilshire 5000 index. We compute monthly log returns by dividing successive end-of-month values of the total market index (Wilshire 5000 Total Market Index, FRED series WILL5000IND), which includes dividend reinvestments, and then taking natural logarithms. As market returns are based on leveraged equity returns, this procedure yields a series $\hat{r}_t^{E,\text{leverage}}$ of leveraged monthly log returns for equity.

Based on these data series, we construct the sample estimates for $\mathbb{E}\left[d\bar{r}^E - dr^{\mathcal{B}}\right]$ and $\sigma\left(d\bar{r}^E - dr^{\mathcal{B}}\right)$ reported in Table 2 as follows. We first define for leveraged returns:

$$\begin{aligned}
\mathbb{E}\left[d\bar{r}^{E,\text{leverage}}\right] &= 12 \cdot \text{sample mean}\left(\hat{r}^{E,\text{leverage}}\right) + \frac{12}{2} \cdot \text{sample var}\left(\hat{r}^{E,\text{leverage}}\right), \\
\mathbb{E}\left[dr^{\mathcal{B}}\right] &= 12 \cdot \text{sample mean}\left(\hat{r}^{\mathcal{B}}\right) + \frac{12}{2} \cdot \text{sample var}\left(\hat{r}^{\mathcal{B}}\right), \\
\sigma^2\left(d\bar{r}^{E,\text{leverage}} - dr^{\mathcal{B}}\right) &= 12 \cdot \text{sample var}\left(\hat{r}^{E,\text{leverage}} - \hat{r}^{\mathcal{B}}\right).
\end{aligned}$$

However, the model counterpart $d\bar{r}^E$ of the market equity return is closer to a delevered equity return. The theoretical relationship between the delevered equity return $d\bar{r}^E$ and the leveraged return $d\bar{r}^{E,\text{leverage}}$ is

$$d\bar{r}^E = dr^{\mathcal{B}} + \frac{1}{\ell}\left(d\bar{r}^{E,\text{leverage}} - dr^{\mathcal{B}}\right),$$

where $\ell \geq 1$ is financial leverage as measured by the ratio of total assets to equity. We therefore define:

$$\mathbb{E}\left[d\bar{r}^{E} - dr^{\mathcal{B}}\right] = \frac{1}{\ell}\left(\mathbb{E}\left[d\bar{r}^{E,\text{leverage}}\right] - \mathbb{E}\left[dr^{\mathcal{B}}\right]\right),$$
$$\sigma\left(d\bar{r}^{E} - dr^{\mathcal{B}}\right) = \frac{1}{\ell}\sigma\left(d\bar{r}^{E,\text{leverage}} - dr^{\mathcal{B}}\right).$$

We use $\ell = 1.5$ to compute delevered equity returns.

For the real risk-free rate we also use monthly data from February 1971 to December 2019. We approximate the nominal risk-free rate by the (annualized) 3-month Treasury Bill secondary market rate (DTB3). We convert nominal rates to real rates using realized inflation based on the consumer price index for all urban consumers (CPIAUCSL_PC1).[28] We compute $\mathbb{E}[r^f]$ and $\sigma(r^f)$ based on sample means and variance of the logged risk-free rate series in the same way as for other financial returns (but without the factor 12 given that the returns are already annualized).

C.1.2 Calibration of the Exogenous $\tilde{\sigma}_t$ Process

We estimate the coefficients $\tilde{\sigma}^0$, ψ, and σ of the idiosyncratic risk process (19) such that it matches the observed CIV series. Here, we first describe the details of the estimation procedure and then explain why CIV is a suitable data counterpart for idiosyncratic risk $\tilde{\sigma}_t^2$ in the model.

Parameters Estimation. We use a maximum likelihood estimation (MLE) to determine $\tilde{\sigma}^0$, ψ, and σ based on a monthly CIV sample from January 1946 to December 2019. MLE is straightforward here because the conditional density of the CIR process $\tilde{\sigma}_t^2$ has a known closed-form expression (e.g. Aït-Sahalia (1999), equation (20)).

While not directly targeted by MLE, the estimated process generates first and second ergodic moments of $\tilde{\sigma}_t$, 0.5078 and 0.1701, respectively, that closely match their empirical counterparts (based on square roots of the CIV sample), 0.4950 and 0.1817, respectively.

CIV as a Model-consistent Measure of $\tilde{\sigma}_t^2$. We briefly outline why CIV indeed measures $\tilde{\sigma}_t^2$. Herskovic et al. (2016) construct CIV as the cross-sectional mean of the id-

[28] Specifically, we first compute realized nominal yearly (gross) returns by chaining 3-month yields and then divide by the realized (gross) inflation over the year. Alternative methods of converting nominal to real rates result in very similar real rate moments.

iosyncratic return variance of individual stocks in their sample. The idiosyncratic return variance of an individual stock, in turn, is defined as the variance of the residual of a factor regression on the market factor.

In our model, this procedure broadly amounts to a (population) regression of the type

$$dr_t^{E,i} - r_t^f dt = \alpha_t^i + \beta_t^i \left(d\bar{r}_t^E - r_t^f dt \right) + \varepsilon_t^i$$

for stocks issued by all agents i. Comparing the return expressions for $dr_t^{E,i}$ and $d\bar{r}_t^E$ stated in Section 2.2, it is clear that this regression yields $\alpha_t^i = 0$, $\beta_t^i = 1$ and $\varepsilon_t^i = \tilde{\sigma}_t d\tilde{Z}_t^i$. The variance of each individual residual ε_t^i therefore exactly equals $\tilde{\sigma}_t^2$, and so does the cross-sectional mean over all residual variances. In other words, if the real-world data was generated by the model, measured CIV at time t would exactly correspond to $\tilde{\sigma}_t^2$.

C.1.3 Calibration of Remaining Model Parameters

The calibration choices for χ and δ are explained in the main text. The remaining nine parameters, γ, ρ, a^0, $\mathfrak{g}$, $\check{\mu}^{\mathcal{B},0}$, α^a, $\alpha^{\mathcal{B}}$, ϕ, ι^0, are chosen to match twelve moments as described in the main text. We briefly explain here (heuristically) how these moments identify the model parameters.

First, given the estimated $\tilde{\sigma}_t$ process, the capital productivity process

$$a_t = a(\tilde{\sigma}_t) = a^0 - \alpha^a(\tilde{\sigma}_t - \tilde{\sigma}^0)$$

is exogenous and fully determined by the two parameters a^0 and α^a. While output $Y_t = a_t K_t$ still contains an endogenous term K_t, the capital stock is slow-moving such that most of the variation in HP-filtered output is due to variation in a_t. Therefore, the parameter α^a is effectively determined by the target moment $\sigma(Y)$.

Second, because $\mathfrak{g}$ is constant, the variability of output left for private uses, $Y - G$, is also determined by the parameter α^a. By the aggregate resource constraint $Y - G = C + I$, so that the choice of α^a also constrains the variation of the sum of consumption and investment. The parameter ϕ effectively controls how much of that variation is absorbed by the individual components of that sum. While in principle the full details of the model matter for the dynamics of investment opportunities, ϕ controls to which extent changes in investment opportunities change actual physical investment as opposed to simply driving up or down capital valuations. For $\phi \to 0$, investment reacts a lot while for $\phi \to \infty$, investment is fixed and only prices react. Therefore, the two relative volatilities $\sigma(C)/\sigma(Y)$ and $\sigma(I)/\sigma(Y)$ effectively determine ϕ.[29]

Third, the ratio of primary surpluses to output is given by

$$S_t/Y_t = -\breve{\mu}_t^{\mathcal{B}} \frac{q_t^B}{a_t} = -\left(\breve{\mu}^{\mathcal{B},0} + \alpha^{\mathcal{B}}(\tilde{\sigma}_t - \tilde{\sigma}^0)\right) \frac{q_t^B}{a_t}.$$

While the dynamics of this variable depend on the endogenous price q_t^B, the parameter $\alpha^{\mathcal{B}}$ is nevertheless able to control the overall volatility of S_t/Y_t.[30] The parameter $\alpha^{\mathcal{B}}$ is therefore determined by the moment $\sigma(S/Y)$.

Fourth, the six average ratio targets in the calibration effectively determine the five parameters ρ, a^0, $\mathfrak{g}$, $\breve{\mu}^{\mathcal{B},0}$, and ι^0. To see this, we explain how, in the stochastic steady state of the model, the five parameters map directly into functions of target ratios and how this mapping can be inverted to obtain the parameters. While we do not target the stochastic steady state but the ergodic mean when matching moments, the two are quantitatively very close.

The identity $C + I + G = Y$ and the level targets for C/Y and G/Y imply $I/Y = 1 - C/Y - G/Y$. We can thus write for capital productivity a^0 in the stochastic steady state

$$a^0 = \frac{Y}{K} = \frac{I/K}{I/Y} = \frac{I/K}{1 - C/Y - G/Y}.$$

This determines a^0 as a function of targets. Due to $G = \mathfrak{g}K$, we obtain immediately also

$$\mathfrak{g} = G/Y \cdot a^0.$$

Because G/Y is a target and a^0 has already been determined, this equation determines $\mathfrak{g}$.

Next, ρ represents the ratio of consumption to total wealth in the model, that is

$$\rho = \frac{C}{(q^B + q^K)K} = \frac{C/Y}{q^B K/Y + q^K K/Y}$$

[29] This does not imply that we are always able to pick ϕ in a way that matches both relative volatilities precisely. It just means that if model dynamics are such that they can be matched at all, then this works only for one value of ϕ.

[30] This is not a rigorous theoretical statement but an empirical one based on observed numerical model solutions.

and the right-hand expression is a function of targeted ratios. Hence, the targets also determine ρ.

By the government budget constraint, the policy variable $\breve{\mu}^{\mathcal{B}}$ in the stochastic steady state must satisfy

$$\breve{\mu}^{\mathcal{B},0} = -\frac{s}{q^B} = -\frac{S/Y}{q^B K/Y}$$

and, again, the right-hand expression is a function of targeted ratios.

Finally, the capital price in the stochastic steady state can be related to the capital-output ratio by the equation $q^{K,0} = q^K K/Y \cdot a^0$. Because a^0 is a function of targeted ratios, so is $q^{K,0}$. It is easy to show that the investment rate is $I/K = \iota^0 + \frac{q^{K,0}-1}{\phi}$. This expression only depends on $q^{K,0}$ and the parameters ι^0 and ϕ. For any given parameter ϕ, ι^0 is therefore determined by targets through the equation

$$\iota^0 = I/K - \frac{q^{K,0} - 1}{\phi}.$$

We remark that the six average ratios do not only identify the five parameters ρ, a^0, $\mathfrak{g}$, $\breve{\mu}^{\mathcal{B},0}$, and ι^0 (in the stochastic steady state) but also the average value ϑ^0 of the endogenous variable ϑ_t, namely

$$\vartheta^0 = \frac{q^B}{q^B + q^K} = \frac{q^B K/Y}{q^B K/Y + q^K K/Y}.$$

This generates an implicit target that must be somehow matched by varying parameters other than ρ, a^0, $\mathfrak{g}$, $\breve{\mu}^{\mathcal{B},0}$, and ι^0 in order to match all six average ratios.

Fifth, because ρ, $\bar{\chi}$, and the dynamics of $\breve{\mu}^{\mathcal{B}}$ and $\tilde{\sigma}_t$ are already determined by external calibration choices or the targeted average ratios, the counterpart of equation (10) in Appendix B.4 implies that this implicit target ϑ^0 for the average value of ϑ_t must be matched by a sufficient size of the risk premium terms in that equation. The only "free" variables in these terms are σ_t^v and γ and the former is effectively also determined by γ (once ρ, $\bar{\chi}$, and the dynamics of $\tilde{\sigma}_t$ are fixed). In fact, the risk premium terms are strictly increasing in γ given the remaining parameter choices. Therefore, the implicit target ϑ^0 is only achieved for a specific value of γ. At the same time, γ affects also the average equity premium $\mathbb{E}[d\bar{r}^E - dr^{\mathcal{B}}]$ and the equity sharpe ratio $\mathbb{E}[d\bar{r}^E - dr^{\mathcal{B}}]/\sigma(d\bar{r}^E - dr^{\mathcal{B}})$. The parameter γ is thus certainly identified by the set of target moments, but it is generally not possible to match all of them.

C.1.4 Calculation of Wealth-weighted Risk Exposures Discussed in the Main Text

In this appendix we explain how we compute empirical counterparts for the wealth-weighted total and idiosyncratic risk exposures discussed in Section 7.

Our data for risk exposures by wealth group are from Bach et al. (2020). These authors report the standard deviation of the excess return on gross wealth (Table I, column (2)) and net wealth (Table II, column (3)) for 16 wealth groups categorized by their relative position in the wealth distribution. For the gross wealth data, the authors also report the fraction that is due to idiosyncratic risk (Table I, column (3)) relative to a factor asset pricing model. In lack of other data, we assume that the same fractions also apply to the net wealth figures. We use these observations to compute for each group both the total and the idiosyncratic variance of the excess return on wealth, both for gross wealth and net wealth. As the observations are based on Swedish administrative data, our implicit assumption is that the mapping from wealth groups to these variances is similar for the the US, where no such data are observable.

To match these variances with wealth shares for the US, we take estimates from Smith et al. (2023) who calculate wealth shares using different methodologies for the following five wealth groups (see their Table I): “Full population”, “Top 10%”, “Top 1%”, “Top 0.1%”, “Top 0.01%”.[31] We use both their “baseline” and their “equal returns” estimate for wealth shares.

Unfortunately, the wealth groups formed by Smith et al. (2023) are coarser than the ones reported in Bach et al. (2020). Where the Smith et al. (2023) estimates only tell us the combined wealth share of several groups based on the Bach et al. (2020) split, we allocate the wealth equally across the groups formed in the latter paper.

Table 3 reports the square roots of the resulting wealth-weighted cross-sectional averages for the variances, both for idiosyncratic and total risk exposures. For each type of estimate, we report four values, depending on which wealth share estimate we use and whether we take gross wealth or net wealth figures for risk exposures.

[31] They also have a sixth group “Top 0.001%”. However, the Bach et al. (2020) is not sufficiently fine-grained at the very top of the wealth distribution, so that we ignore this additional group throughout.

Table 3: Wealth-weighted risk exposures

wealth shares	"baseline"		"equal returns"	
gross/net wealth	gross	net	gross	net
idiosyncratic risk	0.09	0.09	0.1	0.1
total risk	0.16	0.18	0.17	0.19

C.2 Alternative Calibration Choices and Robustness

In this appendix we report results for three alternative calibration choices and show that our main conclusions are robust to them.

First, one concern with our calibration may be that it overstates the real effects of variation in idiosyncratic risk $\tilde{\sigma}_t$. This concern arises because we impose a perfectly linear relationship between this variable and productivity a_t and choose the sensitivity α^a of productivity to variation in $\tilde{\sigma}_t$ to match total output volatility. However, empirically, the correlation between measures of (total factor) productivity and volatility are not nearly as strong as imposed in our model, so that some of the empirically observed output volatility is likely due to factors unrelated to variation in (idiosyncratic) risk.

Here, we show that this is not an issue. Theoretically, the dynamics of the endogenous variable ϑ_t matter most for the predictions of our model (see Proposition 1), but, at least in the log utility case, a_t-dynamics do not affect the determination of ϑ_t at all (compare equation (10)). While the same is no longer exactly true for the preferences we use in our calibrated model, we can verify numerically that the parameter α^a is not particularly important for any of our results. We do so by showing that lowering α^a to half its value in the baseline calibration lowers output volatility (by construction) but otherwise has only marginal effects on model predictions. We report the parameters and model moments for this alternative specification in the column "lower α^a" of Tables 4 and 5, respectively.

We remark that we still need $\alpha^a > 0$ to be sufficiently large such that aggregate consumption falls in times of high idiosyncratic risk. Otherwise, our model fails to match the correct sign of all aggregate risk premia.[32]

[32] If, for example, a_t was constant, then consumption would rise in times of high idiosyncratic risk, so that equity and capital would command a negative aggregate risk premium and government bonds a positive one.

Table 4: Alternative Parameter Specifications: Parameters

parameter	baseline	lower α^a	lower debt/GDP target	matching cov$(S/Y, Y)$
$\tilde{\sigma}^0$	0.54	0.54	0.54	0.54
ψ	0.67	0.67	0.67	0.67
σ	0.4	0.4	0.4	0.4
$\bar{\chi}$	0.3	0.3	0.3	0.3
γ	6	6	5.4	5.9
ρ	0.138	0.138	0.138	0.138
a^0	0.63	0.63	0.62	0.63
$\mathfrak{g}$	0.138	0.138	0.136	0.138
$\breve{\mu}^{\mathcal{B},0}$	0.0026	0.0026	0.0042	0.0017
α^a	0.071	0.036	0.071	0.072
$\alpha^{\mathcal{B}}$	0.12	0.12	0.19	0.07
ϕ	8.1	8.1	6.2	8.6
ι^0	-0.022	-0.022	-0.0877	-0.0131
δ	0.055	0.055	0.028	0.057

Table 5: Alternative Parameter Specifications: Moments

moment	baseline	lower α^a	lower debt/GDP target	matching cov$(S/Y, Y)$
$\sigma(Y)$	1.3%	0.7%	1.3%	1.3%
$\sigma(C)/\sigma(Y)$	0.61	0.35	0.60	0.61
$\sigma(I)/\sigma(Y)$	3.35	4.44	3.32	3.37
$\sigma(S/Y)$	1.1%	1.0%	1.1%	0.6%
$\mathbb{E}[C/Y]$	0.58	0.58	0.58	0.58
$\mathbb{E}[G/Y]$	0.22	0.22	0.22	0.22
$\mathbb{E}[S/Y]$	-0.0005	-0.0005	-0.0005	-0.0005
$\mathbb{E}[I/K]$	0.12	0.12	0.12	0.12
$\mathbb{E}[q^K K/Y]$	3.48	3.49	3.72	3.48
$\mathbb{E}[q^B K/Y]$	0.74	0.71	0.48	0.74
$\mathbb{E}[d\bar{r}^E - dr^{\mathcal{B}}]$	3.59%	3.26%	2.83%	3.78%
$\frac{\mathbb{E}[dr^E - dr^{\mathcal{B}}]}{\sigma(dr^E - dr^{\mathcal{B}})}$	0.31	0.28	0.29	0.29
$\rho(Y, C)$	0.98	0.67	0.99	0.98
$\rho(Y, I)$	0.99	0.97	0.99	0.99
$\rho(Y, S/Y)$	0.98	0.97	0.98	0.98
$\sigma(q^B K/Y)$	4.8%	4.7%	2.9%	5.29%
$\mathbb{E}[r^f]$	5.18%	5.41%	4.74%	5.50%
$\sigma(r^f)$	5.47%	5.97%	5.95%	5.31%

Notes: All variables are defined in precisely the same way as in Table 2 in the main text.

A second concern is that our calibration target for the debt-output ratio is too large, not only because we take the average over the last decade (for the reason explained in the main text) but also because we do not account for the fact that a substantial fraction of US government debt is held abroad and that share has risen over our sample period.

We have chosen not to exclude foreign held debt in our baseline calibration because it is not at all clear that this portion should indeed be excluded. This portion of debt is also relevant for the government budget and for pricing total debt. The implicit assumption in our calibration is, however, that foreign holders of US debt have a qualitatively and quantitatively comparable safe asset demand for this debt as domestic holders (so that one should think of them as also being agents in our model). It is unclear whether this is really the case.

For this reason, we report in Tables 4 and 5 in the column "lower debt/GDP target" an alternative calibration that reduces the target for the debt-output ratio by a third, which is approximately the fraction of US federal debt held abroad over the last decade. The new target is therefore 0.47 instead of 0.71 in the baseline calibration. We follow otherwise precisely the same procedure as outlined in the maintext to choose our parameters[33] We find that this modification affects the ability of our model to match the moments only marginally. Specifically, holding the dynamics of idiosyncratic risk constant due to our calibration choices, we need to lower risk aversion γ to reduce safe asset demand for bonds to match the lower debt-output ratio. This leads to a slight reduction in the equity premium and Sharpe ratio relative to the baseline specification. All other moments can still be matched equally well.

A third concern is that our calibration overstates the procyclicality of primary surpluses and therefore underestimates the value of the cash flow component in Figure 2. This concern arises because we target both the volatilities of output and the surplus-output ratio, but operate within a model environment that presumes a next to perfect correlation between the two variables while the empirical correlation is much weaker, 0.60. An alternative choice would be to ignore the empirical volatility of S/Y and instead target the covariance with output as the covariance is more directly related to pricing. This is equivalent to targeting a volatility $\sigma(S/Y)$ that is lowered by the factor $0.60/0.98$, the ratio between the empirical and the model-implied correlation between the two variables.

[33] We choose δ, which does not affect anything of interest, to keep the average growth rate in the model the same as in the baseline calibration.

We provide results for this alternative calibration choice in the column "matching $\text{cov}(S/Y, Y)$" of Tables 4 and 5. The resulting moments are largely identical to the ones for the baseline specification.

Beyond the effects on model moments, we also plot the counterparts of our key Figure 2 that decomposes the value of government debt into a cash flow and a service flow component for the alternative calibration choices. Figure 4 depicts the results for the two calibration choices "lower debt/GDP target" (left panel) and "matching $\text{cov}(S/Y, Y)$" (right panel).[34] The qualitative and quantitative takeaways remain the same as in Section 4.3.

Figure 5 depicts the Debt Laffer Curves for the dynamic models arising from the four alternative specifications. It shows that only lowering the target for the debt-output ratio considerably affects the size of the sustainable permanent deficit. The rationale for reducing the target was that a substantial fraction of US debt is held abroad. One way to interpret the difference between the orange line and the blue line in Figure 5 is therefore that the latter depicts the Laffer curve trade-off if non-domestic demand for US debt as a safe asset continues to absorb a significant fraction of debt issuance whereas the former depicts a Laffer curve that the US would face if US treasury debt lost its status as a global safe asset.

Not shown in Figure 5 as the comparison Laffer curves for the steady state models arising under the alternative specifications. However, the main conclusion from Figure 3 that the negative-β property is quantitatively important for the Laffer curve arise here analogously. In fact, for all but one specification the steady-state Laffer curve is almost identical with the one shown in Figure 3. The exception is the specification "lower debt/GDP target". In this case, there is no public debt bubble in the steady state model such that the maximum sustainable deficit is zero. Clearly, the conclusion that the negative-β property matters holds in this case as well.

[34]In the interest of space, the specification "lower α^a" is omitted. It looks almost identical to Figure 2.

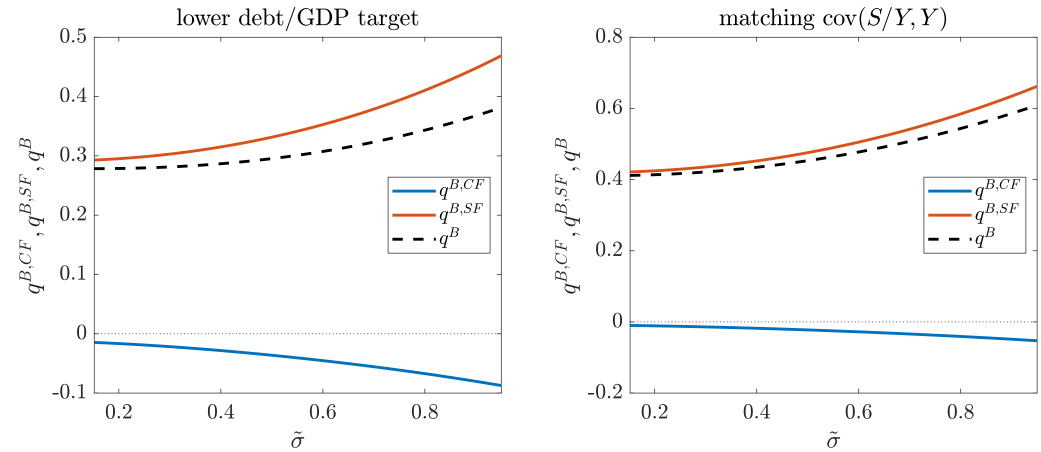

Figure 4: Decomposition of the value of government debt for alternative calibration choices. The description of Figure 2 applies analogously.

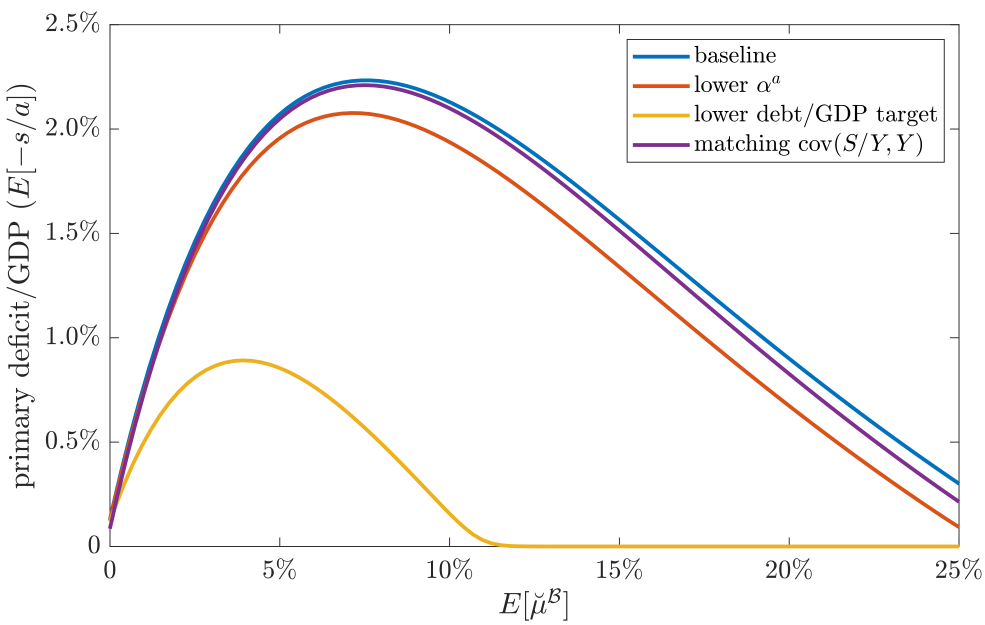

Figure 5: Debt Laffer curves for the alternative parameter specifications. The description of Figure 3 applies analogously to all four lines (for line "dynamic model" in that figure).