ECO529: Modern Macro, Money and (International) Finance

Problem Set 3

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0. Outside Equity vs Debt

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- Suppose experts are unconstrained in OE issuance:

$$r_t^{OE} - r_t = \varsigma_t^e \sigma_t^{r,K} = (\theta_t^{e,K} + \theta_t^{e,OE})(\sigma^{r,K})^2 = (1 - \theta_t^{e,D})(\sigma^{r,K})^2$$
$$r_t^{OE} - r_t = \varsigma_t^h \sigma_t^{r,K} = (\theta_t^{h,K} + \theta_t^{h,OE})(\sigma^{r,K})^2 = (1 - \theta_t^{h,D})(\sigma^{r,K})^2$$

- Must be that $\varsigma_t^e = \varsigma_t^h$

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$$r_t^{OE} - r_t = \varsigma_t^h \sigma_t^{r,K} = (\theta_t^{h,K} + \theta_t^{h,OE})(\sigma^{r,K})^2 = (1 - \theta_t^{h,D})(\sigma^{r,K})^2$$

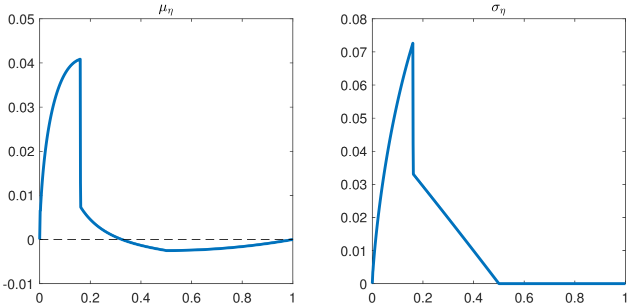
- Must be that $\varsigma_t^e = \varsigma_t^h$
- If experts issue a little bit of debt:
  - $(1 - \theta^{e,D}) \uparrow$, $(1 - \theta^{h,D}) \downarrow$, and $\varsigma_t^e \uparrow$, $\varsigma_t^h \downarrow$
  - Experts demand lower interest rate on debt
  - Households demand higher interest rate on debt
  - Market does not clear

1. Numerical Methods

$$\mu_\eta(\eta) = \eta(1-\eta_t)\left[(\varsigma_t^e - \sigma - \sigma_t^q)(\sigma_t^\eta + \sigma + \sigma_t^q) - (\varsigma_t^h - \sigma - \sigma_t^q)\left(-\frac{\eta_t \sigma_t^\eta}{1-\eta_t} + \sigma + \sigma_t^q\right) - \left(\frac{C_t^e}{N_t^e} - \frac{C_t^h}{N_t^h}\right)\right]$$

$$\sigma_\eta(\eta) = (\chi_t - \eta_t)(\sigma + \sigma_t^q)$$

1. Numerical Methods



- Grid $= \{\eta_1, \eta_2, \ldots, \eta_N\}$
- Constructing the M matrix:
  - Ensure $\mu_1 \geq 0, \mu_N \leq 0, \sigma_1 = \sigma_N = 0$
  - Including $\{0\}$ or $\{1\}$?

1. Numerical Methods

Compute stationary distribution:

1. Iteration method:
   - $g^{i+1} = (I - \Delta t M')^{-1} g^i$
   - Set g^0 to anything that integrates to 1
   - Depending on how M is constructed: avoid $g_N^0 > 0$.
2. Yuliy's trick from the lecture (requires $\mu_N > 0$)

$$0 = M'g \iff \begin{bmatrix} M'_{11} & M'_{1,2:N} \\ M'_{2:N,1} & M'_{2:N,2:N} \end{bmatrix} \begin{bmatrix} g_1 \\ g_{2:N} \end{bmatrix} = 0 \iff g_{2:N} = -(M'_{2:N,2:N})^{-1} M'_{2:N,1} g_1$$

3. Solve for kernel: $M'g = 0$, e.g. `null(full(M'))`

1. Numerical Methods

- Absorbing boundaries ($\eta_1 = 0$, $\eta_N = 1$, $\mu_1 = \mu_N = \sigma_1 = \sigma_N = 0$)
- Have a look at matrix M:

$$M_{1:3,1:4} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1.62 & -6.63 & 5.01 & 0 \\ 0 & 6.15 & -18.85 & 12.70 \end{bmatrix}$$

$$M_{N-2:N,N-3:N} = \begin{bmatrix} 0.02 & -0.02 & 0 & 0 \\ 0 & 0.01 & -0.01 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

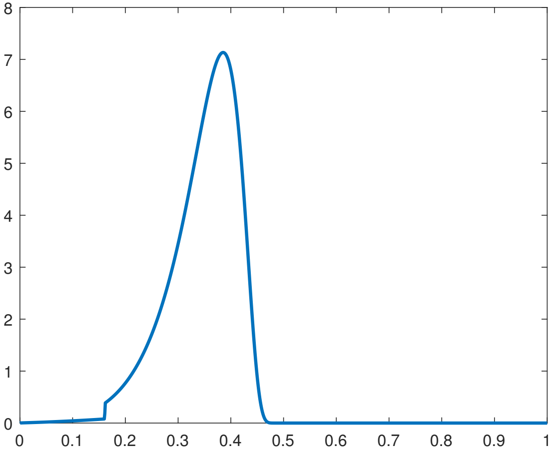
1. Numerical Methods

- Absorbing boundaries ($\eta_1 = 0$, $\eta_N = 1$, $\mu_1 = \mu_N = \sigma_1 = \sigma_N = 0$)
- Adjust matrix M (reflecting boundaries):

$$M_{1:3,1:4} = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 1.62 & -6.63 & 5.01 & 0 \\ 0 & 6.15 & -18.85 & 12.70 \end{bmatrix}$$

$$M_{N-2:N,N-3:N} = \begin{bmatrix} 0.02 & -0.02 & 0 & 0 \\ 0 & 0.01 & -0.01 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

1. Numerical Methods



1. Numerical Methods

$$v(x,t) = E\left[\int_t^T e^{-\rho s} x_s ds \,\middle|\, x_t = x\right]$$

$$dx_t = -x_t dt + (1 - x_t^2) dZ_t$$

$$\rho v(x,t) = \partial_t v(x,t) + x - x\partial_x v(x,t) + \frac{(1-x^2)^2}{2}\partial_{xx} v(x,t)$$

$$v(x,T) = 0$$

- Grid $x \in [-1, 1],\ t \in [0, 1]$

1. Numerical Methods

- Discretize state space: $v(x,t)$ is a vector $\mathbf{v}_t$
- Explicit method:

$$\frac{\mathbf{v}_t - \mathbf{v}_{t-\Delta t}}{\Delta t} = \rho \mathbf{v}_{t} - \mathbf{x} - \mathbf{M}\mathbf{v}_{t}$$

$$\mathbf{v}_{t-\Delta t} = \Delta t \mathbf{x} + ((1-\rho\Delta t)\mathbf{I}_N + \Delta t \mathbf{M})\mathbf{v}_t$$

- Implicit method:

$$\frac{\mathbf{v}_t - \mathbf{v}_{t-\Delta t}}{\Delta t} = \rho \mathbf{v}_{t-\Delta t} - \mathbf{x} - \mathbf{M}\mathbf{v}_{t-\Delta t}$$

$$\mathbf{v}_{t-\Delta t} = ((1+\rho\Delta t)\mathbf{I}_N - \Delta t \mathbf{M})^{-1}(\Delta t \mathbf{x} + \mathbf{v}_t)$$

1. Numerical Methods

- Stability of explicit method:

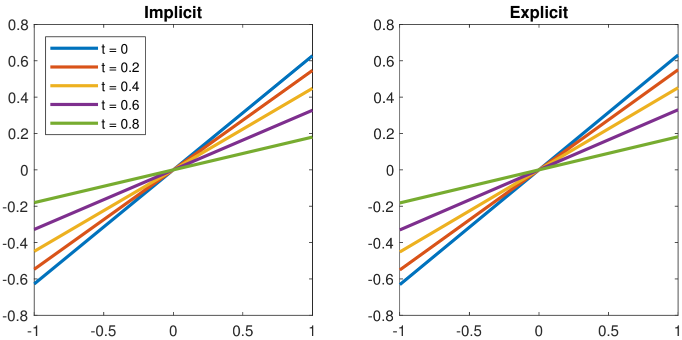
$$\mathbf{v}_{t-\Delta t} = \Delta t \mathbf{x} + \underbrace{\left((1 - \rho\Delta t)\mathbf{I}_N + \Delta t \mathbf{M}\right)}_{\geq 0} \mathbf{v}_t$$

$$1 - \rho\Delta t - \Delta t \left[\frac{|\mu|}{\Delta x} + \frac{\sigma^2}{(\Delta x)^2}\right] \geq 0$$

- To ensure this in the limit as $\Delta x \to 0 : \lim_{\Delta x \to 0} \frac{\Delta t}{(\Delta x)^2} \leq C \implies \Delta t = O((\Delta x)^2)$

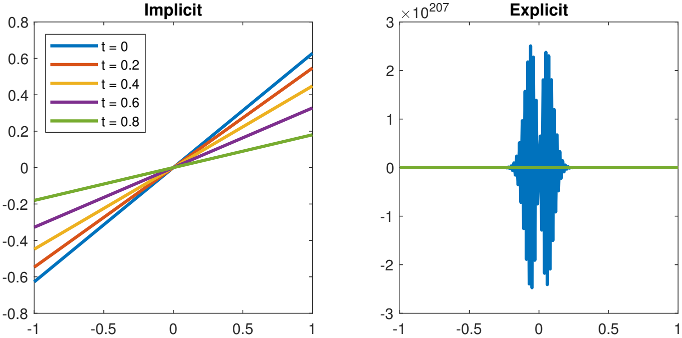
1. Numerical Methods

- $\Delta x = 0.1,\ \Delta t = 0.01$



1. Numerical Methods

- $\Delta x = 0.01,\ \Delta t = 0.01$



2/3. Two Sector Model

- Solution algorithm
  1. Guess value functions $v^e, v^h \Longrightarrow \varsigma^i, \frac{C^i}{N^i}$
  2. Solve inner loop for $q, \kappa, \chi, \iota, \sigma^q \Longrightarrow \mu^{v^i}, \mu^\eta, \sigma^\eta$
  3. Update v^e, v^h via time step

$$\mu_t^{v^i} v_t^i = \partial_t v_t^i + \eta \mu^\eta \partial_\eta v_t^i + \frac{1}{2}(\eta \sigma^\eta)^2 \partial_{\eta\eta} v_t^i$$

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- CRRA: $\mu_t^{v^i} = \rho^i - \frac{c_t^i}{n_t^i} - (1-\gamma)\left(\Phi(\iota_t) - \delta - \gamma\frac{\sigma^2}{2} + \sigma\sigma_t^{v^i}\right)$

$$\rho^i v_t^i = \partial_t v_t^i + \left(\frac{c_t^i}{n_t^i} + (1-\gamma)\left(\Phi(\iota_t) - \delta - \gamma\frac{\sigma^2}{2} + \sigma\sigma_t^{v^i}\right)\right) v_t^i + \eta\mu^\eta \partial_\eta v_t^i + \frac{1}{2}(\eta\sigma^\eta)^2 \partial_{\eta\eta} v_t^i$$

2/3. Two Sector Model

- CRRA
  - $\frac{C^i}{N^i} = \frac{(\eta^i q)^{\frac{1-\gamma}{\gamma}}}{(v^i)^{\frac{1}{\gamma}}}$
  - $\frac{\mathbb{E}\left[dV_t^i / V_t^i\right]}{dt} = \rho^i - \frac{C_t^i}{N_t^i}$
- Epstein-Zin (IES=1)
  - $\frac{C^i}{N^i} = \rho^i$
  - $\frac{\mathbb{E}\left[dV_t^i / V_t^i\right]}{dt} = -\frac{\partial f^i(c,U)}{\partial U} - \frac{C_t^i}{N_t^i}$

2/3. Two Sector Model

- Epstein-Zin (IES=1):

$$f^i(c,U) = (1-\gamma)\rho^i U \left(\log(c) - \frac{1}{1-\gamma}\log((1-\gamma)\rho^i U)\right)$$

$$\frac{\partial f^i(c,U)}{\partial U} = (1-\gamma)\rho^i \left(\log(c) - \frac{1}{1-\gamma}\log((1-\gamma)\rho^i U)\right) - \rho^i$$

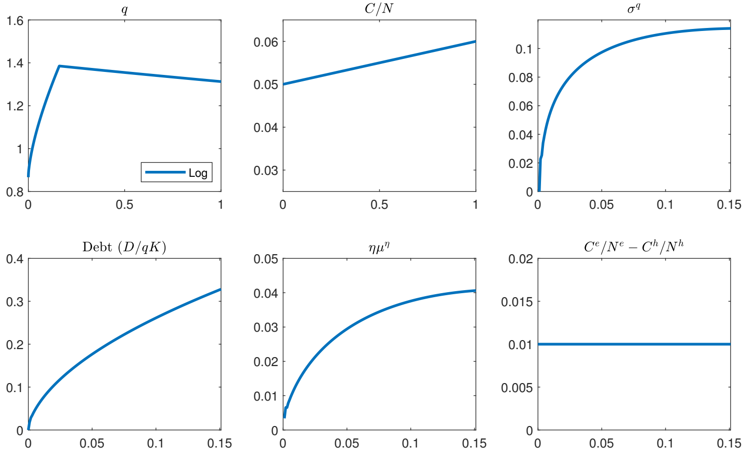
- Recall $U = V^i = \frac{1}{\rho^i}\frac{(\omega^i n^i)^{1-\gamma}}{1-\gamma}$ and $\omega^i = \frac{(\rho^i v^i)^{\frac{1}{1-\gamma}}}{\eta^i q}$

$$\begin{aligned}\frac{\partial f^i(c,U)}{\partial U} &= (1-\gamma)\rho^i \left(\log(\rho^i) - \log(\omega^i)\right) - \rho^i \\ &= (1-\gamma)\rho^i \log(\rho^i \eta^i q) - \rho^i \log(\rho^i v^i) - \rho^i\end{aligned}$$

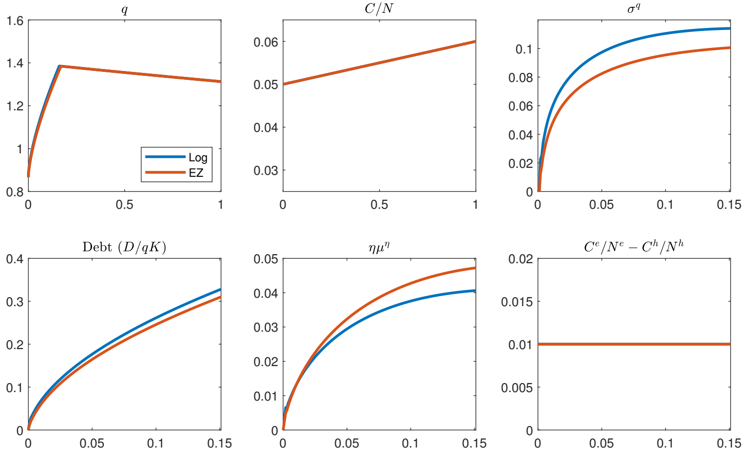
2/3. Two Sector Model

	Log Utility $\longrightarrow$	Epstein-Zin $\longrightarrow$	CRRA
RRA	$\gamma = 1$	$\gamma = 2$	$\gamma = 2$
IES	$\psi = 1$	$\psi = 1$	$\psi = 0.5$

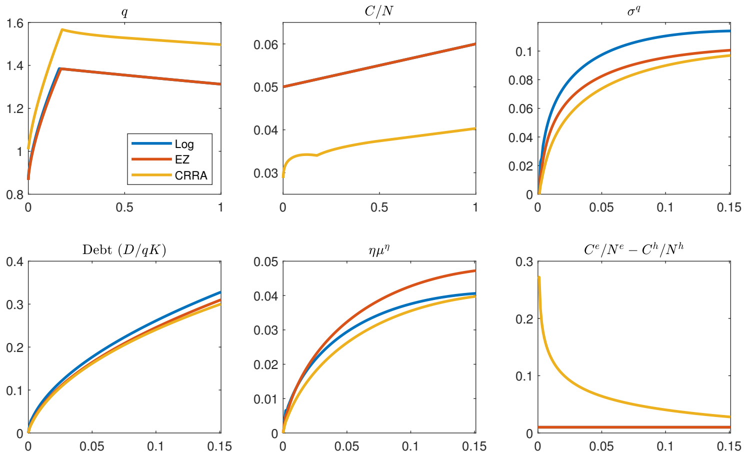
2/3. Two Sector Model



2/3. Two Sector Model



2/3. Two Sector Model



2/3. Two Sector Model