ECO529: Modern Macro, Money and (International) Finance

Problem Set 2

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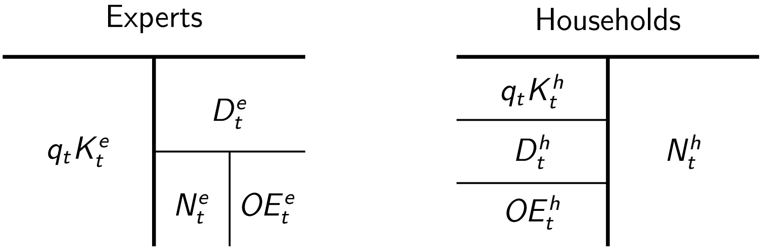
2. Risk Allocation

$$\eta_t = \frac{N_t^e}{N_t}, \quad \kappa_t = \frac{K_t^e}{K_t}$$

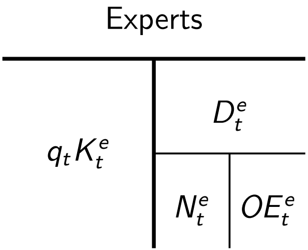
$$\theta_t^{e,K} = \frac{q_t K_t^e}{N_t^e}, \quad \theta_t^{e,OE} = -\frac{OE_t^e}{N_t^e}, \quad \theta_t^{e,D} = -\frac{D_t^e}{N_t^e} = 1 - \theta_t^{e,K} - \theta_t^{e,OE}$$



2. Risk Allocation

$$\chi_t = \frac{\sigma_{N^e,t}}{\sigma_{qK,t}} = \left(\theta_t^{e,K} + \theta_t^{e,OE}\right)\eta_t = \frac{q_t K_t^e - OE_t^e}{q_t K_t}$$

$$\frac{\chi_t}{\kappa_t} = \frac{\theta_t^{e,K} + \theta_t^{e,OE}}{\theta_t^{e,K}} = \frac{q_t K_t^e - OE_t^e}{q_t K_t^e}$$



2. Risk Allocation

$$\chi_t = \frac{\sigma_{N^e,t}}{\sigma_{qK,t}} = \left(\theta_t^{e,K} + \theta_t^{e,OE}\right)\eta_t = \frac{q_t K_t^e - OE_t^e}{q_t K_t}$$

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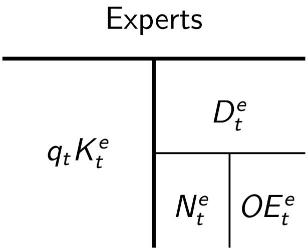


- Total risk: $\sigma_{qK,t} = (\sigma + \sigma_t^q) q_t K_t$
- Risk on experts' asset side: $\sigma_{qK^e,t} = (\sigma + \sigma_t^q) q_t K_t^e$
- Risk the experts hold: $\sigma_{N^e,t} = \chi_t \underbrace{(\sigma + \sigma_t^q) q_t K_t}_{\sigma_{qK,t}} = \frac{\chi_t}{\kappa_t} \underbrace{(\sigma + \sigma_t^q) q_t K_t^e}_{\sigma_{qK^e,t}}$
- Risk offloaded to households:

  $\sigma_{OE^e,t} = \sigma_{qK^e,t} - \sigma_{N^e,t} = \left(1 - \frac{\chi_t}{\kappa_t}\right)(\sigma + \sigma_t^q) q_t K_t^e$

2. Risk Allocation

- Case 1a:
  - $\kappa_t < 1 \implies K_t^h > 0$
  - $\chi_t > \eta_t \implies D_t^e > 0$

Experts

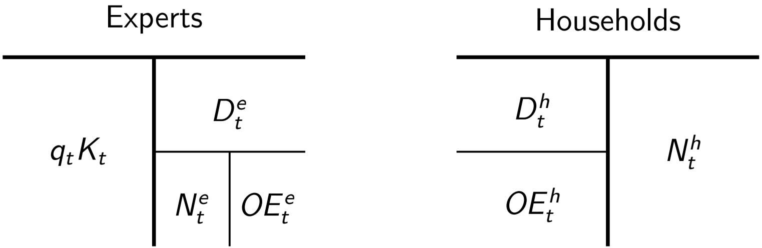
Assets	Liabilities	
$q_t K_t^e$	D_t^e	
	N_t^e	OE_t^e

Households

Assets	Liabilities
$q_t K_t^h$	N_t^h
D_t^h	
OE_t^h	

2. Risk Allocation

- Case 1b:
  - $\kappa_t = 1 \implies K_t^h = 0$
  - $\chi_t > \eta_t \implies D_t^e > 0$



2. Risk Allocation

- Case 2a:
  - $\kappa_t = 1 \implies K_t^h = 0$
  - $\chi_t = \eta_t \implies D_t^e = 0$

Experts

$q_t K_t$	N_t^e	OE_t^e

Households

OE_t^h	N_t^h

2. Risk Allocation

- Shock arrives at $t = 0$, destroys λK_0, $K_0^+ = (1-\lambda)K_0$
- Total wealth $N_0^+ = (1-\lambda)N_0$
- Experts wealth $N_0^{e,+} = N_0^e - \chi_0 \lambda N_0 = (1 - \lambda \frac{\chi_0}{\eta_0})N_0^e$
- Wealth share η:

$$\eta_0^+ = \frac{N_0^{e,+}}{N_0^+} = \frac{(1-\lambda\frac{\chi_0}{\eta_0})N_0^e}{(1-\lambda)N_0} = \frac{(1-\lambda\frac{\chi_0}{\eta_0})}{(1-\lambda)}\eta_0$$

- $\eta_0^+ < \eta_0$ if $\chi_0 > \eta_0$, and $\eta_0^+ = \eta_0$ if $\chi_0 = \eta_0$
- What can we infer about σ^η?

1. Stochastic Maximum Principle

- Experts' problem:

$$\max_{c^e,\theta^{e,K},\theta^{e,OE},\iota^e} \mathbb{E}\left[\int_0^\infty e^{-\rho^e t}\log(c_t^e)dt\right]$$

$$\text{s.t} \quad \frac{dn_t^e}{n_t^e} = -\frac{c_t^e}{n_t^e}dt + r_t dt + \theta_t^{e,K}\left(dr_t^{e,K}(\iota_t^e) - r_t dt\right) + \theta_t^{e,OE}\left(dr_t^{OE} - r_t dt\right)$$

$$\theta_t^{e,OE} + (1-\alpha)\theta_t^{e,K} \geq 0$$

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$$\theta_t^{e,OE} + (1-\alpha)\theta_t^{e,K} \geq 0$$

- Introduce new variable $\tilde{\chi}_t = \frac{\theta_t^{e,K}+\theta_t^{e,OE}}{\theta_t^{e,K}}$

$$\max_{c^e,\theta^{e,K},\tilde{\chi},\iota^e} \mathbb{E}\left[\int_0^\infty e^{-\rho^e t}\log(c_t^e)dt\right]$$

$$\frac{dn_t^e}{n_t^e} = -\frac{c_t^e}{n_t^e}dt + r_t dt + \theta_t^{e,K}\left[\left(dr_t^{e,K}(\iota_t^e) - r_t dt\right) - (1-\tilde{\chi}_t)\left(dr_t^{OE} - r_t dt\right)\right]$$

$$\tilde{\chi}_t \geq \alpha$$

1. Stochastic Maximum Principle

- Hamiltonian for experts:

$$H_t^e = e^{-\rho^e t}\log(c_t^e) + \xi_t^e\left[-c_t^e + n_t^e\left(r_t + \theta_t^{e,K}\left[\left(r_t^{e,K} - r_t\right) - (1-\tilde{\chi}_t)\left(r_t^{OE} - r_t\right)\right]\right)\right]$$
$$- \varsigma_t^e \xi_t^e n_t^e \theta_t^{e,K}\left[\sigma_t^{r,K} - (1-\tilde{\chi}_t)\sigma_t^{r,K}\right] + \xi_t^e n_t^e \lambda_t^e(\tilde{\chi}_t - \alpha)$$

- Experts' FOCs wrt $\theta_t^{e,K}$ and $\tilde{\chi}_t$:

$$\left(r_t^{e,K} - r_t\right) - (1-\tilde{\chi}_t)\left(r_t^{OE} - r_t\right) = \varsigma_t^e \tilde{\chi}_t \sigma_t^{r,K}$$

$$r_t^{OE} - r_t = \varsigma_t^e \sigma_t^{r,K} - \lambda_t^e$$

1. Stochastic Maximum Principle

- Hamiltonian for households:

$$H_t^h = e^{-\rho^h t}\log(c_t^h) + \xi_t^h\left[-c_t^h + n_t^h\left(r_t + \theta_t^{h,K}\left(r_t^{h,K} - r_t\right) + \theta_t^{h,OE}\left(r_t^{OE} - r_t\right)\right)\right]$$
$$- \varsigma_t^h \xi_t^h n_t^h\left[\theta_t^{h,K} + \theta_t^{h,OE}\right]\sigma_t^{r,K} + \xi_t^h n_t^h \lambda_t^h \theta_t^{h,K}$$

- Households' FOCs wrt $\theta_t^{h,K}$ and $\theta_t^{h,OE}$:

$$r_t^{h,K} - r_t = \varsigma_t^h \sigma_t^{r,K} - \lambda_t^h$$
$$r_t^{OE} - r_t = \varsigma_t^h \sigma_t^{r,K}$$

1. Stochastic Maximum Principle

$$\left(r_t^{e,K} - r_t\right) - (1 - \tilde{\chi}_t)\left(r_t^{OE} - r_t\right) = \varsigma_t^e \tilde{\chi}_t \sigma_t^{r,K}$$

$$r_t^{OE} - r_t = \varsigma_t^e \sigma_t^{r,K} - \lambda_t^e$$

$$r_t^{h,K} - r_t = \varsigma_t^h \sigma_t^{r,K} - \lambda_t^h$$

$$r_t^{OE} - r_t = \varsigma_t^h \sigma_t^{r,K}$$

- If $\lambda_t^h = 0$ and $\theta^{h,K} > 0 \Longrightarrow \kappa_t < 1$, $\lambda_t^e > 0$, and $\tilde{\chi}_t = \alpha$, $(\chi_t = \alpha\kappa_t)$
- If $\lambda_t^e = 0$ and $\tilde{\chi}_t > \alpha$, $(\chi_t = \alpha\kappa_t) \Longrightarrow \lambda_t^h > 0$, $\theta^{h,K} = 0$, and $\kappa_t = 1$
- $\lambda_t^e > 0$ and $\lambda_t^h > 0$ possible

Change of Numeraire in SMP

- Let $\hat{x}_t$ be x_t in N_t-numeraire

$$\max_{c^e,\theta^{e,K},\tilde{\chi},\iota^e} \mathbb{E}\left[\int_0^\infty e^{-\rho^e t}\log(c_t^e)dt\right]$$

$$\frac{d\hat{n}_t^e}{\hat{n}_t^e} = -\frac{\hat{c}_t^e}{\hat{n}_t^e}dt + d\hat{r}_t + \theta_t^{e,K}\left[\left(d\hat{r}_t^{e,K}(\iota_t^e) - d\hat{r}_t\right) - (1-\tilde{\chi}_t)\left(d\hat{r}_t^{OE} - d\hat{r}_t\right)\right]$$

$$\tilde{\chi}_t \geq \alpha$$

Change of Numeraire in SMP

- Let $\hat{x}_t$ be x_t in N_t-numeraire

$$\max_{c^e,\theta^{e,K},\tilde{\chi},\iota^e} \mathbb{E}\left[\int_0^\infty e^{-\rho^e t}\log(c_t^e)dt\right]$$

$$\frac{d\hat{n}_t^e}{\hat{n}_t^e} = -\frac{\hat{c}_t^e}{\hat{n}_t^e}dt + d\hat{r}_t + \theta_t^{e,K}\left[\left(d\hat{r}_t^{e,K}(\iota_t^e) - d\hat{r}_t\right) - (1-\tilde{\chi}_t)\left(d\hat{r}_t^{OE} - d\hat{r}_t\right)\right]$$

$$\tilde{\chi}_t \geq \alpha$$

$$\left(\hat{r}_t^{e,K} - \hat{r}_t\right) - (1-\tilde{\chi}_t)\left(\hat{r}_t^{OE} - \hat{r}_t\right) = \hat{\varsigma}_t^e\tilde{\chi}_t\left(\hat{\sigma}_t^{r,K} - \hat{\sigma}_t^r\right)$$

$$\hat{r}_t^{OE} - \hat{r}_t = \hat{\varsigma}_t^e\left(\hat{\sigma}_t^{r,K} - \hat{\sigma}_t^r\right) - \hat{\lambda}_t^e$$

Change of Numeraire in SMP

$$\frac{d\eta_t}{\eta_t} = \frac{d\hat{n}_t^e}{\hat{n}_t^e} = \underbrace{\left[-\rho^e + \hat{r}_t + \theta_t^{e,K} \left[\left(\hat{r}_t^{e,K}(\iota_t^e) - \hat{r}_t \right) - (1 - \tilde{\chi}_t) \left(\hat{r}_t^{OE} - \hat{r}_t \right) \right] \right]}_{\mu_t^\eta} dt$$

$$+ \underbrace{\left[\hat{\sigma}_t^r + \theta_t^{e,K} \tilde{\chi}_t \left(\hat{\sigma}_t^{r,K} - \hat{\sigma}_t^r \right) \right]}_{\sigma_t^\eta} dZ_t$$

$$= \underbrace{\left[-\rho^e + \hat{r}_t + \theta_t^{e,K} \hat{\varsigma}_t^e \tilde{\chi}_t \left(\hat{\sigma}_t^{r,K} - \hat{\sigma}_t^r \right) \right]}_{\mu_t^\eta} dt + \sigma_t^\eta dZ_t \quad \Longrightarrow$$

$$\mu_t^\eta + \rho^e = \hat{r}_t + \hat{\varsigma}_t^e \left(\sigma_t^\eta - \hat{\sigma}_t^r \right)$$

2. Numerical Solution: General Preferences

- Individual choice of ι

$$\frac{1}{q_t} = \Phi'(\iota_t) \tag{1}$$

- Goods market clearing:

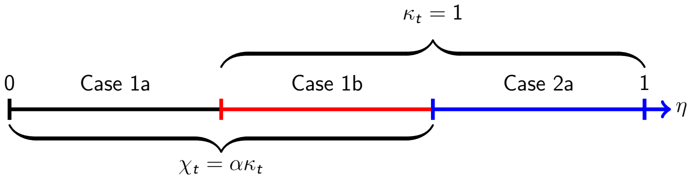
$$\kappa_t a^e + (1-\kappa_t)a^h - \iota_t = q_t \left(\eta_t \frac{C_t^e}{N_t^e} + (1-\eta_t)\frac{C_t^h}{N_t^h} \right) \tag{2}$$

- Volatility of q:

$$\sigma + \sigma_t^q = \frac{\sigma}{1 - \frac{q'(\eta_t)}{q(\eta_t)}(\chi_t - \eta_t)} \tag{3}$$

2. Numerical Solution: General Preferences

- Individual choice of $\theta \implies$ Asset/Risk Allocation χ, κ



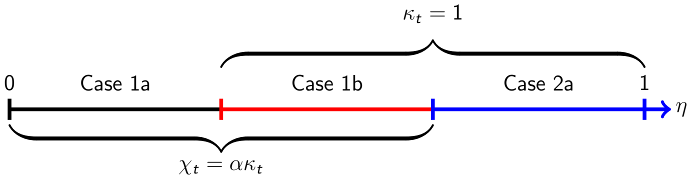
$$\chi_t = \alpha\kappa_t, \qquad a^e - a^h = \alpha(\varsigma_t^e - \varsigma_t^h)(\sigma + \sigma_t^q)q_t \qquad (4.1a + 5.1a)$$

$$\chi_t = \alpha\kappa_t, \qquad \kappa_t = 1 \qquad (4.1b + 5.1b)$$

$$\kappa_t = 1, \qquad (\varsigma_t^e - \varsigma_t^h)(\sigma + \sigma_t^q) = 0 \qquad (4.2a + 5.2a)$$

2. Numerical Solution: Log Utility

- Individual choice of $\theta \implies$ Asset/Risk Allocation χ, κ



$$\chi_t = \alpha\kappa_t, \qquad a^e - a^h = \alpha\frac{\chi_t - \eta_t}{\eta_t(1-\eta_t)}(\sigma + \sigma_t^q)^2 q_t \qquad (4.1a + 5.1a)$$

$$\chi_t = \alpha\kappa_t, \qquad \kappa_t = 1 \qquad (4.1b + 5.1b)$$

$$\kappa_t = 1, \qquad \frac{\chi_t - \eta_t}{\eta_t(1-\eta_t)}(\sigma + \sigma_t^q)^2 = 0 \qquad (4.2a + 5.2a)$$

2. Numerical Solution

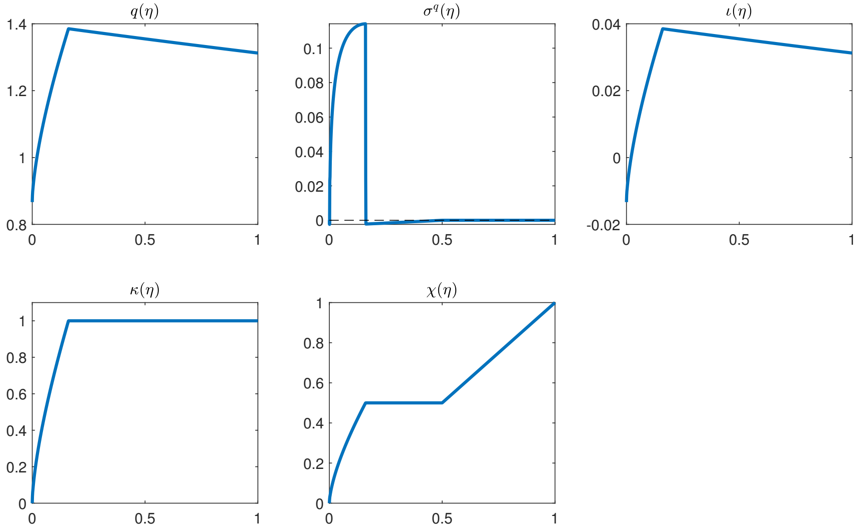
- Absorbing states $\eta = 0$ and $\eta = 1$: $\sigma^q(0) = \sigma^q(1) = 0$, $\chi(0) = \kappa(0) = 0$, $\chi(1) = \kappa(1) = 1$, $q(0) = \frac{1+a^h\phi}{1+\rho^h\phi}$, $q(1) = \frac{1+a^e\phi}{1+\rho^e\phi}$
- Grid $\{\eta_1 = 0.0001, \ldots, \eta_N = 0.9999\}$ and ODE:

$$\sigma + \sigma_t^q = \frac{\sigma}{1 - \frac{q'(\eta_t)}{q(\eta_t)}(\chi_t - \eta_t)}$$

$$\sigma + \sigma_i^q = \frac{\sigma}{1 - \frac{q_i - q_{i-1}}{\eta_i - \eta_{i-1}} \frac{\chi_i - \eta_i}{q_i}}$$

with initial condition $q_0 = \frac{1+a^h\phi}{1+\rho^h\phi}$

2. Numerical Solution



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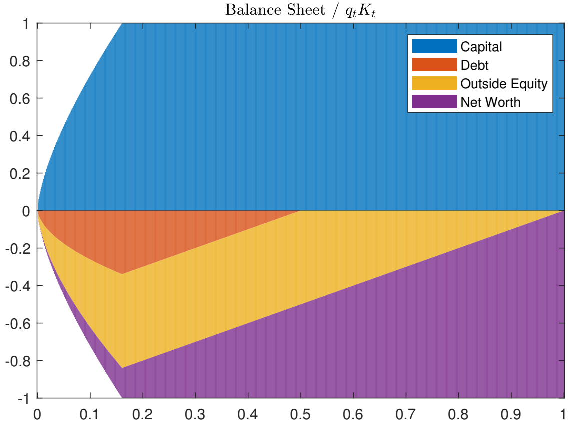

2. Numerical Solution