ECO529: Modern Macro, Money and (International) Finance

Problem Set 4

Andrey Alexandrov

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1. Endogenous Jumps

- Setup:
  - Sunspots arrive with exogenous intensity λ
  - Jump: $q(\eta) \to q(0)$ if self-fulfilling (i.e. if wipes experts out and $\eta \to 0$)

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1. Endogenous Jumps

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  - Jump: $q(\eta) \to q(0)$ if self-fulfilling (i.e. if wipes experts out and $\eta \to 0$)
- Issue: $\eta = 0$ is absorbing
- Fix: introduce transfer τK_t in case of a jump:
  - $q(\eta) \to q(\eta^*)$
  - $N_t^e \to 0$, experts default on debt, receive τK_t
  - $\eta \to \eta^*$ (self-fulfilling)

1. Endogenous Jumps

- Laws of motion:

$$d\eta_t = \mu_{\eta,t} dt + \sigma_{\eta,t} dZ_t + j_t^{\eta} \eta_t^- dJ_t$$

$$dq_t = \mu_{q,t} t dt + \sigma_{q,t} dZ_t + j_t^{q} q_t^- dJ_t$$

- Jump at t: $\eta_t^- \to \eta_t^+$, $d\eta_t = \eta_t^+ - \eta_t^-$ and $dJ_t = 1$

$$d\eta_t = \eta_t^+ - \eta_t^- = j_t^{\eta} \eta_t^-, \qquad j_t^{\eta} = \frac{\eta^* - \eta_t}{\eta_t}$$

$$dq_t = q_t^+ - q_t^- = j_t^{q} q_t^-, \qquad j_t^{q} = \frac{q^* - q_t}{q_t}$$

1. Endogenous Jumps

- Vulnerability region:

$$\underbrace{\chi_t(q_t - q^*)K_t}_{\text{Loss in } N_t^e} \geq \underbrace{\eta_t q_t K_t}_{N_t^e} \iff -j_t^q \chi_t \geq \eta_t$$

- Once N_t^e reaches 0, the rest is absorbed by defaulting debt D_t^e:

$$\underbrace{\chi_t(q_t - q^*)K_t - \eta_t q_t K_t}_{\text{Loss in } D_t^e} = -j_t^{r^D} D_t^{e,-} = -j^{r^D}(\chi_t - \eta_t)q_t K_t$$

$$j_t^{r^D} = \frac{\chi_t j_t^q + \eta_t}{\chi_t - \eta_t}$$

1. Endogenous Jumps

- Transfer τ ensures $\eta_t \to \eta^*$:

$$N_t^{e,+} = \tau K_t \iff \tau = q^* \frac{N_t^{e,+}}{q^* K_t} = q^* \eta^*$$

- Computing j^{N^e}:

$$dN_t^e = j_t^{N^e} N_t^{e,-} = N_t^{e,+} - N_t^{e,-} = \tau K_t - \eta_t q_t K_t$$

$$j_t^{N^e} = \frac{q^* \eta^* K_t - q_t \eta_t K_t}{q_t \eta_t K_t} = \frac{q^* \eta^* - q_t \eta_t}{q_t \eta_t}$$

1. Endogenous Jumps

- Log utility $\Longrightarrow$ add jumps ex-post
- Crucial assumption: jumps always wipe experts out

$$\frac{d\eta_t}{\eta_t} = \mu_t^{\eta} dt + \sigma_t^{\eta} dZ_t + j_t^{\eta} dJ_t$$

$$\mu_t^{\eta} = (1-\eta_t)\Bigg[\hat{\varsigma}_t^e(\sigma_t^{\eta^e} - \sigma - \sigma_t^q) - \hat{\varsigma}_t^h(\sigma_t^{\eta^h} - \sigma - \sigma_t^q) - \left(\frac{C_t^e}{N_t^e} - \frac{C_t^h}{N_t^h}\right)$$

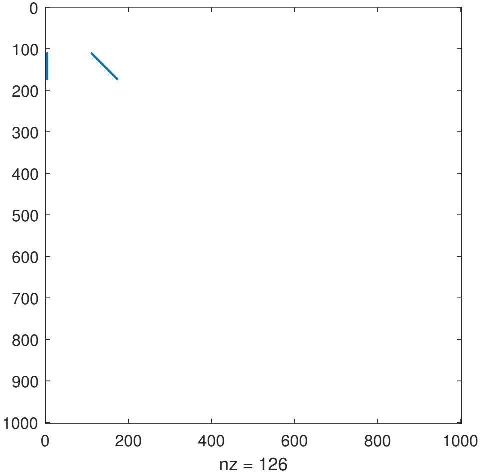
$$+ \lambda_t \left(\hat{\nu}_t^e \left(j_t^{\eta^e} - \frac{j_t^{r^D} - j_t^N}{1+j_t^N}\right) - \hat{\nu}_t^h \left(j_t^{\eta^h} - \frac{j_t^{r^D} - j_t^N}{1+j_t^N}\right)\right)\Bigg] - \lambda_t j_t^{\eta^e}$$

$$\sigma_t^{\eta^i} = \frac{\chi_t^i - \eta_t^i}{\eta_t^i}(\sigma + \sigma_t^q)$$

1. Endogenous Jumps

- Stationary distribution: $\tilde{M}'g = 0$, $\tilde{M} = M + \Lambda$
- The usual matrix M from drift and volatility
- An additional matrix Λ that reflects jumps:
  - Denote vulnerability region by $V_r = \{\eta \mid -j^q(\eta)\chi(\eta) \geq \eta\}$
  - $\Lambda_{i,i} = -\lambda$ if $\eta_i \in V_r$
  - $\Lambda_{i,m} = \lambda$ if $\eta_i \in V_r$ and $\eta_m = \eta^*$

1. Endogenous Jumps



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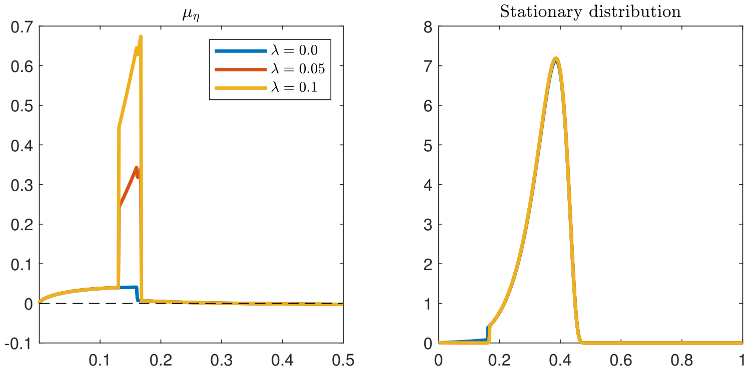

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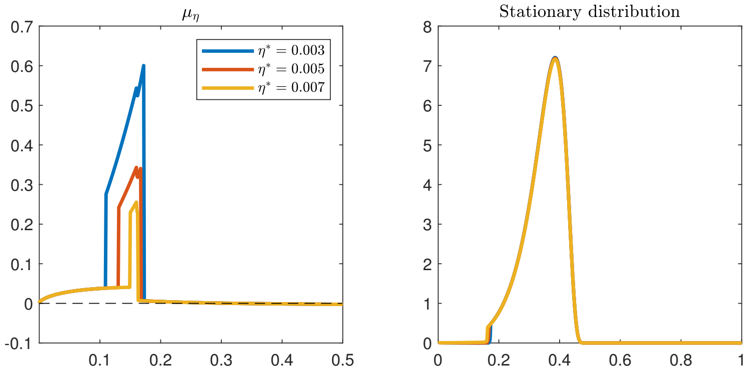

2. Money Model with Stochastic Volatility

- One sector + time-varying idiosyncratic risk

$$\frac{dk_t^{\tilde{i}}}{k_t^{\tilde{i}}} = \left(\Phi(\iota_t^{\tilde{i}}) - \delta\right) dt + \tilde{\sigma}_t d\tilde{Z}_t^{\tilde{i}}$$

$$d\tilde{\sigma}_t = b(\tilde{\sigma}^{ss} - \tilde{\sigma}_t)dt + \nu\sqrt{\tilde{\sigma}_t}dZ_t$$

- No policy ($\mu^B = i = \sigma^B = g = \tau = 0$)

2. Money Model with Stochastic Volatility

- Optimal investment: $q^K = 1 + \phi\iota$
- Goods market clearing: $\rho(q^K + q^B) = a - \iota$
- Bond share in wealth: $\vartheta = \frac{q^B}{q^K + q^B}$

$$q_t^B = q^B(\vartheta_t) = \vartheta_t \frac{1 + \phi a}{(1 - \vartheta_t) + \phi\rho}$$

$$q_t^K = q^K(\vartheta_t) = (1 - \vartheta_t) \frac{1 + \phi a}{(1 - \vartheta_t) + \phi\rho}$$

$$\iota_t = \iota(\vartheta_t) = \frac{(1 - \vartheta_t)a - \rho}{(1 - \vartheta_t) + \phi\rho}$$

2. Money Model with Stochastic Volatility

- Change of numeraire + martingale method:
  - Asset A: capital
  - Asset B: bonds

1. Express returns on A and B in N_t-numeraire
2. Apply martingale pricing formula

2. Money Model with Stochastic Volatility

- Returns on capital and bonds:

$$dr_t^{k,\tilde{i}} = \frac{a - \iota_t}{q_t^K} dt + \frac{d\left(q_t^K k_t^{\tilde{i}}\right)}{q_t^K k_t^{\tilde{i}}}$$

$$dr_t^B = \underbrace{i_t}_{=0} + \frac{d(1/P_t)}{1/P_t}$$

- SDF:

$$\frac{d\xi_t^{\tilde{i}}}{\xi_t^{\tilde{i}}} = -r_t^f dt - \varsigma_t dZ_t - \tilde{\varsigma}_t d\tilde{Z}_t^{\tilde{i}}$$

2. Money Model with Stochastic Volatility

- Recall $\vartheta_t = \frac{q_t^B K_t}{(q_t^K + q_t^B) K_t} = \frac{q_t^B K_t}{N_t}$. In N_t-numeraire:

$$d\hat{r}_t^{k,\tilde{i}} = \frac{a - \iota_t}{q_t^K} dt + \frac{d\left(q_t^K k_t^{\tilde{i}} / N_t\right)}{q_t^K k_t^{\tilde{i}} / N_t} = \frac{a - \iota_t}{q_t^K} dt + \frac{d\left((1 - \vartheta_t)\frac{k_t^{\tilde{i}}}{K_t}\right)}{(1 - \vartheta_t)\frac{k_t^{\tilde{i}}}{K_t}}$$

$$= \left(\frac{a - \iota_t}{q_t^K} + \mu_t^{1-\vartheta}\right) dt + \tilde{\sigma}_t dZ_t^{\tilde{i}} = \left(\frac{\rho}{1 - \vartheta_t} - \mu_t^{\vartheta} \frac{\vartheta_t}{1 - \vartheta_t}\right) dt + \tilde{\sigma}_t dZ_t^{\tilde{i}}$$

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$$d\hat{r}_t^B = \frac{d(1/(P_t N_t))}{1/(P_t N_t)} = \frac{d(\vartheta_t / B_t)}{\vartheta_t / B_t} = \mu_t^{\vartheta} dt + \sigma_t^{\vartheta} dZ_t$$

$$\frac{d\hat{\xi}_t^{\tilde{i}}}{\hat{\xi}_t^{\tilde{i}}} = \frac{d\xi_t^{\tilde{i}} N_t}{\xi_t^{\tilde{i}} N_t} = -(r_t^f - \mu_t^N) dt - (\varsigma_t - \sigma_t^N) dZ_t - \tilde{\varsigma}_t d\tilde{Z}_t^{\tilde{i}}$$

2. Money Model with Stochastic Volatility

- Martingale formula:

$$\frac{\mathbb{E}[d\hat{r}_t^{k,\tilde{i}}]}{dt} - \frac{\mathbb{E}[d\hat{r}_t^{B}]}{dt} = \hat{\varsigma}_t(\sigma_t^{\hat{r}^{k,\tilde{i}}} - \sigma_t^{\hat{r}^{B}}) + \hat{\tilde{\varsigma}}_t(\tilde{\sigma}_t^{\hat{r}^{k,\tilde{i}}} - \tilde{\sigma}_t^{\hat{r}^{B}})$$

$$\frac{\rho}{1-\vartheta_t} - \mu_t^{\vartheta}\frac{\vartheta_t}{1-\vartheta_t} - \mu_t^{\vartheta} = (\varsigma_t - \sigma_t^{N})(-\sigma_t^{\vartheta}) \; + \; \tilde{\varsigma}_t\tilde{\sigma}_t$$

$$\mu_t^{\vartheta} = \rho - \tilde{\varsigma}_t(1-\vartheta_t)\tilde{\sigma}_t$$

$$\mu_t^{\vartheta} = \rho - (1-\vartheta_t)^2\tilde{\sigma}_t^2$$

2. Money Model with Stochastic Volatility

- $\vartheta_t = \vartheta(\tilde{\sigma}_t)$, Ito's formula:

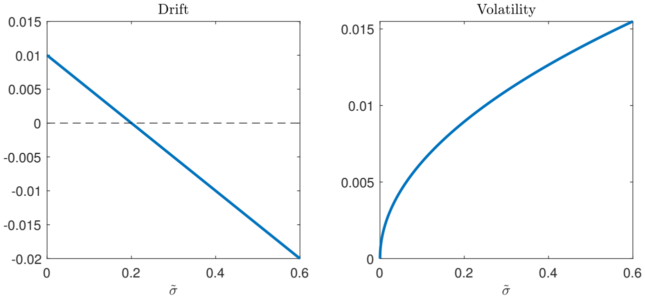
$$d\vartheta_t = \underbrace{\left(\mu_{\tilde{\sigma},t}\vartheta'(\tilde{\sigma}) + \frac{\sigma^2_{\tilde{\sigma},t}}{2}\vartheta''(\tilde{\sigma}) \right)}_{\mu^{\vartheta}_t \vartheta_t} dt + \sigma_{\tilde{\sigma},t}\vartheta'(\tilde{\sigma}) dZ_t$$

$$\rho\vartheta(\tilde{\sigma}) = (1-\vartheta(\tilde{\sigma}))^2\tilde{\sigma}^2\vartheta(\tilde{\sigma}) + b(\tilde{\sigma}^{ss} - \tilde{\sigma})\vartheta'(\tilde{\sigma}) + \frac{\nu^2\tilde{\sigma}}{2}\vartheta''(\tilde{\sigma})$$

$$\rho\vartheta_t(\tilde{\sigma}) = \partial_t\vartheta_t(\tilde{\sigma}) + (1-\vartheta_t(\tilde{\sigma}))^2\tilde{\sigma}^2\vartheta_t(\tilde{\sigma}) + b(\tilde{\sigma}^{ss} - \tilde{\sigma})\vartheta'_t(\tilde{\sigma}) + \frac{\nu^2\tilde{\sigma}}{2}\vartheta''_t(\tilde{\sigma})$$

2. Money Model with Stochastic Volatility

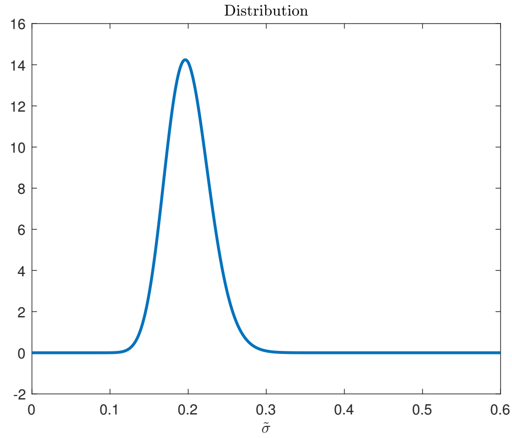
$$d\tilde{\sigma}_t = b(\tilde{\sigma}^{ss} - \tilde{\sigma}_t)dt + \nu\sqrt{\tilde{\sigma}_t}dZ_t$$



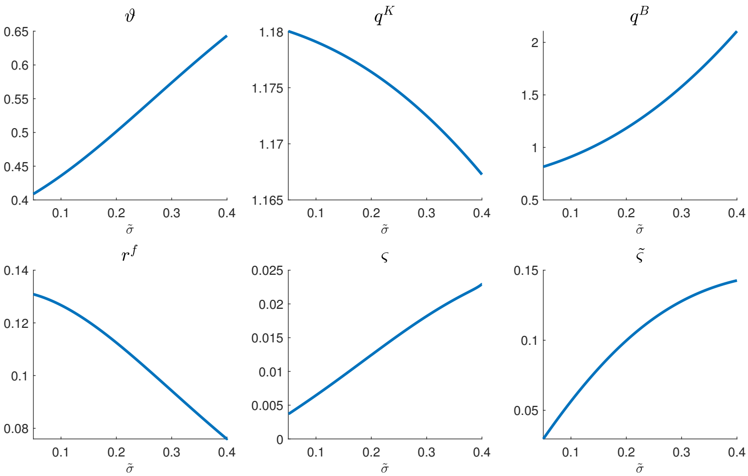
- 'Natural' lower boundary: $\tilde{\sigma} = 0$
- Upper boundary: 'sufficiently' large $\tilde{\sigma}$

2. Money Model with Stochastic Volatility

- LOM of $\tilde{\sigma}$ does not depend on model's solution!
- Compute its distribution:



2. Money Model with Stochastic Volatility



Stationary Distribution of Wealth (Shares)

- Consider the one-sector money model with constant idiosyncratic volatility $\tilde{\sigma}$
- Distribution of wealth share $\eta_t^{\tilde{i}} = \frac{n_t^{\tilde{i}}}{N_t}$ is non-stationary

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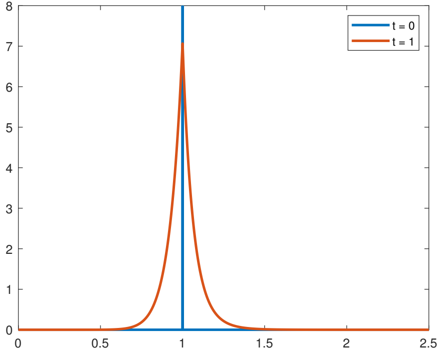

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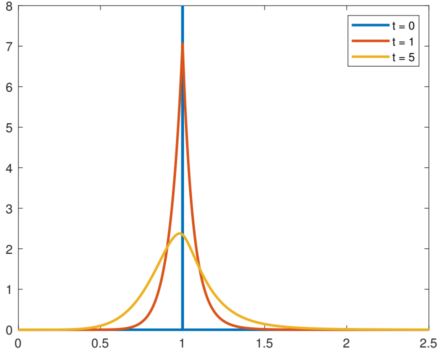

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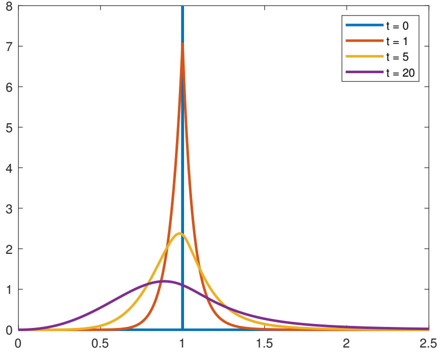

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- Distribution of wealth share $\eta_t^{\tilde{i}} = \frac{n_t^{\tilde{i}}}{N_t}$ is non-stationary
- Can make it stationary with an ad hoc fix:
  - Idiosyncratic Poisson wealth shocks
  - With intensity λ wealth share $\eta^{\tilde{i}}$ is set to η^*
  - Log-utility, same returns $\Longrightarrow$ no effect on equilibrium

$$\frac{dn_t^{\tilde{i}}}{n_t^{\tilde{i}}} = \Big(- \underbrace{\rho}_{c_t^{\tilde{i}}/n_t^{\tilde{i}}} + \underbrace{g}_{r_t^B = \Phi(\iota) - \delta} + (1-\theta_t) \underbrace{\frac{a-\iota}{q^K}}_{E[r_t^{k,\tilde{i}}] - r_t^B} \Big) dt + (1-\theta_t)\tilde{\sigma} dZ_t^{\tilde{i}} + j_t^{n,\tilde{i}} dJ_t^{\tilde{i}}$$

Stationary Distribution of Wealth (Shares)

$$\frac{dn_t^{\tilde{i}}}{n_t^{\tilde{i}}} = \left(-\rho + g + (1-\vartheta_t)\frac{a-\iota}{q^K}\right)dt + (1-\theta_t)\tilde{\sigma}\, dZ_t^{\tilde{i}} + j_t^{n,\tilde{i}} dJ_t^{\tilde{i}}$$

$$\frac{dN_t}{N_t} = g dt$$

$$\frac{d\eta_t^{\tilde{i}}}{\eta_t^{\tilde{i}}} = \underbrace{(-\rho + (1-\vartheta_t)\frac{a-\iota}{q^K})}_{=0} dt + (1-\vartheta_t)\tilde{\sigma}\, dZ_t^{\tilde{i}} + j_t^{n,\tilde{i}} dJ_t^{\tilde{i}}$$

Stationary Distribution of Wealth (Shares)

$$\frac{d\eta_t^{\tilde{i}}}{\eta_t^{\tilde{i}}} = (1-\vartheta)\tilde{\sigma}\, dZ_t^{\tilde{i}} + j_t^{n,\tilde{i}} dJ_t^{\tilde{i}}$$

- Set $j_t^{n,\tilde{i}} = \frac{\eta^* - \eta_t^{\tilde{i}}}{\eta_t^{\tilde{i}}}$
- KFE (for all $\eta \neq \eta^*$) is given by:

$$0 = \frac{(1-\vartheta)^2\tilde{\sigma}^2}{2}\frac{\partial(\eta^2 g(\eta))}{\partial\eta} - \lambda g(\eta)$$

- There is a kink at η^*

Stationary Distribution of Wealth (Shares)

- KFE (for all $\eta \neq \eta^*$) is given by:

$$0 = g''(\eta)\eta^2 + 4g'(\eta)\eta + \left(2 - \frac{2\lambda}{(1-\vartheta)^2\tilde{\sigma}^2}\right) g(\eta)$$

- Euler's equation – has closed-form solutions

$$g(\eta) = C_1\eta^{\alpha_1} + C_2\eta^{\alpha_2} \text{ for } \eta < \eta^*$$

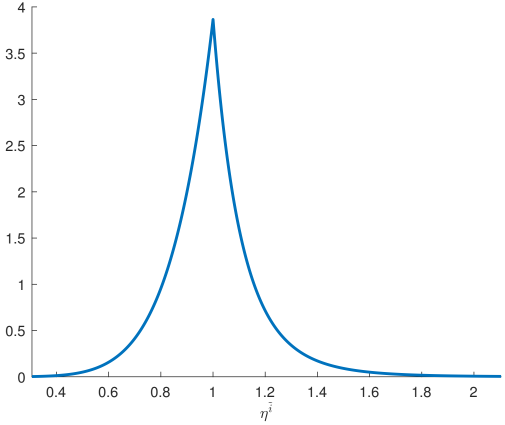
$$g(\eta) = C_3\eta^{\alpha_1} + C_4\eta^{\alpha_2} \text{ for } \eta \geq \eta^*$$

$$\int_0^\infty g(\eta)d\eta = 1, \qquad \lim_{\eta\to 0} g(\eta) = \lim_{\eta\to\infty} g(\eta) = 0$$

- + continuity at η^*, $\alpha_1 = \frac{\alpha-3}{2}$, $\alpha_2 = -\frac{\alpha+3}{2}$, $\alpha = \sqrt{\frac{8\lambda}{(1-\vartheta)^2\tilde{\sigma}^2} + 1}$

Stationary Distribution of Wealth (Shares)

- Solution under $\eta^* = 1$: $C_1 = C_4 = \frac{2\lambda}{(1-\vartheta)^2\tilde{\sigma}^2\alpha}$, $C_2 = C_3 = 0$



A (General) Model

$$\max_{c,\theta} \mathbb{E}\left[\int_0^\infty e^{-\rho t} u(c_t) dt\right]$$

$$dn_t^{\tilde{i}} = \left(-c_t^{\tilde{i}} + y_t^{\tilde{i}}\right) dt + n_t^{\tilde{i}} \left(rdt + (1-\theta_t^{\tilde{i}})(dr_t^{k,\tilde{i}} - rdt)\right)$$

$$dr_t^{k,\tilde{i}} = r^k dt + \tilde{\sigma}^k dZ_t^{k,\tilde{i}}$$

$$dy_t^{\tilde{i}} = -\nu y_t^{\tilde{i}} dt + \tilde{\sigma}^y dZ_t^{y,\tilde{i}}$$

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$$dy_t^{\tilde{i}} = -\nu y_t^{\tilde{i}} dt + \tilde{\sigma}^y dZ_t^{y,\tilde{i}}$$

1. This class: $\tilde{\sigma}^y = 0$
2. Bewley-Huggett-Aiyagari model: $\tilde{\sigma}^k = 0$

Bewley-Huggett-Aiyagari Model

- Key difference: idiosyncratic risk is in endowment, not returns

$$\max_c \mathbb{E}\left[\int_0^\infty e^{-\rho t} u(c_t) dt\right]$$

$$dn_t^{\tilde{i}} = \left(-c_t^{\tilde{i}} + rn_t^{\tilde{i}} + y_t^{\tilde{i}}\right) dt$$

$$dy_t^{\tilde{i}} = -\nu y_t^{\tilde{i}} dt + \tilde{\sigma} dZ_t^{\tilde{i}}$$

- Risk does not scale with wealth $\Longrightarrow$ C/N (and portfolio) depends on wealth

Bewley-Huggett-Aiyagari Model

- Go to HJB directly

$$\rho v(y,n) = \max_c \left[u(c) + (-c + rn + y)\,\partial_n v(y,n) - \nu y \partial_y v(y,n) + \frac{\tilde{\sigma}^2}{2}\partial_{yy} v(y,n) \right]$$

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- FOC: $\partial_c u(c) = \partial_n v(y,n) \implies c(v,y,n) = (u')^{-1}(\partial_n v(y,n))$

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$$\rho v(y,n) = u(v,y,n) + (-c(v,y,n) + rn + y)\, \partial_n v(y,n) - \nu y \partial_y v(y,n) + \frac{\tilde{\sigma}^2}{2} \partial_{yy} v(y,n)$$

$$\rho \mathbf{v} = \mathbf{u}(\mathbf{v}) + \mathbf{M}(\mathbf{v})\mathbf{v}$$