

Macro, Money and (International) Finance – Problem Set 2

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Prepared by Andrey Alexandrov (alexandrov@princeton.edu). Please let me know if any tasks are unclear or you find mistakes in the problem descriptions.

Please submit to alexandrov@princeton.edu by 1pm on Monday, September 26 (Princeton time).

1 Stochastic Maximum Principle

The goal of this exercise is to derive the martingale pricing conditions and the risk and capital allocation conditions (due to Fisher Separating Theorem) of the model in Lecture 5 using the Stochastic Maximum Principle. To ease notation, let $dr_t^{K,i}$ be the return on capital for sector $i \in \{e, h\}$, dr_t^{OE} – return on outside equity, and $r_t dt$ – on riskless debt:

$$\begin{aligned} dr_t^{K,i} &= r_t^{K,i} dt + \sigma_t^{r,K} dZ_t \\ dr_t^{OE} &= r_t^{OE} dt + \sigma_t^{r,K} dZ_t \\ \frac{dn_t^i}{n_t^i} &= -\frac{c_t^i}{n_t^i} dt + r_t dt + \theta_t^{i,K} \left(dr_t^{K,i} - r_t dt \right) + \theta_t^{i,OE} \left(dr_t^{OE} - r_t dt \right) \end{aligned}$$

1. Set up Hamiltonians for both agent types and derive the martingale pricing conditions. Remember that experts face equity issuing constraint $\theta_t^{e,OE} \geq (\alpha - 1)\theta_t^{e,K}$ and households can not short capital ($\theta^{h,K} \geq 0$).¹ Keep the assumption of log utility and take $c_t^i = \rho^i n_t^i$ as given.
2. Combine households' and experts' pricing conditions to recover the equilibrium risk and capital allocations. Describe the three cases (1a, 1b, 2a) in terms of portfolio weights.

¹Of course there are other constraints, but only these two turn out to be occasionally binding, whereas the rest are always slack.

2 Risk Allocation

1. Denote the value of outside equity issued by experts as OE_t^e such that $OE_t^e \geq 0$, and the value of debt issued as $D_t^e \geq 0$. In this case:

$$\eta_t = \frac{N_t^e}{N_t}, \quad \kappa_t = \frac{K_t^e}{K_t}$$

$$\theta_t^{e,K} = \frac{q_t K_t^e}{N_t^e}, \quad \theta_t^{e,OE} = -\frac{OE_t^e}{N_t^e}, \quad \theta_t^{e,D} = -\frac{D_t^e}{N_t^e} = 1 - \theta_t^{e,K} - \theta_t^{e,OE}$$

Using the definition of $\chi_t = \frac{\sigma_{N^e,t}}{\sigma_{qK,t}}$, derive:

- (a) Mapping from portfolio weights to χ_t and κ_t
- (b) χ_t as a function of q_t, K_t, K_t^e, OE_t^e
- (c) $\frac{\chi_t}{\kappa_t}$ as a function of q_t, K_t^e, OE_t^e
- (d) Experts' Debt-to-Equity ratio $\frac{D_t^e}{N_t^e}$ as a function of χ_t and η_t
- (e) Experts' Loan-to-Value ratio $\frac{D_t^e}{q_t K_t^e}$ as a function of χ_t, κ_t and η_t .

What is the economic interpretation of χ_t and $\frac{\chi_t}{\kappa_t}$?

2. Depict or describe the balance sheets of experts and households in three regions: Case 1a, Case 1b and Case 2a. You can use your answer from Problem 1.3 or solve the model in Problem 3.1 and simply plot the balance sheets.
3. Suppose a sudden shock destroys a (small) fraction λ of aggregate capital such that capital losses are distributed proportionately to capital holdings (experts lose λK_t^e and households lose λK_t^h). Express the wealth losses of experts and households at the instant when the shock arrives² using the pre-shock values of aggregate variables. Derive the effect on η . What does it depend on and why?

3 Numerical Solution

In this exercise you will solve the model from Lecture 5 numerically, under the assumption of log utility.

1. Our goal is to construct functions $q(\eta), \iota(\eta), \chi(\eta), \kappa(\eta)$ and $\sigma^a(\eta)$ on the $[0, 1]$ grid. Slide 68 provides the parameter values (ignore γ), and slide 65 provides the set of equations and the algorithm.
 - (a) Solve the model at the boundaries: for $\eta = 0$ and $\eta = 1$.
 - (b) Create a uniform grid for $\eta \in [0.0001, 0.9999] = \{\eta_1 = 0.0001, \eta_2, \dots, \eta_N = 0.9999\}$.
 - (c) Using the implicit method with the one-step Newton's algorithm, solve the system of equations on slide 65 (with $\chi = \alpha\kappa$) for η_1, η_2, \dots and so on.
 - (d) Stop once you reach $\kappa \geq 1$. From here on, set $\kappa = 1$ and $\chi = \max\{\alpha\kappa, \eta\}$, solve for q and σ^a .

²That is: after the shock reduced capital, but before prices adjusted.

- (e) Verify your solution by plotting $q(\eta)$ and $\sigma^q(\eta)$ and comparing it with the graph on slide 68 (you won't get an exact match since on slide 68 $\gamma = 2$, but the shape will be similar). Do your functions converge to the boundary solution for $\eta = 1$ that you obtained in (a) as $\eta \rightarrow 1$?
- (f) Plot the remaining variables: $\iota(\eta), \kappa(\eta), \chi(\eta)$.
- (g) We can also look at the experts' balance sheet: derive expressions for the scaled versions of issued debt and outside equity: $\frac{D_t^e}{q_t K_t}, \frac{OE_t^e}{q_t K_t}$ and plot them against η .

2. Recall from the lecture that drift and volatility of η in the general case are given by:

$$\mu_t^\eta = (1 - \eta_t) \left[(\zeta_t^e - \sigma - \sigma_t^q)(\sigma_t^\eta + \sigma + \sigma_t^q) - (\zeta_t^h - \sigma - \sigma_t^q) \left(-\frac{\eta_t}{1 - \eta_t} \sigma_t^\eta + \sigma + \sigma_t^q \right) - \left(\frac{C_t^e}{N_t^e} - \frac{C_t^h}{N_t^h} \right) \right]$$

$$\sigma_t^\eta = \frac{\chi_t - \eta_t}{\eta_t} (\sigma + \sigma_t^q)$$

- (a) Which terms in the above equations can we simplify/substitute because of log utility and why? Perform these substitutions and derive the drift and volatility of η under log utility.
- (b) Verify your solution by plotting $\eta\mu^\eta(\eta)$ and $\eta\sigma^\eta(\eta)$ and comparing them with the graph on slide 69 (you can plot for $\sigma = 0.1$ only and should expect a similar shape).