Modern Macro, Money, and International Finance

Eco529

Lecture 02: Why Continuous Time Modeling?

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Cts.-time Macro: Macro-Finance vs HANK

Agents:	**Heterogenous investor focus** **- Net worth distribution (often discrete)**	**Heterogenous consumer focus** **- Net worth distribution (often cts.)**
Tradition:	Finance (Merton) *PORTFOLIO AND CONSUMPTION CHOICE*	DSGE (Woodford) *CONSUMPTION CHOICE*
	Full/global dynamical system - focused on non-linearities away from steady state (crisis …) - Length of recession is stochastic	Transition dynamics back to steady state - Zero probability shock - Length of recession is deterministic
Money due to:	Risk & financial frictions	Price stickiness
Risk:	Risk & financial frictions	No aggregate risk (in HANK paper)
Price of risk:	Idiosyncratic & aggregate risk	
Assets:	Capital, money, bonds with different risk profile - Risk-return trade-off - Liquidity-return trade-off - Flight to safety	All assets are risk-free - No risk-return trade-off - Liquidity-return trade-off

Financial Frictions and Distortions

- Belief distortions
  - Match “belief surveys”

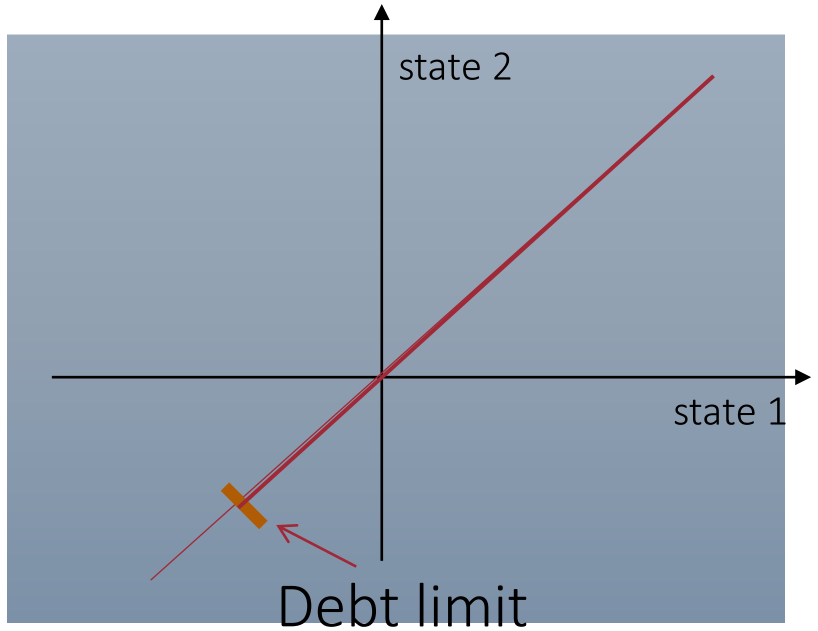
- Incomplete markets
  - “natural” leverage constraint *(BruSan)*
  - Costly state verification *(BGG)*

- + Leverage constraints
  (no “liquidity creation”)
  - Exogenous limit *(Bewley/Ayagari)*

  - Collateral constraints
    - Next period’s price *(KM)*

      $$Rb_t \leq q_{t+1}k_t$$

    - Next periods volatility *(VaR, JG)*
    - Current price



Debt limit
can depend on
prices/volatility

- Search Friction *(DGP)*

Why Continuous Time Modeling?

- Time aggregation
  - Data come in different frequency
    - GDP quarterly
    - High frequency financial data

- Consumption
  - Same IES within and across periods
  - Discrete time consumption
    - IES/RA within period $= \infty$, but across periods $= 1/\gamma$

- Optimal Stopping problems – no integer issues

- Sharp distinction between stock and flow (rate)
  - Beginning of period = end of period wealth
    - E.g. consumption = time-preference rate * end of period wealth

Why Continuous Time Modeling?

- Ito processes… fully characterized by drift and volatility

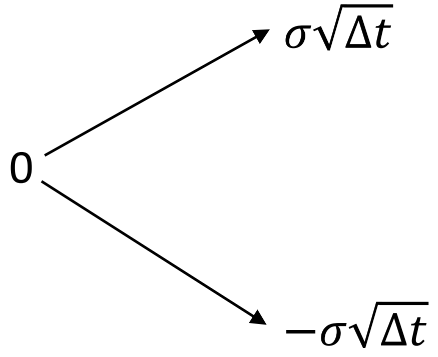
$$dX_t = \mu(X_t, t)dt + \sigma(X_t, t)dZ_t$$

  - Arithmetic Ito Process $dX_t = \mu_t^X dt \quad + \sigma_t^X dZ_t$
  - Geometric Ito Process $dX_t = \mu_t^X X_t dt + \sigma_t^X X_t dZ_t$

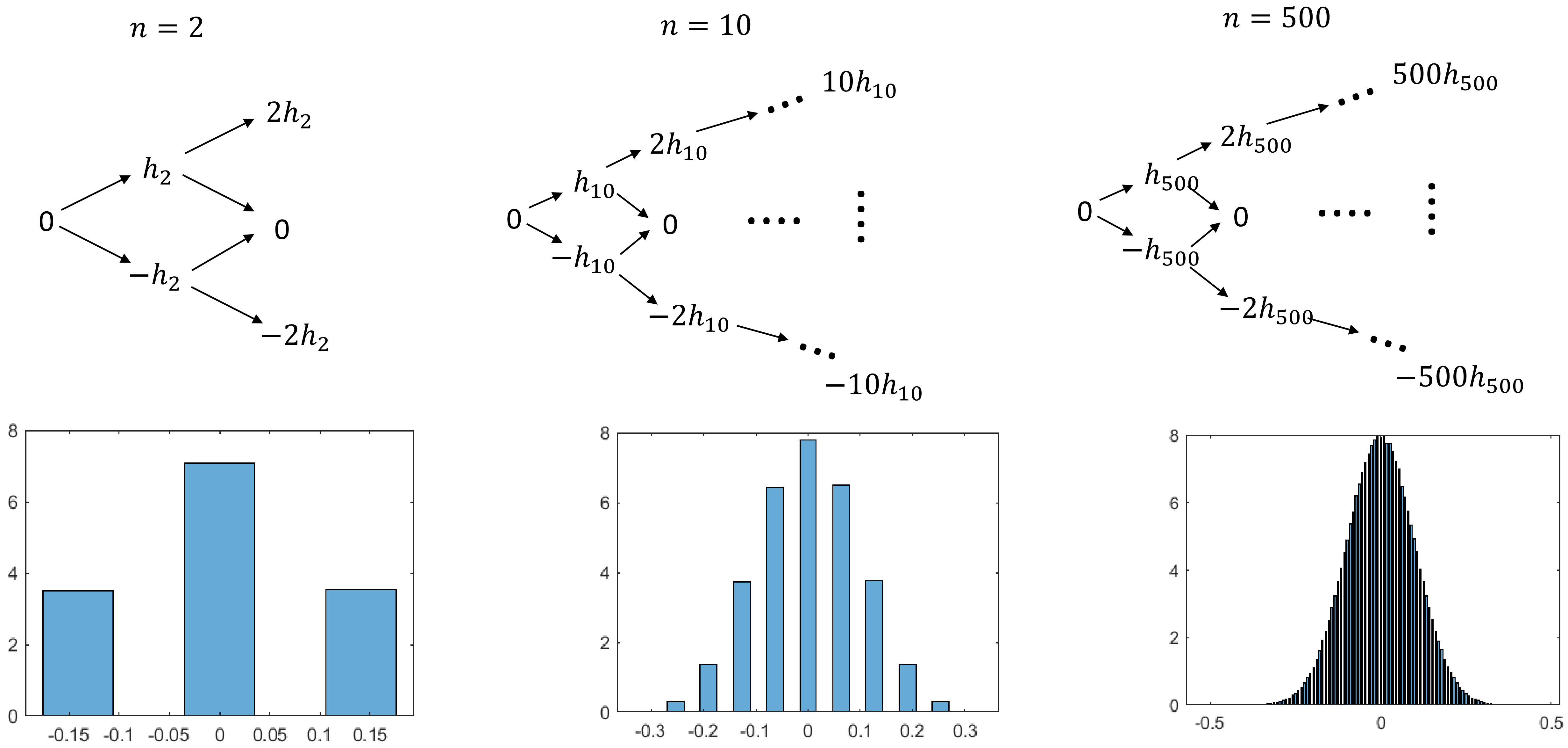
- Characterization for full volatility dynamics on Prob.-space
  - Discrete time: Probability-loading on states
    conditional expectations $E[X|Y]$ difficult to handle
  - Cts. time: Loading on a Brownian Motion dZ_t (captured by σ)

Why Continuous Time Modeling?

- Brownian Motion as a binomial tree over Δt



- More steps with shrinking step size: $h_n = \sigma\sqrt{\Delta t/n}$



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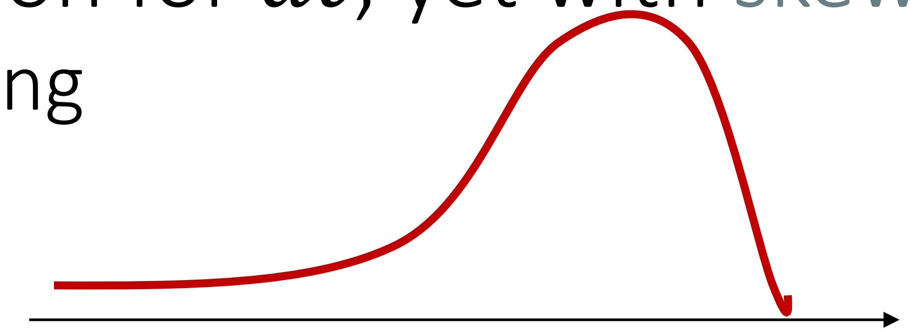
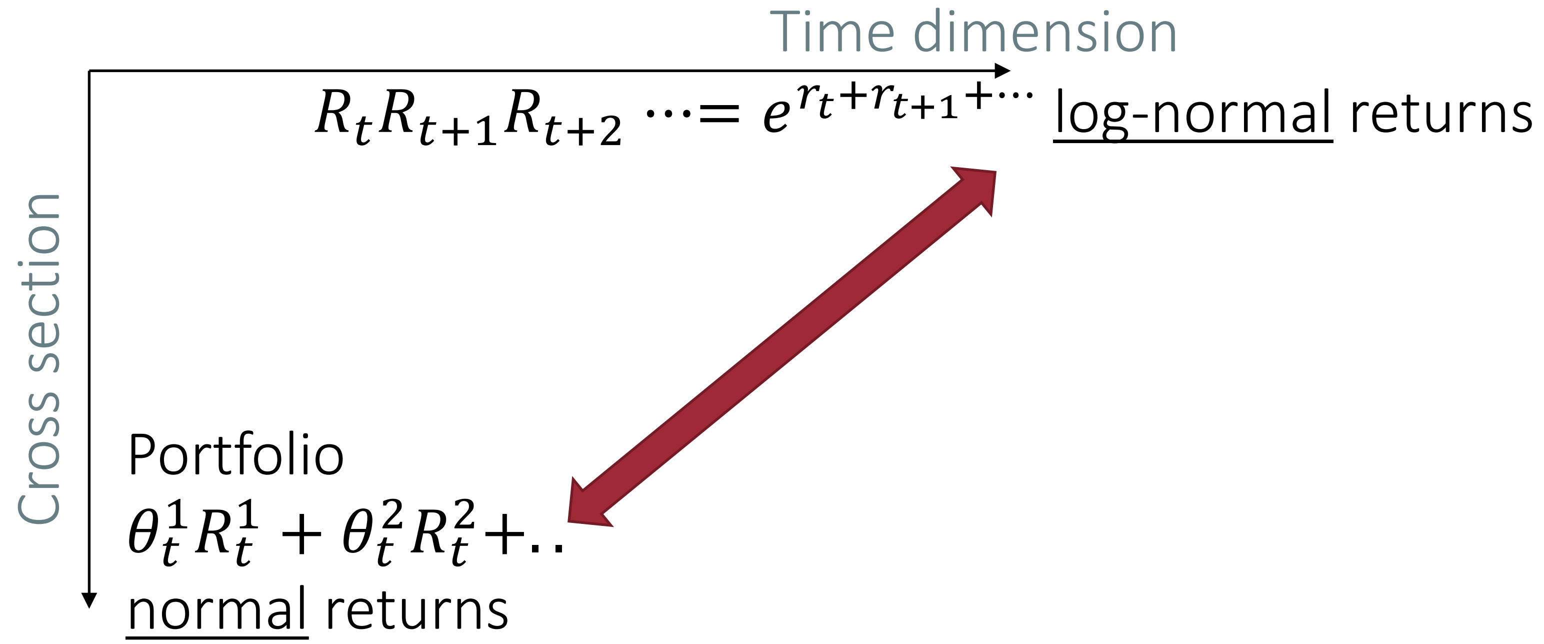
    conditional expectations $E[X|Y]$ difficult to handle
  - Cts. time: Loading on a Brownian Motion dZ_t (captured by σ)
- Normal distribution for dt, yet with skewness for $\Delta t > 0$
  - If σ_t is time-varying



  - E.g. from normal-dt to log-normal-Δt and vice versa (geom dX_t.)

Why Continuous Time Modeling?

- Portfolio choice – tension in discrete time



  - Linearize: kills σ-term, all assets are equivalent
  - 2nd order approximation: kills time-varying σ
  - Log-linearize a la Campbell-Shiller

- As $\Delta t \rightarrow 0$ (log) returns converge to normal distribution
  - Constantly adjust the approximation point
  - Nice formula for portfolio choice for Ito process

Why Continuous Time Modeling?

- Consumption choice
  - Nice process
    - consumption/wealth ratio is constant for log-utility, e.g. for log-utility $C_t = \rho N_t$
    - Beginning = end of period net worth/wealth

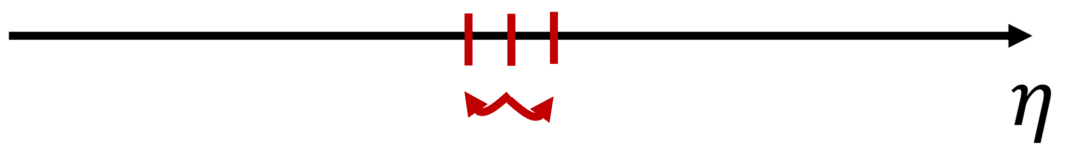
- Evolution of wealth (shares)/distribution
  - Nice characterization
  - Evolution of distributions (e.g. wealth distribution) characterized by Kolmogorov Forward Equation (Fokker-Planck equation)

Why Continuous Time Modeling with Ito?

- Continuous path
  - Information arrives continuously "smoothly" – not in lumps
  - Implicit assumption: can react continuously to continuous info flow
  - Never jumps over a specific point, e.g. insolvency point
  - Simplifies numerical analysis:
    - Only need change from grid-point to grid-point (since one never jumps beyond the next grid-points)
  - No default risk
    - Can continuously delever as wealth declines
      - Might embolden investors ex-ante
  - Collateral constraint
    - Discrete time: $b_t R_{t,t+1} \leq \min\{q_{t+1}\} k_t$
    - Cts. time: $b_t \leq (p_t + \underbrace{dp_t}_{\to 0}) k_t$
      - For short-term debt – not for long-term debt … or if there are jumps
- Levy processes… with jumps
  - Still price of risk * risk, but not linear

Why Continuous Time Modeling with Ito?

- $E[dV(\eta)] = V'(\eta)\mu^{\eta}\eta dt + \frac{1}{2}V''(\eta)(\sigma^{\eta})^2\eta^2 dt$



Just need the two neighboring grid points (not whole function)

- More analytical steps
  - Return equations
    - Next instant returns are essentially normal (easy to take expectations)
  - Explicit net worth and state variable dynamics
    - Continuous: only slope of price function determines amplification
    - Discrete: need whole price function (as jump size can vary)