Modern Macro, Money, and International Finance

Eco 529

Lecture 10: One Sector Monetary Model with Heterogenous Agents

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Roadmap

- Intuition for different “Monetary Theories”

- Monetary Model with one sector with constant idiosyncratic risk
  - Safe Asset and Service Flows
  - Bubble (mining) or not
  - 2 Different Asset Pricing Perspectives/SDFs

- Monetary model with time-varying idiosyncratic risk
  - Safe asset has negative CAPM-β
  - Calibration:
    Debt valuation puzzle, Debt Laffer Curve, Flight-to-Safety and Equity excess volatility

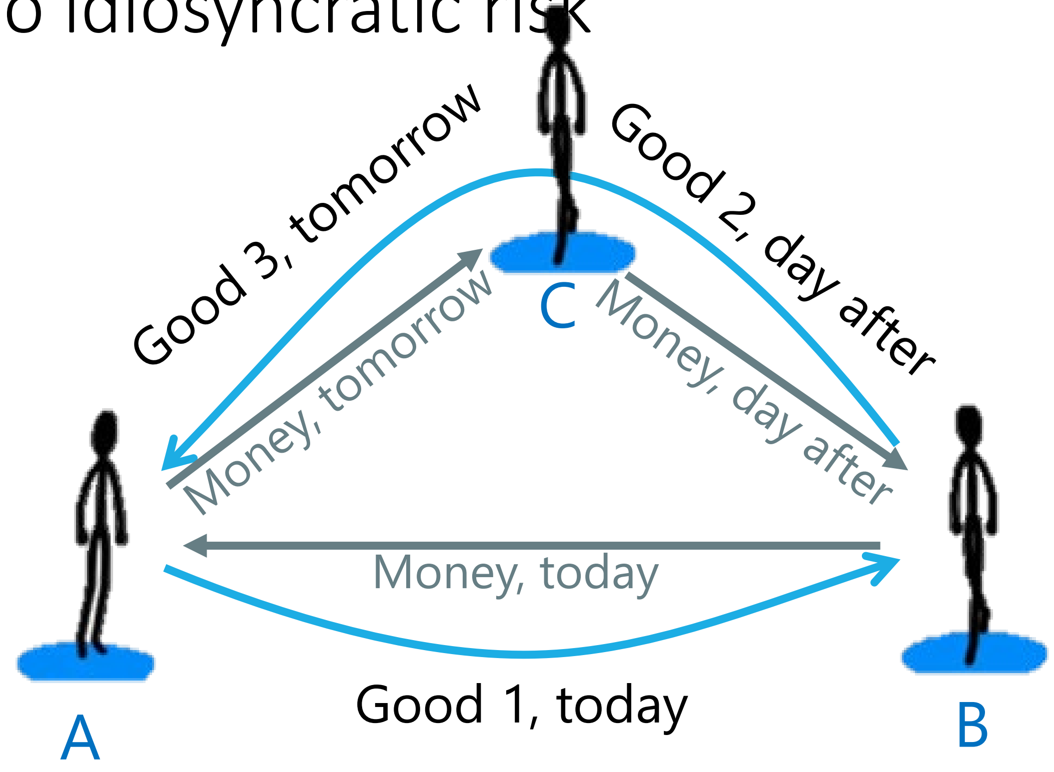
- Medium of Exchange Role

The 3 Roles of Money

- Unit of account
  - Intratemporal: Numeraire — bounded rationality/price stickiness
  - Intertemporal: Debt contract — incomplete markets

- Store of value
  - "I Theory of Money without I"
    Less risky than other "capital" – no idiosyncratic risk
  - Fiscal theory of the price level

- Medium of exchange
  - Overcome double-coincidence of wants problem



- *Record keeping device – money is memory*
  - *Virtual ledger*

Money versus Credit

- Credit zero net supply

- Money positive net supply

- “Medium of Exchange Money” (double-coincidence of wants)
  - Credit renders money useless

- “Store of value Money” (safe asset)
  - Money/gov. debt still useful if bubble since it “partially completes markets”
    - Incomplete markets
    - OLG

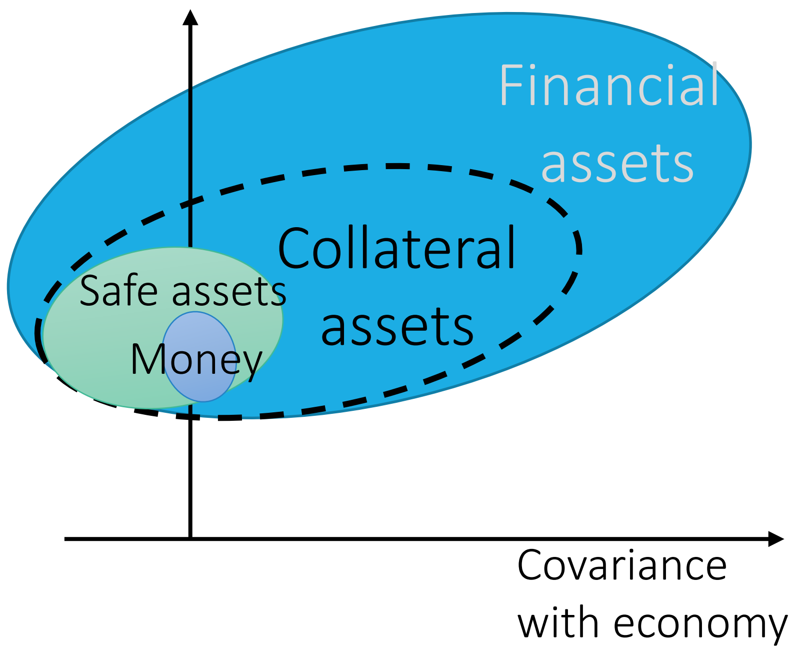
Money, Reserves, and Government Debt

- Cash: extra convenience yield and zero interest ⇒ lower return by Δi
    - Erodes due to Fintech revolution
- Reserves: Interest bearing $M \quad (\Delta i)$
    - Special form of
        - Infinite maturity more like equity (no rollover risk)
        - Zero duration more like overnight debt
        - Banking system can't offload it
            - Financial Repression

- Is QE simply swapping one debt for another one (reserves)?

(Narrow) Money, Gov. Debt, Safe Assets

- Service flows ⊇ convenience yield
  1. Safe asset: [good friend analogy]
     - When one needs funds, one can sell at stable price … since others buy
     - Partial insurance through retrading - market liquidity
  2. Collateral: relax constraints (Lagrange multiplier)
  3. Money (narrow): relax double-coincidence of wants



- Higher Asset Price = lower expected return - **exorbitant privilege**

Precautionary savings/self-insurance

$$\underbrace{\underset{\text{Preference rate}}{\rho} + \underset{\text{Ramsey term}}{\gamma\mathbb{E}[g_c]} - \tfrac{1}{2}\gamma(\gamma+1)\{\underset{\textit{Aggregate Risk}}{Var_t[g_c]} + \underset{\textit{Idiosyncratic Risk}}{Var_t[\tilde{g}_c]}\}}_{r^f} - \{\,\lambda(\text{Collateral Constr})\} - \Delta i + \underset{\text{(inflation risk + loss of safe asset status)}}{\text{risk permium}}$$

- Problem: safe asset + money status might burst like a bubble
  - Multiple equilibria: [safe asset tautology]

Inflation Theories

- Fiscal Theory of the Price Level (FTPL) *store of value*
  - Price level is determined by $\frac{\mathcal{B}_t}{\wp_t} = E_t[PV(\text{primary surpluses})] + \cdots$
  - Fiscal vs. Monetary Dominance + Financial Dominance
- Monetarism – assumes Monetary Dominance
  - Fiscal implications of monetary policy induces government to change lump-sum taxes
    - so that per-period budget constraint holds (is unaltered)
  - Money $\mathcal{M}_t/\wp_t$ serves as *medium of exchange*
    - Cash-in-advance constraint, transaction cost, shopping time model, …
    - $\Rightarrow \; \Delta i$ (convenience yield)
  - Price level is determined by $\mathcal{M}_t v = \wp_t Y_t$ (if velocity, v is constant)
- (New) Keynesianism – assumes Monetary Dominance
  - Cashless limit
  - Interest rate policy
    - All price/inflation paths are explosive except for one
      - (Cochrane: What's wrong with explosive nominal paths?)

Price Stickiness and Phillips Curve

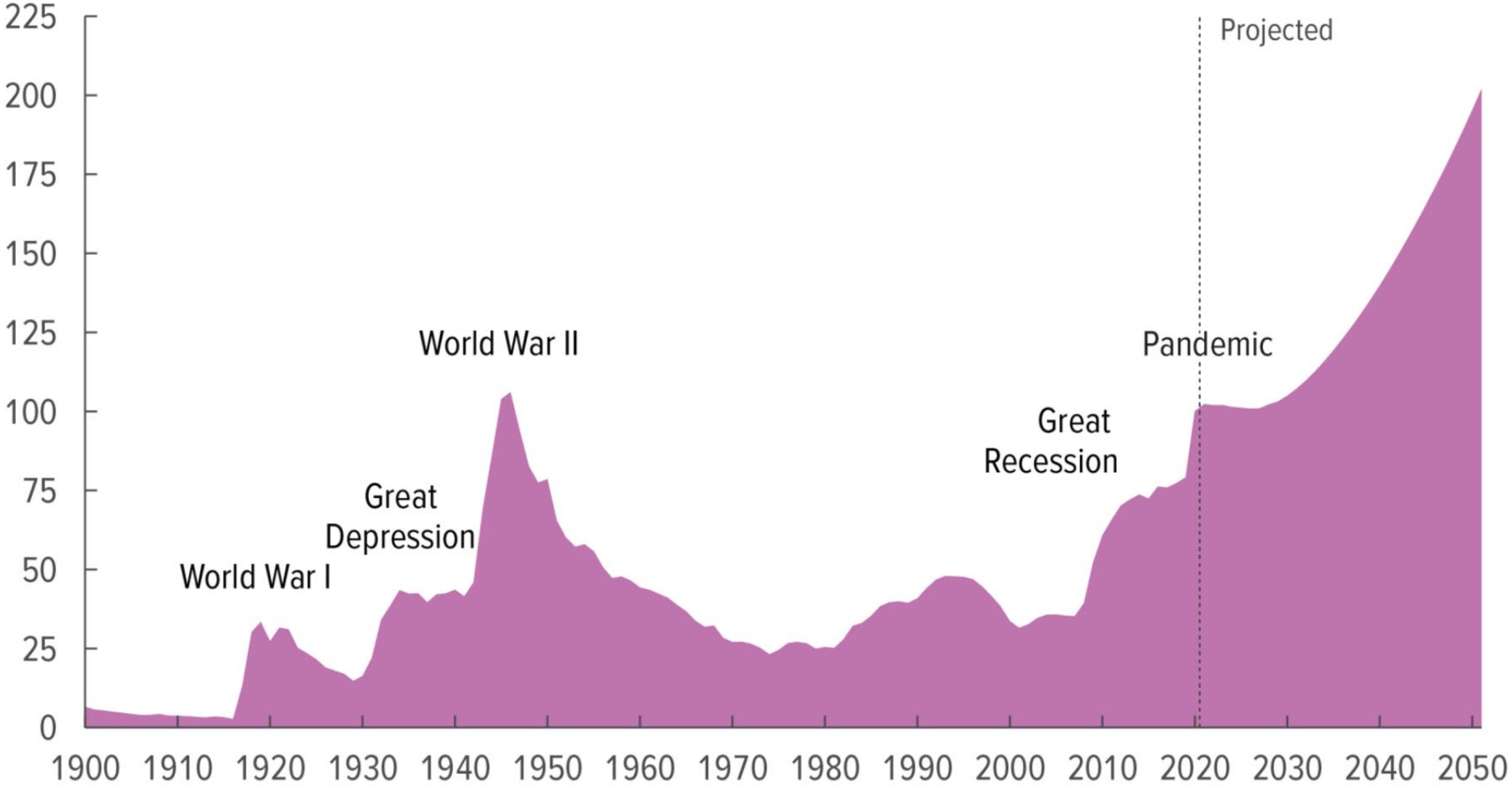
- Flexible prices: Prices adjust immediately

- Sticky prices:
  - Since prices adjust sluggishly, output has to adjust
    - Inflation pressure: prices too low during transition period,
      output (demand) overshoots natural (= flexible price) level
    - Deflation pressure: prices too high during transition period,
      output (demand) undershoots natural level
  - Sticky price models smooth out adjustment dynamics relative to equivalent flex price models

Inflation & Gov. Debt

Percentage of Gross Domestic Product

CBO projection, May 2021



FTPL: US: Inflation – Fiscal Link

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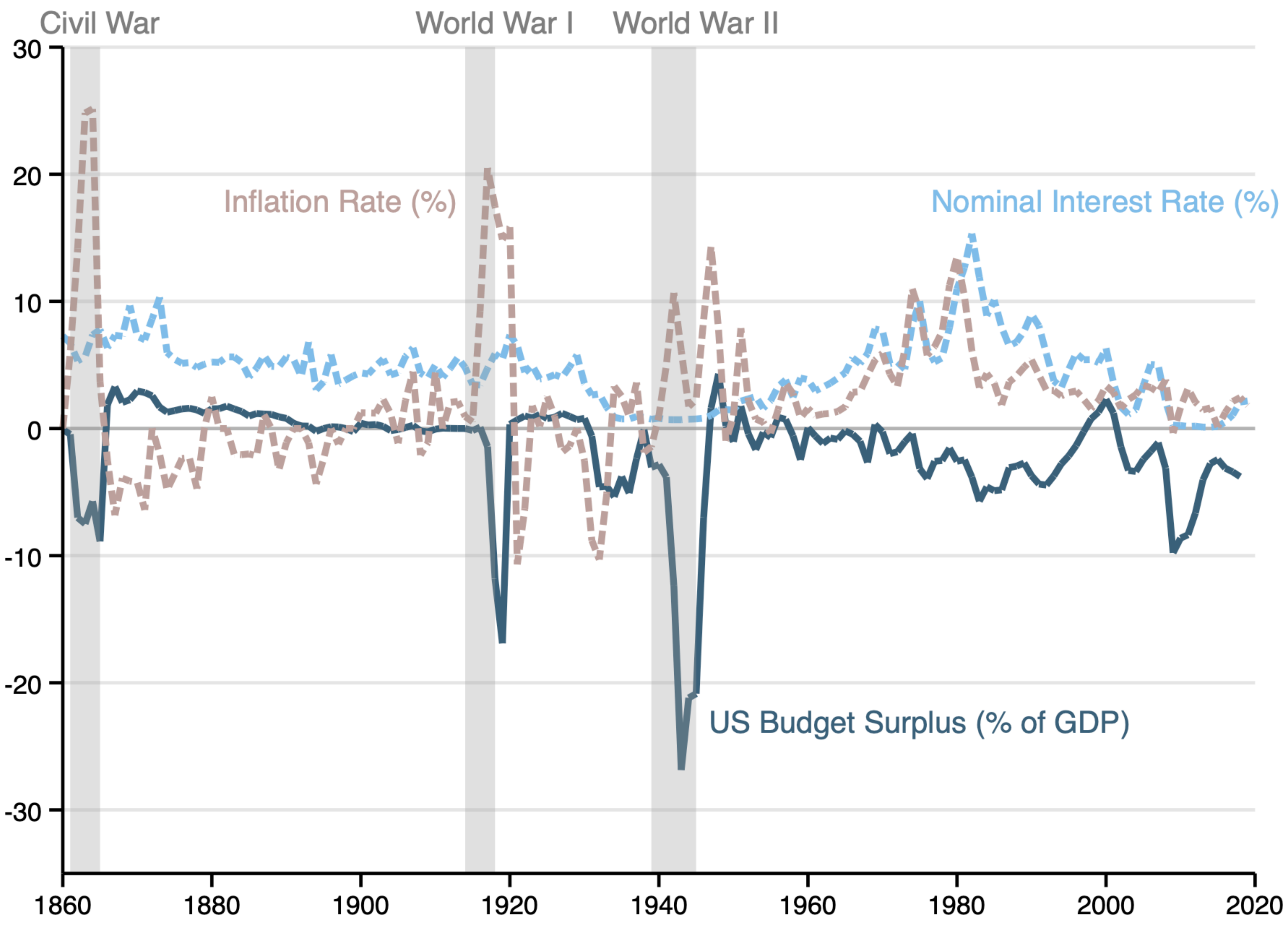
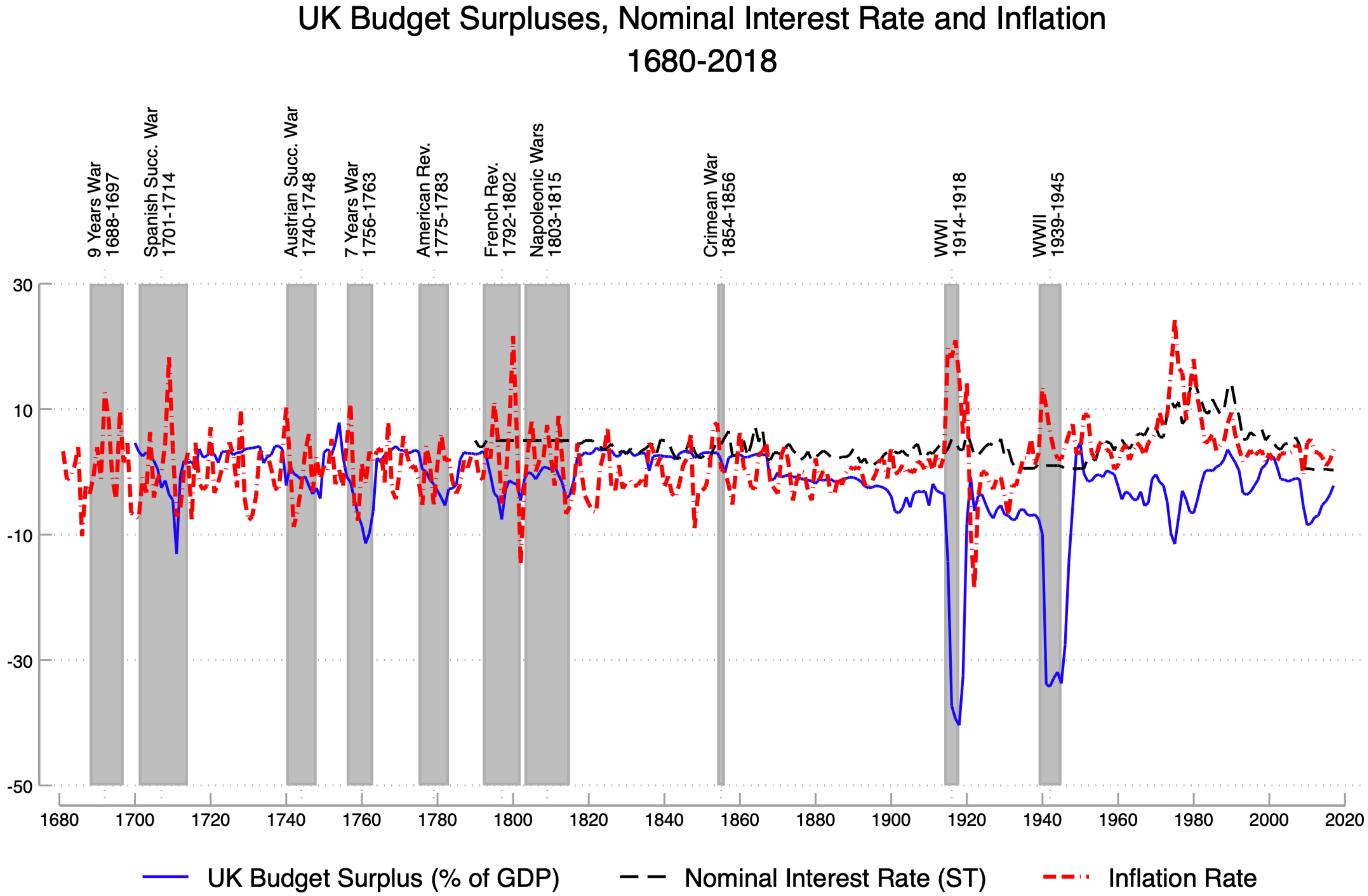


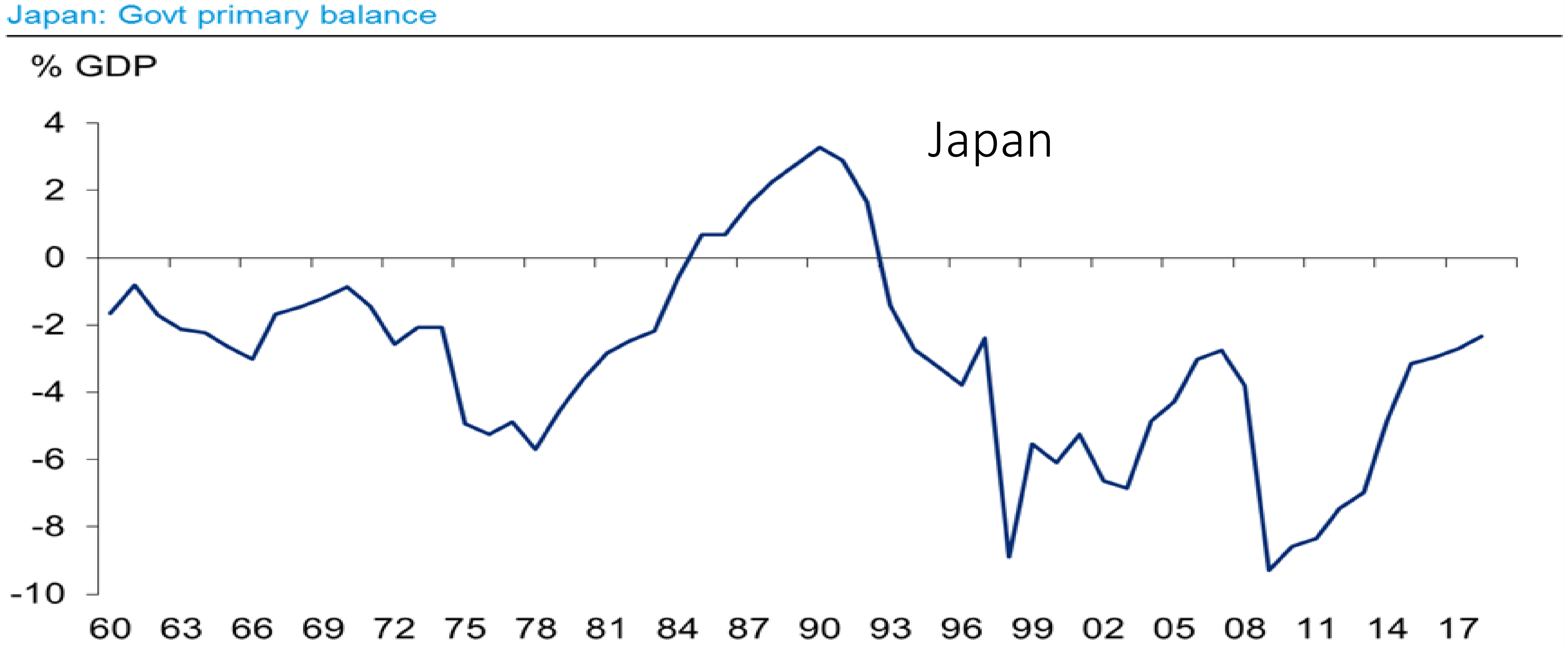
FTPL: UK: inflation-fiscal link + wars



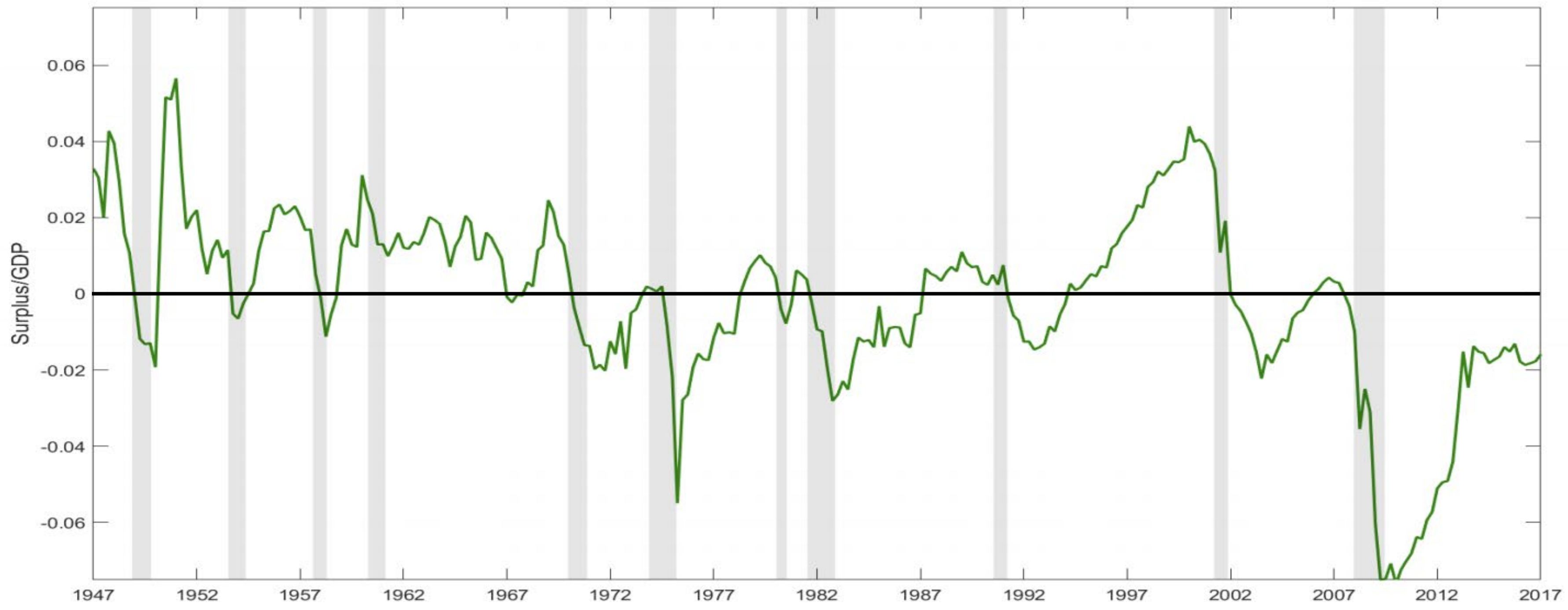
Source: ukpulicrevenues.co.uk, MeasuringWorth.com, Young (1925), Maddison (2010), Schmelzing (2020)

FTPL: Valuating Government Debt: Japan

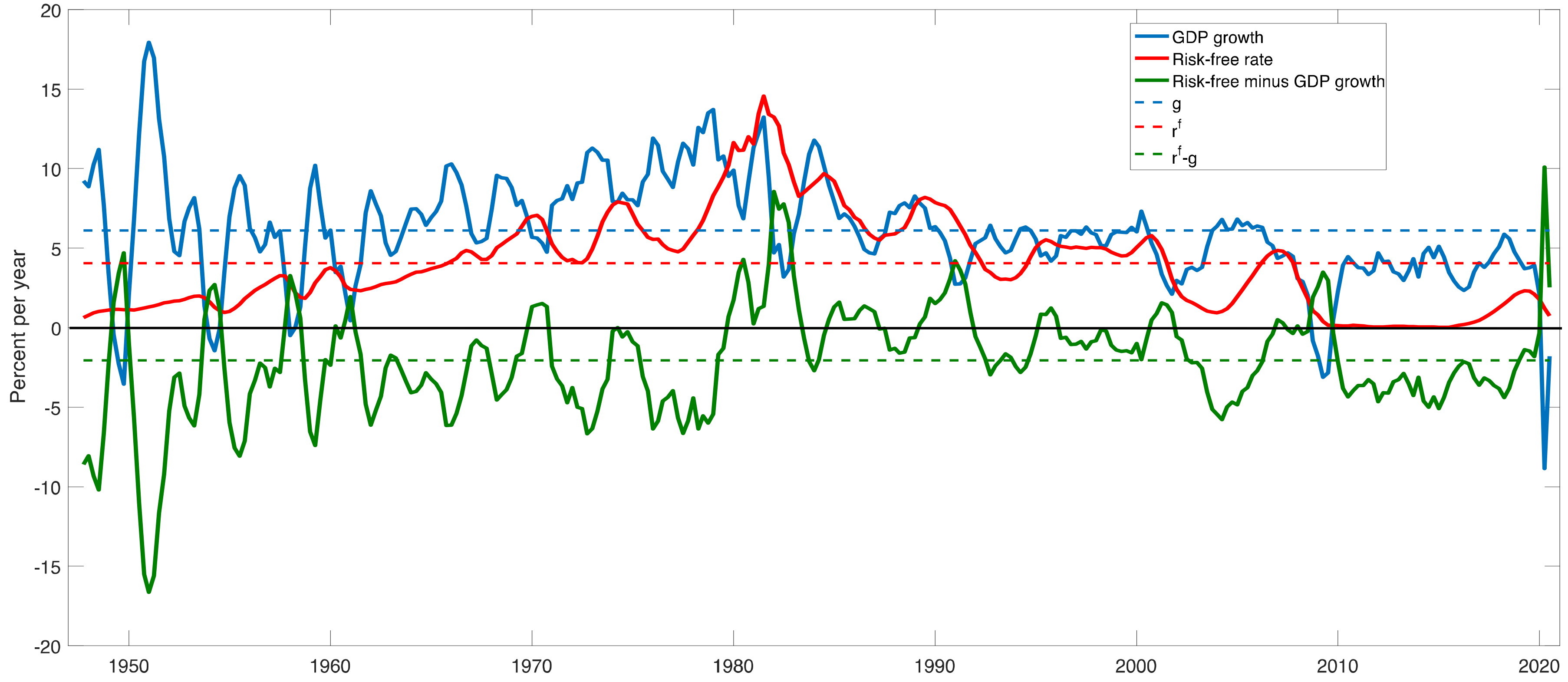
- Think of a representative agent holding all gov. debt
  - His cash flow is primary surplus
  - $\frac{\mathcal{B}_t}{\wp_t} = E_t[PV_r(\text{primary surpluses})] + \boldsymbol{Bubble}$ [FTPL]
  - … link to inflation
  - Can surpluses be negative forever? Yes, if $r < g$ (e.g. due to safe asset nature)



FTPL: Primary surplus, r and g for United States



- Primary surplus/GDP
  - Negative surplus in recession



- g GDP growth
- r
- $r - g$

FTPL Equation: Negative primary surplus forever?

- without creating inflation (devaluing debt)?
- Yes, if $r < g$

- $\frac{\mathcal{B}_t}{\wp_t} = E_t[PV_r(\text{primary surpluses})] \; + \underbrace{\lim_{T\to\infty} PV_r \frac{\mathcal{B}_T}{\wp_T}}_{\text{Bubble}}$

discount at r
(agents' SDF)

grows at g with
constant deficit/GDP

$\rightarrow -\infty$

$\rightarrow +\infty$

To determine real value of gov. debt and price level
FTPL equation is not enough
(goods market clearing and wealth effect)

FTPL Equation: Negative primary surplus forever?

- without creating inflation (devaluing debt)?
- Yes, if $r < g$
- $\frac{\mathcal{B}_t}{\wp_t} = E_t[PV_r(\text{primary surpluses})] + \underbrace{\lim_{T\to\infty} PV_r \frac{\mathcal{B}_T}{\wp_T}}_{\text{Bubble}}$

discount at r
(agents' SDF)

grows at g with
constant deficit/GDP

$\rightarrow -\infty$

$\rightarrow +\infty$

- Discount at a different rate $r^{**} > g$ instead, so that

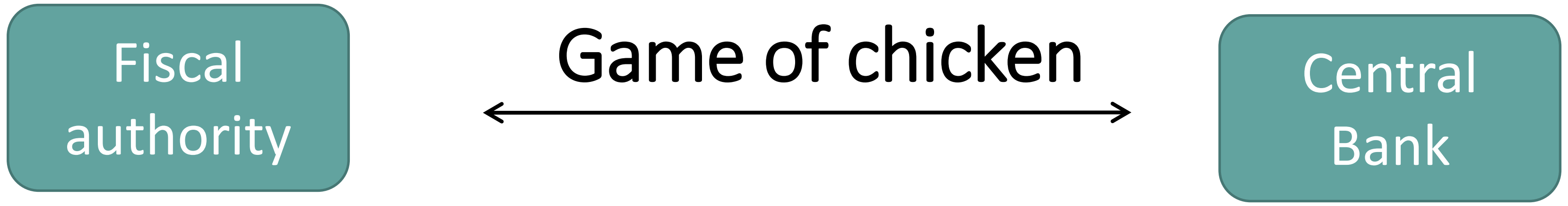
$$\frac{\mathcal{B}_t}{\wp_t} = E_t[PV_{r^{**}}(\text{primary surpluses})] + E_t[PV_{r^{**}}(\text{service flow})]$$

$> -\infty$

$< +\infty$

  - Both terms meaningful
  - Discount rate r^{**}= representative agents' risk-free rate $\neq m$ (Reis)

FTPL: Monetary vs. Fiscal Dominance



Fiscal authority

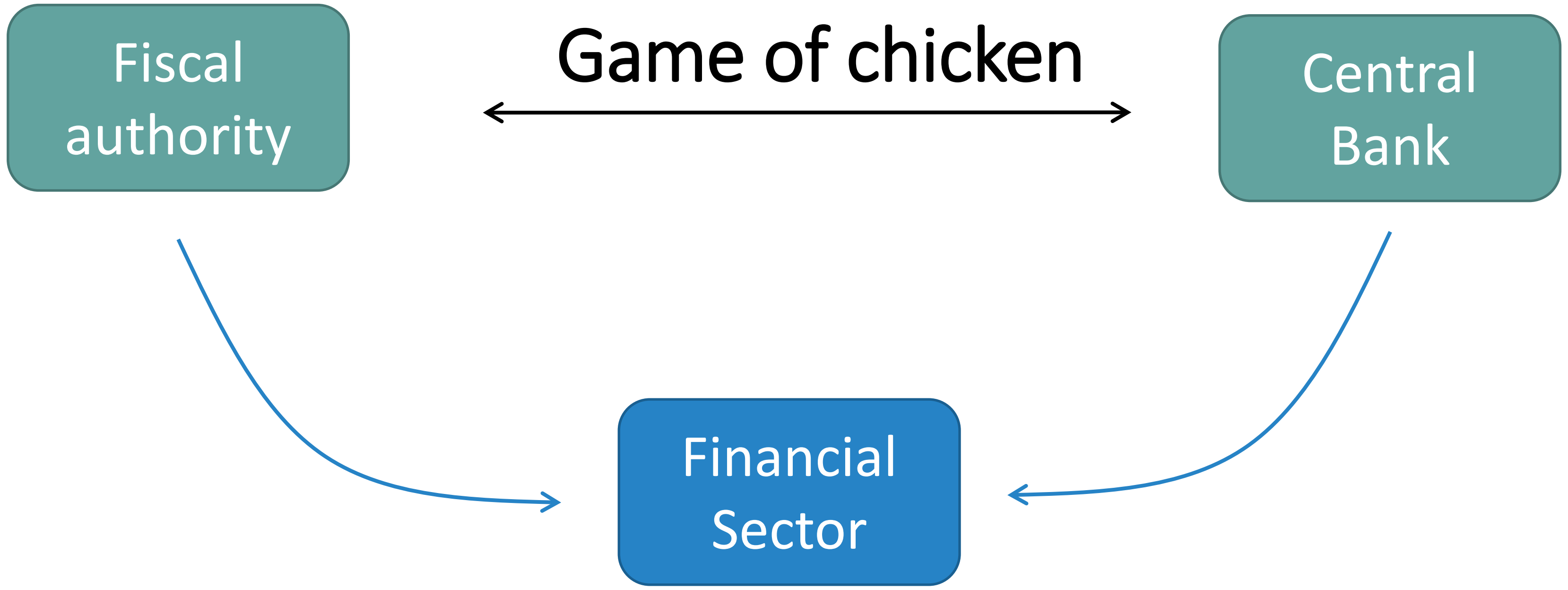
Game of chicken

Central Bank

- Monetary dominance
    - Monetary tightening leads fiscal authority to reduce fiscal deficit
- Fiscal dominance
    - Interest rate increase does not reduce primary fiscal deficit
    - … only lead to higher inflation

See YouTube video 4, minute 4:15

FTPL: Monetary vs. Fiscal Dominance



Fiscal authority

Game of chicken

Central Bank

Financial Sector

- Monetary dominance
  - Monetary tightening leads fiscal authority to reduce fiscal deficit
- Fiscal dominance
  - Interest rate increase does not reduce primary fiscal deficit
  - … only lead to higher inflation
- Financial dominance
  - Monetary tightening causes havoc in financial markets

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Simplify to One Sector Model

Expert sector

- Output: $y_t^e = a^e k_t^e$
- Consumption rate: c_t^e
- Investment rate: ι_t^e

$$\frac{dk_t^{\tilde{\iota},e}}{k_t^{\tilde{\iota},e}} = \left(\Phi\left(\iota_t^{\tilde{\iota},e}\right) - \delta\right)dt + d\Delta_t^{k,\tilde{\iota},e} + \sigma dZ_t + \tilde{\sigma} d\tilde{Z}_t^{\tilde{\iota}}$$

- $E_0[\int_0^\infty e^{-\rho^e t} \frac{(c_t^e)^{1-\gamma}}{1-\gamma} dt]$

Can only issue

- Risk-free debt
- Equity, but most hold $\chi_t^e \geq \alpha \kappa_t$

~~Household sector~~

- ~~Output: $y_t^h = a^h k_t^h$~~
- ~~Consumption rate: c_t^h~~
- ~~Investment rate: ι_t^h~~

~~$\frac{dk_t^{\tilde{\iota},h}}{k_t^{\tilde{\iota},h}} = \left(\Phi\left(\iota_t^{\tilde{\iota},h}\right) - \delta\right)dt + d\Delta_t^{k,\tilde{\iota},h} + \sigma dZ_t + \tilde{\sigma} d\tilde{Z}_t^{\tilde{\iota}}$~~

- ~~$E_0[\int_0^\infty e^{-\rho^h t} \frac{(c_t^h)^{1-\gamma}}{1-\gamma} dt]$~~

Model Overview

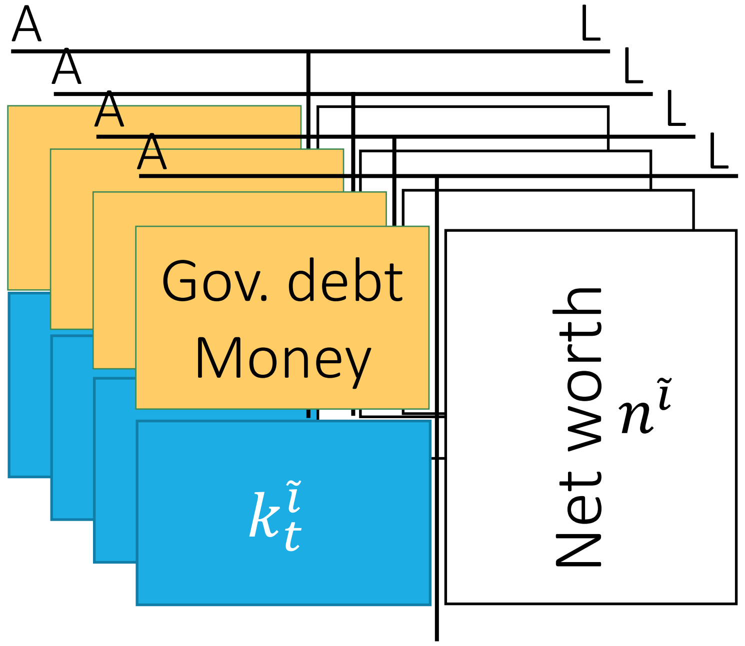
- Continuous time, infinite horizon, one consumption good
- Continuum of agents
  - Operate capital with time-varying idiosyncratic risk, AK production technology
  - Can trade capital and government bond, Extension: add diversified equity claims
- Government
  - Exogenous spending
  - Taxes output
  - Issues (nominal) bonds
- Financial Frictions: incomplete markets
  - Agents cannot trade idiosyncratic risk
  - Extension with equity: must retain skin in the game
- Aggregate risk: fluctuations in volatility of idio risk (& capital productivity)

Model with Capital + Safe Asset

- Each heterogenous citizen $\tilde\iota \in [0,1]$

$$\mathbb{E}_t\left[\int_t^\infty e^{-\rho s}\left(\frac{c_s^{1-\gamma}}{1-\gamma}+f(\mathscr{g}K_s)\right)\,ds\right] \qquad \text{where } K_t := \int k_t^{\tilde\iota}\,d\tilde\iota$$

$$\text{s.t. } \frac{dn_t^{\tilde\iota}}{n_t^{\tilde\iota}} = -\frac{c_t^{\tilde\iota}}{n_t^{\tilde\iota}}dt + dr_t^{\mathcal{B}} + \left(1-\theta_t^{\tilde\iota}\right)\left(dr_t^{K,\tilde\iota}(\iota_t^{\tilde\iota}) - dr_t^{\mathcal{B}}\right)$$



- Each citizen operates physical capital $k_t^{\tilde\iota}$
  - Output (net investment) $y_t^{\tilde\iota} = a_t k_t^{\tilde\iota} - \iota_t^{\tilde\iota} k_t^{\tilde\iota} dt$
  - Output tax $\tau a_t k_t^{\tilde\iota} dt$
  - $\frac{dk_t^{\tilde\iota}}{k_t^{\tilde\iota}} = \left(\Phi(\iota_t^{\tilde\iota}) - \delta\right)dt + \tilde\sigma_t d\tilde Z_t^{\tilde\iota} + d\Delta_t^{k,\tilde\iota}$
    - $d\tilde Z_t^{\tilde\iota}$ idiosyncratic Brownian
- Aggregate risk dZ_t : Heston model (time-varying idiosyncratic risk)
  - $d\tilde\sigma_t^2 = -\psi\left(\tilde\sigma_t^2 - \left(\tilde\sigma^0\right)^2\right)dt - \sigma\tilde\sigma_t dZ_t$
  - $a_t = a(\tilde\sigma_t)$
- Financial Friction: Incomplete markets: no $d\tilde Z_t^{\tilde\iota}$ claims

Government: Taxes, Bond/Money Supply, Gov. Budget

- Policy Instruments
  - Government spending $g K_t$ (with exogenous g)
  - Proportional output tax $\tau a K_t$
  - Nominal value of total government debt supply $d\mathcal{B}_t = \mu_t^{\mathcal{B}} \mathcal{B}_t dt$
  - Floating nominal interest rate i_t on outstanding bonds
- Government budget constraint (BC)

$$\underbrace{\left(\mu_t^{\mathcal{B}} - i_t\right)}_{\breve{\mu}_t^{\mathcal{B}} :=} \mathcal{B}_t + \wp_t K_t \underbrace{(\tau a - g)}_{s :=} = 0$$

Primary surplus (per K_t)

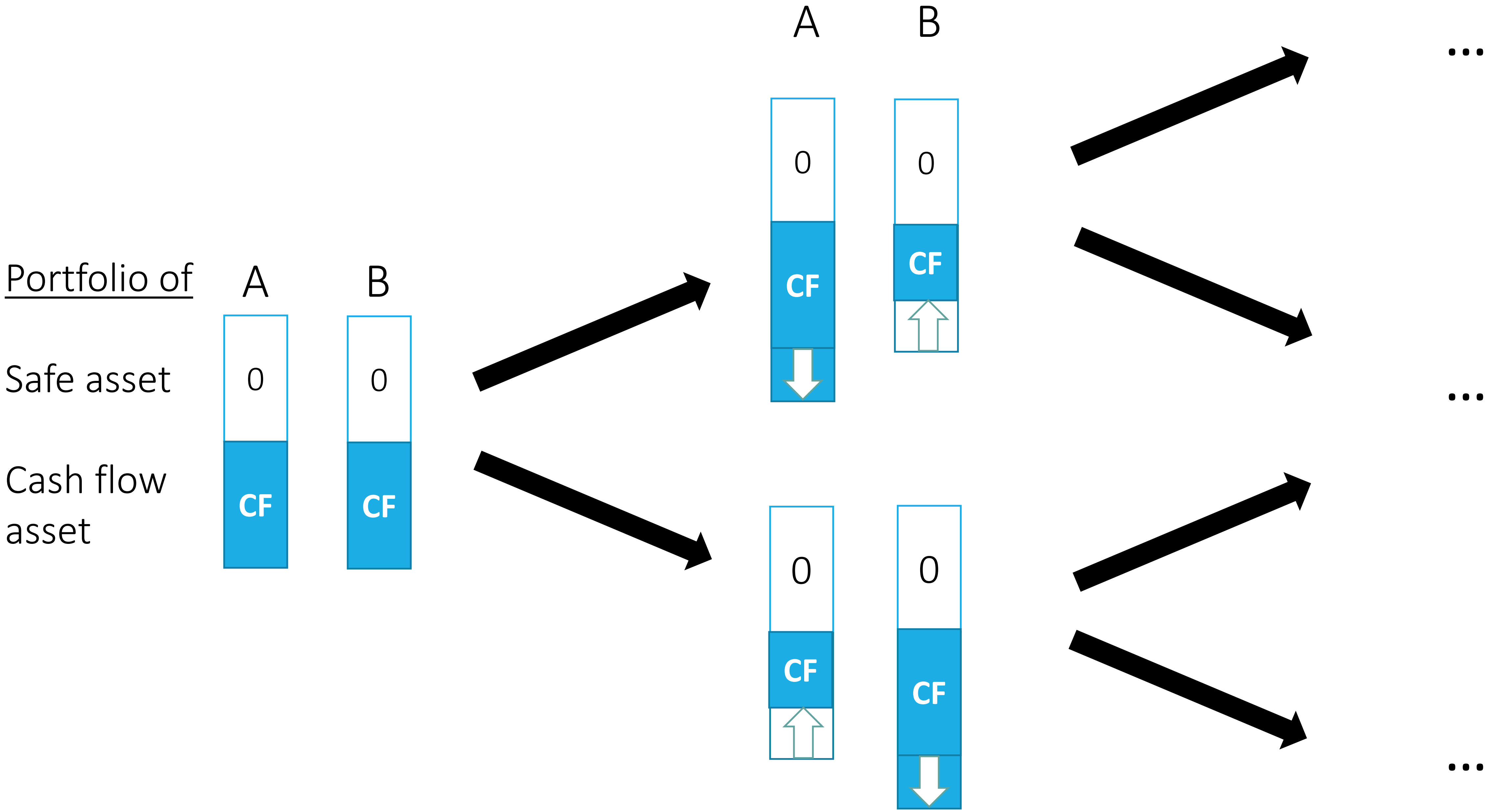
What's a Safe Asset? What is its Service Flow?

- $\frac{\mathcal{B}_t}{\wp_t} = E_t\left[PV_{\xi^{**}}(\text{primary surpluses})\right] + E_t\left[PV_{\xi^{**}}(\text{service flow})\right]$ (FTPL*)

What’s a Safe Asset? What is its Service Flow?

- $\frac{\mathcal{B}_t}{\wp_t} = \mathbb{E}_t\left[PV_{\xi^{**}}(\text{primary surpluses})\right] + \mathbb{E}_t\left[PV_{\xi^{**}}(\text{service flow})\right]$ (FTPL**)

Example: = 0



Portfolio of A B

Safe asset 0 0

Cash flow asset CF CF

A B

0 0

CF CF

0 0

CF CF

…

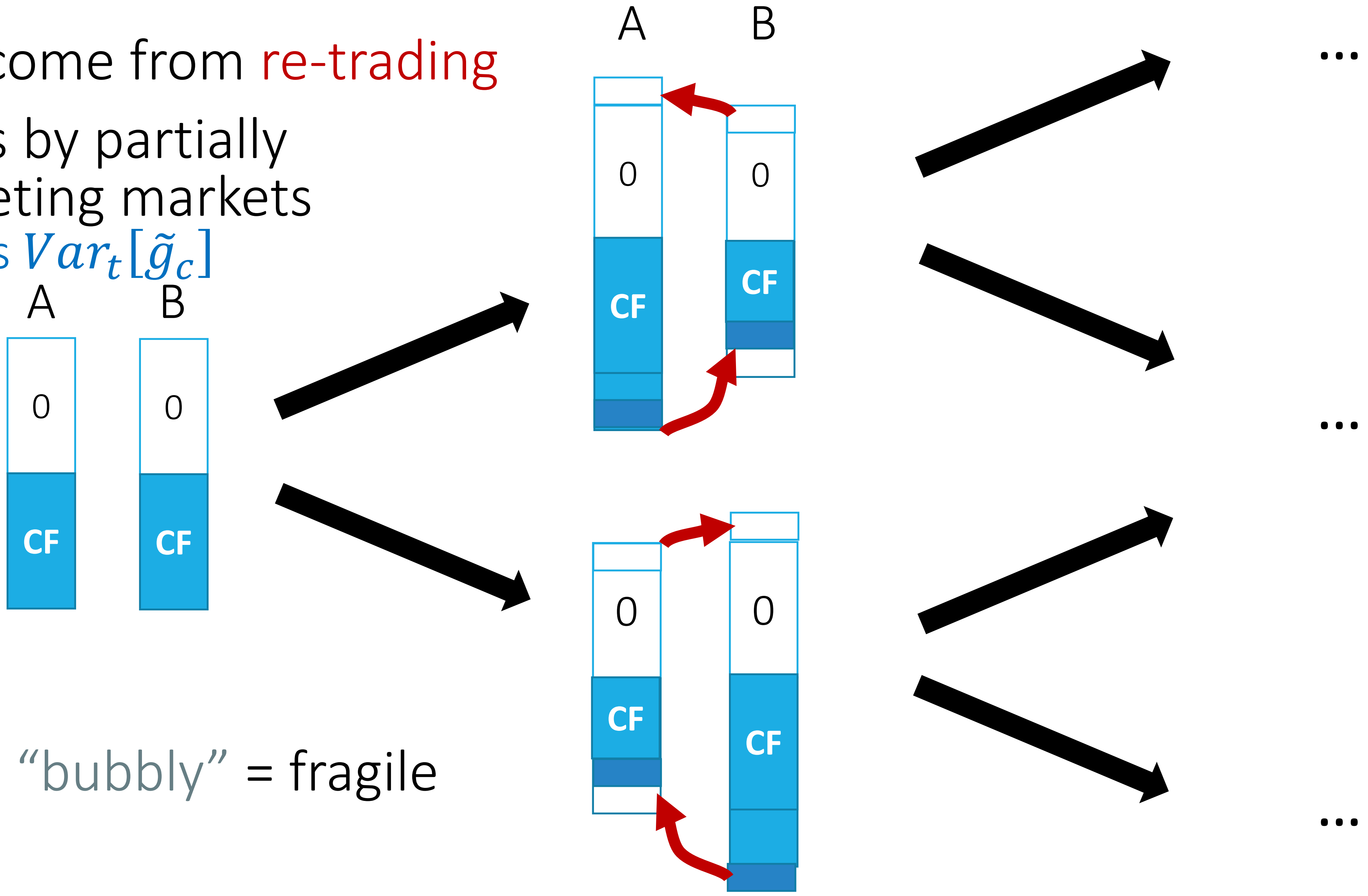
…

…

What's a Safe Asset? What is its Service Flow?

- $\frac{\mathcal{B}_t}{\wp_t} = \mathbb{E}_t\left[PV_{\xi^{**}}(\text{primary surpluses})\right] + \mathbb{E}_t\left[PV_{\xi^{**}}(\text{service flow})\right]$ (FTPL**)

- Value come from re-trading
- Insures by partially completing markets
  Reduces $Var_t[\tilde{g}_c]$



- Can be "bubbly" = fragile

What's a Safe Asset?

- In incomplete markets setting (Bewley, Aiyagari, BruSan, …)
- Good friend analogy (Brunnermeier Haddad, 2012)
    - When one needs funds, one can sell at stable price… since others buy
        - Idiosyncratic shock: *Partial insurance through retrading - low bid-ask spread*
        - Aggregate (volatility) shock: *Appreciate in value in times of crises*
- **Safe asset definition**
    - Tradeable: no asymmetric info – info insensitive
        - Service flow is derived from "dynamic re-trading"
    - Individual $\beta_t^i = -\frac{Cov_t\left[d\xi_t^i/\xi_t^i, dr_t\right]}{Var_t\left[d\xi_t^i/\xi_t^i\right]} \leq 0$

where ξ_t^i is SDF of agent $\boldsymbol{i}$

Note: $-Cov_t\left[d\xi_t^i/\xi_t^i, dr_t\right] = \varsigma_t^i \sigma_t^r + \tilde{\varsigma}_t^i \tilde{\sigma}_t^{r,i}$

where $d\xi_t^i/\xi_t^i \quad = -r_t^f dt - \varsigma_t^i dZ_t - \tilde{\varsigma}_t^i d\tilde{Z}_t^{\tilde{\imath}}$

Solving MacroModels Step-by-Step

0. Postulate aggregates, price processes & obtain return processes
1. For given C/N-ratio and SDF processes for each i *finance block*
   a. Real investment ι + Goods market clearing *(static)*
   - *Toolbox 1:* Martingale Approach, HJB vs. Stochastic Maximum Principle Approach

   b. Portfolio choice θ + Asset market clearing or
   Asset allocation κ & risk allocation χ
   - *Toolbox 2:* "price-taking social planner approach" – Fisher separation theorem
   - *Toolbox 3:* Change in numeraire to total wealth (including SDF)
     - "Money evaluation/FTPL equation" ϑ
2. Evolution of state variable ~~η (and~~ K) *forward equation*
3. Value functions *backward equation*
   a. Value fcn. as fcn. of individual investment opportunities ω
   - *Special cases: log-utility, constant investment opportunities*

   b. Separating value fcn. $V^i\left(n^{\tilde{\imath}};\eta,K\right)$ into $v^i(\eta)\left(\tilde{\eta}^{\tilde{\imath}}\right)^{1-\gamma}u(K)\left(n^{\tilde{\imath}}/n^i\right)^{1-\gamma}$
   c. Derive $\check{\rho}=C/N$-ratio and ς price of risk
4. Numerical model solution
   a. Transform BSDE for separated value fcn. $v^i(\eta)$ into PDE
   b. Solve PDE via value function iteration
5. KFE: Stationary distribution, Fan charts

Assets, Aggregate Resource Constraint, and Markets

- Assets: capital and bonds
  - q_t^K Capital price
  - $q_t^{\mathcal{B}} := \frac{\mathcal{B}_t}{\wp_t}/K_t$ value of the bonds per unit of capital
  - $\vartheta_t \coloneqq \frac{\mathcal{B}_t/\wp_t}{q_t^K K_t + \mathcal{B}_t/\wp_t} = \frac{q_t^{\mathcal{B}}}{q_t^K + q_t^{\mathcal{B}}}$ Share of bond wealth
  - Postulate Ito price processes

$$dq_t^K/q_t^K = \mu_t^{q,K} dt + \sigma_t^{q,K} dZ_t, \; dq_t^{\mathcal{B}}/q_t^{\mathcal{B}} = \mu_t^{q,\mathcal{B}} dt + \sigma_t^{q,\mathcal{B}} dZ_t,$$

$$d\vartheta_t/\vartheta_t = \mu_t^{\vartheta} dt + \sigma_t^{\vartheta} dZ_t$$

  - SDF for each $\tilde{\imath}$ agent $d\xi_t^{\tilde{\imath}}/\xi_t^{\tilde{\imath}} = -r_t^{f,\cancel{\tilde{\imath}}} dt - \varsigma_t^{\cancel{\tilde{\imath}}} dZ_t - \tilde{\varsigma}_t^{\cancel{\tilde{\imath}}} d\tilde{Z}_t^{\tilde{\imath}}$

- Aggregate resource constraints:
  - Output: $C_t + \iota_t K_t + g K_t = a K_t$
  - Capital: $\int \left(k_t^{\tilde{\imath}} d\Delta_t^{k,\tilde{\imath}}\right) d\tilde{\imath} = 0$

Poll 34: Why do risk-free rate and price of risk not depend on individual $\tilde{\imath}$?
a) risk-free bond can be traded
b) aggregate risk can be traded
c) CRRA utility for all agents with same γ

- Markets: Walrasian goods, bonds, and capital markets

0. Return on Gov. Bond/Money

- Number of Bonds/coins follows:

$$\frac{d\mathcal{B}_t}{\mathcal{B}_t} = (\check{\mu}_t^{\mathcal{B}} + i_t)dt + \sigma_t^{\mathcal{B}} dZ_t$$

  - Where i_t is interest paid on government bonds/outside money (reserves)
- Return on Gov. Bond/Money: in output numeraire

$$dr_t^{\mathcal{B}} = i_t dt + \underbrace{\frac{d\left(q_t^{\mathcal{B}} K_t/\mathcal{B}_t\right)}{q_t^{\mathcal{B}} K_t/\mathcal{B}_t}}_{-inflation}$$

$$= \frac{d\left(q_t^{\mathcal{B}} K_t\right)}{q_t^{\mathcal{B}} K_t} - \check{\mu}_t^{\mathcal{B}} dt - \sigma_t^{\mathcal{B}} dZ_t + \sigma_t^{\mathcal{B}} \left(\sigma_t^{\mathcal{B}} - \sigma - \sigma_t^{q,\mathcal{B}}\right) dt$$

- Seigniorage (excluding interest paid to money holders)

0. Distribution of "Seigniorage"

1. Proportionally to bond/money holdings
   - No real effects, only nominal
2. Proportionally to capital holdings
   - Bond/Money return decreases with $d\mathcal{B}_t$ (change in debt level/money supply)
   - Capital return increases
   - Pushes citizens to hold more capital
3. Proportionally to net worth
   - Fraction of seigniorage goes to capital - same as 2.
   - Rest of seigniorage goes to money holders - same as 1.
4. Per capita
   - No real effects:
     people simply borrow against the transfers they expect to receive

0. Return on Capital (with seigniorage rebate terms)

Simplified case:
$\sigma^B = 0$

- $dr_t^{K,\tilde{\imath}} = \frac{a\,(1-\tau)-\iota_t^{\tilde{\imath}}}{q_t^K} dt + \frac{d\left(q_t^K k_t^{\tilde{\imath}}\right)}{q_t^K k_t^{\tilde{\imath}}}$

$$= \left(\frac{a\,(1-\tau)-\iota_t^{\tilde{\imath}}}{q_t^K} + \Phi\left(\iota_t^{\tilde{\imath}}\right) - \delta + \mu_t^{q^K}\right) dt + \sigma_t^{q^K} dZ_t + \tilde{\sigma} d\tilde{Z}_t^{\tilde{\imath}}$$

- Use government budget constraint to substitute out τ (and $\mathcal{B}_t/\wp_t = q_t^{\mathcal{B}} K_t$)

$$\underbrace{\left(\mu_t^{\mathcal{B}} - i_t\right)}_{\breve{\mu}_t^{\mathcal{B}}:=} q_t^{\mathcal{B}} + \underbrace{(\tau\, a - g)}_{s:=} = 0$$

$$dr_t^{K,\tilde{\imath}} = \left(\frac{\overbrace{a-g}^{\breve{a}:=}-\iota_t^{\tilde{\imath}}}{q_t^K} + \Phi\left(\iota_t^{\tilde{\imath}}\right) - \delta + \mu_t^{q^K} + \frac{q_t^{\mathcal{B}}}{q_t^K}\breve{\mu}_t^{\mathcal{B}}\right) dt + \sigma_t^{q^K} dZ_t + \tilde{\sigma} d\tilde{Z}_t^{\tilde{\imath}}$$

1. Optimal Choices

- Optimal investment rate
  - $\phi\iota_t = q_t^K - 1$
- Consumption
  - $\frac{c_t}{n_t} =: \check{\rho}_t \quad \Rightarrow C_t = \check{\rho}_t\left(q_t^{\mathcal{B}} + q_t^K\right)K_t$

  - Looking ahead to Step 3:
    When is $\frac{c}{n}$ constant? Recall $\frac{c}{n} = \rho^{1/\gamma}\omega^{1-1/\gamma}$
    - Log utility, $\gamma = 1$: $\check{\rho} = \rho$
    - In steady state:
      ω investment opportunity/net worth multiplier is constant

1. Optimal Choices & Market Clearing

- Optimal investment rate
  - $\phi\iota_t = q_t^K - 1$
- Consumption

Goods market

  - $\frac{c_t}{n_t} =: \check{\rho}_t \quad \Rightarrow C_t = \check{\rho}_t\left(q_t^{\mathcal{B}} + q_t^K\right)K_t \quad = (a - \iota_t - g)K_t$
- Portfolio

Capital market

  - Solve for $\boldsymbol{\theta_t}$ later

$1 - \theta_t = 1 - \vartheta_t$

Bond market

clears by Walras law

Equilibrium (before solving for portfolio choice)

Equilibrium
$q_t^{\mathcal{B}} = \vartheta_t \dfrac{1+\phi\check{a}}{(1-\vartheta_t)+\phi\check{\rho}_t}$
$q_t^K = (1-\vartheta_t)\dfrac{1+\phi\check{a}}{(1-\vartheta_t)+\phi\check{\rho}_t}$
$\iota_t = \dfrac{(1-\vartheta_t)\check{a}-\check{\rho}_t}{(1-\vartheta_t)+\phi\check{\rho}_t}$

- Moneyless equilibrium with $q_t^{\mathcal{B}} = 0 \Rightarrow \vartheta_t = 0$
- Next, determine portfolio choice.

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        - "Money evaluation/FTPL equation" ϑ
2. Evolution of state variable ~~η (and~~ K) *forward equation*
3. Value functions *backward equation*
    a. Value fcn. as fcn. of individual investment opportunities ω
    - *Special cases: log-utility, constant investment opportunities*

    b. Separating value fcn. $V^i(n^{\tilde{\imath}};\eta,K)$ into $v^i(\eta)\left(\tilde{\eta}^{\tilde{\imath}}\right)^{1-\gamma}u(K)\left(n^{\tilde{\imath}}/n^i\right)^{1-\gamma}$
    c. Derive $\check{\rho} = C/N$-ratio and ς price of risk
4. Numerical model solution
    a. Transform BSDE for separated value fcn. $v^i(\eta)$ into PDE
    b. Solve PDE via value function iteration
5. KFE: Stationary distribution, Fan charts

1.b Portfolio choice θ: Bond/Money Evaluation/FTPL Equation

- Recall: Expected return: $\mu_t^A = r_t^i + \varsigma_t^i \sigma_t^A$
  - Excess expected return to risky asset B: $\boxed{\mu_t^A - \mu_t^B = \varsigma_t^i(\sigma_t^A - \sigma_t^B)}$

- Alternative derivations:
  - In consumption numeraire
    i. Expected excess return of capital w.r.t. bond return
    ii. Expected excess return of net worth (portfolio) w.r.t. bond return

  - In total net worth numeraire
    iii. Expected excess return of capital w.r.t. bond return
    iv. Expected excess return of individual net worth (=net worth share) w.r.t. bond return (per bond)

1.b Portfolio choice θ: Bond/Money Evaluation/FTPL Equation

Price capital relative to money in consumption numeraire

- For stationary setting with $\boldsymbol{\sigma} = \boldsymbol{\sigma}_t^B = \mathbf{0}$
- Asset pricing equation (martingale method): $\boxed{\mu_t^A - \mu_t^B = \varsigma_t^i(\sigma_t^A - \sigma_t^B)}$

$$\frac{\mathbb{E}_t\left[dr_t^{K,\tilde{i}}\right]}{dt} = \frac{\check{a} - \iota_t}{q_t^K} + \frac{q_t^{\mathcal{B}}}{q_t^K}\check{\mu}^{\mathcal{B}} + \Phi(\iota_t) - \delta + \mu_t^{q^K} + \sigma\sigma_t^{q^K} = r_t^f + \varsigma_t\sigma_t^{q^K} + \tilde{\varsigma}_t\tilde{\sigma}$$

$$\frac{\mathbb{E}_t\left[dr_t^{\mathcal{B}}\right]}{dt} = \Uparrow \quad \Uparrow -\check{\mu}^{\mathcal{B}} + \Phi(\iota_t) - \delta + \mu_t^{q^{\mathcal{B}}} + \sigma\sigma_t^{q^{\mathcal{B}}} = r_t^f + \varsigma_t\sigma_t^{q^{\mathcal{B}}} \quad \Uparrow$$

$$\frac{\mathbb{E}_t\left[dr_t^{K,\tilde{i}}\right]}{dt} - \frac{\mathbb{E}_t\left[dr_t^{\mathcal{B}}\right]}{dt} = \frac{\check{a} - \iota_t}{q_t^K} + \frac{1}{1 - \vartheta_t}\check{\mu}^{\mathcal{B}} + \underbrace{\mu_t^{q^K} - \mu_t^{q^{\mathcal{B}}}}_{= -\mu_t^{\vartheta}/(1-\vartheta_t)} + \sigma\left(\sigma_t^{q^K} - \sigma_t^{q^{\mathcal{B}}}\right) = \varsigma_t(\sigma_t^{q^K} - \sigma_t^{q^{\mathcal{B}}}) + \tilde{\varsigma}_t\tilde{\sigma}$$

Hint: $d(q^K + q^{\mathcal{B}}) = dq^K + dq^{\mathcal{B}}$

$$\mu^{q^K+q^{\mathcal{B}}} = \frac{\mu_{q^K+q^{\mathcal{B}}}}{q^K+q^{\mathcal{B}}} = \frac{\mu^{q^K}q^K + \mu^{q^{\mathcal{B}}}q^{\mathcal{B}}}{q^K+q^{\mathcal{B}}} = (1-\vartheta)\mu^{q^K} + \vartheta\mu^{q^{\mathcal{B}}}.$$

$$\sigma^{q^K+q^{\mathcal{B}}} = \frac{\sigma_{q^K+q^{\mathcal{B}}}}{q^K+q^{\mathcal{B}}} = \frac{\sigma^{q^K}q^K + \sigma^{q^{\mathcal{B}}}q^{\mathcal{B}}}{q^K+q^{\mathcal{B}}} = (1-\vartheta)\sigma^{q^K} + \vartheta\sigma^{q^{\mathcal{B}}}.$$

- Goods market clearing: $\check{\rho}(q_t^{\mathcal{B}} + q_t^K)K_t = (\check{a} - \iota_t)K_t \Rightarrow \frac{\check{a}-\iota_t}{q_t^K} = \frac{\check{\rho}}{1-\vartheta_t}$
- Price of idiosyncratic risk: $\tilde{\varsigma}_t = \gamma\tilde{\sigma}_t^n = (1 - \theta_t)\gamma\tilde{\sigma}$
- Capital market clearing: $1 - \theta_t = 1 - \vartheta_t$

Poll 43: Why is the price of idiosyncratic risk simply $\gamma\tilde{\sigma}_t^n$ for CRRA?

a) because idiosyncratic risk for $\boldsymbol{\sigma} = 0$ there is no idiosyncratic investment opportunity

b) idiosyncratic risk is always myopic (like for log-u)

1.b Portfolio choice θ: Bond/Money Evaluation/FTPL Equation

Price capital relative to money in consumption numeraire

- For stationary setting with $\boldsymbol{\sigma = \sigma_t^B = 0}$
- Asset pricing equation (martingale method): $\boxed{\mu_t^A - \mu_t^B = \varsigma_t^i(\sigma_t^A - \sigma_t^B)}$

$$\frac{\mathbb{E}_t\left[dr_t^{K,\tilde{i}}\right]}{dt} = \frac{\check{a} - \iota_t}{q_t^K} + \frac{q_t^{\mathcal{B}}}{q_t^K}\check{\mu}^{\mathcal{B}} + \Phi(\iota_t) - \delta + \mu_t^{q^K} + \sigma\sigma_t^{q^K} = r_t^f + \varsigma_t\sigma_t^{q^K} + \tilde{\varsigma}_t\tilde{\sigma}$$

$$\frac{\mathbb{E}_t\left[dr_t^{\mathcal{B}}\right]}{dt} = \quad \Uparrow \qquad \Uparrow -\check{\mu}^{\mathcal{B}} + \Phi(\iota_t) - \delta + \mu_t^{q^{\mathcal{B}}} + \sigma\sigma_t^{q^{\mathcal{B}}} = r_t^f + \varsigma_t\sigma_t^{q^{\mathcal{B}}} \qquad \Uparrow$$

$$\frac{\mathbb{E}_t\left[dr_t^{K,\tilde{i}}\right]}{dt} - \frac{\mathbb{E}_t\left[dr_t^{\mathcal{B}}\right]}{dt} = \frac{\check{a} - \iota_t}{q_t^K} + \frac{1}{1 - \vartheta_t}\check{\mu}^{\mathcal{B}} + \underbrace{\mu_t^{q^K} - \mu_t^{q^{\mathcal{B}}}}_{= -\mu_t^{\vartheta}/(1-\vartheta_t)} + \sigma\left(\sigma_t^{q^K} - \sigma_t^{q^{\mathcal{B}}}\right) = \varsigma_t(\sigma_t^{q^K} - \sigma_t^{q^{\mathcal{B}}}) + \tilde{\varsigma}_t\tilde{\sigma}$$

- Goods market clearing: $\check{\rho}(q_t^{\mathcal{B}} + q_t^K)K_t = (\check{a} - \iota_t)K_t \Rightarrow \frac{\check{a}-\iota_t}{q_t^K} = \frac{\check{\rho}}{1-\vartheta_t}$
- Price of idiosyncratic risk: $\tilde{\varsigma}_t = \gamma\tilde{\sigma}_t^n = (1 - \theta_t)\gamma\tilde{\sigma}$
- Capital market clearing: $1 - \theta_t = 1 - \vartheta_t$
- Money valuation Equation: $\mu_t^{\vartheta} = \rho + \check{\mu}_t^{\mathcal{B}} - (1 - \vartheta_t)^2\gamma\tilde{\sigma}^2$
  - In steady state $\mu_t^{\vartheta} = 0$: $(1 - \vartheta) = \sqrt{\check{\rho} + \check{\mu}^{\mathcal{B}}}/(\sqrt{\gamma}\tilde{\sigma})$

1.b Deriving FTPL - traditional

- Money valuation equation for log utility $\gamma = 1$

$$\vartheta_t \mu_t^{\vartheta} = \vartheta_t(\underbrace{\rho + \overbrace{g}^{\Phi(\iota)-\delta} - (1-\vartheta_t)^2\tilde{\sigma}^2}_{=r^f} - g + \check{\mu}_t^{\mathcal{B}})$$

- Integrate forward
- $\vartheta_0 = \mathbb{E}\int_0^\infty e^{-r^f t} e^{gt}(-\check{\mu}_t^{\mathcal{B}})\vartheta_t dt$ recall gov. budget constraint $\check{\mu}_t^{\mathcal{B}} = -s/q_t^{\mathcal{B}}$

$$= \mathbb{E}\int_0^\infty e^{-r^f t} e^{gt} \frac{s}{q_t^{\mathcal{B}}} \vartheta_t dt$$

$$= \mathbb{E}\int_0^\infty e^{-(r^f-g)t} \frac{sK_t}{N_t} dt$$

- Multiply by N_0: $\vartheta_0 N_0 = \mathbb{E}[\int_0^\infty e^{-(r^f-g)t} \underbrace{\frac{N_0}{N_t}}_{e^{-gt}} sK_t]dt = \frac{\mathcal{B}_0}{\wp_0} = q_0^{\mathcal{B}} K_0 = \vartheta_0 N_0$

- FTPL equation: $\boldsymbol{\frac{\mathcal{B}_0}{\wp_0} = \mathbb{E}\left[\int_0^\infty e^{-r^f t} sK_t dt\right]}$ if $g < r^f$ since $K_t = e^{gt} K_0$

1.b Deriving FTPL – separating service flow with SDF ξ_t^{**}

- Money valuation equation for log utility $\gamma = 1$

$$\vartheta_t \mu_t^{\vartheta} = \vartheta_t(\rho - (1-\vartheta_t)^2 \tilde{\sigma}^2 + \check{\mu}_t^{\mathcal{B}})$$

- Integrate forward
- $$\begin{aligned} \vartheta_0 &= \mathbb{E}\int_0^\infty e^{-\rho t}\left(-\check{\mu}_t^{\mathcal{B}} + (1-\vartheta_t)^2\tilde{\sigma}^2\right)\vartheta_t dt \\ &= \mathbb{E}\int_0^\infty e^{-\rho t}\frac{s}{q_t^{\mathcal{B}}}\vartheta_t dt + \mathbb{E}\int_0^\infty e^{-\rho t}(1-\vartheta_t)^2\tilde{\sigma}^2\vartheta_t dt \\ &= \mathbb{E}\int_0^\infty e^{-\rho t}\frac{sK_t}{N_t} dt \quad + \mathbb{E}\int_0^\infty e^{-\rho t}(1-\vartheta_t)^2\tilde{\sigma}^2\frac{\mathcal{B}_t}{\wp_t N_t} dt \end{aligned}$$
- Multiply by N_0

$$\vartheta_0 N_0 = \frac{\mathcal{B}_0}{\wp_0} = \mathbb{E}[\int_0^\infty \underbrace{e^{-\rho t}\frac{N_0}{N_t}}_{\xi_t^{**} := \int \xi_t^{\tilde{i}}\eta_t^{\tilde{i}} d\tilde{i}} sK_t dt] + \mathbb{E}[\int_0^\infty \underbrace{e^{-\rho t}\frac{N_0}{N_t}}_{\xi_t^{**} := \int \xi_t^{\tilde{i}}\eta_t^{\tilde{i}} d\tilde{i}} \underbrace{(1-\vartheta_t)^2\tilde{\sigma}^2\frac{\mathcal{B}_t}{\wp_t}}_{\text{Service flow}} dt]$$

What is Quasi-SDF $\xi_t^{**} = \int \xi_t^i \eta_t^i di$?

- $\xi_t^{**} := \int \xi_t^{\tilde{\imath}} \eta_t^{\tilde{\imath}} d\tilde{\imath}$

$$= \int e^{-\rho t} \frac{u'\left(c_t^{\tilde{\imath}}\right)}{u'\left(c_0^{\tilde{\imath}}\right)} \eta_t^{\tilde{\imath}} d\tilde{\imath} = \int e^{-\rho t} \left(\frac{c_t^{\tilde{\imath}}}{c_0^{\tilde{\imath}}}\right)^{-\gamma} \eta_t^{\tilde{\imath}} d\tilde{\imath} = \int e^{-\rho t} \left(\frac{\cancel{\check{\rho}} n_t^{\tilde{\imath}}}{\cancel{\check{\rho}} n_0^{\tilde{\imath}}}\right)^{-\gamma} \eta_t^{\tilde{\imath}} d\tilde{\imath}$$

- For log utility $\gamma = 1$: $\xi_t^{**} = \int e^{-\rho t} \left(\frac{n_0^{\tilde{\imath}}}{n_t^{\tilde{\imath}}}\right) \eta_t^{\tilde{\imath}} d\tilde{\imath} = e^{-\rho t} \frac{N_0}{N_t}$

- Total net worth (incl. bubble wealth) $= N_t = \mathbb{E}_t[\int_t^\infty \underbrace{\frac{\int \xi_s^i \eta_s^i di}{\int \xi_t^i \eta_t^i di}}_{\frac{\xi_s^{**}}{\xi_t^{**}}} C_s ds]$
    - Net worth share weighted SDF
    - "Representative agent SDF"
    - Complete markets: $\xi_t^{**} = \xi_t$

Eliciting the service flow term - 2 Asset pricing perspectives

Agent i's SDF, ξ_t^i: $d\xi_t^i/\xi_t^i \quad = -r_t^f\, dt - \varsigma_{\cancel{t}} dZ_t - \tilde{\varsigma}_t^i d\tilde{Z}_t^i$

- Buy and Hold Perspective:
  - $\text{market cap} = P_0 = \lim_{T\to\infty}\left(\mathbb{E}\left[\int_0^T \xi_t^i \underset{\text{of whole issuance}}{\text{AssetCashflow}_t}\; dt\right] + \underbrace{\mathbb{E}\left[\xi_T^i P_T\right]}_{\text{Bubble term}}\right)$
    - If all agents i are marginal investors of aggregate risk asset

First aggregate and then iterate (over time)

Eliciting the service flow term - 2 Asset pricing perspectives

Agent i's SDF, ξ_t^i: $d\xi_t^i/\xi_t^i \quad = -r_t^f\,dt - \cancel{\varsigma_t} dZ_t - \tilde{\varsigma}_t^i d\tilde{Z}_t^i$

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    - If all agents i are marginal investors of aggregate risk asset

First aggregate and then iterate (over time)

- Dynamic Trading Perspective:
  - Dynamic trading strategy leads to cashflows conditional on idiosyncratic risks
  - Denote η^i the *share of asset* held by agent i
  - $$= \lim_{T\to\infty}\left(\int \mathbb{E}\left[\int_0^\infty \xi_t^i\ (\eta_t^i \text{AssetCashflow}_t + \eta_t^i \text{TradingCashflow}_t) dt\right] di + \cdots\right)$$
  - $$= \mathbb{E}[\int_0^\infty \underbrace{\int \xi_t^i \eta_t^i di}_{\xi_t^{**}:=}\ \text{AssetCashflow}_t]dt + \underbrace{\mathbb{E}[\int_0^\infty \underbrace{\int \xi_t^i \eta_t^i di}_{\xi_t^{**}:=}\ \text{TradingCashflow}_t\ dt]}_{\text{Service flow term}}$$
    - Discount rate $E[dr^\eta]/dt = r^f + \tilde{\varsigma}\tilde{\sigma}$
    - ξ^i and η^i are negatively correlated ⇒ depresses weighted "Quasi-SDF" (higher discount rate)

First iterate (over time) then aggregate

FTPL Equation with Bubble: 2 Perspectives - Intuition

- Buy and Hold Perspective:

  Expected bond return

  Ramsey term *Precautionary savings/self-insurance*

$$= \underbrace{\rho + \gamma\mu^c - \frac{1}{2}\gamma(\gamma+1)\{(\sigma^c)^2 + (\tilde{\sigma}^c)^2\}}_{\text{Risk-free rate } r^f \ =} + risk\ premium - convience\ yield$$

- Dynamic Trading Perspective:

  Expected bond return

$$= \underbrace{\rho + \gamma\mu^c - \frac{1}{2}\gamma(\gamma+1)\{(\sigma^c)^2\}}_{\text{Risk-free rate } r^{f**} =} + risk\ premium - \underbrace{\{\frac{1}{2}\gamma(\gamma+1)(\tilde{\sigma}^c)^2 + convience\ yield\}}_{\text{“Service Flow”}}$$

  - r^{f**} = “representative agent’s” risk-free rate

Recall: Equilibrium (before solving for portfolio choice)

- $\mu_t^{\vartheta} = \rho + \check{\mu}_t^{\mathcal{B}} - (1-\vartheta_t)^2 \gamma \tilde{\sigma}^2$
    - In steady state $\mu_t^{\vartheta} = 0$: $(1-\vartheta) = \sqrt{\check{\rho} + \check{\mu}^{\mathcal{B}}} / (\sqrt{\gamma}\tilde{\sigma})$

Equilibrium
$q_t^{\mathcal{B}} = \vartheta_t \dfrac{1+\phi\check{a}}{(1-\vartheta_t)+\phi\check{\rho}_t}$
$q_t^{K} = (1-\vartheta_t) \dfrac{1+\phi\check{a}}{(1-\vartheta_t)+\phi\check{\rho}_t}$
$\iota_t = \dfrac{(1-\vartheta_t)\check{a}-\check{\rho}_t}{(1-\vartheta_t)+\phi\check{\rho}_t}$

Two Stationary Equilibria

Non-Monetary	Monetary
$q_0^{\mathcal{B}} = 0$	$q^{\mathcal{B}} = \frac{\left(\sqrt{\gamma}\tilde{\sigma} - \sqrt{\breve{\rho} + \breve{\mu}^{\mathcal{B}}}\right)(1 + \phi\breve{a})}{\sqrt{\breve{\rho} + \breve{\mu}^{\mathcal{B}}} + \phi\sqrt{\gamma}\tilde{\sigma}\breve{\rho}} = \frac{\left(\sqrt{\gamma}\tilde{\sigma} - \sqrt{\breve{\rho} - s/q^{\mathcal{B}}}\right)(1 + \phi\breve{a})}{\sqrt{\breve{\rho} - s/q^{\mathcal{B}}} + \phi\sqrt{\gamma}\tilde{\sigma}\breve{\rho}}$
$q_0^K = \frac{1 + \phi\breve{a}}{1 + \phi\breve{\rho}_0}$	$q^K = \frac{\sqrt{\breve{\rho}+\breve{\mu}^{\mathcal{B}}}\ (1+\phi\breve{a})}{\sqrt{\breve{\rho}+\breve{\mu}^{\mathcal{B}}}+\phi\sqrt{\gamma}\tilde{\sigma}\breve{\rho}}$
$\iota = \frac{\breve{a} - \breve{\rho}_0}{1 + \phi\breve{\rho}_0}$	$\iota = \frac{\breve{a}\sqrt{\breve{\rho}+\breve{\mu}^{\mathcal{B}}}-\sqrt{\gamma}\tilde{\sigma}\breve{\rho}}{\sqrt{\breve{\rho}+\breve{\mu}^{\mathcal{B}}}+\phi\sqrt{\gamma}\tilde{\sigma}\breve{\rho}}$

- For log utility
  - $\breve{\rho} = \breve{\rho}_0 = \rho$
  - $\gamma = 1$

ρ time preference rate
ϕ adjustment cost for investment rate
$\breve{\mu}_t^{\mathcal{B}} = \mu_t^{\mathcal{B}} - i_t$ bond issuance rate beyond interest rate
$\breve{a} = a - \mathcal{g}$ part of TFP not spend on gov.

Remarks

- Real risk-free rate
  - $r^f = \underbrace{\left(\Phi\left(\iota(\check{\mu}^{\mathcal{B}})\right) - \delta\right)}_{=g} - \check{\mu}^{\mathcal{B}}$

- $\check{\mu}^{\mathcal{B}} = 0 \Rightarrow s = 0$ no primary surplus (no cash payoff for bond)
  - $q^{\mathcal{B}} K = \frac{\mathcal{B}}{P} > 0$ bond trades at a **bubble** due to service flow

- $\check{\mu}^{\mathcal{B}} > 0 \Rightarrow s < 0$ primary deficit (constant fraction of GDP)
  - As long as $q^{\mathcal{B}} > 0$ "mine the bubble"

- $\check{\mu}^{\mathcal{B}} < 0 \Rightarrow s > 0$ and $r > g$ primary surplus (constant fraction of GDP)
  - $q^{\mathcal{B}} K_t = E_t[PV_{r^f}(sK_t)]$ no bubble, but service flow

Service Flow Term, Convenience Yield, Ponzi Scheme

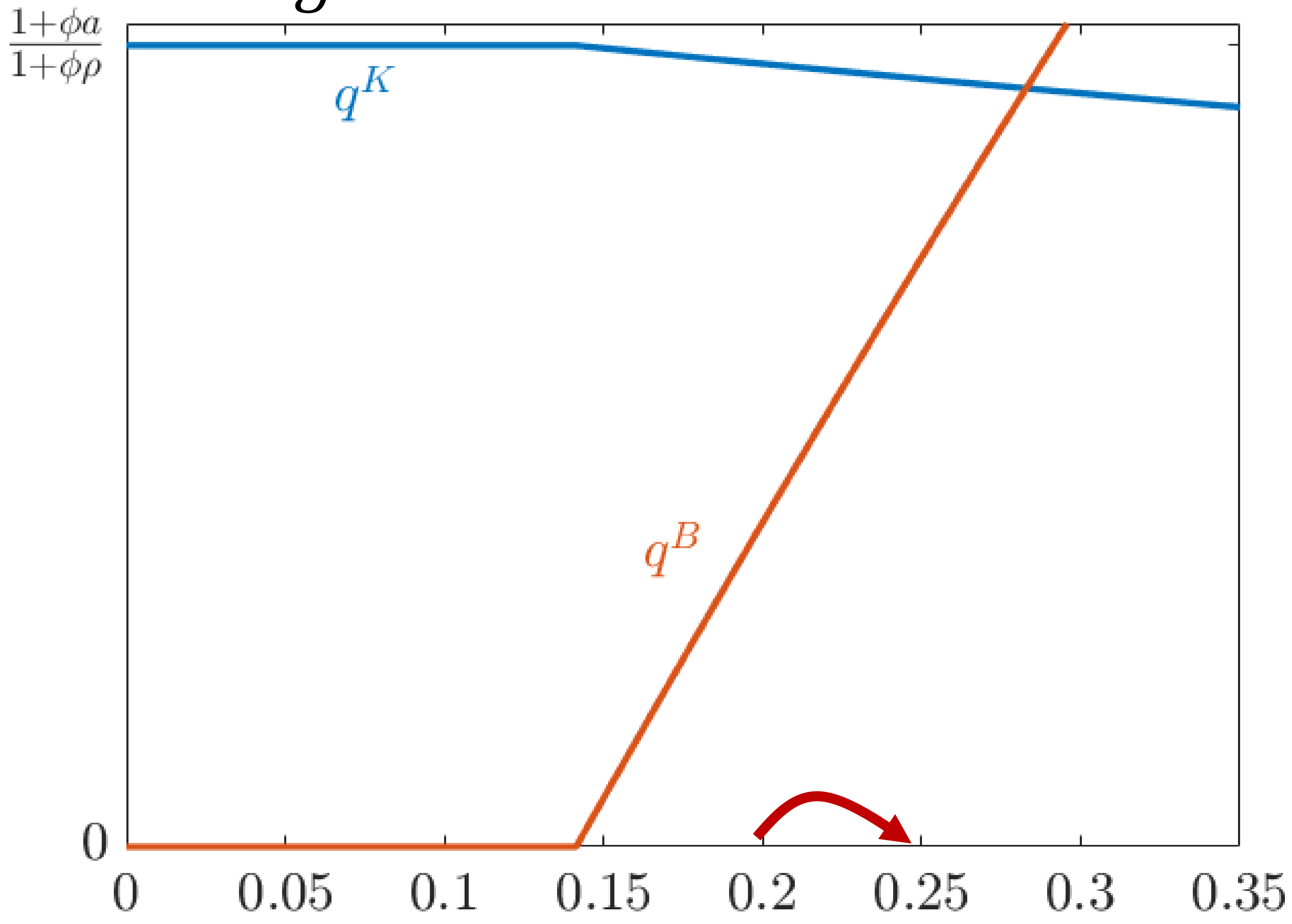
- Service flow
  - Convenience yield: relaxes collateral constraint or CIA constraint (money)
    - Traditional measure: BAA-US Treasury spread

  - Here: Partially completing markets through retrading
    - Low interest rate (cash flow) asset can be issued by everyone
      Hence, corporate-Treasury spread = 0

- Ponzi scheme is not feasibly for everyone
  No Ponzi constraint may be binding
  - Who can run a Ponzi scheme? **exorbitant privilege**
    … assigned by equilibrium selection
    - Likely to government, private entities are subject to solvency constraint
      - … still there is a Debt Laffer Curve

Flight to Safety: Comparative static w.r.t. $\tilde{\sigma}$

- Flight to safety into bubbly gov. debt
  - $q^{\mathcal{B}}$ rises (disinflation)
  - q^K falls and so does ι and g



- Similar with stochastic idiosyncratic volatility

Extra: 1.b Portfolio choice/Bond/Money Evaluation/FTPL Equation

- Recall: Expected return: $\mu_t^A = r_t^i + \varsigma_t^i \sigma_t^A$
  - Excess expected return to risky asset B: $\mu_t^A - \mu_t^B = \varsigma_t^i(\sigma_t^A - \sigma_t^B)$

- Alternative derivations:
  - In consumption numeraire
    i. Expected excess return of capital w.r.t. bond return
    ii. Expected excess return of net worth (portfolio) w.r.t. bond return (homework!)

  - In total net worth numeraire
    iii. Expected excess return of capital w.r.t. bond return (homework!)
    iv. Expected excess return of individual net worth (=net worth share) w.r.t. bond return (per bond)

Extra: 1.b Recall: Change to total net worth numeraire N_t

- SDF in consumption numeraire

$$\frac{d\xi_t^{\tilde{\imath}}}{\xi_t^{\tilde{\imath}}} = -r_t^f dt - \varsigma_t dZ_t - \tilde{\varsigma}_t d\tilde{Z}_t^{\tilde{\imath}}$$

- SDF in N_t-numeraire

$$\frac{d\hat{\xi}_t^{\tilde{\imath}}}{\hat{\xi}_t^{\tilde{\imath}}} = \frac{d(\xi_t^{\tilde{\imath}} N_t)}{(\xi_t^{\tilde{\imath}} N_t)} = -(r_t^f - \mu_t^N + \varsigma_t \sigma_t^N)dt - (\varsigma_t - \sigma_t^N)dZ_t - \tilde{\varsigma}_t d\tilde{Z}_t^{\tilde{\imath}}$$

- Return in consumption numeraire:

$$dr_t^j = \mu_t^{r^j} dt + \sigma_t^{r^j} dZ_t - \tilde{\sigma}_t^{r^j} d\tilde{Z}_t^{\tilde{\imath}}$$

- Return in N_t-numeraire

$$dr_{t,N}^j = \left(\mu_t^{r^j} - \mu_t^N - \sigma_t^N \left(\sigma_t^{r^j} - \sigma_t^N\right)\right) dt + \left(\sigma_t^{r^j} - \sigma_t^N\right) dZ_t - \tilde{\sigma}_t^{r^j} d\tilde{Z}_t^{\tilde{\imath}}$$

  - Value of self-financing strategy investing in asset in the consumption numeraire, e.g. x^j satisfies $dx_t^j / x_t^j = dr_t^j$. The same holds in the N_t-numeraire, but now the value is x_t^j / N_t.

Extra: 1.b Alternative iii: Portfolio choice θ (N_t-numeraire)

Total net worth N_t relative to a single bond/coin of money

- Asset pricing equation (martingale method)

$$\frac{\mathbb{E}\left[dr_t^{\tilde{\eta}^{\tilde{\imath}}}\right]}{dt} = \check{\rho}_t \qquad = \left(r_t^f - (\Phi(\iota_t) - \delta) - \mu_t^{q^K+q^{\mathcal{B}}} - \sigma\sigma_t^{q^K+q^{\mathcal{B}}} + \varsigma_t\sigma_t^{q^K+q^{\mathcal{B}}}\right) + (\varsigma_t - \sigma_t^N)0 + \tilde{\varsigma}_t(1-\theta_t)\tilde{\sigma}$$

$$\frac{\mathbb{E}\left[dr_t^{\vartheta/\mathcal{B}}\right]}{dt} = i_t + \mu_t^{\vartheta/\mathcal{B}} = \underbrace{\left(r_t^f - (\Phi(\iota_t) - \delta) - \mu_t^{q^K+q^{\mathcal{B}}} - \sigma\sigma_t^{q^K+q^{\mathcal{B}}} + \varsigma_t\sigma_t^{q^K+q^{\mathcal{B}}}\right)}_{risk-free\ rate\ in\ N_t-numeraire} + \underbrace{(\varsigma_t - \sigma_t^N)\sigma_t^{\vartheta/\mathcal{B}}}_{\substack{price\ of\ risk \\ in\ N_t-numeraire}}$$

$$\check{\rho}_t - i_t - \mu_t^{\vartheta/\mathcal{B}} = \quad -(\varsigma_t - \sigma_t^N)\sigma_t^{\vartheta/\mathcal{B}} + \tilde{\varsigma}_t(1-\theta_t)\tilde{\sigma}$$

- Remark: $\vartheta/\mathcal{B}$ = wealth share per bond
  - Value of a single bond/coin in N_t-numeraire

$$\frac{d(\vartheta_t/\mathcal{B}_t)}{\vartheta_t/\mathcal{B}_t} = \mu_t^{\vartheta}dt + \sigma_t^{\vartheta}dZ_t - \mu_t^{\mathcal{B}}dt - \sigma_t^{\mathcal{B}}dZ_t + \sigma_t^{\mathcal{B}}(\sigma_t^{\mathcal{B}} - \sigma_t^{\vartheta})dt$$

$$= \mu_t^{\vartheta/\mathcal{B}}dt + \sigma_t^{\vartheta/\mathcal{B}}dZ_t \text{ (defining return-drift and volatility)}$$

  - Terms are shifted into risk-free rate in N_t-numeraire, which drop out when differencing

Recall
$\mu_t^{\vartheta/\mathcal{B}} = \mu_t^{\vartheta} - \mu_t^{\mathcal{B}} + \sigma_t^{\mathcal{B}}(\sigma_t^{\mathcal{B}} - \sigma_t^{\vartheta})$
$\check{\mu}^{\mathcal{B}} = \mu_t^{\mathcal{B}} - i_t$

Solving MacroModels Step-by-Step

0. Postulate aggregates, price processes & obtain return processes
1. For given C/N-ratio and SDF processes for each i *finance block*
   a. Real investment ι + Goods market clearing *(static)*
   - *Toolbox 1:* Martingale Approach, HJB vs. Stochastic Maximum Principle Approach

   b. Portfolio choice θ + Asset market clearing or
   Asset allocation κ & risk allocation χ
   - *Toolbox 2:* "price-taking social planner approach" – Fisher separation theorem
   - *Toolbox 3:* Change in numeraire to total wealth (including SDF)
     - "Money evaluation/FTPL equation" ϑ
2. Evolution of state variable ~~η (and~~ K) *forward equation*
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   a. Value fcn. as fcn. of individual investment opportunities ω
   - *Special cases: log-utility, constant investment opportunities*

   b. Separating value fcn. $\boldsymbol{V^i(n^{\tilde{i}};\eta,K)}$ into $\boldsymbol{v^i(\eta)\left(\tilde{\eta}^{\tilde{i}}\right)^{1-\gamma}u(K)\left(n^{\tilde{i}}/n\right)^{1-\gamma}}$
   c. Derive $\check{\rho} = C/N$-ratio and ς price of risk
4. Numerical model solution
   a. Transform BSDE for separated value fcn. $v^i(\eta)$ into PDE
   b. Solve PDE via value function iteration
5. KFE: Stationary distribution, Fan charts

3a.+b. + Isolating Idio. Risk

For CRRA utility fcn

- Rephrase the conjecture value function as

$$V_t^{\tilde{i}} = \frac{\left(\omega_t^i n_t^{\tilde{i}}\right)^{1-\gamma}}{1-\gamma} = \underbrace{\left(\omega_t \frac{N_t}{K_t}\right)^{1-\gamma}}_{=:v_t} \underbrace{\left(\frac{n_t^{\tilde{i}}}{N_t}\right)^{1-\gamma}}_{=:\left(\tilde{\eta}_t^{\tilde{i}}\right)^{1-\gamma}} \frac{K_t^{1-\gamma}}{(1-\gamma)}$$

- v_t^i depend only on aggregate state η_t
- Ito's quotation rule

$$\frac{d\tilde{\eta}_t^{\tilde{i}}}{\tilde{\eta}_t^{\tilde{i}}} = \frac{d\left(n_t^{\tilde{i}}/N_t\right)}{n_t^{\tilde{i}}/N_t} = \left(\mu_t^{n^{\tilde{i}}} - \mu_t^N + \left(\sigma_t^N\right)^2 - \sigma^N \sigma^{n^{\tilde{i}}}\right) dt + \left(\sigma_t^{n^{\tilde{i}}} - \sigma_t^N\right) dZ_t + \tilde{\sigma}^{n^{\tilde{i}}} d\tilde{Z}_t^{\tilde{i}} \ = \tilde{\sigma}^{n^{\tilde{i}}} d\tilde{Z}_t^{\tilde{i}}$$

- Ito's Lemma

$$\frac{d\left(\tilde{\eta}_t^{\tilde{i}}\right)^{1-\gamma}}{\left(\tilde{\eta}_t^{\tilde{i}}\right)^{1-\gamma}} = -\frac{1}{2}\gamma(1-\gamma)\left(\tilde{\sigma}^{n^{\tilde{i}}}\right)^2 dt + (1-\gamma)\tilde{\sigma}^{n^{\tilde{i}}} d\tilde{Z}_t^{\tilde{i}}$$

3b. BSDE for v_t^i

$$\frac{dV_t^{\tilde{i}}}{V_t^{\tilde{i}}} = \frac{d\left(v_t\left(\tilde{\eta}_t^{\tilde{i}}\right)^{1-\gamma}(K_t)^{1-\gamma}\right)}{v_t\left(\tilde{\eta}_t^{\tilde{i}}\right)^{1-\gamma}(K_t)^{1-\gamma}}$$

- By Ito's product rule

$$= \left(\mu_t^v + (1-\gamma)(\Phi(\iota_t) - \delta) - \frac{1}{2}\gamma(1-\gamma)\left(\sigma^2 + \left(\tilde{\sigma}^{n^{\tilde{i}}}\right)^2\right) + (1-\gamma)\sigma\sigma_t^v\right)dt + volatility\ terms$$

- Recall by consumption optimality

  $\frac{dV_t^{\tilde{i}}}{V_t^{\tilde{i}}} - \rho dt + \frac{c_t^{\tilde{i}}}{n_t^{\tilde{i}}} dt$ follows a martingale

  - Hence, drift above $= \rho - \frac{c_t^{\tilde{i}}}{n_t^{\tilde{i}}}$

- BSDE:

$$\mu_t^v + (1-\gamma)(\Phi(\iota_t) - \delta) - \frac{1}{2}\gamma(1-\gamma)\left(\sigma^2 + \left(\tilde{\sigma}_t^{n^{\tilde{i}}}\right)^2\right) + (1-\gamma)\sigma\sigma_t^v = \rho - \frac{c_t^{\tilde{i}}}{n_t^{\tilde{i}}}$$

3. Deriving C/N-ratio $\breve{\rho}$ in stationary setting

- In stationary equilibrium

$$\cancel{\underbrace{\mu_t^v}_{=0}} + (1-\gamma)(\Phi(\iota_t)-\delta) - \frac{1}{2}\gamma(1-\gamma)\left(\sigma^2 + \left(\tilde{\sigma}^{n^{\tilde{i}}}\right)^2\right) + \cancel{\underbrace{(1-\gamma)\sigma\sigma_t^v}_{=0}} = \rho - \underbrace{\frac{c_t^{\tilde{i}}}{n_t^{\tilde{i}}}}_{=\breve{\rho}}$$

- Recall and plug in
  - $\tilde{\sigma}^{n^{\tilde{i}}} = (1-\vartheta)\tilde{\sigma} = \sqrt{\breve{\rho}+\mu^{\mathcal{B}}}/\sqrt{\gamma}$ using $(1-\vartheta) = \sqrt{\breve{\rho}+\mu^{\mathcal{B}}}/(\sqrt{\gamma}\tilde{\sigma})$
  - $\iota = \frac{a\sqrt{\breve{\rho}+\mu^{\mathcal{B}}}-\sqrt{\gamma}\tilde{\sigma}\breve{\rho}}{\sqrt{\breve{\rho}+\mu^{\mathcal{B}}}+\kappa\sqrt{\gamma}\tilde{\sigma}\breve{\rho}}$

  yields an equation for $\breve{\rho}$

$$(1-\gamma)\left(\frac{1}{\kappa}\log\frac{\sqrt{\breve{\rho}+\mu^{\mathcal{B}}}\ (1+\phi a)}{\sqrt{\breve{\rho}+\mu^{\mathcal{B}}}+\phi\sqrt{\gamma}\tilde{\sigma}\breve{\rho}} - \delta\right) - \frac{1}{2}\gamma(1-\gamma)\left(\sigma^2 + \frac{\breve{\rho}+\mu^{\mathcal{B}}}{\gamma^2}\right) = \rho - \breve{\rho}$$

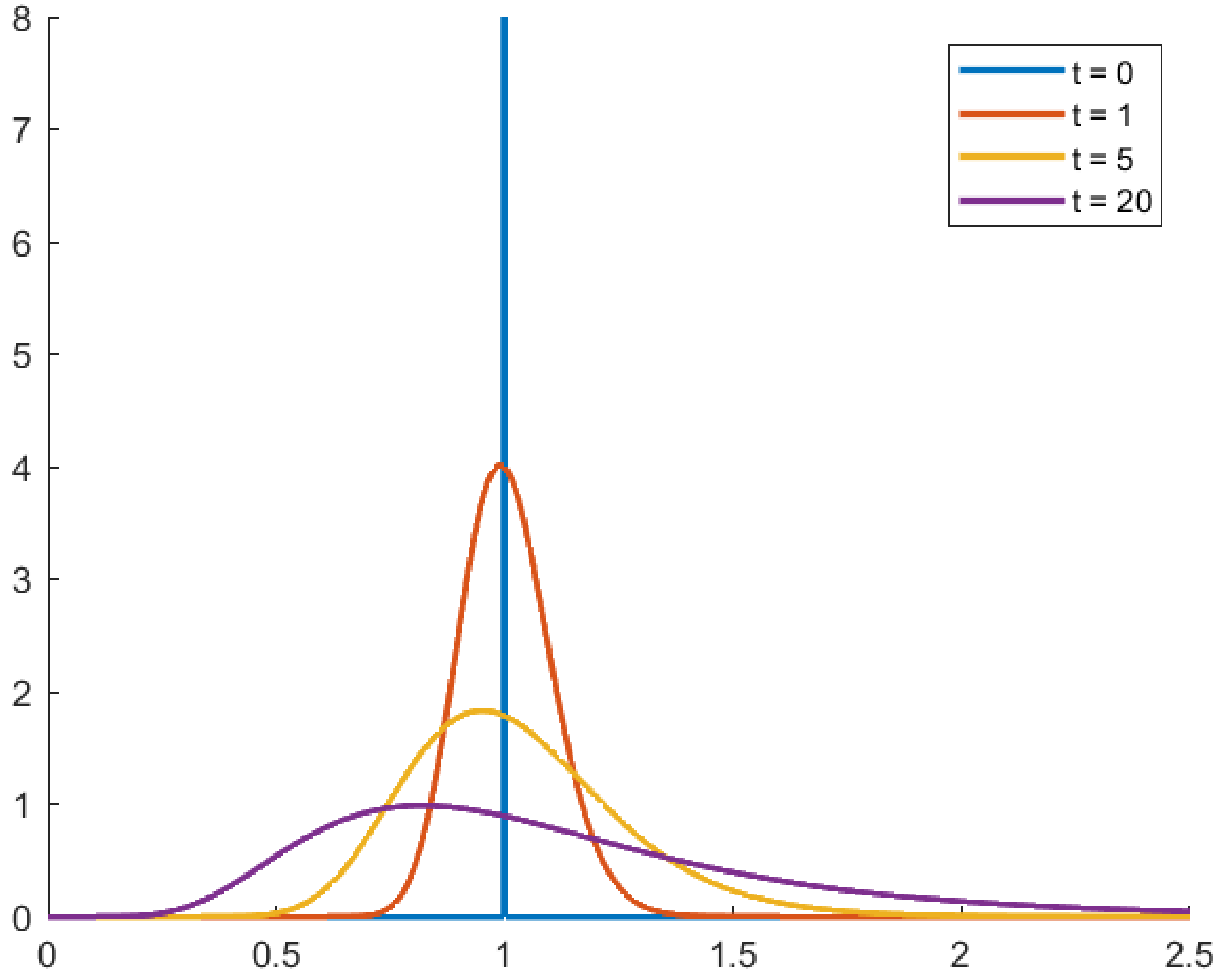
  - For $\gamma = 1$: $\breve{\rho} = \rho$

4. Numerical Solution

- Simpler than previous lectures since there is no state variable
- Generalization with $\tilde{\sigma}_t$ as a state variable.

Cross-sectional net worth distribution

- $\tilde{\eta}_t^{\tilde{\imath}} = \frac{\tilde{n}_t^{\tilde{\imath}}}{N_t}$ is non-stationary … and log-normally distributed



- Next: Extend model with net worth reset jumps to η^*

Aside 1: Model with idiosyncratic net worth reset jumps

- With Poisson intensity λ net worth $\tilde{\eta}_t^{\tilde{\imath}-}$ jumps from to
- Log-utility ⇒ same returns, no impact on equilibrium

$$\frac{dn_t^{\tilde{\imath}}}{n_t^{\tilde{\imath}}} = \Big(-\underbrace{\rho}_{c_t^{\tilde{\imath}}/n_t^{\tilde{\imath}}} + \underbrace{g}_{r_t^B = \Phi(\iota) - \delta} + (1-\theta_t) \underbrace{\frac{a-\iota}{q^K}}_{E[r_t^{k,\tilde{\imath}}] - r_t^B} \Big) dt + (1-\theta_t)\tilde{\sigma} dZ_t^{\tilde{\imath}} + j_t^{n,\tilde{\imath}} dJ_t^{\tilde{\imath}}$$

$$\frac{dn_t^{\tilde{\imath}}}{n_t^{\tilde{\imath}}} = \left(-\rho + g + (1-\vartheta_t)\frac{a-\iota}{q^K} \right) dt + (1-\theta_t)\tilde{\sigma} dZ_t^{\tilde{\imath}} + j_t^{n,\tilde{\imath}} dJ_t^{\tilde{\imath}}$$

$$\frac{dN_t}{N_t} = g dt$$

$$\frac{d\eta_t^{\tilde{\imath}}}{\eta_t^{\tilde{\imath}}} = \underbrace{(-\rho + (1-\vartheta_t)\frac{a-\iota}{q^K})}_{=0} dt + (1-\vartheta_t)\tilde{\sigma} dZ_t^{\tilde{\imath}} + j_t^{n,\tilde{\imath}} dJ_t^{\tilde{\imath}}$$

$$\frac{d\eta_t^{\tilde{\imath}}}{\eta_t^{\tilde{\imath}}} = (1-\vartheta)\tilde{\sigma} dZ_t^{\tilde{\imath}} + j_t^{n,\tilde{\imath}} dJ_t^{\tilde{\imath}}$$

- Set $j_t^{n,\tilde{\imath}} = \frac{\eta^* - \eta_t^{\tilde{\imath}}}{\eta_t^{\tilde{\imath}}}$
- KFE (for all $\eta \neq \eta^*$) is given by:

$$0 = \frac{(1-\vartheta)^2\tilde{\sigma}^2}{2}\frac{\partial(\eta^2 g(\eta))}{\partial\eta} - \lambda g(\eta)$$

- There is a kink at η^*

Aside 1: Model Extension with idiosyncratic reset jumps

$$\frac{d\eta_t^{\tilde{i}}}{\eta_t^{\tilde{i}}} = (1-\vartheta)\tilde{\sigma}\, dZ_t^{\tilde{i}} + j_t^{n,\tilde{i}}\, dJ_t^{\tilde{i}}$$

- Set $j_t^{n,\tilde{i}} = \frac{\eta^* - \eta_t^{\tilde{i}}}{\eta_t^{\tilde{i}}}$
- KFE (for all $\eta \neq \eta^*$) is given by:

$$0 = \frac{(1-\vartheta)^2\tilde{\sigma}^2}{2}\frac{\partial(\eta^2 g(\eta))}{\partial\eta} - \lambda g(\eta)$$

- There is a kink at η^*

- Solution under $\eta^* = 1$:

$$C_1 = C_4 = \frac{2\lambda}{(1-\varsigma)^2\tilde{\sigma}^2\alpha}, C_2 = C_3 = 0$$

- KFE (for all $\eta \neq \eta^*$) is given by:

$$0 = g''(\eta)\eta^2 + 4g'(\eta)\eta + \left(2 - \frac{2\lambda}{(1-\vartheta)^2\tilde{\sigma}^2}\right) g(\eta)$$

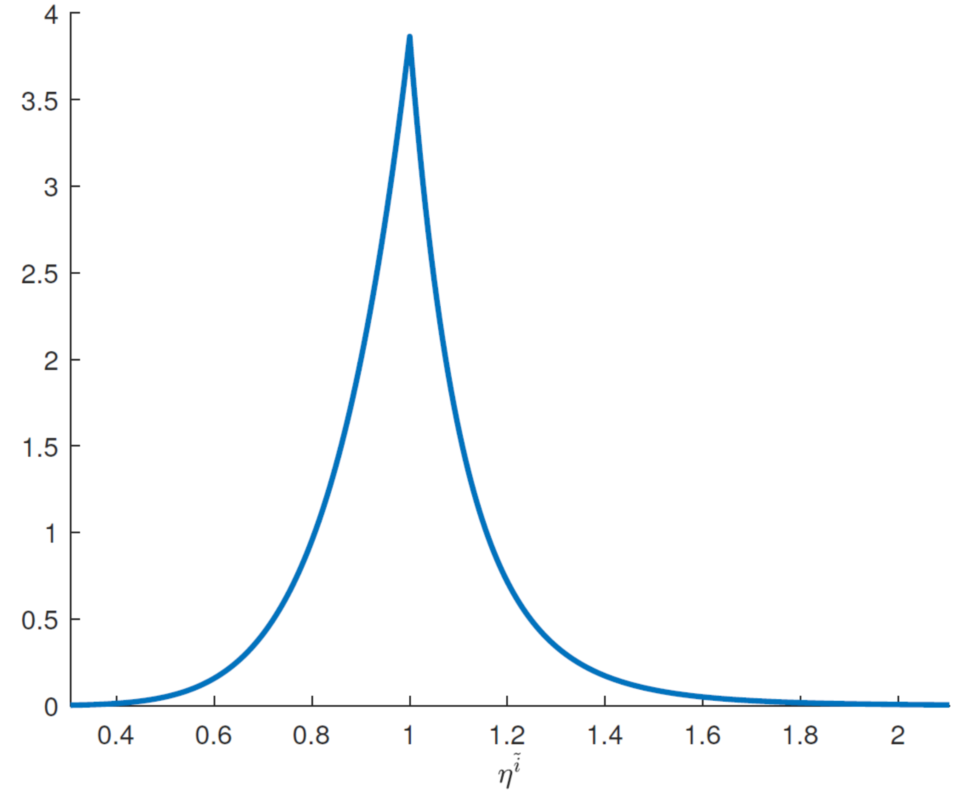
- Euler's equation – has closed-form solutions

$$g(\eta) = C_1\eta^{\alpha_1} + C_2\eta^{\alpha_2} \text{ for } \eta < \eta^*$$

$$g(\eta) = C_3\eta^{\alpha_1} + C_4\eta^{\alpha_2} \text{ for } \eta \geq \eta^*$$

$$\int_0^\infty g(\eta)d\eta = 1, \qquad \lim_{\eta\to 0} g(\eta) = \lim_{\eta\to\infty} g(\eta) = 0$$

- + continuity at η^*, $\alpha_1 = \frac{\alpha-3}{2}$, $\alpha_2 = -\frac{\alpha+3}{2}$, $\alpha = \sqrt{\frac{8\lambda}{(1-\vartheta)^2\tilde{\sigma}^2} + 1}$



Models on Money as Store of Value

\Friction	OLG	Incomplete Markets + idiosyncratic risk	
Risk	deterministic	endowment risk borrowing constraint	return risk Risk tied up with Individual capital
Only money	Samuelson	Bewley	
			“I Theory without I” Brunnermeier-Sannikov (AER PP 2016)
With capital	Diamond	Aiyagari	

Aside 2: BruSan meets Bewley-Huggett-Aiyagari

- $\max_{c,\theta} \mathbb{E} \int_0^\infty e^{-\rho t} u(c_t) dt$
  - $dn_t^{\tilde{\imath}} = (-c_t^{\tilde{\imath}} + y_t^{\tilde{\imath}})dt + n_t^{\tilde{\imath}}\left(rdt + (1-\theta)\left(dr_t^{k,\tilde{\imath}} - rdt\right)\right)$
  - $dr_t^{k,\tilde{\imath}} = r^k dt + \tilde{\sigma}^k dZ_t^{k,\tilde{\imath}}$
  - $dy_t^{k,\tilde{\imath}} = -\nu y_t^{\tilde{\imath}} dt + \tilde{\sigma}^y dZ_t^{y,\tilde{\imath}}$

Partial insurance via retrading

- BruSan: $\tilde{\sigma}^y = 0$ … retrade capital and safe asset + smooth consumption
- Bewley-Huggett-Aiyagari: $\tilde{\sigma}^k = 0$ … smooth consumption
  - Risk does not scale with net worth ⇒ $\frac{c}{n}$ and portfolio θ depends on net worth

Roadmap

- Intuition for different “Monetary Theories”

- Monetary Model with one sector with constant idiosyncratic risk
  - Safe Asset and Service Flows
  - Bubble (mining) or not
  - 2 Different Asset Pricing Perspectives/SDFs

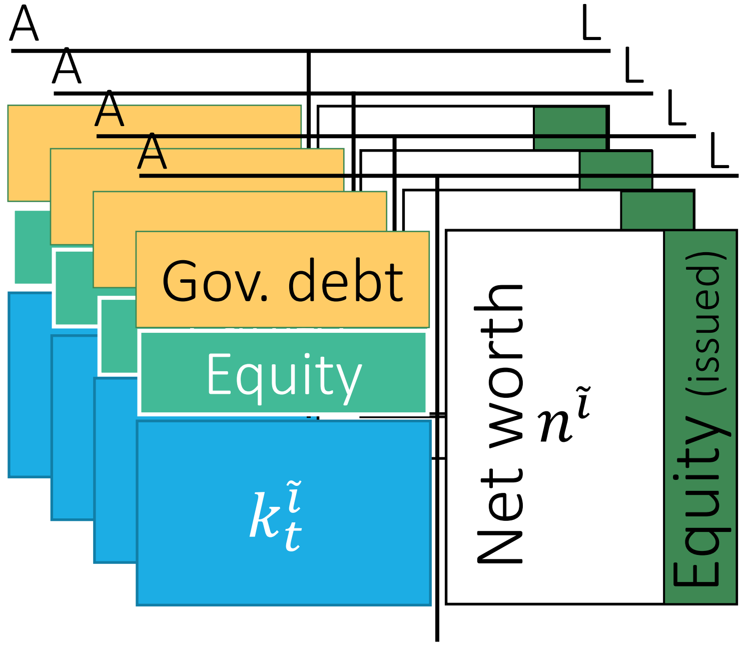
- Monetary model with time-varying idiosyncratic risk
  - Safe asset has negative CAPM-β
  - Calibration:
    Debt valuation puzzle, Debt Laffer Curve, Flight-to-Safety and Equity excess volatility

- Medium of Exchange Role

Time-varying Idio Risk + Equity Markets + Epstein-Zin

- Equity Market
  - Each citizen $\tilde{\imath}$ can sell off a fraction $(1-\bar{\chi})$ of capital risk to outside equity holders
  - Return $dr_t^{E,\tilde{\imath}}$
    - Same risk as $dr_t^{K,\tilde{\imath}}$
    - But $\mathbb{E}_t\left[dr_t^{E,\tilde{\imath}}\right] < \mathbb{E}_t\left[dr_t^{K,\tilde{\imath}}\right]$… due to insider premium
  - Prop.: Model equations as before but replace $\tilde{\sigma}$ *with* $\bar{\chi}\tilde{\sigma}$
- Aggregate risk dZ_t: Heston model (time-varying idiosyncratic risk)
  - $d\tilde{\sigma}_t^2 = -\psi\left(\tilde{\sigma}_t^2 - \left(\tilde{\sigma}^0\right)^2\right)dt - \sigma\tilde{\sigma}_t dZ_t, \quad a_t = a(\tilde{\sigma}_t)$

    CIR – ensures that $\tilde{\sigma}$ stays positive
- Monetary/bond issuing policy: $d\mathcal{B}_t/\mathcal{B}_t = \mu_t^{\mathcal{B}}dt + \sigma_t^{\mathcal{B}}dZ_t$
- Epstein-Zin preferences for calibration (EIS=1)

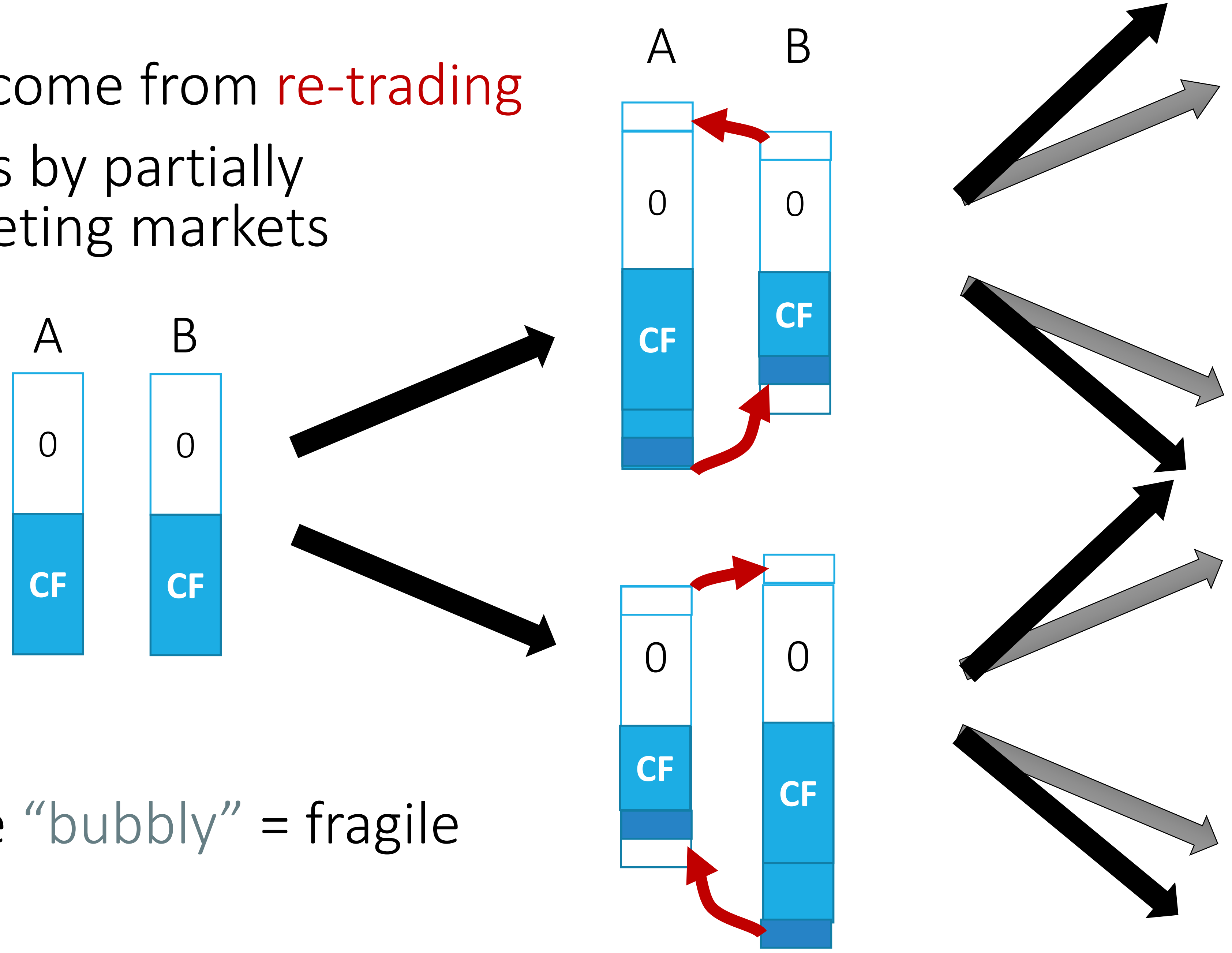


$$V_t^i = E_t\left[\int_t^\infty (1-\gamma)\rho V_s^i\left(\log(c_s^i) - \frac{1}{1-\gamma}\log\left((1-\gamma)V_s^i\right)\right)ds\right]$$

What's a Safe Asset? What is its Service Flow?

- $\frac{\mathcal{B}_t}{\wp_t} = \mathbb{E}_t\left[PV_{\xi^{**}}(\text{primary surpluses})\right] + \mathbb{E}_t\left[PV_{\xi^{**}}(\text{service flow})\right]$

- Value come from re-trading
- Insures by partially completing markets



In recessions:

Risk is higher

- Service flow is more valuable
- Cash flows are lower
  (depends on fiscal policy)

- Can be "bubbly" = fragile

Government: Taxes, Bond/Money Supply, Gov. Budget

- $\sigma_t^{\mathcal{B}} \neq 0$ leads to stochastic “seigniorage revenue” (state contingent)
- Relabel tax revenue process to $\frac{d\tau_t}{\tau_t} = \mu_t^{\tau} dt + \sigma_t^{\tau} dZ_t$
  - Or should we label s (primary surplus) as a process
- Government budget constraint (BC) [REDEFINE]

$$d\mathcal{B}_t - i_t \mathcal{B}_t dt + \wp_t K_t \underbrace{(d\tau_t\, a_t - g dt)}_{s_t :=} = 0$$

Primary surplus (per K_t)

- Return on Gov. Bond/Money: in output numeraire

$$dr_t^{\mathcal{B}} = idt + \underbrace{\frac{d\left(q_t^{\mathcal{B}} K_t/\mathcal{B}_t\right)}{q_t^{\mathcal{B}} K_t/\mathcal{B}_t}}_{-inflation} = \frac{d\left(q_t^{\mathcal{B}} K_t\right)}{q_t^{\mathcal{B}} K_t} - \breve{\mu}_t^{\mathcal{B}} dt - \sigma_t^{\mathcal{B}} dZ_t + \sigma_t^{\mathcal{B}} \left(\sigma_t^{\mathcal{B}} - \sigma - \sigma_t^{q,\mathcal{B}}\right) dt$$

2.b+c Alternative iii: Portfolio choice θ (N_t-numeraire)

Total net worth N_t relative to single bond/coin

- Asset pricing equation (martingale method)

$$\frac{\mathbb{E}\left[dr_t^{\tilde{\eta}^{\tilde{i}}}\right]}{dt} = \check{\rho}_t = \left(r_t^f - (\Phi(\iota_t) - \delta) - \mu_t^{q^K+q^{\mathcal{B}}} - \sigma\sigma_t^{q^K+q^{\mathcal{B}}} + \varsigma_t\sigma_t^{q^K+q^{\mathcal{B}}}\right) + (\varsigma_t - \sigma_t^N)0 + \tilde{\varsigma}_t(1-\theta_t)\tilde{\sigma}$$

$$\frac{\mathbb{E}\left[dr_t^{\vartheta/\mathcal{B}}\right]}{dt} = \mu_t^{\vartheta/\mathcal{B}} = \underbrace{\left(r_t^f - (\Phi(\iota_t) - \delta) - \mu_t^{q^K+q^{\mathcal{B}}} - \sigma\sigma_t^{q^K+q^{\mathcal{B}}} + \varsigma_t\sigma_t^{q^K+q^{\mathcal{B}}}\right)}_{risk-free\ rate\ in\ N_t-numeraire} + \underbrace{(\varsigma_t - \sigma_t^N)\sigma_t^{\vartheta/\mathcal{B}}}_{\substack{price\ of\ risk \\ in\ N_t-numeraire}}$$

2.b+c Alternative iii: Portfolio choice θ (N_t-numeraire)

Total net worth N_t relative to a single bond/coin of money

- Asset pricing equation (martingale method)

$$\frac{\mathbb{E}\left[dr_t^{\tilde{\eta}^{\tilde{i}}}\right]}{dt} = \check{\rho}_t = \left(r_t^f - (\Phi(\iota_t) - \delta) - \mu_t^{q^K+q^{\mathcal{B}}} - \sigma\sigma_t^{q^K+q^{\mathcal{B}}} + \varsigma_t\sigma_t^{q^K+q^{\mathcal{B}}}\right) + (\varsigma_t - \sigma_t^N)0 + \tilde{\varsigma}_t(1-\theta_t)\tilde{\sigma}$$

$$\frac{\mathbb{E}\left[dr_t^{\vartheta/\mathcal{B}}\right]}{dt} = \mu_t^{\vartheta/\mathcal{B}} = \underbrace{\left(r_t^f - (\Phi(\iota_t) - \delta) - \mu_t^{q^K+q^{\mathcal{B}}} - \sigma\sigma_t^{q^K+q^{\mathcal{B}}} + \varsigma_t\sigma_t^{q^K+q^{\mathcal{B}}}\right)}_{risk-free\ rate\ in\ N_t-numeraire} + \underbrace{(\varsigma_t - \sigma_t^N)\sigma_t^{\vartheta/\mathcal{B}}}_{\substack{price\ of\ risk \\ in\ N_t-numeraire}}$$

$$\check{\rho}_t - \mu_t^{\vartheta/\mathcal{B}} = -(\varsigma_t - \sigma_t^N)\sigma_t^{\vartheta/\mathcal{B}} + \tilde{\varsigma}_t(1-\theta_t)\tilde{\sigma}$$

- Remark:
  - Value of a single bond/coin in N_t-numeraire

$$\frac{d(\vartheta_t/\mathcal{B}_t)}{\vartheta_t/\mathcal{B}_t} = \mu_t^{\vartheta} + \sigma_t^{\vartheta}dZ_t - \mu_t^{\mathcal{B}}dt - \sigma_t^{\mathcal{B}}dZ_t + \sigma_t^{\mathcal{B}}\left(\sigma_t^{\mathcal{B}} - \sigma_t^{\vartheta}\right)dt$$

$$= \mu_t^{\vartheta/\mathcal{B}}dt + \sigma_t^{\vartheta/\mathcal{B}}dZ_t$$ (defining return-drift and volatility)

  - Terms are shifted into risk-free rate in N_t-numeraire, which drop out when differencing

2.b+c Alternative iii: Portfolio choice θ (N_t-numeraire)

Total net worth N_t relative to single bond/coin of money

- Asset pricing equation (martingale method)

$$\frac{\mathbb{E}\left[dr_t^{\tilde{\eta}^{\tilde{i}}}\right]}{dt} = \check{\rho}_t = \left(r_t^f - (\Phi(\iota_t) - \delta) - \mu_t^{q^K+q^{\mathcal{B}}} - \sigma\sigma_t^{q^K+q^{\mathcal{B}}} + \varsigma_t\sigma_t^{q^K+q^{\mathcal{B}}}\right) + (\varsigma_t - \sigma_t^N)0 + \tilde{\varsigma}_t(1-\theta_t)\tilde{\sigma}$$

$$\frac{\mathbb{E}\left[dr_t^{\vartheta/\mathcal{B}}\right]}{dt} = \mu_t^{\vartheta/\mathcal{B}} = \underbrace{\left(r_t^f - (\Phi(\iota_t) - \delta) - \mu_t^{q^K+q^{\mathcal{B}}} - \sigma\sigma_t^{q^K+q^{\mathcal{B}}} + \varsigma_t\sigma_t^{q^K+q^{\mathcal{B}}}\right)}_{risk-free\ rate\ in\ N_t-numeraire} + \underbrace{(\varsigma_t - \sigma_t^N)\sigma_t^{\vartheta/\mathcal{B}}}_{\substack{price\ of\ risk \\ in\ N_t-numeraire}}$$

$$\check{\rho}_t - \mu_t^{\vartheta/\mathcal{B}} = -(\varsigma_t - \sigma_t^N)\sigma_t^{\vartheta/B} + \tilde{\varsigma}_t(1-\theta_t)\tilde{\sigma}$$

- Price of Risk: $\varsigma_t = -\sigma_t^v + \sigma_t^{p+q} + \gamma\sigma$, $\tilde{\varsigma}_t = \gamma\tilde{\sigma}_t^n = \gamma(1-\theta_t)\tilde{\sigma}$

$$\check{\rho}_t - \mu_t^{\vartheta/\mathcal{B}} = (\sigma_t^v - (\gamma-1)\sigma)\sigma_t^{\vartheta/\mathcal{B}} + \gamma(1-\theta_t)^2\tilde{\sigma}^2$$

- Capital market clearing: $1-\theta = 1-\vartheta$

Recall

$\mu_t^{\vartheta/\mathcal{B}} = \mu_t^{\vartheta} - \mu_t^{\mathcal{B}} + \sigma_t^{\mathcal{B}}(\sigma_t^{\mathcal{B}} - \sigma_t^{\vartheta})$

$\sigma_t^{\vartheta/\mathcal{B}} = \sigma_t^{\vartheta} - \sigma_t^{\mathcal{B}}$

FTPL Equation with Bubble: 2 Perspectives

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- Agent $\tilde{\imath}$'s SDF, $\xi_t^{\tilde{\imath}}$: $d\xi_t^{\tilde{\imath}}/\xi_t^{\tilde{\imath}} \quad = -r_t^f\, dt - \varsigma_{\cancel{t}} dZ_t - \tilde{\varsigma}_t^{\tilde{\imath}} d\tilde{Z}_t^{\tilde{\imath}}$
- Buy and Hold Perspective:

$$\frac{\mathcal{B}_0}{\mathcal{P}_0} = \lim_{T\to\infty}\left(\mathbb{E}\left[\int_0^T \xi_t^i s_t K_t dt\right] + \mathbb{E}\left[\xi_T^i \frac{\mathcal{B}_T}{\mathcal{P}_T}\right]\right)$$

  - Bubble is possible: $\lim_{T\to\infty} \mathbb{E}[\bar{\xi}_t \frac{\mathcal{B}_T}{\wp_T}] > 0$ if $r_t^f + \varsigma_t \sigma_t^{q,B} \leq g_t$ (on average) $g - \check{\mu}^B$ = discount rate
- Dynamic Trading Perspective:
  - Value cash flow from individual bond portfolios, including trading cash flows
  - Integrate over citizens weighted by net worth share η_t^i
  - Bond as part of a dynamic trading strategy

$$\frac{\mathcal{B}_0}{\mathcal{P}_0} = \mathbb{E}\left[\int_0^\infty \underbrace{\left(\int \xi_t^i \eta_t^i di\right)}_{=\xi_t^{**}} s_t K_t dt\right] + \mathbb{E}\left[\int_0^\infty \underbrace{\left(\int \xi_t^i \eta_t^i di\right)}_{=\xi_t^{**}} (\tilde{\sigma}_t^c)^2 \frac{\mathcal{B}_t}{\mathcal{P}_t} dt\right]$$

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  - Discount rate $E[dr^\eta]/dt = r^f + \tilde{\varsigma}\tilde{\sigma}$
  - ξ^i and η^i are negatively correlated ⇒ depresses weighted "Quasi-SDF" (higher discount rate)

Numerical Steps

$\vartheta_t = \vartheta(\tilde{\sigma}_t)$, Ito's formula:

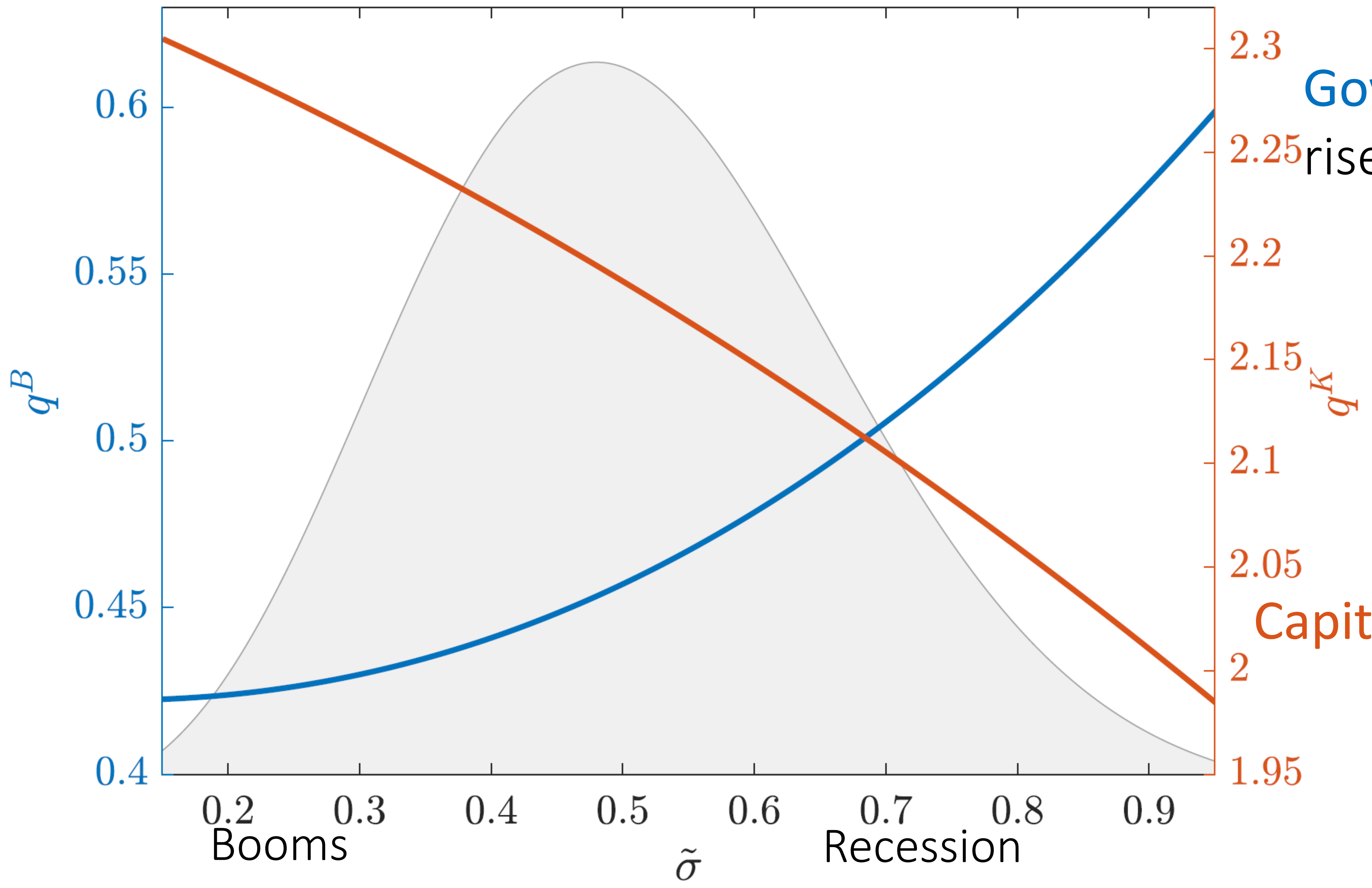
$$d\vartheta_t = \underbrace{\left(\mu_{\tilde{\sigma},t}\vartheta'(\tilde{\sigma}) + \frac{\sigma^2_{\tilde{\sigma},t}}{2}\vartheta''(\tilde{\sigma})\right)}_{\mu_t^{\vartheta}\vartheta_t} dt + \sigma_{\tilde{\sigma},t}\vartheta'(\tilde{\sigma})dZ_t$$

$$\rho\vartheta(\tilde{\sigma}) = (1-\vartheta(\tilde{\sigma}))^2\tilde{\sigma}^2\vartheta(\tilde{\sigma}) + b(\tilde{\sigma}^{ss}-\tilde{\sigma})\vartheta'(\tilde{\sigma}) + \frac{\nu^2\tilde{\sigma}}{2}\vartheta''(\tilde{\sigma})$$

$$\rho\vartheta_t(\tilde{\sigma}) = \partial_t\vartheta_t(\tilde{\sigma}) + \underbrace{(1-\vartheta_t(\tilde{\sigma}))^2\tilde{\sigma}^2\vartheta_t(\tilde{\sigma})}_{u(v)} + \underbrace{b(\tilde{\sigma}^{ss}-\tilde{\sigma})\vartheta_t'(\tilde{\sigma}) + \frac{\nu^2\tilde{\sigma}}{2}\vartheta_t''(\tilde{\sigma})}_{Mv}$$

- For $\tilde{\sigma}$-grid: from 0 to $\tilde{\sigma}$ large enough (look at stationary distribution of $\tilde{\sigma}$)

Bond and Capital Value for time-varying idiosyncratic risk $\tilde{\sigma}_t$



Gov. debt value rises in recessions

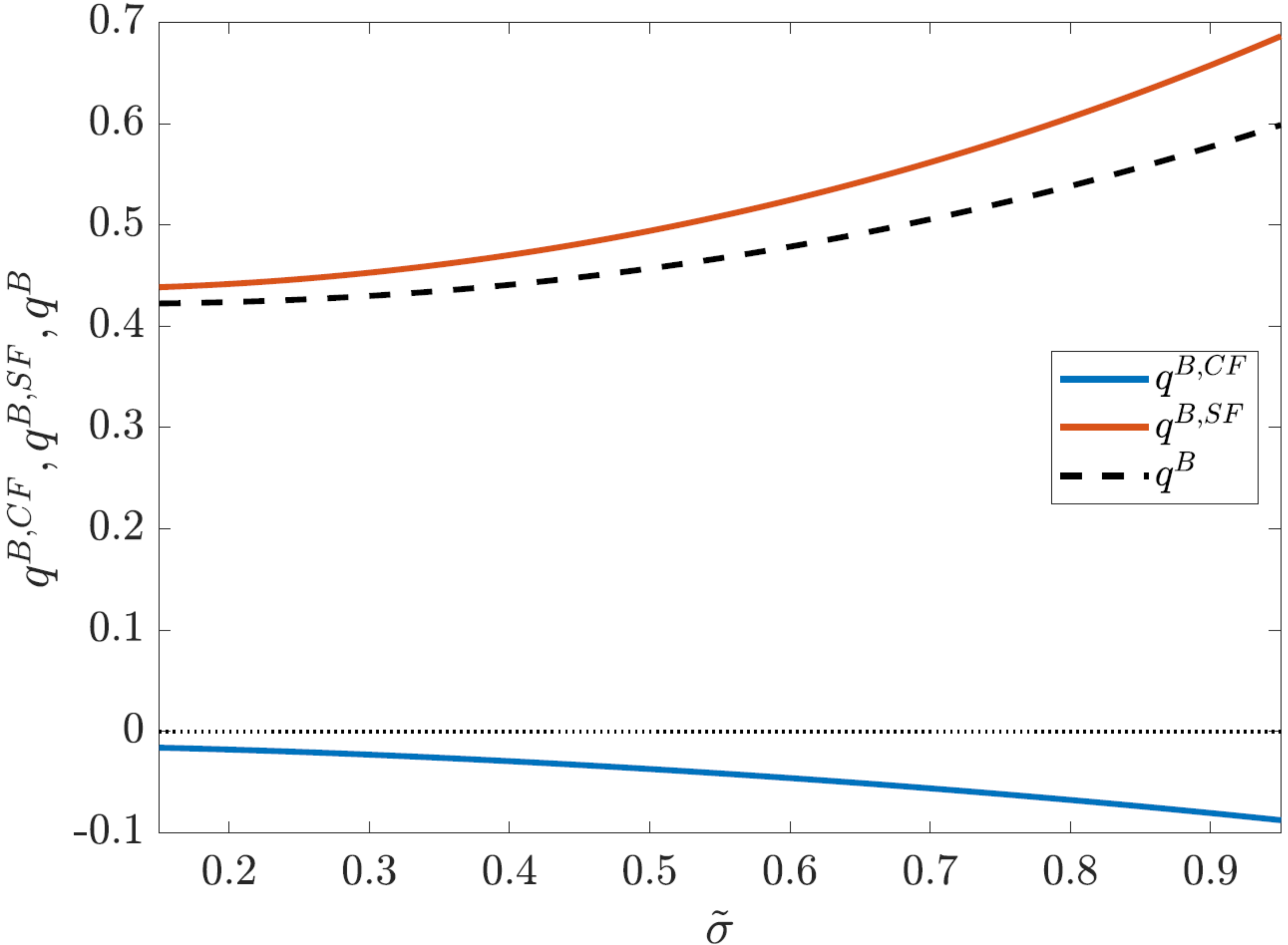
Capital price

Booms

Recession

Dynamic Trading Perspective Decomposition

$$\frac{\mathcal{B}_0}{\mathcal{P}_0} = \overbrace{\mathbb{E}\left[\int_0^\infty \xi_t^{**} s_t K_t dt\right]}^{\mathbb{E}\mathrm{PV}(\text{cash flow})} + \overbrace{\mathbb{E}\left[\int_0^\infty \xi_t^{**} \gamma \left(\tilde{\sigma}_t^c\right)^2 \frac{\mathcal{B}_t}{\mathcal{P}_t} dt\right]}^{\mathbb{E}\mathrm{PV}(\text{service flow})}$$



Calibration

- Exogenous processes:
  Recessions feature high idiosyncratic risk and low consumption
  - $\tilde{\sigma}_t$: Heston (1993) model of stochastic volatility

  $$d\tilde{\sigma}_t^2 = -\psi\left(\tilde{\sigma}_t^2 - \left(\tilde{\sigma}^0\right)^2\right)dt - \sigma^{\tilde{\sigma}}\tilde{\sigma}_t dZ_t$$

  CIR – ensures that $\tilde{\sigma}$ stays positive
  - a_t: $a_t = a(\tilde{\sigma}_t)$

  $$a_t(\tilde{\sigma}_t) = a^0 - \alpha^a(\tilde{\sigma}_t - \tilde{\sigma}^0)$$

  - $\mathscr{g}_t = 0$

- Government (bubble-mining policy)

$$\check{\mu}_t^{\mathcal{B}} = \check{\mu}_t^{\mathcal{B},0} + \alpha^{\mathcal{B}}(\tilde{\sigma}_t - \tilde{\sigma}^0)$$

- Calibration to US data (1970-2019, period length is one year)

Parameters

parameter	description	value	target
			external calibration
$\tilde{\sigma}^0$	$\tilde{\sigma}_t$ stoch. steady state	0.54	MLE targeting common idiosyncratic volatility (Herskovic et al. 2018)
ψ	$\tilde{\sigma}_t^2$ mean reversion	0.67	
σ	$\tilde{\sigma}_t^2$ volatility	0.4	
$\bar{\chi}$	undiversifiable idio. risk	0.3	Heaton, Lucas (1996, 2000, 2001), Angeletos (2007) (range $[0.2, 0.6]$)
			calibration to match model moments
γ	risk aversion	6	chosen jointly to match (approximately) - volatility of Y, C, I, S/Y - average C/Y, G/Y, S/Y, I/K, $q^K K/Y$, $q^B K/Y$ - mean equity premium - equity Sharpe ratio
ρ	time preference	0.138	
a^0	a_t stoch. steady state	0.63	
$\mathfrak{g}$	gov. expenditures	0.138	
$\breve{\mu}^{\mathcal{B},0}$	$\breve{\mu}_t^{\mathcal{B}}$ stoch. steady state	0.0023	
α^a	a_t slope	0.071	
$\alpha^{\mathcal{B}}$	$\breve{\mu}_t^{\mathcal{B}}$ slope	0.12	
ϕ	capital adjustment cost	8.5	
			other parameters
δ	depreciation rate	0.055	economic growth rate (*ultimately irrelevant for all results*)

Quantitative Model Fit

moment		model	data
symbol	description		
$\sigma(Y)$	output volatility	1.3%	1.3%
$\sigma(C)/\sigma(Y)$	relative consumption volatility	0.63	0.64
$\sigma(S/Y)$	surplus volatility	1.1%	1.1%
$\mathbb{E}[S/Y]$	average surplus-output ratio	-0.0004	-0.0005
$\mathbb{E}[q^K K/Y]$	average capital-output ratio	3.48	3.73
$\mathbb{E}[q^B K/Y]$	average debt-output ratio	0.74	0.71
$\mathbb{E}[d\bar{r}^E - dr^{\mathcal{B}}]$	average (unlevered) equity premium	3.62%	3.40%
$\frac{\mathbb{E}[dr^E - dr^{\mathcal{B}}]}{\sigma\left(dr^E - dr^{\mathcal{B}}\right)}$	equity sharpe ratio	0.31	0.31

Two Debt Valuation Puzzles

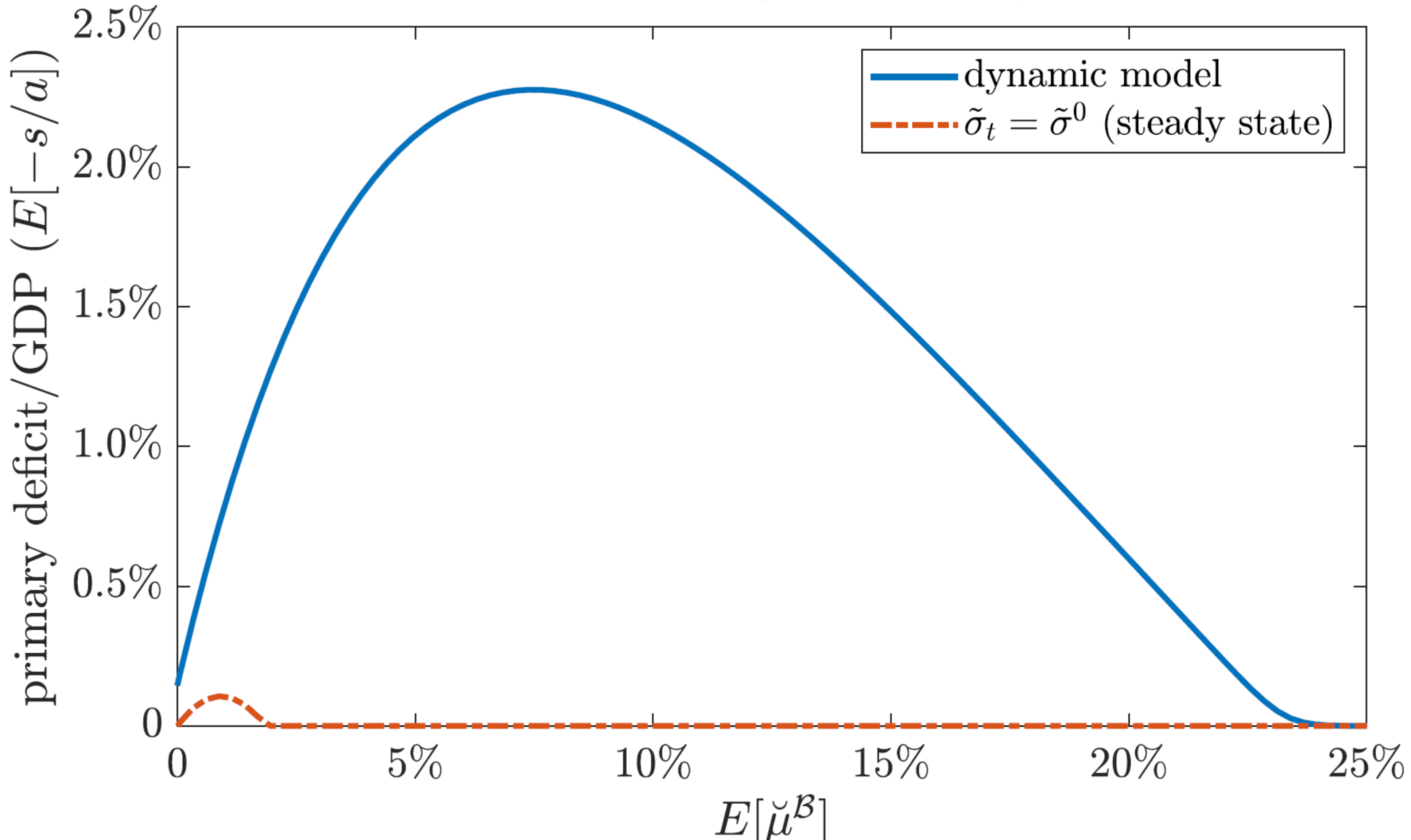
- Properties of US primary surpluses
  - Average surplus ≈ 0
  - Procyclical surplus (> 0 in booms, < 0 in recessions)

- Two valuation puzzles from standard perspective:
  (Jiang, Lustig, van Nieuwerburgh, Xiaolan, 2019, 2020)

1. “Public Debt Valuation Puzzle”
   - Empirical: $E[PV(surpluses)] < 0$, yet $\frac{\mathcal{B}}{\wp} > 0$
   - Our model: bubble/service flow component overturns results
2. “Gov. Debt Risk Premium Puzzle”
   - Debt should be positive β asset, but market don’t price it this way
   - Our model: can be rationalized with countercyclical bubble/service flow

Debt Laffer Curve ≠ MMT Debt Sustainability Analysis 1

- Issue bonds at a faster rate $\breve{\mu}^B$ (esp. in recessions)
  - ⇒ tax precautionary self insurance ⇒ tax rate ↑
  - ⇒ real value of bonds, $\frac{\mathcal{B}}{\wp}$, ↓ ⇒ "tax base" ↓
    - Less so in recession due to flight-to-safety



Sizeable revenue only if
Gov. debt has negative β

Service Flow Term, Convenience Yield, Ponzi Scheme

- Ponzi scheme is not feasibly for everyone
  No Ponzi constraint may be binding
  - Who can run a Ponzi scheme? **exorbitant privilege**
    … assigned by equilibrium selection
    - Likely to government, private entities are subject to solvency constraint
      - … still there is a Debt Laffer Curve

- Service flow
  - Convenience yield: relaxes collateral constraint or CIA constraint (money)
    - Traditional measure: BAA-US Treasury spread

  - Here: Partially completing markets through retrading
    - Low interest rate (cash flow) asset can be issued by everyone
      Hence, corporate-Treasury spread = 0

Why Does Safe Asset Survive in Presence of ETFs?

- Diversified stock portfolio is free of idiosyncratic risk
  - Trading in stocks (ETF) can also self-insure idiosyncratic risk
    - Good friend in idiosyncratically bad times

- But: poor hedge against aggregate risk, losses value in recessions
  - Positive β
    - Bad friend in aggregate bad times

- Why positive β? (after all r^f goes down in recessions, lowers discount rate)
  - Equity are claims to capital, but marginal capital holder is insider
  - Insider bears idiosyncratic risk, must be compensated
  - $\tilde{\sigma}_t \uparrow \Rightarrow$ insider premium $E_t[dr_t^K] - E_t[dr_t^E] \uparrow \Rightarrow$ payouts to stockholders fall

Stock Market Volatility due to Flight to Safety

- “Aggregate Intertemporal Budget Constraint

$$\underbrace{q_t^K K_t + q_t^B K_t}_{\text{total (net) wealth}} = \mathbb{E}_t[\int_t^\infty \underbrace{\frac{\int \xi_s^i \eta_s^i di}{\int \xi_t^i \eta_t^i di}}_{\frac{\xi_s^{**}}{\xi_t^{**}}} C_s ds] \quad (*)$$

- Lucas-type models: $q^B = 0$ (also $C_t = Y_t$, no idiosyncratic risk)
    - Value of equity (Lucas tree) = PV of consumption claim
    - Volatility equity values require volatile RHS of (*)
- This model: even for constant RHS of (*), $q_t^K K_t$ can be volatile due to flight to safety:
    - increase in $\tilde{\sigma}_t$ $\Rightarrow$ Portfolio reallocation from capital to bonds, $q_t^K K_t \downarrow, \mathcal{B}_t / \wp_t \uparrow$,
- Quantitatively relevant? Yes
    - Excess return volatility
        - 2.9% in equivalent bondless model ($s = 0$ and no bubble)
        - 12.9% in our model

Loss of Safe Asset Status – Equilibrium selection

- When government debt has a (stationary) bubble, other equilibria possible
  - Stationary no bubble equilibrium
  - Nonstationary equilibria that converge to the no bubble equilibrium

- Implies fragility: bubbles may pop, loss of safe asset status

- Are there policies to prevent a loss of safe asset status?
  1. Create a "fundamentally safe asset"
     - Raise (positive) surpluses to generate safe cash flow component $q_t^{B,CF}$
     - If surpluses always exceed a (positive) fraction of total output, no bubble
     - But: gives up revenues from bubble mining
  2. Off-equilibrium tax backing
     - Sufficient to (credibly) promise policy 1 off equilibrium
     - See "FTPL with a Bubble"

Roadmap

- Intuition for different “Monetary Theories”

- Monetary Model with one sector with constant idiosyncratic risk
  - Safe Asset and Service Flows
  - Bubble (mining) or not
  - 2 Different Asset Pricing Perspectives/SDFs

- Monetary model with time-varying idiosyncratic risk
  - Safe asset has negative CAPM-β
  - Calibration:
    Debt valuation puzzle, Debt Laffer Curve, Flight-to-Safety and Equity Excess Volatility

- Medium of Exchange Role

ADD “Medium of Exchange” to Store of Value

- Fiscal Theory of the Price Level (FTPL) *store of value*
  - SDF is time-varying + Bubble term

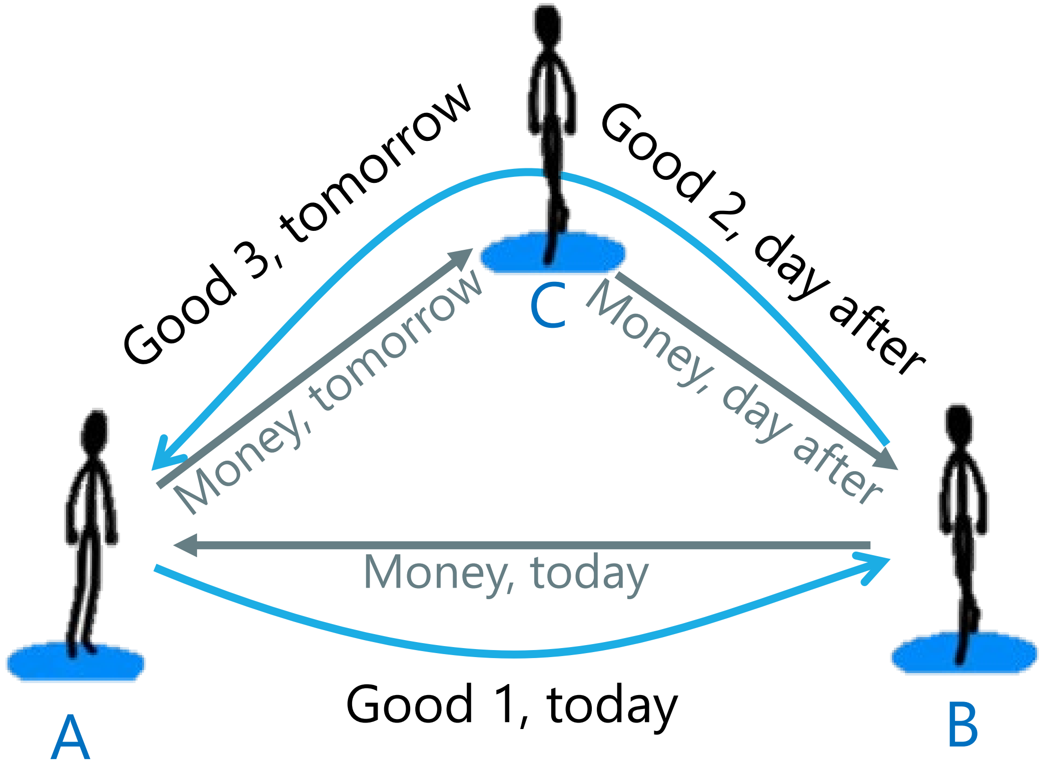
- ADD *medium of exchange*
    - Cash-in-advance constraint, transaction cost, shopping time model, …
    - $\Rightarrow\ \Delta i$ (convenience yield)
  - Price level is determined by $\mathcal{M}_t v(\cdot) = \wp_t Y_t$

The 4 Roles of Money

- Unit of account
  - Intratemporal: Numeraire bounded rationality/price stickiness
  - Intertemporal: Debt contract incomplete markets

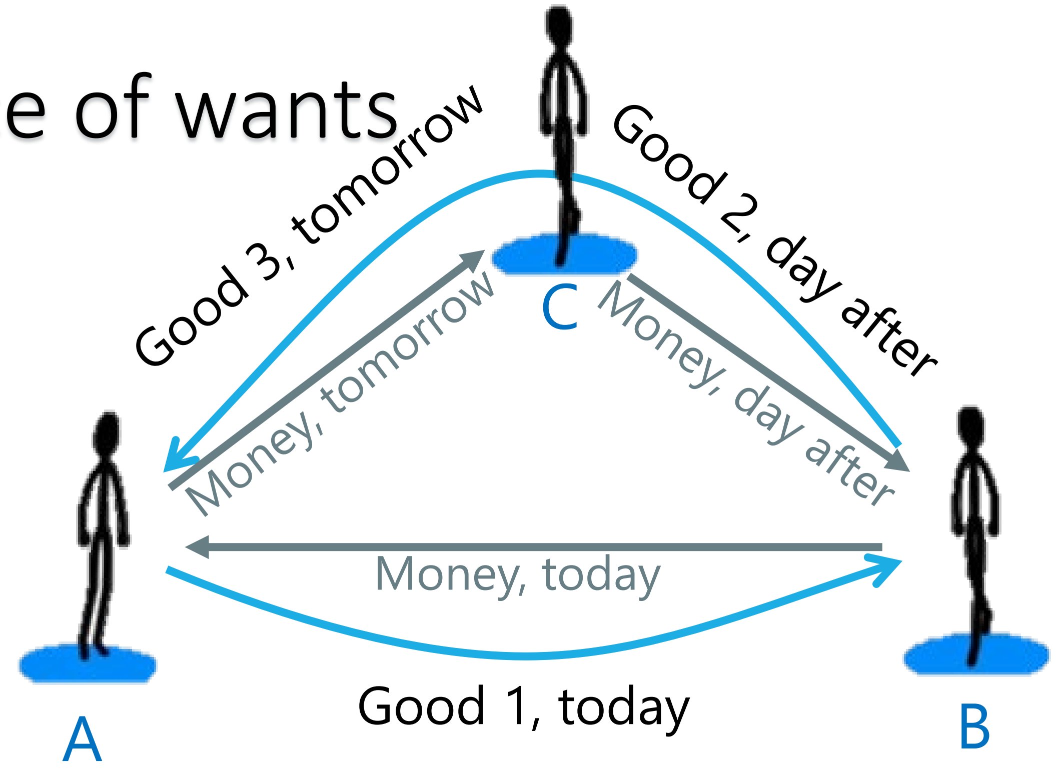
- Store of value
  - "I Theory of Money without I"
  - Fiscal theory of the price level

- Medium of exchange
  - Overcome double-coincidence



Medium of Exchange – Transaction Role

- Overcome double-coincidence of wants



- Quantity equation: $\boxed{\wp_t T_t = \nu M_t}$
  - ν (nu) is velocity (Monetarism: ν exogenous, constant)
  - T transactions

▪ Consumption	C	$\left.\begin{matrix} \\ \\ \end{matrix}\right\} Y$	
▪ New investment production	ιK		produce own machines
▪ Transaction of physical capital	$d\Delta^k$		infinite velocity
▪ Transaction of financial claims	$\boldsymbol{d\theta^{j\notin M}}$		infinite velocity

Models of Medium of Exchange

- Reduced form models
  - Cash in advance $T_t = \nu \frac{M_t}{\wp_t}$

    Only assets $j \in M$ with money-like features

    $c_t \leq \sum_{j \in M} \nu^j \theta^j n_t$
  - Shopping time models $c = (c^c, l)$
  - Money in the utility function consume money CES
    - New Keynesian Models
    - No satiation point
  - New Monetary Economics

For generic setting encompassing all models:
see Brunnermeier-Niepelt 2018

Cash in Advance

- Liquidity/cash in advance constraint
  - $c_t \leq \sum_{j \in M} \nu^j \theta^j n_t$ Lagrange multiplier $\hat{\lambda}_t$
  - Asset $j \in M$ which relaxes liquidity/CIA constraint

- Price of liquid/money asset

Relaxes constraint

Value of money yields extra "liquidity service"

$$q_t^{j \in M} = E_t\left[\frac{\xi_{t+\Delta}}{\xi_t}\left(x_{t+\Delta} + q_{t+\Delta}^{j \in M}\right)\right] - \hat{\lambda}_t \nu^j q_t^{j \in M}$$

$$q_t^{j \in M} = E_t\left[\frac{\xi_{t+\Delta}}{\xi_t} \underbrace{\frac{1}{1 + \hat{\lambda}_t \nu^j}}_{\Lambda_{t+\Delta}^j / \Lambda_t^j :=} \left(x_{t+\Delta} + q_{t+\Delta}^j\right)\right]$$

$$q_t^{j \in M} = \lim_{T \to \infty} \mathbb{E}_t\left[\sum_{\tau=1}^{(T-t)/\Delta} \frac{\xi_{t+\tau\Delta}}{\xi_t} \frac{\Lambda^j{}_{t+\tau\Delta}}{\Lambda^j{}_t} x_{t+\tau\Delta}\right] + \lim_{T \to \infty} \mathbb{E}_t\left[\frac{\xi_T}{\xi_t} \frac{\Lambda^j{}_T}{\Lambda^j{}_t} q_T\right]$$

As if SDF is multiplied by "liquidity multiplier" (Brunnermeier-Niepelt)

Cash in Advance

- Liquidity/cash in advance constraint
  - $c_t \leq \sum_{j \in M} \nu^j \theta^j n_t$ Lagrange multiplier $\hat{\lambda}_t$
  - Asset $j \in M$ which relaxes liquidity/CIA constraint

$$q_t^{j \in M} = \lim_{T \to \infty} E_t \left[\int_t^T \frac{\xi_\tau}{\xi_t} \frac{\Lambda^j{}_\tau}{\Lambda^j{}_t} x_\tau d\tau \right] + \lim_{T \to \infty} E_t \left[\frac{\xi_T}{\xi_t} \frac{\Lambda^j{}_T}{\Lambda^j{}_t} q_T \right]$$

  - "Money bubble" easier to obtain due to liquidity service
    - Condition absent aggregate risk: $r^M < g$ easier to obtain since $r^M < r^f$
- Stochastic Maximum Principle approach (with constraints)

$$\mu_t^{r,j} = r_t^f + \varsigma_t \sigma_t^{r,j} + \tilde{\varsigma}_t \tilde{\sigma}^{r,j} - \lambda_t \nu_t$$

where $\lambda_t = \hat{\lambda}_t / V'(n_t)$

↑ (Shadow) risk-free rate of illiquid asset

Add Cash in Advance to BruSan Model

- Return on money (no bonds)
  - Store of value – as before
  - Liquidity service (medium of exchange)

$$\frac{E\left[dr_t^M\right]}{dt} = \Phi(\iota_t) - \delta + \mu_t^p + \sigma\sigma_t^p - \mu^M = r_t^f + \varsigma_t\left(\sigma + \sigma_t^p\right) - \lambda_t \nu^M$$

- In steady state

$$\Phi(\iota) - \delta - \underbrace{\left(\mu^M - \lambda\nu^M\right)}_{\ddot{\mu}^M :=} = r^f + \varsigma\sigma$$

- Solving the model as before …
  - By simply replace μ^M with $\mu^M - \lambda_t \nu_t^M$
  - Special case: $\ddot{\mu}^M = 0$, i.e. $\mu^M = \lambda\nu^M$, $\gamma = 1$ ⇒ explicit solution as fcn of $\check{\rho}$
    - Same q^K and $q^{\mathcal{B}}$ as a function of ζ,
    - But $\check{\rho} \neq \rho$ if CIA constraint binds in steady state, otherwise $\check{\rho} = \rho$
      1. Assume it binds, i.e. $\zeta = \nu\vartheta$
      2. Recall from slide 21 for $\hat{\mu}^M = 0$ and $\gamma = 1$, $\vartheta = \frac{\tilde{\sigma} - \sqrt{\zeta}}{\tilde{\sigma}}$
      3. Equate 1. and 2. to obtain quadratic solution for $\check{\rho}$
         1. If $< \rho$, then solution equals $\check{\rho}$
         2. If $> \rho$, then $\check{\rho} = \rho$ and hence CIA doesn't bind, $\lambda = 0$, above solution
- "Occasionally" binding CIA constraint (outside of steady state)
- for sufficiently high $\tilde{\sigma}$, store of value (insurance motive) ⇒ $\lambda_t = 0$

Add Money in Utility to BruSan Model

- Money in utility function $u(c, M/\wp) = u(c, \theta n)$

- Can be expressed as equality constraint
  - Difference to CIA inequality: No satiation point

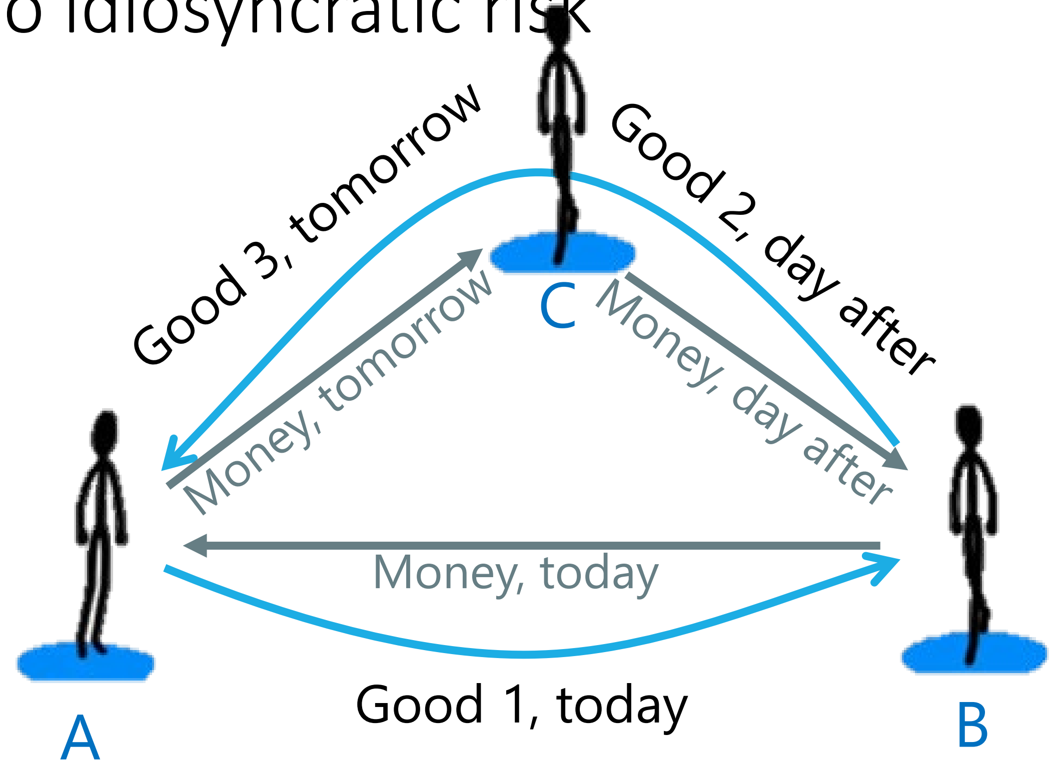
- DiTella add MiU to BruSan 2016 AER PP
  - https://web.stanford.edu/~sditella/Papers/Di_Tella_Liquidity_Traps.pdf

The 4 Roles of Money

- Unit of account
  - Intratemporal: Numeraire — bounded rationality/price stickiness
  - Intertemporal: Debt contract — incomplete markets

- Store of value
  - "I Theory of Money without I"
    Less risky than other "capital" – no idiosyncratic risk
  - Fiscal theory of the price level

- Medium of exchange
  - Overcome double-coincidence of wants problem



- Record keeping device – money is memory
  - Virtual ledger

Extra Slides

Related Literature

- Safe Asset:
  - Gorton-Pennachi (1990), Dang et al (2017), …
  - Brunnermeier et al. (2017) , ESBies,
- Equity premium
  - Constantinides-Duffie (1996) – imposes “aggregate” transversality condition
- Public Debt Evaluation Puzzles:
  - Jiang et al. (2020,2021)
- Fiscal debt sustainability $\boldsymbol{r}$ vs. $\boldsymbol{g}$:
  - OLG: Bohn (1995), Samuelson (1958), Diamond (1965), Tirole (1985), Blanchard (2019), Martin-Ventura (2018)
  - Incomplete markets: Bewley (1980), Aiyagari-McGrattan (1998), Angeletos (2007)

No capital

Capital is a safe asset (risk-free)

No debt

  - Bassetto-Cui (2018), Reis (2020)

m instead of ξ^{**}/r^{**}

Deriving FTPL equation (in cts time)

- Nominal government budget constraint

$$\left(\mu_t^{\mathcal{B}}\mathcal{B}_t + \mu_t^{\mathcal{M}}\mathcal{M}_t + \wp_t T_t\right)dt = (i_t\mathcal{B}_t + i_t^m\mathcal{M}_t + \wp_t G_t)dt$$

- Multiply by nominal SDF $\xi_t/\wp_t$, rearrange

$$[\left(\mu_t^{\mathcal{B}} - i_t\right)\tfrac{\xi_t}{\wp_t}\mathcal{B}_t + \left(\mu_t^{\mathcal{M}} - i_t\right)\tfrac{\xi_t}{\wp_t}\mathcal{M}_t]dt = -\xi_t(T_t - G_t - \underbrace{(i_t - i_t^m)}_{\Delta i_t :=}\tfrac{\mathcal{M}_t}{\wp_t})dt$$

- Suppose $\xi_t/\wp_t$ prices the nominal bond
  - Then $E_t\left[\frac{d(\xi_t/\wp_t)}{(\xi_t/\wp_t)}\right] = i_t dt$
  - Substitute into above, use product rule, take expectations

$$E_t\left[d\left(\frac{\xi_t}{\wp_t}(\mathcal{B}_t + \mathcal{M}_t)\right)\right] = -E_t[\xi_t\left(T_t - G_t - \Delta i_t\frac{\mathcal{M}_t}{\wp_t}\right)dt]$$

Hint:

$$d\left(\frac{\xi_t}{\mathcal{P}_t}\mathcal{B}_t\right) = \left(\mu_t^{\mathcal{B}} - i_t\right)\frac{\xi_t}{\mathcal{P}_t}\mathcal{B}_t dt + \frac{\xi_t}{\mathcal{P}_t}\mathcal{B}_t\left(\frac{d\left(\xi_t/\mathcal{P}_t\right)}{\xi_t/\mathcal{P}_t} + i_t dt\right),$$

$$d\left(\frac{\xi_t}{\mathcal{P}_t}\mathcal{M}_t\right) = \left(\mu_t^{\mathcal{M}} - i_t\right)\frac{\xi_t}{\mathcal{P}_t}\mathcal{M}_t dt + \frac{\xi_t}{\mathcal{P}_t}\mathcal{M}_t\left(\frac{d\left(\xi_t/\mathcal{P}_t\right)}{\xi_t/\mathcal{P}_t} + i_t dt\right)$$

- In integral form

$$\frac{\mathcal{B}_t + \mathcal{M}_t}{\wp_t} = E_t\int_t \frac{\xi_s}{\xi_t}(T_s - G_s)ds + E_t\int_t \frac{\xi_s}{\xi_t}\Delta i_s\frac{\mathcal{M}_s}{\wp_s}ds + \frac{\xi_T}{\xi_t}\frac{\mathcal{B}_T + \mathcal{M}_T}{\wp_T}$$

Deriving FTPL equation (in cts time)

- Take limit $T \to \infty$

$$\frac{\mathcal{B}_t + \mathcal{M}_t}{\wp_t} = E_t \int_t^\infty \frac{\xi_s}{\xi_t}(T_s - G_s)ds + E_t \int_t^\infty \frac{\xi_s}{\xi_t} \Delta i_s \frac{\mathcal{M}_s}{\wp_s} ds + \lim_{T\to\infty} E_t \frac{\xi_T}{\xi_t} \frac{\mathcal{B}_T + \mathcal{M}_T}{\wp_T}$$

- Remark 1:
  - Literature focuses on settings, in which private-sector transversality eliminates the bubble term
  - Here: fiscal theory in setting, in which where transversality does not rule out bubbles
- Remark 2:
  - The sum of the three limits on the right may not be well-defined mathematically, because they can be infinite with opposite signs
  - The limit of the sum may nevertheless exist and be finite
    - This is what matters economically (cannot separately trade the bubble and fundamental components)

3 Forms of Seigniorage

$$\frac{\mathcal{B}_t + \mathcal{M}_t}{\wp_t} = E_t \int_t^\infty \frac{\xi_s}{\xi_t}(T_s - G_s)ds + E_t \int_t^\infty \frac{\xi_s}{\xi_t}\Delta i_s \frac{\mathcal{M}_s}{\wp_s} ds + \lim_{T\to\infty} E_t \frac{\xi_T}{\xi_t}\frac{\mathcal{B}_T + \mathcal{M}_T}{\wp_T}$$

1. Surprise devaluation
   - Irrational expectations
   - Small (Hilscher, Raviv, Reis 2014)
     - Inflation options imply likelihood of exceeding 5% of GDP is less than 1%

2. Exploiting liquidity benefits of "narrow" cash
   - Only for "narrow" cash that provides medium-of-exchange services
   - $\Delta i = i - i^M$
   - 0.36 % of GDP, NPV = 20% (at most 30%) of GDP, (Reis 2019)

3. "Money bubble mining"