Modern Macro, Money, and International Finance

Eco529

Lecture 05: Endogenous Risk Dynamics in Real Macro Model with Heterogenous Agents

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Course Overview

Real Macro-Finance Models with Heterogeneous Agents

1. A Simple Real Macro-finance Model
2. Endogenous (Price of) Risk Dynamics
3. A Model with Jumps due to Sudden Stops/Runs

Money Models

1. A Simple Money Model
2. Cashless vs. Cash Economy and "The I Theory of Money"
3. Welfare Analysis & Optimal Policy
    1. Fiscal, Monetary, and Macroprudential Policy

International Macro-Finance Models

1. International Financial Architecture

Digital Money

Risk premia, price of risk

- Risk premia = price of risk * (endogenous + exogenous risk)

  - Exogenous risk – shock from outside
  - Endogenous risk
    - Amplification: adverse feedback loops
    - Multiple equilibria: Run, Sudden Stops

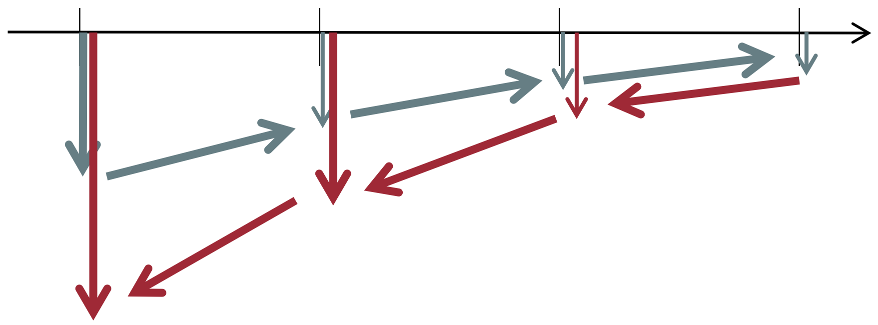
- Non-linearities are key for financial stability
  - Around vs. away from steady state

Desired Model Properties

- Normal regime: stable around steady state
  - Experts are adequately capitalized
  - Experts can absorb macro shock
- Endogenous risk and price of risk
  - Fire-sales, liquidity spirals, fat tails
  - Spillovers across assets and agents
  - Market and funding liquidity connection
  - SDF vs. cash-flow news
- Volatility paradox
- (Financial innovation less stable economy)
- ("Net worth trap" double-humped stationary distribution)

Persistence Leads to Dynamic Amplification

- *Static* amplification occurs because fire-sales of capital from productive sector to less productive sector depress asset prices
  - Importance of *market liquidity* of physical capital
- *Dynamic* amplification occurs because a temporary shock translates into a persistent decline in output and asset prices
  - Forward grow net worth
  - Backward asset pricing

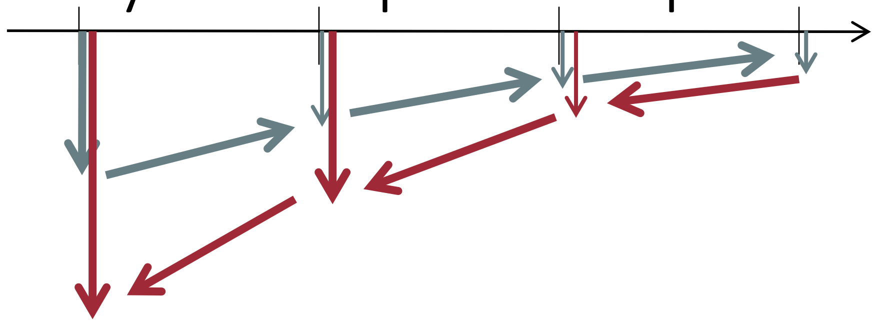


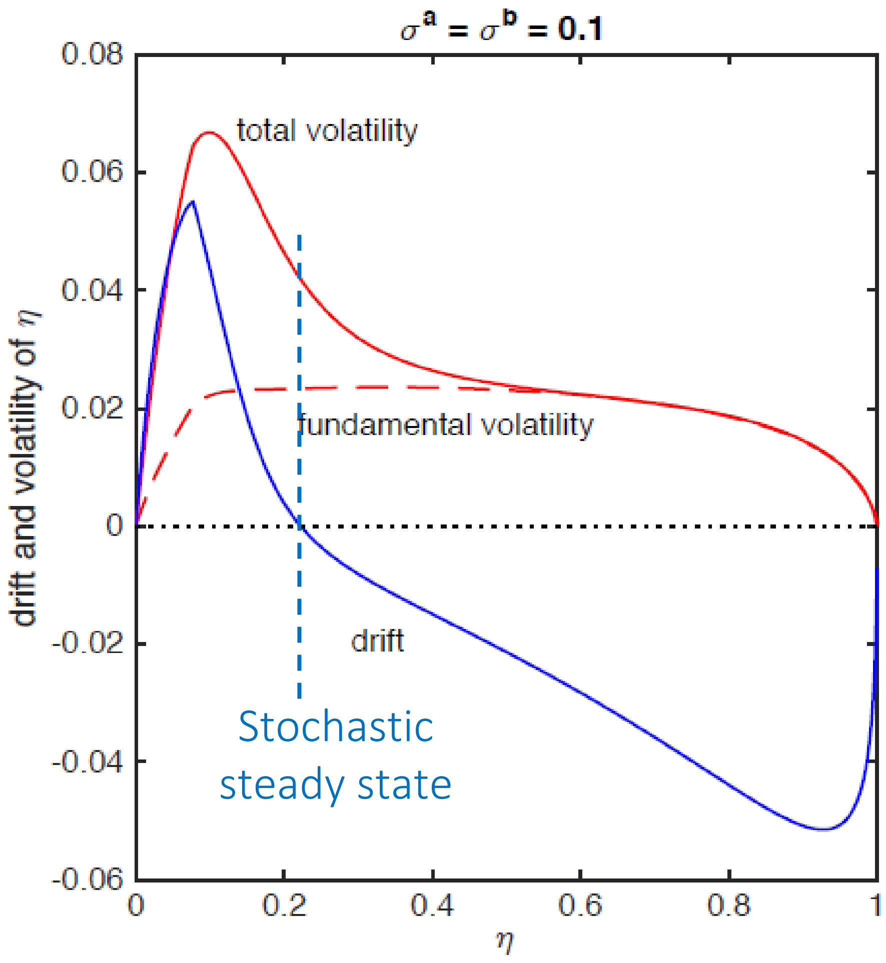
“Single Shock Critique”

- Critique: After the shock all agents in the economy know that the economy will deterministically return to the steady state.
  - Length of slump is deterministic (and commonly known)
    - No safety cushion needed
  - In reality an adverse shock may be followed by additional adverse shocks
    - Build-up extra safety cushion for an additional shock in a crisis
- Impulse response vs. volatility dynamics

Endogenous Volatility & Volatility Paradox

- Endogenous Risk/Volatility Dynamics in BruSan
  - Beyond Impulse responses





  - Input: constant volatility
  - Output: endogenous risk
    time-varying volatility

⇒Precautionary savings
  - Role for money/safe asset
    - *Later: in Money lecture*

⇒ Nonlinearities in crisis ⇒ endogenous fait tails, skewness

- Volatility Paradox
  - Low exogenous (measured) volatility leads to
    high build-up of (hidden) endogenous volatility (Minksy)

Toolboxes: Technical Innovations

- Occasionally binding equity issuance constraint
  (in addition to natural borrowing limit due to risk aversion)

- Price setting social planner to find capital and risk allocation

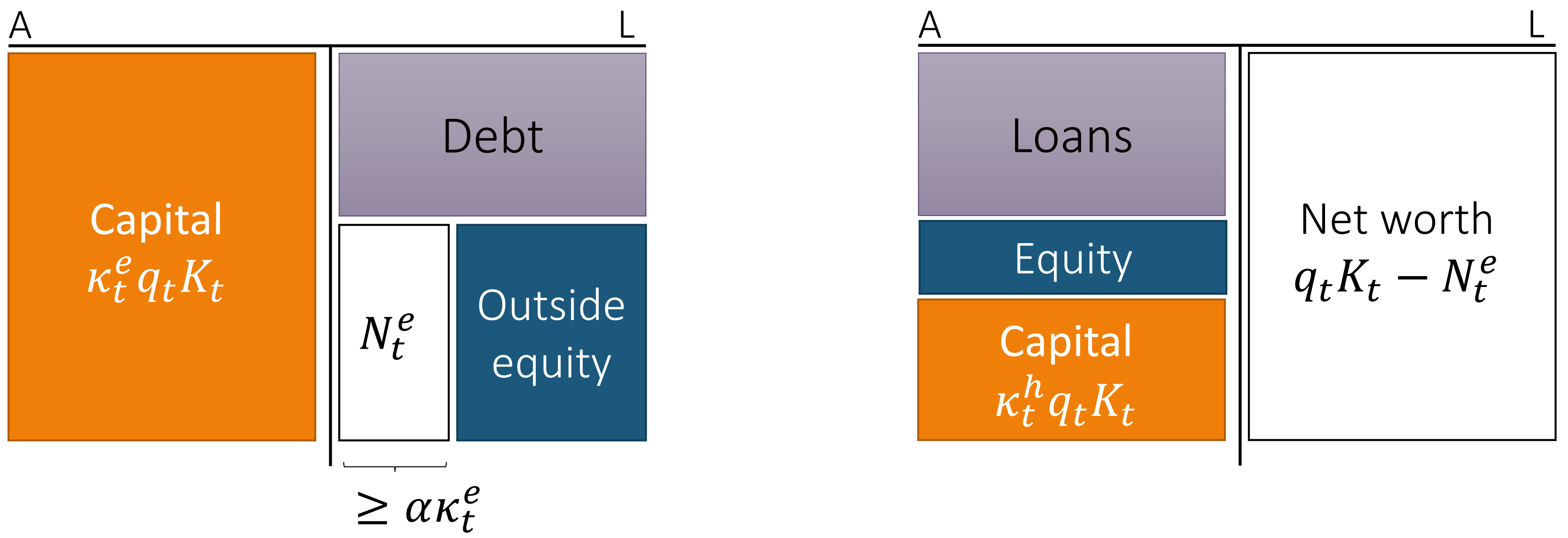
- Change of numeraire
  - Easily incorporate aggregate fluctuations
  - To use martingale methods more broadly

- Newton Method to solve log-utility numerical example

Two Type/Sector Model with Outside Equity

BruSan 2017: Handbook of Macroeconomics, Lecture Notes, Chatper 3

- Expert sector   Household sector



- Skin in the Game Constraint:
  Experts must hold fraction $\chi_t^e = \frac{\sigma_{N^e,t}}{\sigma_{qK,t}} \geq \alpha\kappa_t^e$ of aggregate capital risk
  with $\alpha \in (0,1)$ ($\chi_t^e > \kappa_t^e$ never happens in equilibrium)
- Return on inside equity N_t can differ from outside equity
  - Issue outside equity at required return from HH
  - In related model, He and Krishnamurthy 2013 impose that inside and outside equity have same return

Financial Frictions and Distortions UPDATE!

- Skin in the game constraint
  - Retain certain fraction of risk

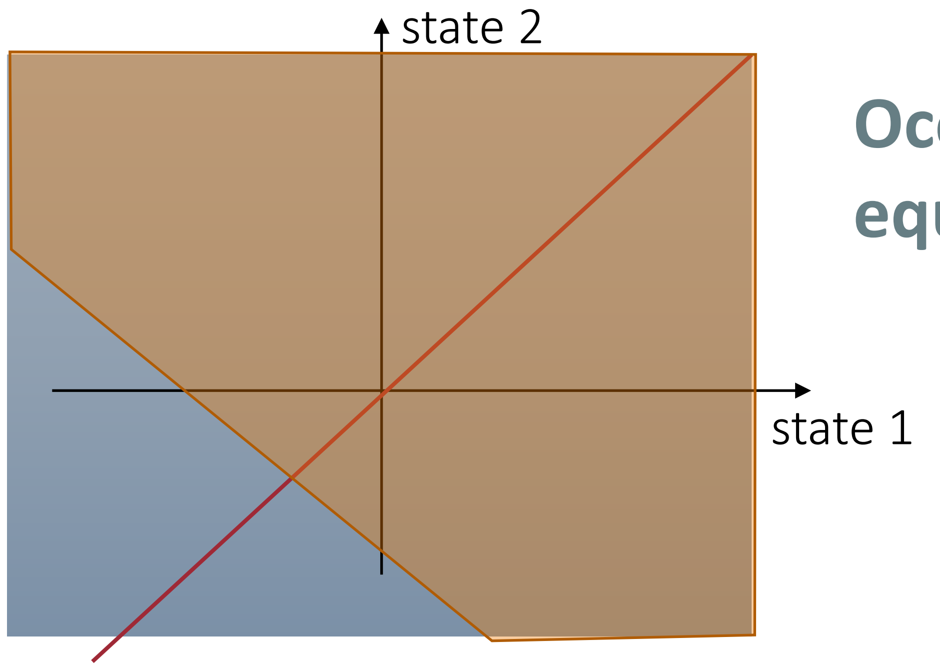
- Incomplete markets
  - "natural" leverage constraint *(BruSan)*
  - Costly state verification *(BGG)*

- + Leverage constraints
  (no "liquidity creation")
  - Exogenous limit *(Bewley/Ayagari)*

  - Collateral constraints
    - Next period's price *(KM)*

      $$Rb_t \leq q_{t+1}k_t$$

    - Next periods volatility *(VaR, JG)*
    - Current price



Occasionally binding equity constraint

Two Type Model Setup

Expert sector

- Output: $y_t^e = a^e k_t^e$

$a^e \geq a^h$

Household sector

- Output: $y_t^h = a^h k_t^h$

$$A(\boldsymbol{\kappa}) = \kappa^e a^e + (1 - \kappa^e) a^h$$

Poll 11: Why is it important that households can hold capital?

a) to capture fire-sales

b) for households to speculate

c) to obtain stationary distribution

Two Type Model Setup

Expert sector

- Output: $y_t^e = a^e k_t^e$ $\quad a^e \geq a^h$
- Consumption rate: c_t^e
- Investment rate: ι_t^e

$$\frac{dk_t^{\tilde{\imath},e}}{k_t^{\tilde{\imath},e}} = \left(\Phi\left(\iota_t^{\tilde{\imath},e}\right) - \delta\right) dt + \sigma dZ_t + d\Delta_t^{k,e}$$

Household sector

- Output: $y_t^h = a^h k_t^h$
- Consumption rate: c_t^h
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Physical capital evolution absent market transactions/fire-sales

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Poll 13: What are the modeling tricks to obtain stationary distribution?

a) switching types

b) agents die, OLG/perpetual youth models (without bequest motive)

c) different preference discount rates

Two Type Model Setup

Expert sector

- Output: $y_t^e = a^e k_t^e$ $\quad a^e \geq a^h$
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- $E_0[\int_0^\infty e^{-\rho^e t} \frac{(c_t^e)^{1-\gamma}}{1-\gamma} dt]$ $\quad \rho^e \geq \rho^h$

Household sector

- Output: $y_t^h = a^h k_t^h$
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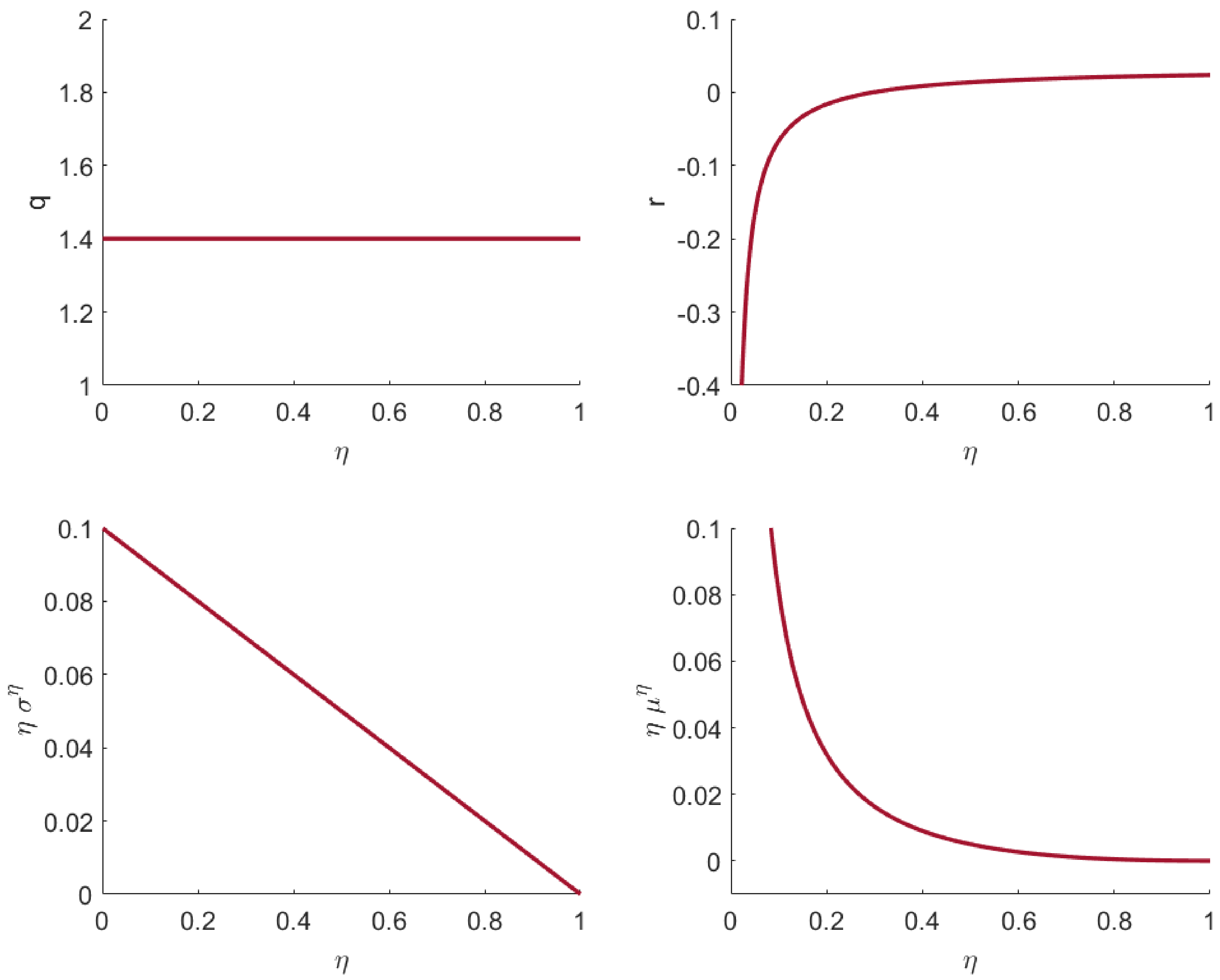
- $E_0[\int_0^\infty e^{-\rho^h t}\frac{(c_t^h)^{1-\gamma}}{1-\gamma}dt]$

Friction: Can only issue

- Risk-free debt
- Equity, but must hold $\chi_t^e \geq \alpha \kappa_t$, i.e. $\theta_t^{e,K} + \theta_t^{e,OE} \geq \alpha \theta_t^{e,K}$

Recall Previous Lecture: HH can't hold capital or equity

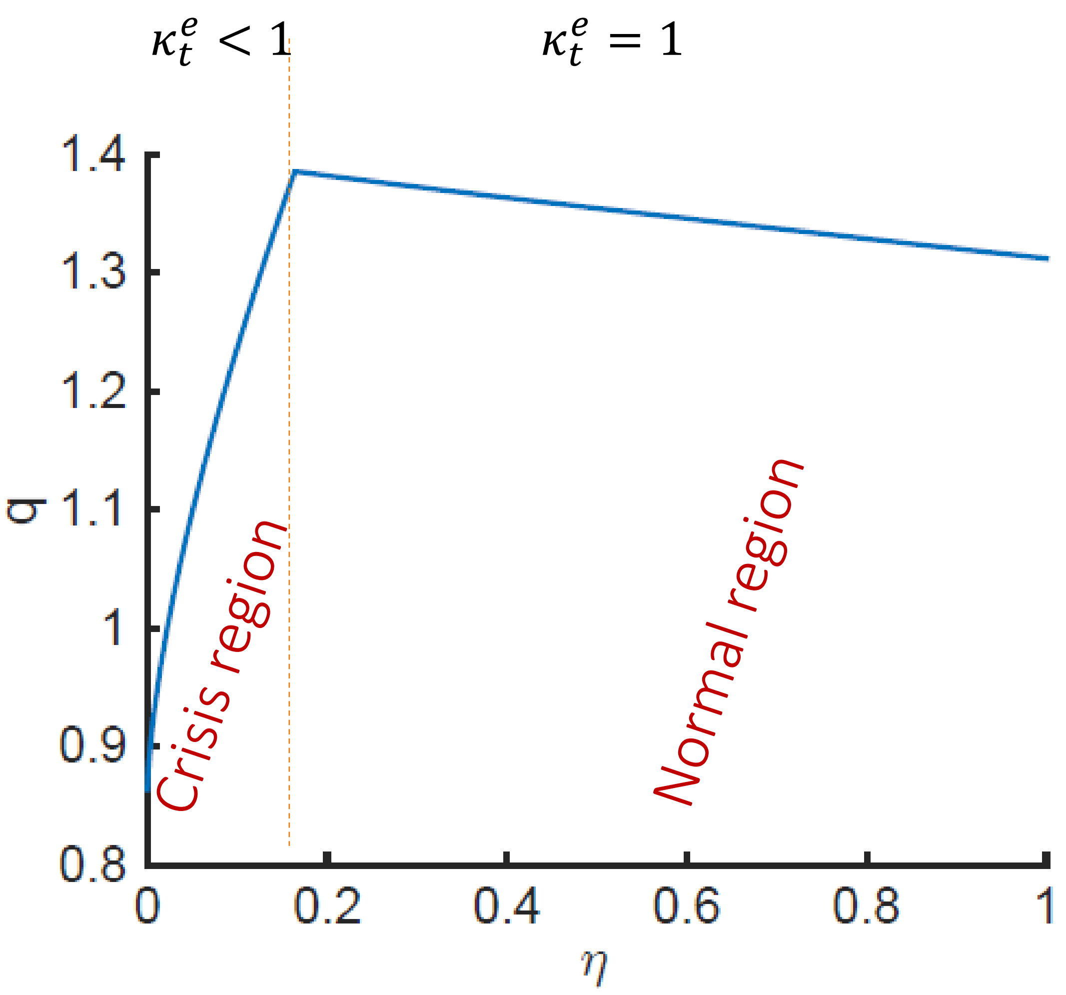


Basak-Cuco

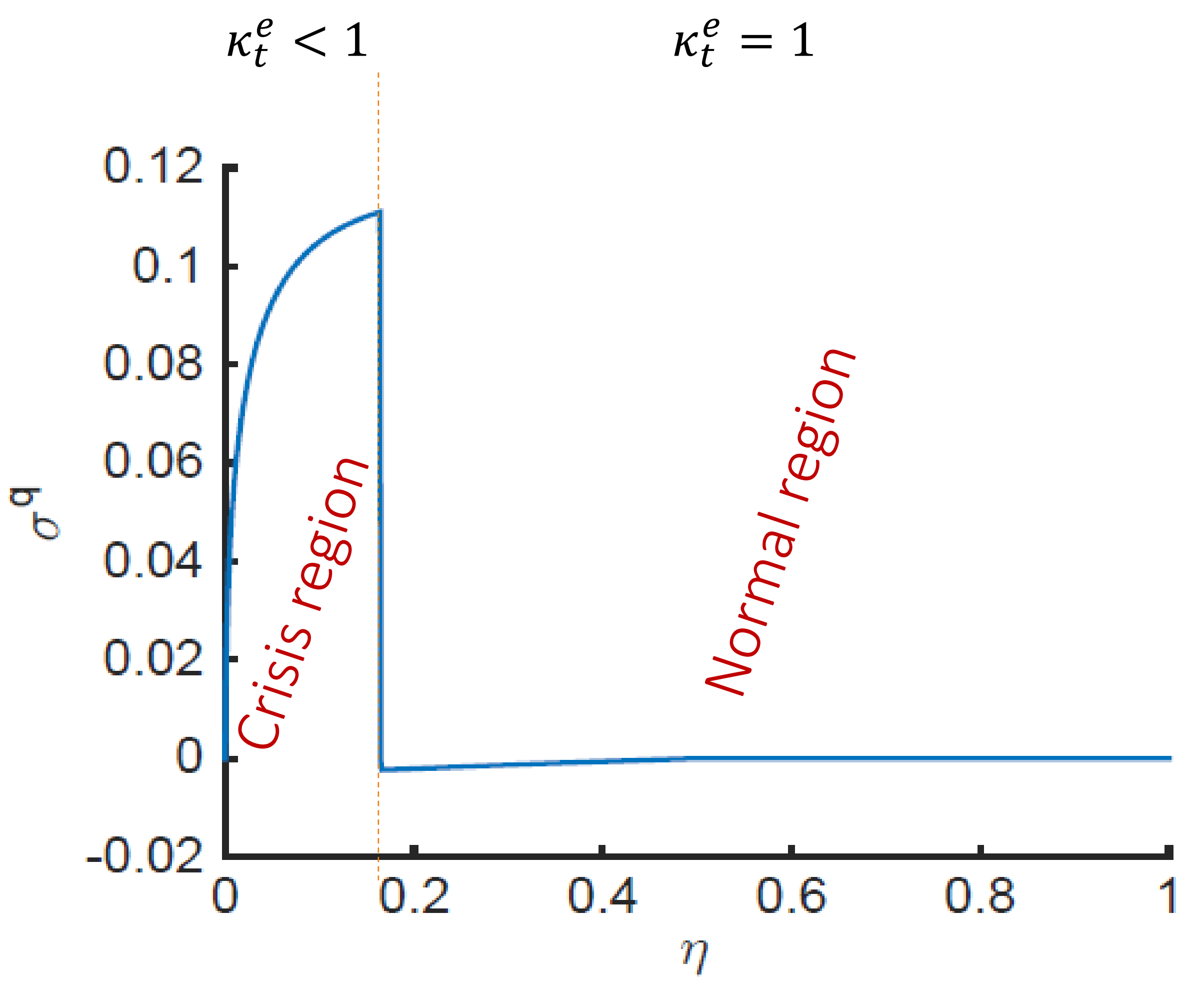
$$a = .11, \rho = 5\%, \sigma = .1, \Phi(\iota) = \frac{\log(\phi\iota + 1)}{\phi}, \phi = 10$$

Preview of new, extended model

- Price of capital



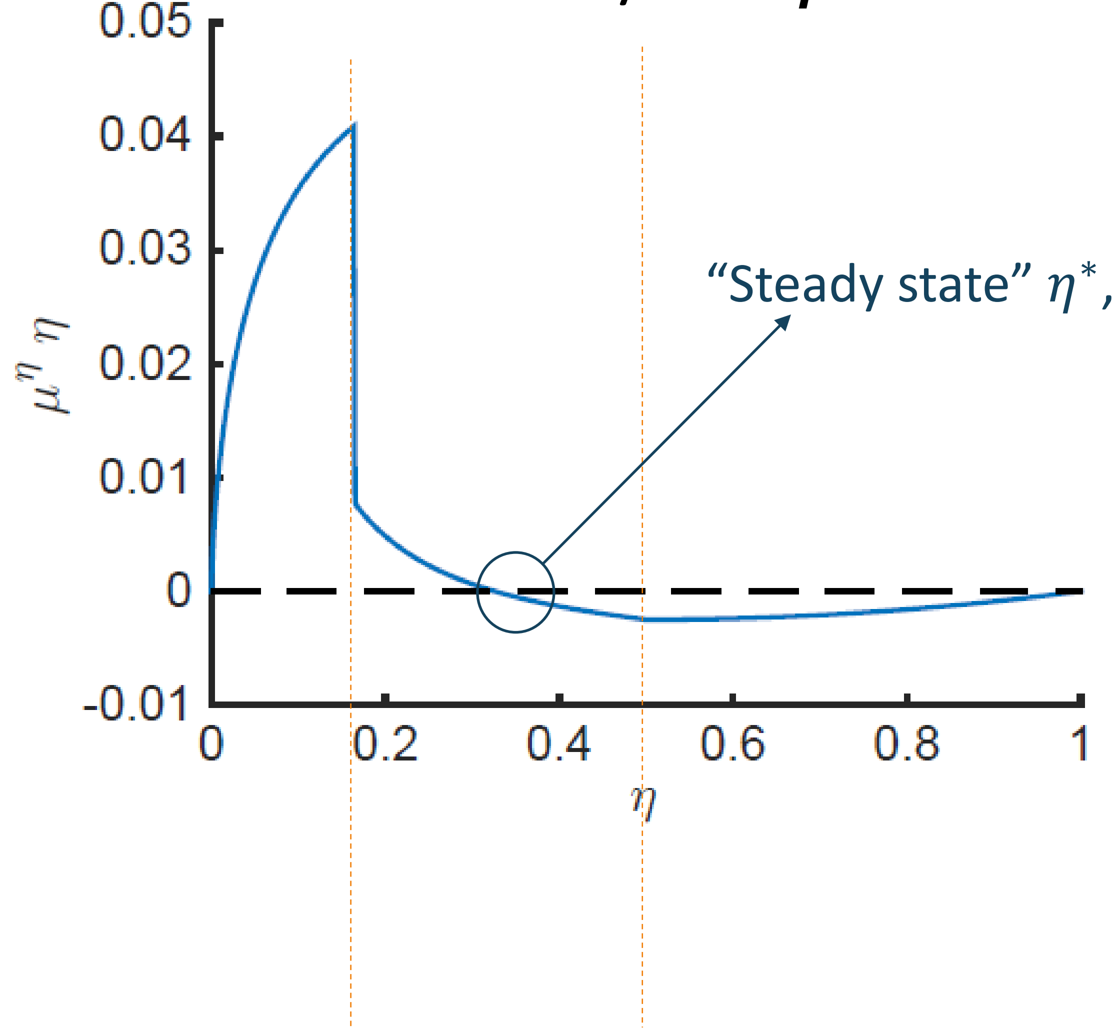
Amplification

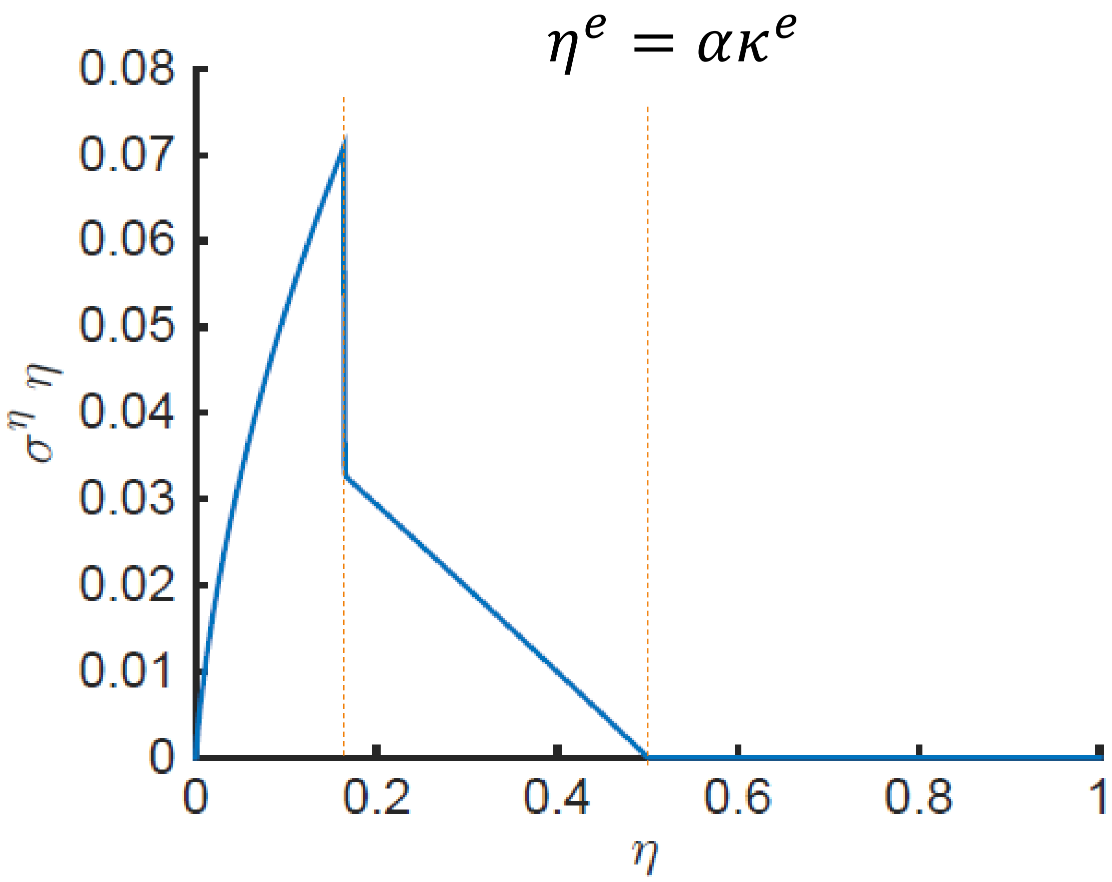


Parameters: $\rho^e = .06, \rho^h = .05, a^e = .11, a^h = .03,$
$\delta = .05, \sigma = .1, \alpha = .50, \gamma = 2, \phi = 10$

Preview $\mu^{\eta^e}(\eta^e)$ & $\sigma^{\eta^e}(\eta^e)$

- Drift and Volatility of η^e





Solving MacroModels Step-by-Step

0. Postulate aggregates, price processes & obtain return processes
1. For given C/N-ratio and SDF processes for each i *finance block*
    a. Real investment ι + Goods market clearing *(static)*
    - *Toolbox 1:* Martingale Approach, HJB vs. Stochastic Maximum Principle Approach

    b. Portfolio choice θ + Asset market clearing or
    Asset allocation κ & risk allocation χ
    - *Toolbox 2:* "price-taking social planner approach" – Fisher separation theorem

    c. ~~"Money evaluation equation" ϑ~~
    - *Toolbox 3:* Change in numeraire to total wealth (including SDF)
2. Evolution of state variable η (and K) *forward equation*
3. Value functions *backward equation*
    a. Value fcn. as fcn. of individual investment opportunities ω
    - *Special cases: log-utility, constant investment opportunities*

    b. Separating value fcn. $V^i(n^{\tilde{i}}; \eta, K)$ into $v^i(\eta)u(K)\left(n^{\tilde{i}}/n^i\right)^{1-\gamma}$
    c. Derive C/N-ratio and ς price of risk
4. Numerical model solution
    a. Transform BSDE for separated value fcn. $v^i(\eta)$ into PDE
    b. Solve PDE via value function iteration
5. KFE: Stationary distribution, Fan charts

0. Postulate Aggregates and Processes

- Individual capital evolution:

$$\frac{dk_t^{\tilde{\imath},i}}{k_t^{\tilde{\imath},i}} = \left(\Phi\left(\iota^{\tilde{\imath},i}\right) - \delta\right)dt + \sigma dZ_t + d\Delta_t^{k,\tilde{\imath},i}$$

  - Where $\Delta_t^{k,\tilde{\imath},i}$ is the individual cumulative capital purchase process

(c is numeraire)

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- Capital aggregation:
  - Within sector i: $K_t^i \equiv \int k_t^{\tilde{\imath},i} d\tilde{\imath}$
  - Across sectors: $K_t \equiv \sum_i K_t^i$
  - Capital share: $\kappa_t^i \equiv K_t^i / K_t$

$$\frac{dK_t}{K_t} = \left(\Phi\left(\iota_t^i\right) - \delta\right)dt + \sigma dZ_t$$

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$$\frac{dK_t}{K_t} = \left(\Phi\left(\iota_t^i\right) - \delta\right)dt + \sigma dZ_t$$

- Net worth aggregation:
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  - Across sectors: $N_t \equiv \sum_i N_t^i$
  - Wealth share: $\eta_t^i \equiv N_t^i / N_t$

(c is numeraire)

0. Postulate Aggregates and Processes

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- Value of capital stock: $q_t K_t$

  Postulate $dq_t/q_t = \mu_t^q dt + \sigma_t^q dZ_t$

Poll 23: How many Brownian motions span prob. space?
a) one
b) two
c) one + number of sectors
d) two + number of sectors

(c is numeraire)

0. Postulate Aggregates and Processes

- Individual capital evolution:

$$\frac{dk_t^{\tilde{\iota},i}}{k_t^{\tilde{\iota},i}} = \left(\Phi\left(\iota^{\tilde{\iota},i}\right) - \delta\right)dt + \sigma dZ_t + d\Delta_t^{k,\tilde{\iota},i}$$

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Same Brownian

(c is numeraire)

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  Postulate $dq_t / q_t = \mu_t^q dt + \sigma_t^q dZ_t$
- Postulated SDF-process: $\frac{d\xi_t^i}{\xi_t^i} = \underbrace{\mu_t^\xi}_{\equiv - r_t} dt + \underbrace{\sigma_t^{\xi^i}}_{\equiv - \varsigma_t^i} dZ_t$ (c is numeraire)

0. Postulate Aggregates and Processes

- … from price processes to return processes (using Ito)
  - Use Ito product rule to obtain (in absence of purchases/sales)
    - Define $\check{k}_t^{\tilde{\imath}}$: $\frac{d\check{k}_t^{\tilde{\imath},i}}{\check{k}_t^{\tilde{\imath},i}} = \left(\Phi\left(\iota_t^{\tilde{\imath},i}\right) - \delta\right)dt + \sigma dZ_t + d\Delta_t^{k,\tilde{\imath},i}$ without purchases/sales

Dividend yield — E[Capital gain rate]= $\frac{d(q_t \check{k}^i{}_t)}{(q_t \check{k}_t^i)}$

$$dr_t^k\left(\iota_t^{\tilde{\imath},i}\right) = \left(\frac{a^i - \iota_t^i}{q} + \Phi(\iota_t^i) - \delta + \mu_t^q + \sigma\sigma_t^q\right)dt + (\sigma + \sigma_t^q)dZ_t$$

For aggregate capital return, Replace a^i with $A(\kappa)$

- Postulate SDF-process: (Example: $\xi_t^i = e^{-\rho t}V'(n_t^i)$.)

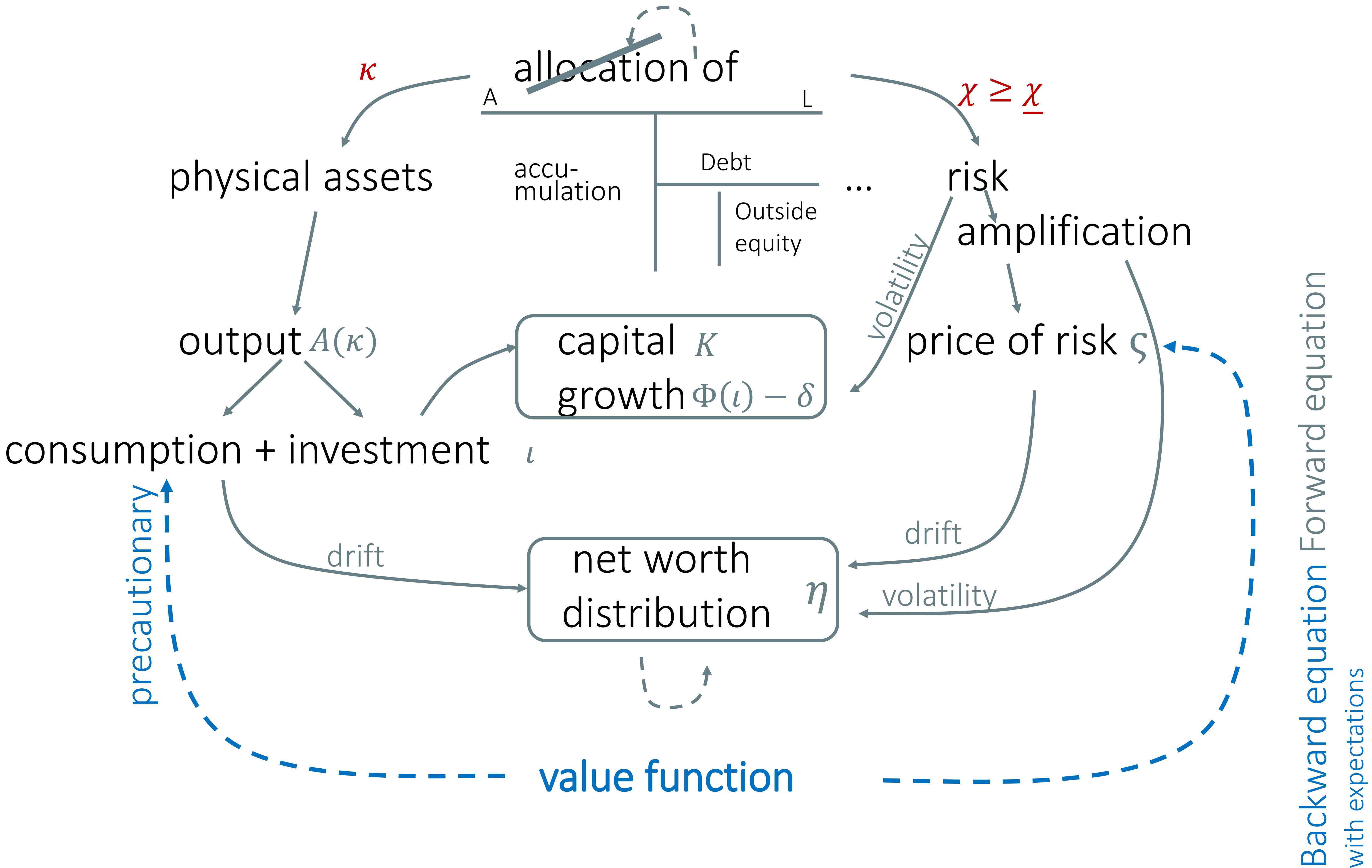
$$\frac{d\xi_t^i}{\xi_t^i} = -r_t dt - \varsigma_t^i dZ_t$$

Price of risk (ς_t^i)

Recall discrete time $e^{-r^F} = E[SDF]$

Poll 26: Why does drift of SDF equal risk-free rate
a) no idio risk
b) $e^{-r^F} = E[SDF]*1$
c) no jump in consumption

The Big Picture



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1a. Individual Agent Choice of ι

- Choice of ι is static problem (and separable) for each t
- $\max_{\iota_t^i} dr_t^k(\iota_t^i)$

$$= \max_{\iota_t^i} \left(\frac{a^i - \iota_t^i}{q_t} + \Phi(\iota_t^i) - \delta + \mu^q + \sigma\sigma^q \right)$$

For aggregate capital return,
Replace a^i with $A(\kappa)$

- FOC: $\frac{1}{q_t} = \Phi'(\iota_t^i)$ Tobin's q
  - All agents $\iota_t^i = \iota_t \Rightarrow \frac{dK_t}{K_t} = (\Phi(\iota_t) - \delta)\, dt + \sigma dZ_t$
  - Special functional form:
    - $\Phi(\iota) = \frac{1}{\phi}\log(\phi\iota + 1) \Rightarrow \phi\iota = q - 1$
- Goods market clearing: $(A(\kappa) - \iota_t)K_t = \sum_i C_t^i$.

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1b. Individual Agent Choice of $\theta \Rightarrow$ asset/risk allocation

- Approach 1: Portfolio optimization
  - Step 1: Optimization e.g. via Martingale Approach – recall: $\mu_t^A = r_t^i + \varsigma_t^i \sigma_t^A$
    - Of experts with outside equity issuance (after plugging in households' outside equity choice)

$$\frac{a^e - \iota_t}{q_t} + \Phi(\iota_t) - \delta + \mu_t^q + \sigma\sigma_t^q =$$

$$r_t + [\varsigma_t^e \chi_t^e / \kappa_t^e + \varsigma_t^h (1 - \chi_t^e / \kappa_t^e)](\sigma + \sigma^q)$$

new compared to Basak-Cuoco

    - Of households' capital choice

$$\frac{a^h - \iota_t}{q_t} + \Phi(\iota_t) - \delta + \mu_t^q + \sigma\sigma_t^q \leq r_t + \varsigma_t^h (\sigma + \sigma^q)$$

with equality if $\kappa_t^e < 1$

  - Step 2: Capital market clearing to obtain asset/risk allocation κ_t^e, χ_t^e from portfolio weights θs
- Approach 2: Price-taking Social Planner Approach

1b. *Toolbox:* Price Taking Social Planner ⇒ Asset/Risk Allocation

- Price-Taking Planner's Theorem:
  A social planner that takes prices as given chooses a physical asset allocation, κ_t, and risk allocation, χ_t, that coincides with the choices implied by all individuals' portfolio choices.

$$\boldsymbol{\varsigma}_t = (\varsigma_t^1, \dots, \varsigma_t^I)$$
$$\boldsymbol{\chi}_t = (\chi_t^1, \dots, \chi_t^I)$$
$$\boldsymbol{\sigma}(\boldsymbol{\chi}_t) = (\boldsymbol{\chi}_t^1 \sigma^N, \dots, \boldsymbol{\chi}_t^I \sigma^N)$$

- Planner's problem

Return on total wealth

$$\max_{\{\boldsymbol{\kappa}_t, \boldsymbol{\chi}_t\}} E_t[dr_t^N(\kappa_t)]/dt - \boldsymbol{\varsigma}_t \sigma(\boldsymbol{\chi}_t)$$

$= dr^F/dt$ in equilibrium

subject to friction: $F(\boldsymbol{\kappa}_t, \boldsymbol{\chi}_t) \leq 0$

  - Example:
    1. $\chi_t = \kappa_t$ (if one holds capital, one has to hold risk)
    2. $\chi_t \geq \alpha \kappa_t$ (skin in the game constraint, outside equity up to a limit)

1b. *Toolbox:* Price Taking Social Planner ⇒ Asset/Risk Allocation

- Sketch of Proof of Theorem

1. Fisher Separation Theorem: (delegated portfolio choice by firm)
   - FOC yield the martingale approach solution
   - Each individual agent $(i, \tilde{\imath})$ portfolio maximization is equivalent to the maximization problem of a firm

$$\max_{\{\boldsymbol{\theta}^{j,i}\}} E_t\left[dr^{n^{(i,\tilde{\imath})}}\right]/dt - \varsigma\sigma^{r^n}$$

   - $dr^{n^{(i,\tilde{\imath})}} = \sum_j \theta^{j,i} dr^j = \sum_j \theta^{j,i} E[dr^j] + \sum_j \theta^{j,i}\sigma^j dZ_t$ is linear in θs
     - Either bang-bang solution for θs s.t. portfolio constraints bind
     - Or prices/returns/risk premia are s.t. that firm is indifferent

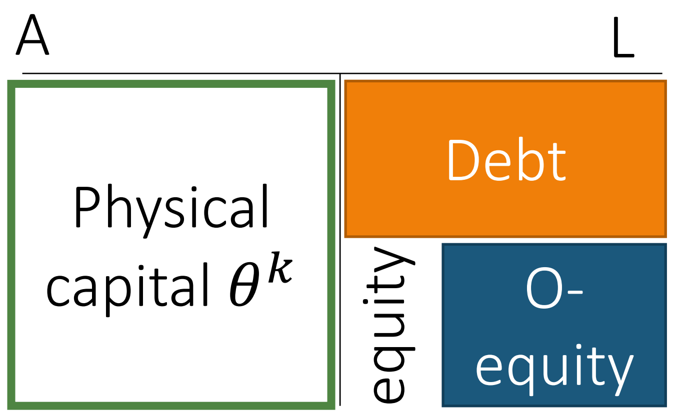
2. Aggregate
   - Taking η-weighted sum to obtain return on aggregate wealth

3. Use market clearing to relate θs to κs & χs (incl. θ-constraint)

1b. *Toolbox:* Price Taking Social Planner ⇒ Asset/Risk Allocation

2 Types

- Expert: $\boldsymbol{\theta^e} = (\theta^{e,K}, \theta^{e,OE}, \theta^{e,D})$ for capital, outside equity, debt
  - Restrictions: $\theta^{e,K} \geq 0,$



$\theta^{e,OE} \leq 0,$ only issue outside equity

$\theta^{e,OE} \geq -(1-\alpha)\theta^{e,K}$ skin in the game

maximize

$$\theta_t^{e,K} E[dr_t^{e,K}]/dt + \theta_t^{e,OE} E[dr_t^{OE}]/dt + \theta_t^{e,D} r_t - \varsigma_t^e (\theta_t^{e,K} + \theta_t^{e,OE}) \sigma^{r^{e,K}}$$

- Household: $\boldsymbol{\theta^h} = (\theta^{h,K}, \theta^{h,OE}, \theta^{h,D})$

$\theta^{h,K} \geq 0$

$\theta^{h,OE} \geq 0$

maximize

$$\theta^{h,K} E[dr_t^{h,K}]/dt + \theta^{h,OE} E[dr_t^{OE}]/dt + \theta^{h,D} r_t - \varsigma_t^e (\theta_t^{h,K} + \theta_t^{h,OE}) \sigma^{r^{h,K}}$$

1b. *Toolbox:* Price Taking Social Planner ⇒ Asset/Risk Allocation

2 Types

- Aggregate η-weighted sum of expert + HH max problem

$$\eta^e\{\dots\} + \eta^h\{\dots\}$$

- $$\underbrace{\eta_t^e\theta_t^{e,K}}_{\kappa_t^e :=} E\left[dr_t^{e,K}\right]/dt + \underbrace{\eta_t^h\theta_t^{hK}}_{\kappa_t^h :=} E\left[dr_t^{h,K}\right]/dt +$$

$$\underbrace{\left(\eta_t^e\theta_t^{e,OE} + \eta_t^h\theta_t^{h,OE}\right)}_{=0} E\left[dr_t^{OE}\right]/dt + \underbrace{\left(\eta_t^e\theta_t^{e,D} + \eta_t^h\theta_t^{h,D}\right)}_{=0} r_t$$

$$-\varsigma_t^e \underbrace{\eta_t^e\left(\theta_t^{e,K} + \theta_t^{e,OE}\right)}_{=:\chi_t^e} \sigma_t^{r^K} - \varsigma_t^h \underbrace{\eta_t^h\left(\theta_t^{h,K} + \theta_t^{h,OE}\right)}_{=:\chi_t^h} \sigma_t^{r^k}$$

1b. *Toolbox:* Price Taking Social Planner ⇒ Asset/Risk Allocation

2 Types

- Aggregate η-weighted sum of expert + HH max problem

$$\eta^e\{\dots\} + \eta^h\{\dots\}$$

- $$\underbrace{\eta_t^e\theta_t^{e,K}}_{\kappa_t^e :=} E\left[dr_t^{e,K}\right]/dt + \underbrace{\eta_t^h\theta_t^{hK}}_{\kappa_t^h :=} E\left[dr_t^{h,K}\right]/dt +$$
$$\underbrace{\left(\eta_t^e\theta_t^{e,OE} + \eta_t^h\theta_t^{h,OE}\right)}_{=0} E\left[dr_t^{OE}\right]/dt + \underbrace{\left(\eta_t^e\theta_t^{e,D} + \eta_t^h\theta_t^{h,D}\right)}_{=0} r_t$$
$$-\varsigma_t^e \underbrace{\eta_t^e\left(\theta_t^{e,K} + \theta_t^{e,OE}\right)}_{=:\chi_t^e} \sigma_t^{r^K} - \varsigma_t^h \underbrace{\eta_t^h\left(\theta_t^{h,K} + \theta_t^{h,OE}\right)}_{=:\chi_t^h} \sigma_t^{r^k}$$

Poll 36: Why $=0$ *?*
a) because marginal benefits = marginal costs at optimum
b) due to martingale behavior
c) because outside equity and debt are in zero net supply

1b. *Toolbox:* Price Taking Social Planner ⇒ Asset/Risk Allocation

2 Types

- Translate constraints:
  - $\chi_t^e \leq \kappa_t^e$ experts cannot buy outside equity of others

    only important for the case with idio risk
  - $\chi_t^e = \underbrace{\eta_t^e \theta_t^{e,K}}_{\kappa_t^e} + \underbrace{\eta_t^e \theta_t^{e,OE}}_{\geq -\kappa_t^e(1-\alpha)} \geq \alpha \kappa_t^e$
- Price-taking social planers problem

$$\max_{\{\kappa_t^e, \kappa_t^h = 1-\kappa_t^e, \chi_t^e \in [\alpha\kappa_t^e, \kappa_t^e], \chi_t^h = 1-\chi_t^e\}} \left[\frac{\kappa_t^e a^e + \kappa_t^h a^h - \iota_t}{q_t} + \Phi(\iota_t) - \delta\right] - (\varsigma_t^e \chi_t^e + \varsigma_t^h \chi_t^h)\sigma_t^{r^K}$$

End of Proof. Q.E.D.

- Linear objective (if frictions take form of constraints)
  - Price of risk adjust such that objective becomes flat *or*
  - Bang-bang solution hitting constraints

1b. *Toolbox:* Price Taking Social Planner ⇒ Asset/Risk Allocation

2 Types

- Example 1: 2 Types + no outside equity $(\alpha = 1)$

$$\max_{\{\kappa_t^e, \chi_t^e\}} \left[\frac{\kappa_t^e a^e + (1-\kappa_t^e) a^h - \iota_t}{q_t} + \Phi(\iota_t) - \delta \right] - \left(\chi_t^e \varsigma_t^e + (1-\chi_t^e) \varsigma_t^h \right) \left(\sigma + \sigma_t^q \right)$$

s.t. friction $\chi_t^e = \kappa_t^e$ if no outside equity can be issued

- FOC_χ: $\frac{a^e - a^h}{q_t} = (\varsigma_t^e - \varsigma_t^h)(\sigma + \sigma_t^q)$

- May hold only with inequality (≥), if at constraint $\kappa_t^e = 1$

1b. Price Taking Social Planner ⇒ Asset/Risk Allocation

2 Types

- Example 2: 2 Types + with outside equity

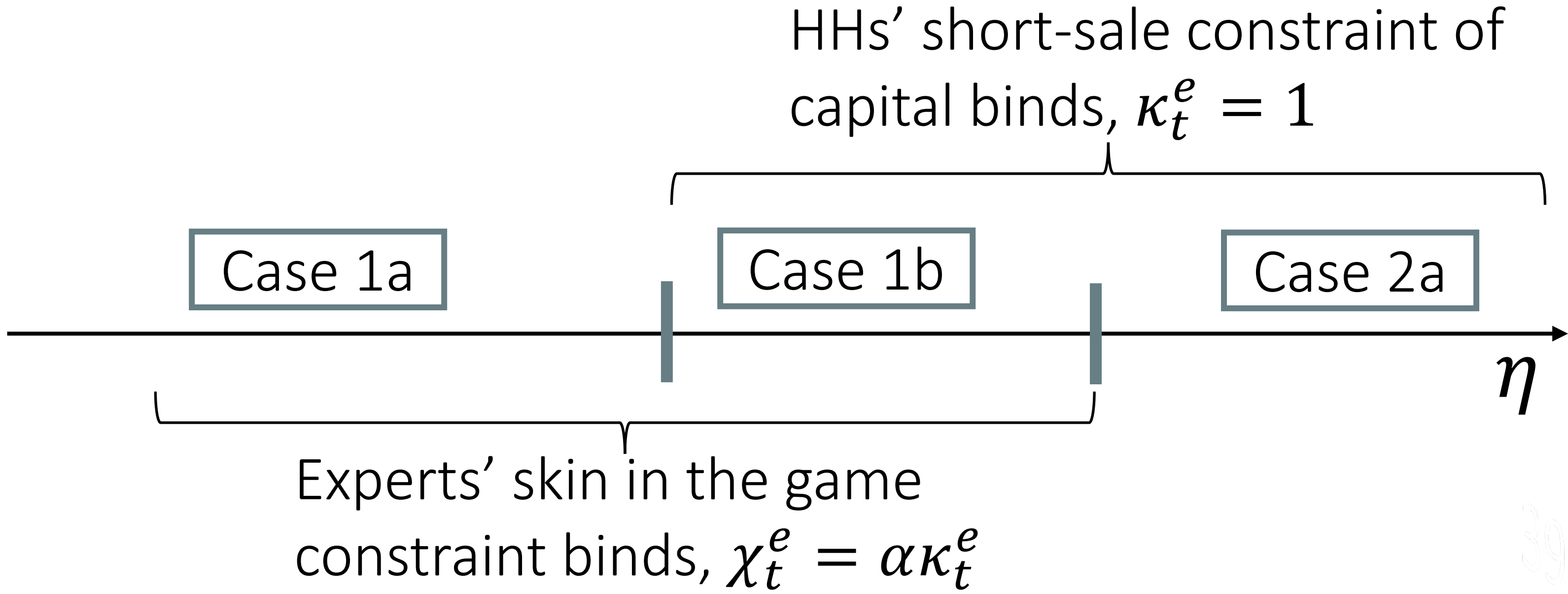
$$\max_{\{\kappa_t^e,\chi_t^e\}} \left[\frac{\kappa_t^e a^e + (1-\kappa_t^e)a^h - \iota_t}{q_t} + \Phi(\iota_t) - \delta\right] - \left(\chi_t^e \varsigma_t^e + (1-\chi_t^e)\varsigma_t^h\right)(\sigma + \sigma_t^q)$$

- FOC_χ: Case 1: $\varsigma_t^e(\sigma + \sigma_t^q) > \varsigma_t^h(\sigma + \sigma_t^q) \Rightarrow \chi_t^e = \alpha\kappa_t^e$

  Case 2: $=$ $\chi_t^e > \alpha\kappa_t^e$

  - Case 1: plug $\chi_t^e = \alpha\kappa_t^e$ in objective
    a. $FOC_\kappa: \frac{a^e - a^h}{q_t} = \alpha(\varsigma_t^e - \varsigma_t^h)(\sigma + \sigma_t^q) \Rightarrow \kappa_t^e < 1$
    b. $>$ $\Rightarrow \kappa_t^e = 1$
  - Case 2:
    a. $FOC_\kappa: \frac{a^e - a^h}{q_t} > 0$ $\Rightarrow \kappa_t^e = 1$
    b. $= 0 \Rightarrow \kappa_t^e < 1$ impossible

Occasionally binding constraint
(skin in the game constraint)

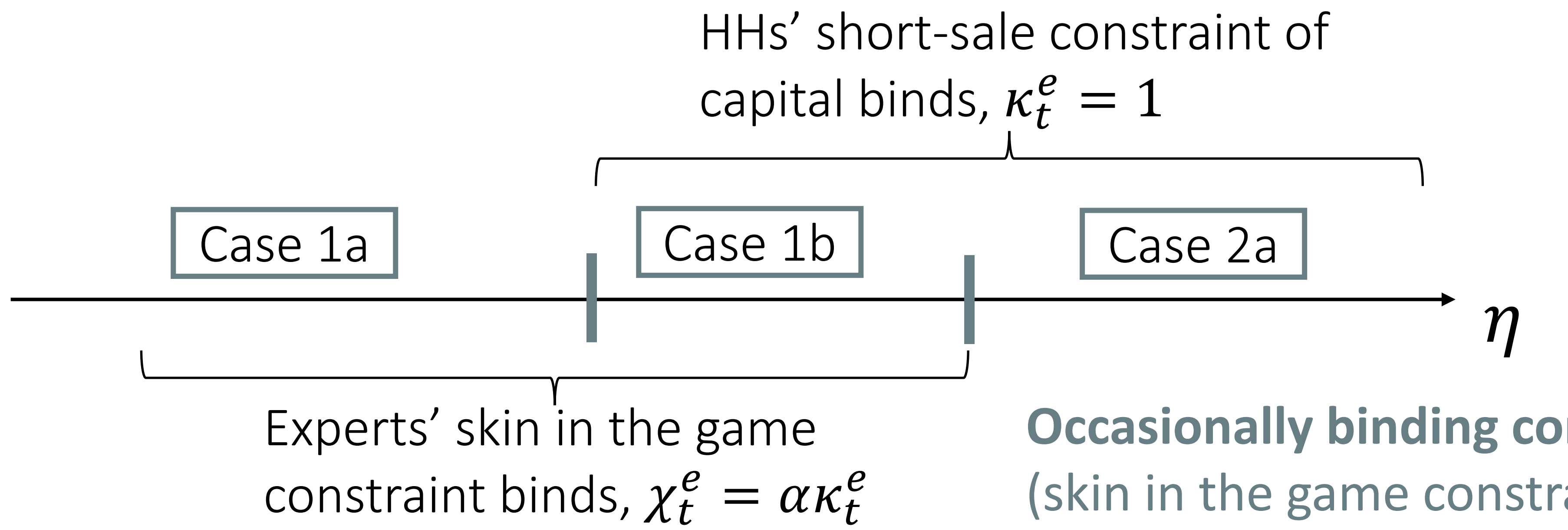


1b. Price Taking Social Planner ⇒ Asset/Risk Allocation

2 Types

- Summarizing previous slide (2 types with outside equity)

Cases	$\chi_t^e \geq \alpha\kappa_t^e$	$\kappa_t^e \leq 1$	$\frac{(a^e - a^h)}{q_t} \geq \alpha(\varsigma_t^e - \varsigma_t^h)(\sigma + \sigma_t^q)$ Shift a capital unit to expert Benefit: LHS Cost: RHS	$(\varsigma_t^e - \varsigma_t^h)(\sigma + \sigma_t^q) \geq \mathbf{0}$ Required risk premium of experts vs. HH
1a	=	<	=	>
1b	=	=	>	>
2a	>	=	>	=
impossible				



Occasionally binding constraint
(skin in the game constraint)

Solving MacroModels Step-by-Step

0. Postulate aggregates, price processes & obtain return processes
1. For given C/N-ratio and SDF processes for each i *finance block*
   a. Real investment ι + Goods market clearing *(static)*
   - *Toolbox 1:* Martingale Approach, HJB vs. Stochastic Maximum Principle Approach (previous lecture)

   b. Portfolio choice θ + Asset market clearing or
      Asset allocation κ & risk allocation χ
   - *Toolbox 2:* "price-taking social planner approach" – Fisher separation theorem

   c. ~~"Money evaluation equation" ϑ~~
   - *Toolbox 3:* Change in numeraire to total wealth (including SDF)
2. Evolution of state variable η (and K) *forward equation*
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   a. Value fcn. as fcn. of individual investment opportunities ω
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   b. Separating value fcn. $V^i(n^{\tilde{i}};\eta,K)$ into $v^i(\eta)u(K)\left(n^{\tilde{i}}/n^i\right)^{1-\gamma}$
   c. Derive C/N-ratio and ς price of risk
4. Numerical model solution
   a. Transform BSDE for separated value fcn. $v^i(\eta)$ into PDE
   b. Solve PDE via value function iteration
5. KFE: Stationary distribution, Fan charts

Toolbox 3: Change of Numeraire

- x_t^A is a value of a self-financing strategy/asset in \$
- Y_t price of € in \$ (exchange rate)

$$\frac{dY_t}{Y_t} = \mu_t^Y dt + \sigma_t^Y dZ_t$$

- x_t^A / Y_t value of the self-financing strategy/asset in €

$$\underbrace{e^{-\rho t} u'(c_t)}_{=\xi_t} Y_t \frac{x_t^A}{Y_t} \text{ follows a martingale}$$

Recall $\mu_t^A - \mu_t^B = \underbrace{(-\sigma_t^\xi)}_{=\varsigma_t} \underbrace{(\sigma^A - \sigma_t^B)}_{risk}$

$$\mu_t^{A/Y} - \mu_t^{B/Y} = \underbrace{(-\sigma_t^\xi - \sigma_t^Y)}_{price\ of\ risk} \underbrace{(\sigma^A - \cancel{\sigma_t^Y} - \sigma_t^B + \cancel{\sigma_t^Y})}_{risk}$$

- Price of risk $\varsigma^€ = \varsigma^\$ - \sigma^Y$

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- Price of risk $\varsigma^€ = \varsigma^\$ - \sigma^Y$

Poll 44: Why does the price of risk change, though real risk remains the same

a) because risk-free rate might not stay risk-free

b) because covariance structure changes

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2. GE: Markov States and Equilibria

- Equilibrium is a map

  Histories of shocks → prices $q_t, \varsigma_t^i, \iota_t^i, \theta_t^i$

  $\{\boldsymbol{Z}_s, s \in [0, t]\}$

  net worth distribution

  $$\eta_t^e = \frac{N_t^e}{q_t K_t} \in (0,1)$$

  net worth share

- All agents maximize utility
  - Choose: portfolio, consumption, technology
- All markets clear
  - Consumption, capital, money, outside equity

2. Law of Motion of Wealth Share η_t

- Method 1: Using Ito's quotation rule $\eta_t^i = N_t^i/(q_t K_t)$
  - Recall
    $$\frac{dN_t^i}{N_t^i} = r_t dt + \underbrace{\frac{\chi_t^i}{\eta_t^i}(\sigma+\sigma_t^q)}_{risk} \underbrace{\varsigma_t^i}_{price\ of\ risk} dt + \frac{\chi_t^i}{\eta_t^i}(\sigma+\sigma_t^q) dZ_t - \frac{C_t^i}{N_t^i} dt$$
  - $\frac{d\eta_t^i}{\eta_t^i} = \ldots$ (lots of algebra)

- Method 2: Change of numeraire + Martingale Approach
  - New numeraire: Total wealth in the economy, N_t
  - Apply Martingale Approach for value of i's portfolio
    - Simple algebra to obtain drift of η_t^i: $\mu_t^{\eta^i}$
      Note that <u>change of numeraire does not affect ratio η^i</u>!

2. μ^η Drift of Wealth Share: Many Types

- New Numeraire
  - "Total net worth" in the economy, N_t (without superscript)
  - Type i's portfolio net worth = net worth share
- Martingale Approach with new numeraire
  - Asset $A = i$'s portfolio return in terms of total wealth,

$$\left(\underbrace{\frac{C_t^i}{N_t^i}}_{\text{Dividend yield}} + \underbrace{\mu_t^{\eta^i}}_{\text{E[capital gains] rate}}\right) dt + \sigma_t^{\eta^i} dZ_t + \tilde{\sigma}_t^{\eta^i} d\tilde{Z}_t$$

  - Asset B (benchmark asset that everyone can hold, e.g. risk-free asset or money (in terms of total economy wide wealth as numeraire))

$$r_t^m dt + \sigma_t^m dZ_t$$

Poll 48: Is risk-free asset, risk free in the new numeraire?
a) Yes
b) No

  - Apply our martingale asset pricing formula

$$\mu_t^A - \mu_t^B = \varsigma_t^i(\sigma_t^A - \sigma_t^B)$$

2. μ^{η} Drift of Wealth Share: Many Types

- Asset pricing formula (relative to benchmark asset)

$$\mu_t^{\eta^i} + \frac{C_t^i}{N_t^i} - r_t^m = (\varsigma_t^i - \sigma_t^N)\left(\sigma_t^{\eta^i} - \sigma_t^m\right)$$

due to change in numeraire

- Add up across types (weighted),

(capital letters without superscripts are aggregates for total economy)

$$\underbrace{\sum_{i'}^{I} \eta_t^{i'} \mu_t^{\eta^{i'}}}_{=0} + \frac{C_t}{N_t} - r_t^m = \sum_{i'} \eta_t^{i'} \left(\varsigma_t^{i'} - \sigma_t^N\right)\left(\sigma_t^{\eta^{i'}} - \sigma_t^m\right)$$

- *Poll 49: Why $= 0$?*
  a) *Because we have stationary distribution*
  b) *Because ηs sum up to 1*
  c) *Because ηs follow martingale*

Benchmark asset everyone can trade

$$\sigma_t^m = -\sigma_t^N$$

2. μ^{η} Drift of Wealth Share: 2 Types

For benchmark asset: risk-free debt
$$\sigma_t^m = -\sigma_t^N$$

- Asset pricing formula (relative to benchmark asset)

$$\mu_t^{\eta^i} + \frac{C_t^i}{N_t^i} - r_t^m = (\varsigma_t^i - \sigma_t^N)\left(\sigma_t^{\eta^i} - \sigma_t^m\right)$$

- Add up across types (weighted),

(capital letters without superscripts are aggregates for total economy)

$$\underbrace{(\eta_t^e \mu_t^{\eta^e} + \eta_t^h \mu_t^{\eta^h})}_{=0} + \frac{C_t}{N_t} - r_t^m$$

$$= \eta_t^e \left(\varsigma_t^e - \sigma_t^N\right)\left(\sigma_t^{\eta^e} - \sigma_t^m\right) + \eta_t^h \left(\varsigma_t^h - \sigma_t^N\right)\left(\sigma_t^{\eta^h} - \sigma_t^m\right)$$

- Subtract from each other yield net worth share dynamics

$$\mu_t^{\eta^e} = (1-\eta_t^e)(\varsigma_t^e - \sigma_t^N)\left(\sigma_t^{\eta^e} - \sigma_t^m\right) - (1-\eta_t^e)\left(\varsigma_t^h - \sigma_t^N\right)\left(\sigma_t^{\eta^h} - \sigma_t^m\right)$$

$$-\left(\frac{C_t^e}{N_t^e} - \frac{C_t}{q_t K_t}\right)$$

2. σ^{η} Volatility of Wealth Share

- Recall Ito ratio rule (only volatility term)
- Since $\eta_t^e = N_t^e / N_t$,

$$\sigma_t^{\eta^e} = \sigma_t^{N^e} - \sigma_t^{N} = \sigma_t^{N^i} - \sum_{i'} \eta_t^{i'} \sigma_t^{N^{i'}} = (1 - \eta_t^i)\sigma_t^{N^i} - \sum_{i^- \neq i} \eta_t^{i^-} \sigma_t^{N^{i^-}}$$

- Note for

$$\sigma_t^{\eta^e} = (1 - \eta_t^e)(\sigma_t^{n^e} - \sigma_t^{n^h})$$

Change in notation in 2 type setting
Type-net worth is $n^i = N^i$

$$\sigma_t^{n^e} = \underbrace{\chi_t^e / \eta_t^e}_{= \theta^{e,K} + \theta^{e,OE}} (\sigma + \sigma_t^q) \qquad \sigma_t^{n^h} = \frac{\chi_t^h}{\eta_t^h}(\sigma + \sigma_t^q) = \frac{1 - \chi_t^e}{1 - \eta_t^e}(\sigma + \sigma_t^q)$$

Hence,

$$\boldsymbol{\sigma_t^{\eta^e} = \frac{\chi_t^e - \eta_t^e}{\eta_t^e} (\sigma + \sigma_t^q)}$$

- Note also, $\eta_t^e \sigma_t^{\eta^e} + \eta_t^h \sigma_t^{\eta^h} = 0 \Rightarrow \sigma_t^{\eta^h} = -\frac{\eta_t^e}{\eta_t^h} \sigma_t^{\eta^e} = -\frac{\eta_t^e}{1 - \eta_t^e} \sigma_t^{\eta^e}$

2. Amplification Formula: Loss Spiral

- Recall $$\sigma_t^{\eta^e} = \underbrace{\frac{\chi_t^e - \eta_t^e}{\eta_t^e}}_{\text{leverage}} (\sigma + \sigma_t^q)$$

- By Ito's Lemma on $q(\eta^e)$ $$\sigma_t^q = \frac{q'(\eta_t^e)}{q(\eta_t^e)} \eta_t^e \sigma_t^{\eta^e}$$

$$\sigma_t^q = \frac{q'(\eta_t^e)}{\underbrace{q/\eta_t^e}_{elasticity}} \frac{\chi_t^e - \eta_t^e}{\eta_t^e} (\sigma + \sigma_t^q)$$

- Total volatility

$$\sigma + \sigma_t^q = \frac{\sigma}{1 - \frac{q'(\eta_t^e)}{q/\eta_t^e} \frac{\chi_t^e - \eta_t^e}{\eta_t^e}}$$

- Loss spiral
  - Market illiquidity (price impact elasticity)

2. Amplification Formula: Loss Spiral

- Recall
$$\sigma_t^{\eta^e} = \underbrace{\frac{\chi_t^e - \eta_t^e}{\eta_t^e}}_{\text{leverage}} (\sigma + \sigma_t^q)$$

- By Ito's Lemma on $q(\eta^e)$
$$\sigma_t^q = \frac{q'(\eta_t^e)}{q(\eta_t^e)} \eta_t^e \sigma_t^{\eta^e}$$

$$\sigma_t^q = \underbrace{\frac{q'(\eta_t^e)}{q/\eta_t^e}}_{elasticity} \frac{\chi_t^e - \eta_t^e}{\eta_t^e} (\sigma + \sigma_t^q)$$

- Total volatility
$$\sigma + \sigma_t^q = \frac{\sigma}{1 - \frac{q'(\eta_t^e)}{q/\eta_t^e} \frac{\chi_t^e - \eta_t^e}{\eta_t^e}}$$

Poll 53: Where is the spiral?
a) Sum of infinite geometric series (denominator)
b) in q', since with constant price, no spiral

- Loss spiral
  - Market illiquidity (price impact elasticity)

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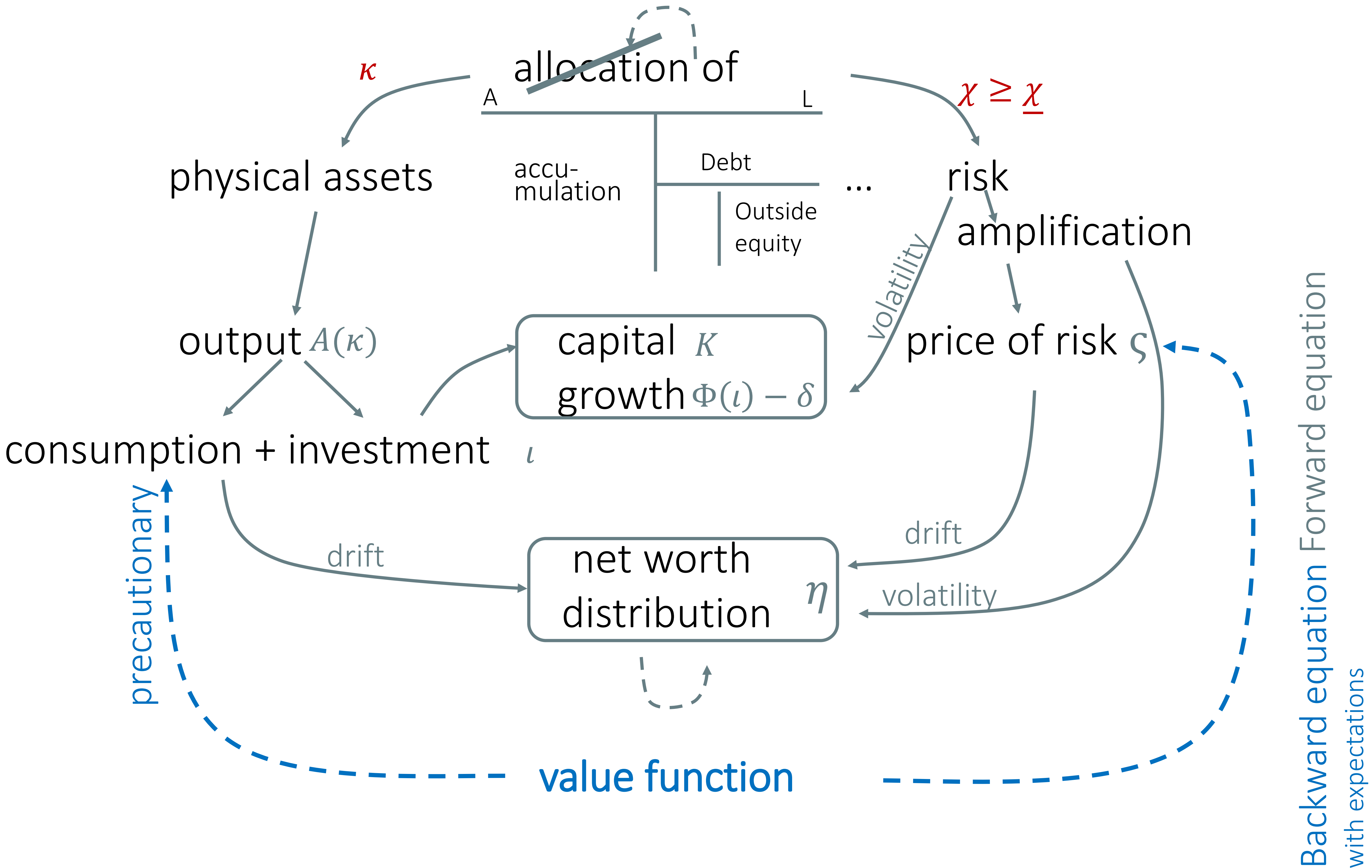
The Big Picture



3a. CRRA Value Function Applies separately for each type of agent

- Martingale Approach: works best in endowment economy
- Here: mix Martingale approach with value function (envelop condition)

- $V^i(n_t^i; \boldsymbol{\eta}_t, K_t)$ for individuals i
- For CRRA/power utility $u(c_t^i) = \frac{(c_t^i)^{1-\gamma}-1}{1-\gamma}$

$\Rightarrow$ increase net worth by factor, optimal c^i for all future states increases by this factor $\Rightarrow \left(\frac{c_t^i}{n_t^i}\right)$-ratio is invariant in n_t^i

- $\Rightarrow$ value function can be written as $V^i(n_t^i; \boldsymbol{\eta}_t, K_t) = \frac{u\left(\omega^i(\boldsymbol{\eta}_t, K_t) n_t^i\right)}{\rho^i}$
- ω_t^i Investment opportunity/ "net worth multiplier"
  - $\omega^i(\boldsymbol{\eta}_t, K_t)$-function turns out to be independent of K_t
  - Change notation from $\omega^i(\boldsymbol{\eta}_t, K_t)$-function to ω_t^i-process

3a. CRRA Value Function: relate to ω

- $\Rightarrow$ value function can be written as $\frac{u(\omega_t^i n_t^i)}{\rho}$, that is

$$= \frac{1}{\rho^i}\frac{(\omega_t^i n_t^i)^{1-\gamma}-1}{1-\gamma} = \frac{1}{\rho^i}\frac{(\omega_t^i)^{1-\gamma}(n_t^i)^{1-\gamma}-1}{1-\gamma}$$

- $\frac{\partial V}{\partial n^i} = u'(c^i)$ by optimal consumption (if no corner solution)

$$\frac{(\omega_t^i)^{1-\gamma}(n_t^i)^{-\gamma}}{\rho^i} = (c_t^i)^{-\gamma} \Leftrightarrow \frac{c_t^i}{n_t^i} = (\rho^i)^{1/\gamma}(\omega_t^i)^{1-1/\gamma}$$

- For log utility $\gamma = 1$
  - Consumption choice: $c_t^i = \rho^i n_t^i$
    - ω_t does not matter $\Rightarrow$ income and substitution effect cancel out
  - Portfolio choice: myopic (no Mertonian hedging demand)
    - Volatility of investment of opportunity/net worth multiplier does not matter $\Rightarrow$ Myopic price of risk $\varsigma_t^i = \sigma_t^{n^i} = \sigma_t^{c^i}$

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   - *Toolbox 2:* "price-taking social planner approach" – Fisher separation theorem

   c. ~~"Money evaluation equation" ϑ~~
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   a. Value fcn. as fcn. of individual investment opportunities ω
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   b. Separating value fcn. $V^i(n^{\tilde{i}};\eta,K)$ into $v^i(\eta)u(K)\left(n^{\tilde{i}}/n^i\right)^{1-\gamma}$

   c. Derive C/N-ratio and ς price of risk
4. Numerical model solution
   a. Transform BSDE for separated value fcn. $v^i(\eta)$ into PDE

   b. Solve PDE via value function iteration
5. KFE: Stationary distribution, Fan charts

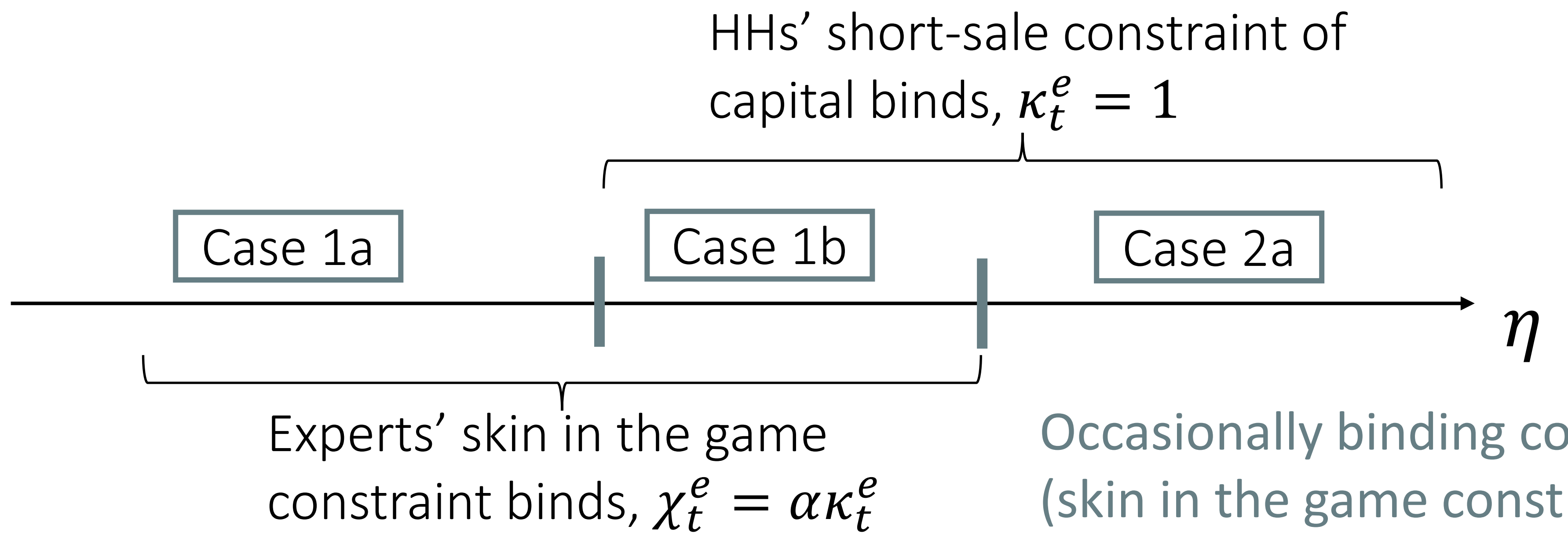
4a. Replacing ι_t

- Recall from optimal re-investment $\Phi'(\iota_t) = 1/q_t$
  - For $\Phi(\iota) = \frac{1}{\phi}\log(\phi\iota + 1) \Rightarrow \boxed{\phi\iota = q - 1}$

4a. Replacing χ, obtain κ for good mkt clearing

- Recall from planner's problem (Step 1b)

Cases	$\chi_t^e \geq \alpha\kappa_t^e$	$\kappa_t^e \leq 1$	$\frac{(a^e - a^h)}{q_t} \geq \alpha(\varsigma_t^e - \varsigma_t^h)(\sigma + \sigma_t^q)$ Shift a capital unit to expert Benefit: LHS Cost: RHS	$(\varsigma_t^e - \varsigma_t^h)(\sigma + \sigma_t^q) \geq 0$ Required risk premium of experts vs. HH
1a	=	<	=	>
1b	=	=	>	>
2a	>	=	>	=
impossible				



Occasionally binding constraint (skin in the game constraint)

4a. Replacing χ, obtain κ for good mkt clearing

- Determination of κ_t
  - Based on difference in risk premia $(\varsigma_t^e - \varsigma_t^h)(\sigma + \sigma_t^q)$
  - For log utility: $\left(\sigma_t^{n^e} - \sigma_t^{n^h}\right)(\sigma + \sigma_t^q) = \frac{\chi_t^e - \eta_t^e}{(1-\eta_t^e)\eta_t^e}(\sigma + \sigma_t^q)$
    - = since $\sigma_t^{\eta^e} = \frac{\chi_t^e - \eta_t^e}{\eta_t^e}(\sigma + \sigma_t^q)$, $\sigma_t^{\eta^h} = -\frac{\eta_t^e}{1-\eta_t^e}\sigma_t^{\eta^e}$ and $\sigma_t^{n^e} - \sigma_t^{n^h} = \sigma_t^{\eta^e} - \sigma_t^{\eta^h}$
  - Hence,

$$(a^e - a^h)/q_t \geq \alpha \frac{\chi_t^e - \eta_t^e}{(1-\eta_t^e)\eta_t^e}(\sigma + \sigma_t^q)^2$$

with equality if $\kappa_t^e < 1$

- Determination of χ_t^e

$$\chi_t^e = \max\{\alpha\kappa_t^e, \eta_t^e\}$$

4a. Replacing χ, obtain κ for good mkt clearing

- Need to determine diff in risk premia $(\varsigma_t^e - \varsigma_t^h)(\sigma + \sigma_t^q)$:
- Recall for log utility $\left(\sigma_t^{n^e} - \sigma_t^{n^h}\right)(\sigma + \sigma_t^q)$
  - diff in price of risk: $\varsigma_t^e - \varsigma_t^h = -\sigma_t^{v^e} + \sigma_t^{v^h} + \frac{\sigma_t^{\eta^e}}{1-\eta_t^e}$
  - By Ito's lemma $\sigma_t^{v^e} = \frac{\partial_\eta v_t^e}{v_t^e}\eta_t^e \sigma_t^{\eta^e}$ and $\sigma_t^{v^h} = \frac{\partial_\eta v_t^h}{v_t^h}\eta_t^e \sigma_t^{\eta^e}$

$$\Rightarrow (\varsigma_t^e - \varsigma_t^h)(\sigma + \sigma_t^q) = \left(-\frac{\partial_\eta v_t^e}{v_t^e} + \frac{\partial_\eta v_t^h}{v_t^h} + \frac{1}{(1-\eta_t^e)\eta_t^e}\right)\eta_t^e \sigma_t^{\eta^e}(\sigma + \sigma_t^q)$$

$$= \left(-\frac{\partial_\eta v_t^e}{v_t^e} + \frac{\partial_\eta v_t^h}{v_t^h} + \frac{1}{(1-\eta_t^e)\eta_t^e}\right)(\chi_t^e - \eta_t^e)(\sigma + \sigma_t^q)^2$$

$$\sigma_t^{\eta^e} = \frac{\chi_t^e - \eta_t^e}{\eta_t^e}(\sigma + \sigma_t^q)$$

$$\sigma_t^{\eta^h} = -\frac{\eta_t^e}{1-\eta_t^e}\sigma_t^{\eta^e}$$

Hence,

$$\sigma_t^{n^e} - \sigma_t^{n^h} = \frac{1}{1-\eta_t^e}\sigma_t^{\eta^e}$$

$$\sigma_t^{n^e} - \sigma_t^{n^h} = \frac{\chi_t^e - \eta_t^e}{(1-\eta_t^e)\eta_t^e}(\sigma + \sigma_t^q)$$

- Note, since $-\frac{\partial_\eta v_t^e}{v_t^e} + \frac{\partial_\eta v_t^h}{v_t^h} + \frac{1}{(1-\eta_t^e)\eta_t^e} > 0$,

$$\chi_t^e > \eta_t^e \Leftrightarrow \alpha\kappa_t^e > \eta_t^e$$

4a. Market Clearing

- Output good market

$$\left(\kappa_t^e a^e + (1-\kappa_t^e)a^h - \iota_t\right)K_t = C_t$$

… jointly restricts κ_t and q_t

$$\kappa_t a^e + (1-\kappa_t)a^h - \iota(q_t) = q_t\left[\eta_t \rho^e + (1-\eta_t)\rho^h\right]$$

$$= \underbrace{\left(\frac{\eta_t^e q_t}{v_t^e}\right)^{1/\gamma}}_{C_t^e/K_t} + \underbrace{\left(\frac{(1-\eta_t^e)q_t}{v_t^h}\right)^{1/\gamma}}_{C_t^h/K_t}$$

4a. Market Clearing

- Output good market

$$\left(\kappa_t^e a^e + (1-\kappa_t^e)a^h - \iota_t\right)K_t = C_t,$$

$$\kappa_t a^e + (1-\kappa_t)a^h - \iota(q_t) = q_t\left[\eta_t \rho^e + (1-\eta_t)\rho^h\right]$$

… jointly restricts κ_t and q_t

- Capital market is taken care off by price taking social planner approach

$$\theta_t^{e,K} = \frac{\kappa_t^e q_t K_t}{\eta_t^e q_t K_t}$$

- Risk-free debt market also taken care off by price taking social planner approach
  (would be cleared by Walras Law anyways)

4a. $\sigma^q(q, q')$

- Recall from "amplification slide" – Step 2

$$\sigma + \sigma_t^q = \frac{\sigma}{1 - \frac{q'(\eta_t^e)}{q/\eta_t^e} \frac{\chi_t^e - \eta_t^e}{\eta_t^e}}$$

$$\sigma^q = \frac{q'(\eta_t^e)}{q(\eta_t^e)} (\chi_t^e - \eta_t^e)(\sigma + \sigma_t^q)$$

- Now all red terms are replaced, and we can solve …

4b. Algorithm – Static Step

- Suppose we know functions $v^e(\eta^e), v^h(\eta)$, have five static conditions:

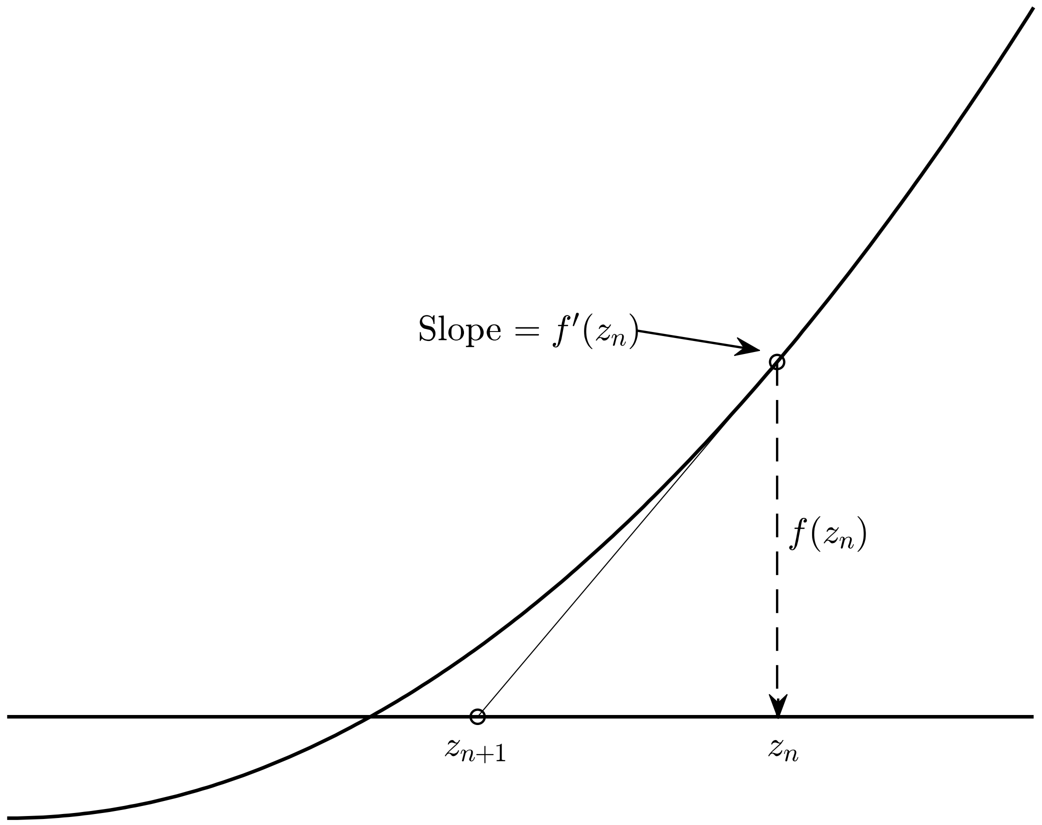
1. $\phi\iota_t = q_t - 1$
2. Planner condition for κ_t^e: $(a^e - a^h)/q_t \geq \alpha \frac{\chi_t^e - \eta_t^e}{(1-\eta_t^e)\eta_t^e}(\sigma + \sigma_t^q)^2$
3. Planner condition for $\chi_t^e = \max\{\alpha\kappa_t^e, \eta_t^e\}$
4. $\kappa_t^e a^e + (1 - \kappa_t^e)a^h - \iota(q_t) = q_t[\eta_t\rho^e + (1 - \eta_t)\rho^h]$
5. $\sigma^q = \frac{q'(\eta_t^e)}{q(\eta_t^e)}(\chi_t^e - \eta_t^e)(\sigma + \sigma_t^q)$

$\Rightarrow$ Get $q(\eta^e)$, $\kappa^e(\eta^e)$, $\sigma^q(\eta^e)$

- Start at $q(0)$, solve to the right, use different procedure for two η regions depending on κ:

1. While $\kappa^e < 1$, solve ODE for $q(\eta^e)$:
   - For given $q(\eta)$, plug optimal investment (1) into (4)
   - Plug planner condition (3) into (2) and (5)
   - Solve ODE using three equilibrium condition (2),(4) and (5) via Newton's method (see next slide)
2. When $\kappa = 1$, (2) is no longer informative, since $\kappa^e = 1$, solve (1) and (4) for $q(\eta)$

4b. *Aside:* Newton's Method



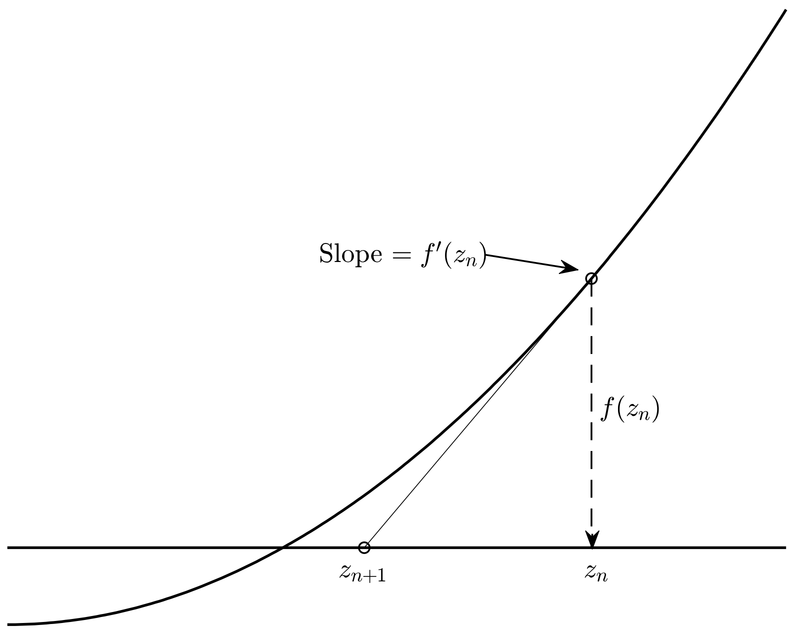
- Find the root of equation system $F(\boldsymbol{z}_n) = 0$ via iterative method

$$z_{n+1} = \mathrm{z_n} - J_n^{-1} F(\boldsymbol{z}_n)$$

Where J_n is the Jacobian matrix, i.e., $J_{ij} = \partial f_i(\boldsymbol{z}) / \partial z_j$.

- Newton's method does not guarantee global convergence.
- commonly take several-step iteration.

4b. *Aside:* Newton's Method



$$\mathbf{z}_n = \begin{bmatrix} q_t \\ \kappa_t^e \\ \sigma + \sigma_t^q \end{bmatrix},$$

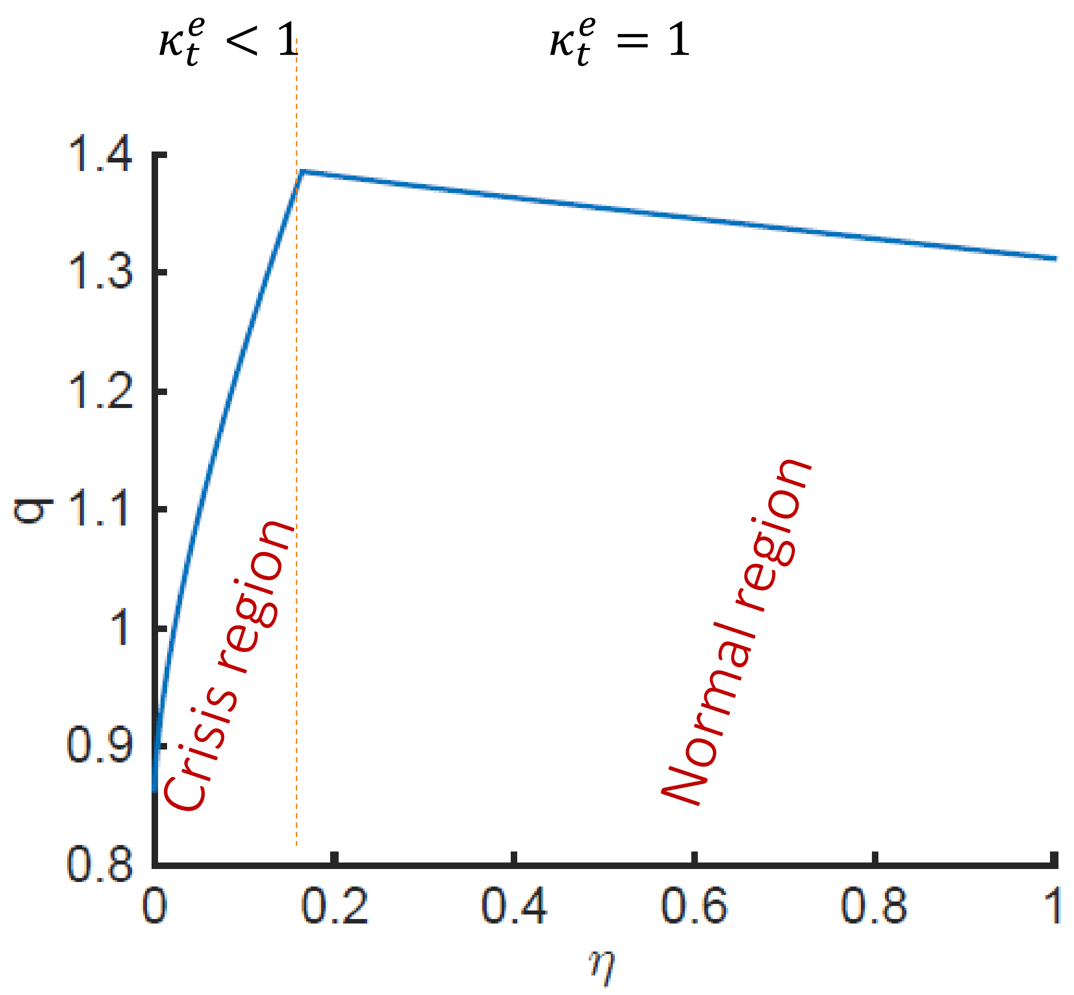
$$F(\mathbf{z}_n) = \begin{bmatrix} \kappa_t^e a^e + (1-\kappa_t^e)a^h - \iota(q_t) - q_t\left[\eta_t \rho^e + (1-\eta_t)\rho^h\right] \\ q'(\eta_t^e)(\chi_t^e - \eta_t^e)(\sigma + \sigma_t^q) - \sigma^q q(\eta_t^e) \\ (a^e - a^h) - \alpha q_t \frac{\chi_t^e - \eta_t^e}{(1-\eta_t^e)\eta_t^e}(\sigma + \sigma_t^q)^2 \end{bmatrix}$$

$$\begin{bmatrix} \text{market clearing condtion} \\ \text{amplification condition} \\ \text{Planner condition for } \kappa_t^e \end{bmatrix}$$

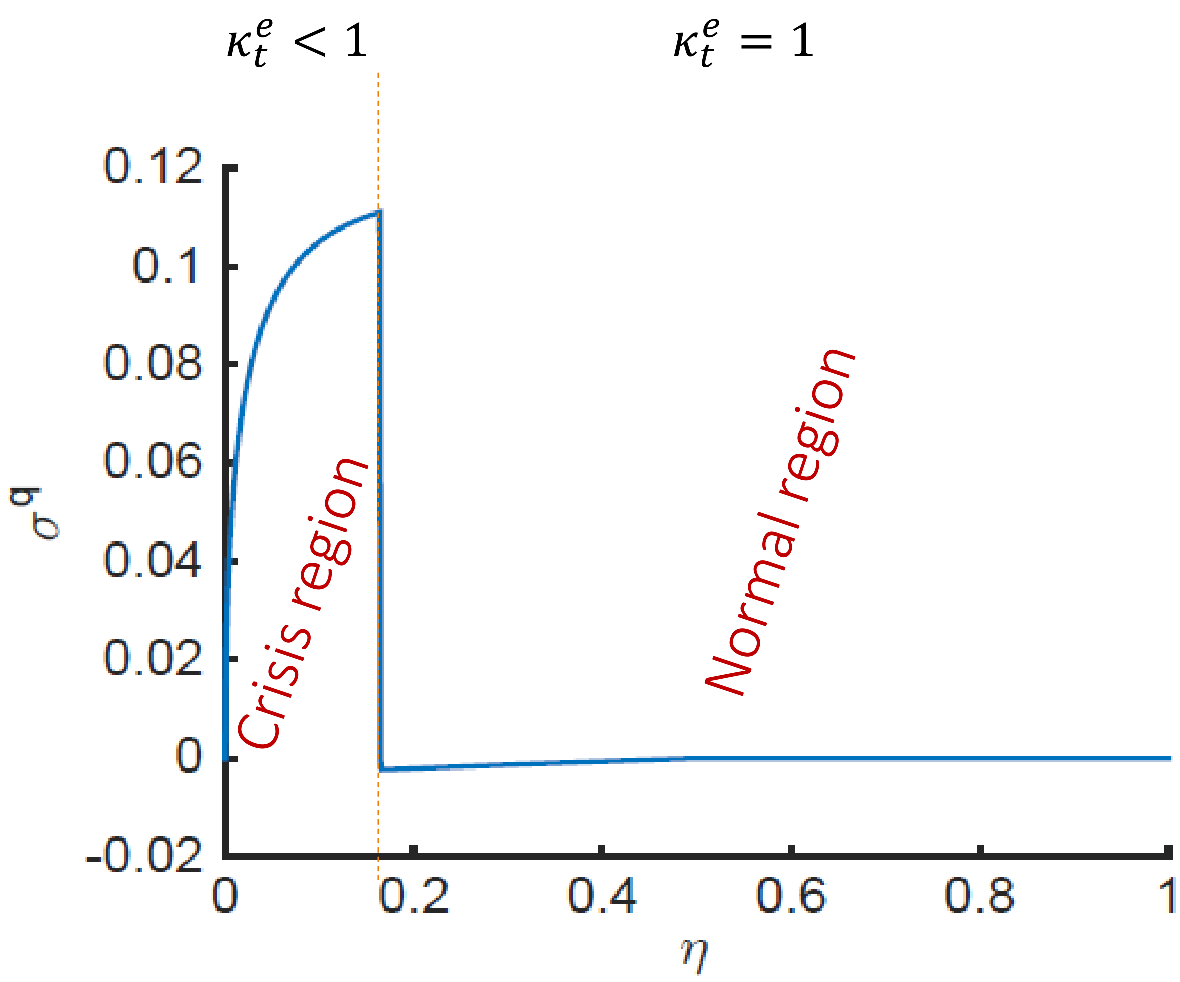
Plug in blue terms from optimal investment and Planner condition for χ_t^e

Solution

- Price of capital



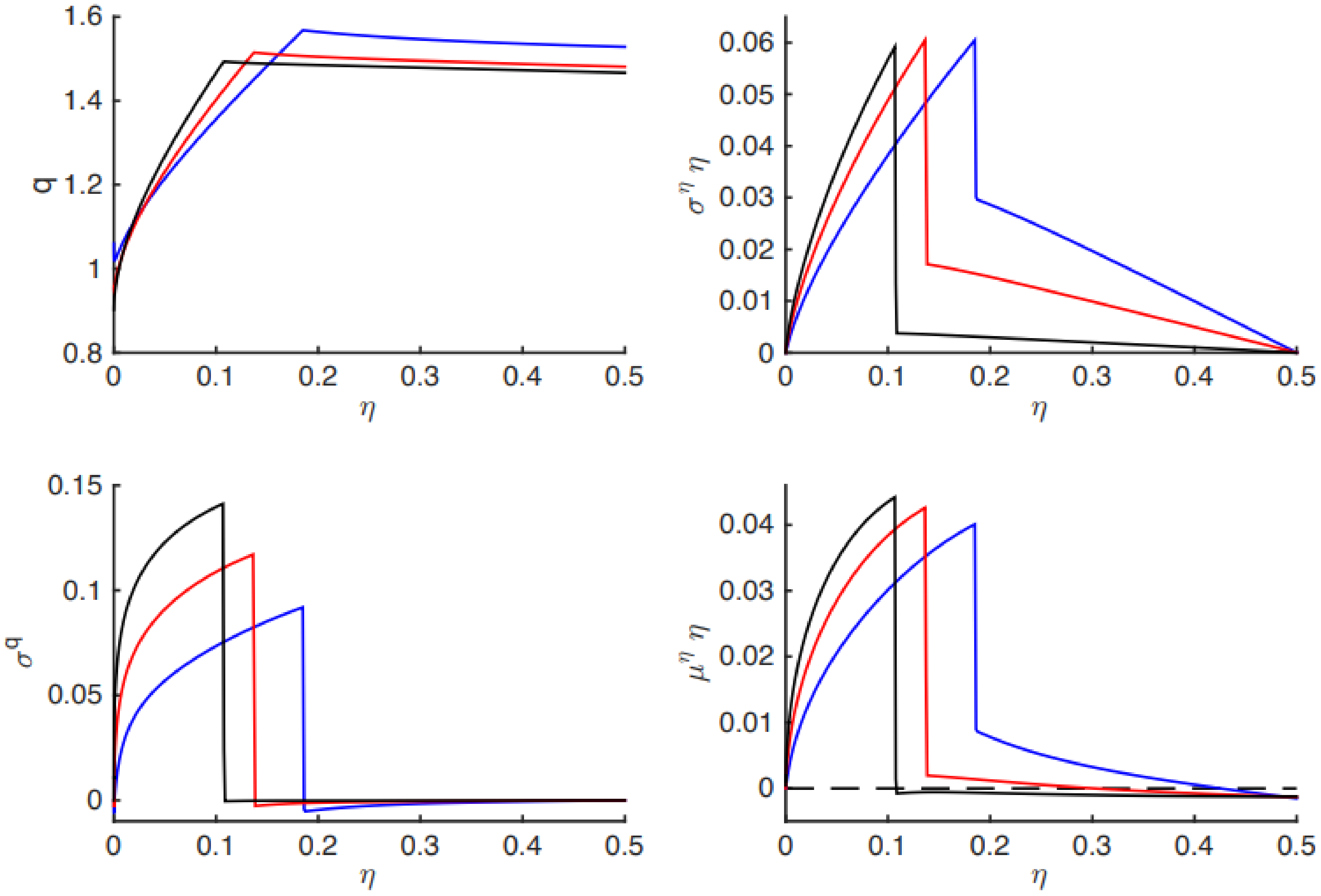
Amplification



Parameters: $\rho^e = .06, \rho^h = .05, a^e = .11, a^h = .03,$
$\delta = .05, \sigma = .1, \alpha = .50, \gamma = 2, \phi = 10$

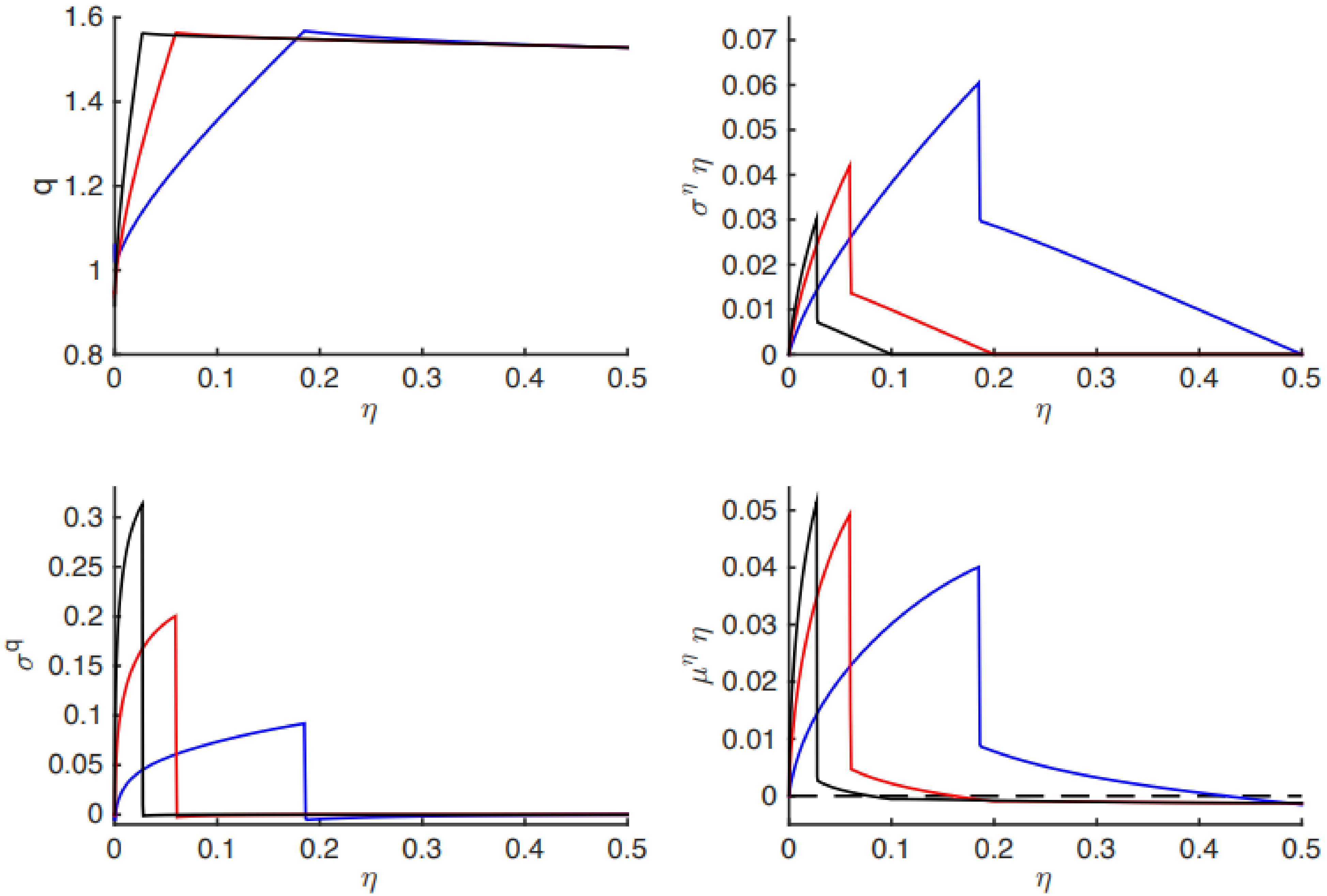
Volatility Paradox

- Comparative Static w.r.t. $\sigma = .01, .05, .1$



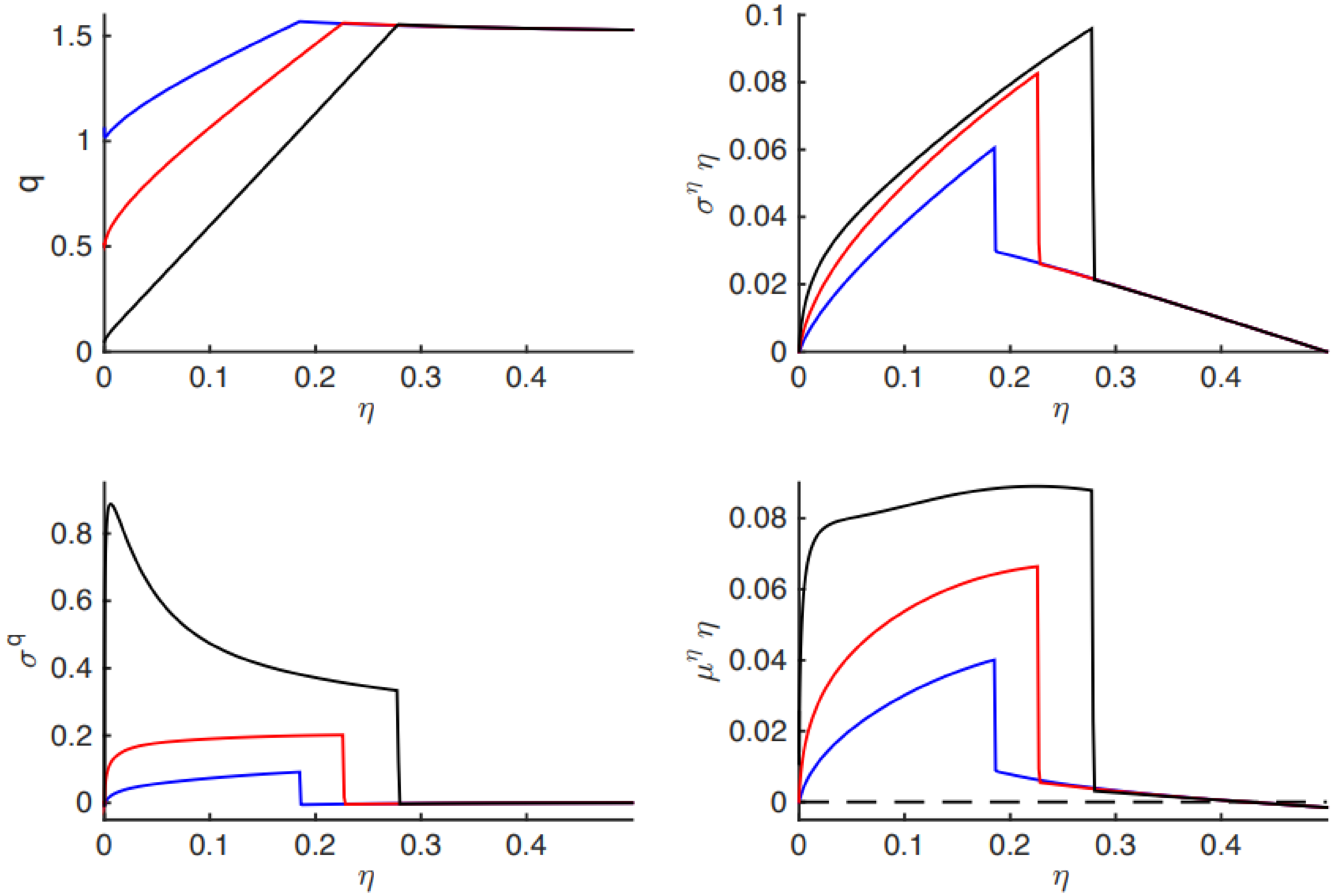
Risk Sharing via Outside Equity

- Comparative Static w.r.t. Risk sharing $\alpha = .1, .2, .5$ (skin the game constraint)



Market Liquidity

- Comparative static w.r.t. $a^h = .03, -.03, -.09$



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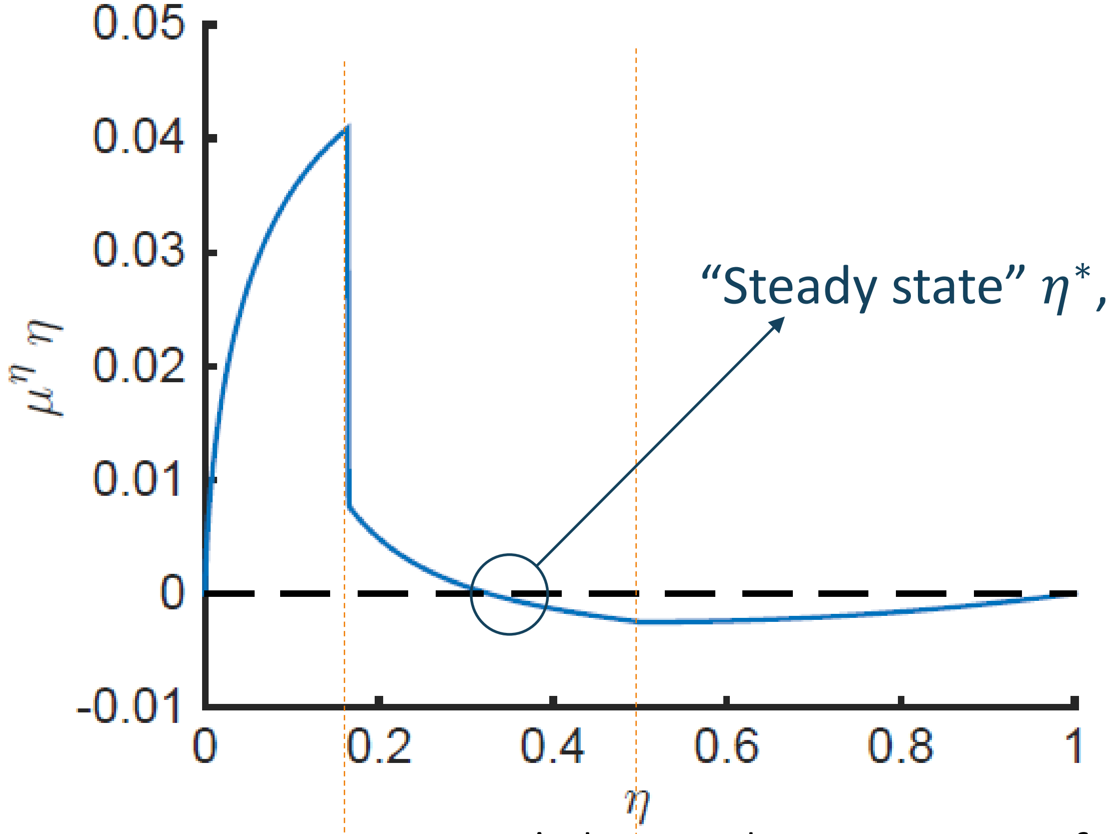
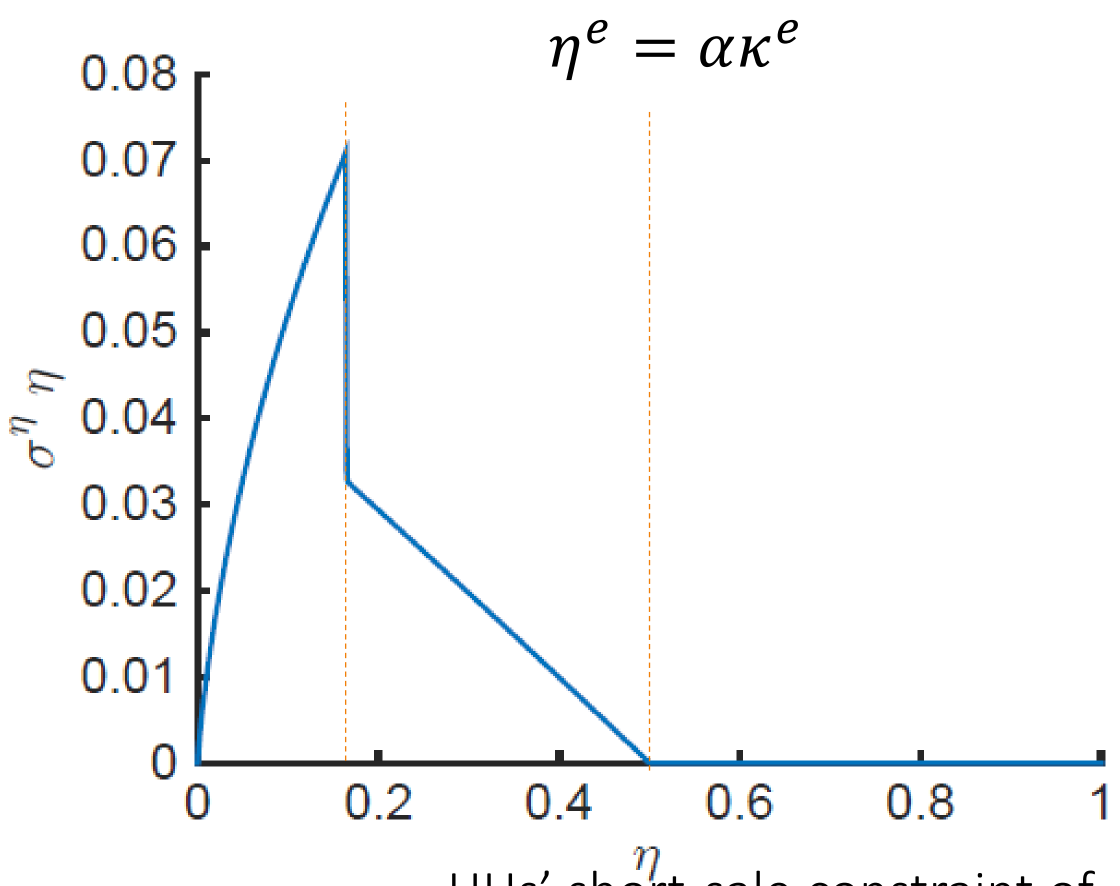
5. KFE: Stationary distribution, Fan charts

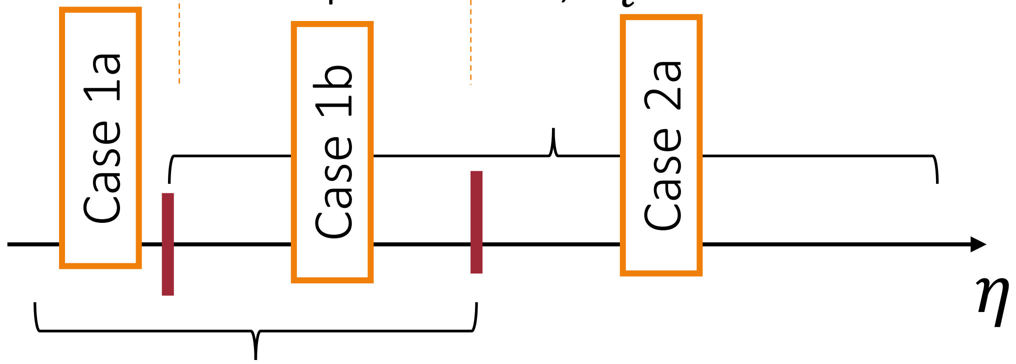
From $\mu^{\eta^e}(\eta^e)$ & $\sigma^{\eta^e}(\eta^e)$ to Stationary Distribution

- Drift and Volatility of η^e



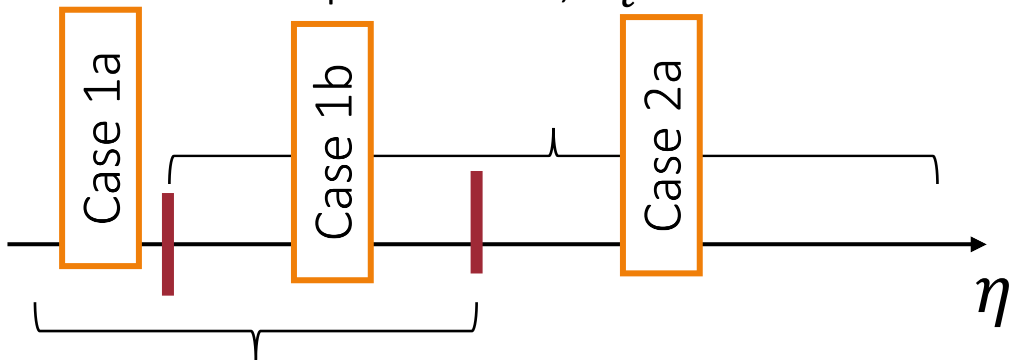


HHs' short-sale constraint of capital binds, $\kappa_t^e = 1$



Experts' skin in the game constraint binds, $\chi_t^e = \alpha\kappa_t^e$

HHs' short-sale constraint of capital binds, $\kappa_t^e = 1$



Experts' skin in the game constraint binds, $\chi_t^e = \alpha\kappa_t^e$

5. Kolmogorov Forward Equation

- Given an initial distribution $f(\eta, 0) = f_0(\eta)$, the density diffusion follows PDE

$$\frac{\partial f(\eta,t)}{\partial t} = -\frac{\partial[f(\eta,t)\mu(\eta)]}{\partial \eta} + \frac{1}{2}\frac{\partial^2[f(\eta,t)\sigma^2(\eta)]}{\partial \eta^2}$$

- “Kolmogorov Forward Equation” is in physics referred to as “Fokker-Planck Equation”

- Corollary: if stationary distribution $f(\eta)$ exists, it satisfies the ODE

$$0 = -\frac{\partial[f(\eta,t)\mu(\eta)]}{\partial \eta} + \frac{1}{2}\frac{\partial^2[f(\eta,t)\sigma^2(\eta)]}{\partial \eta^2}$$

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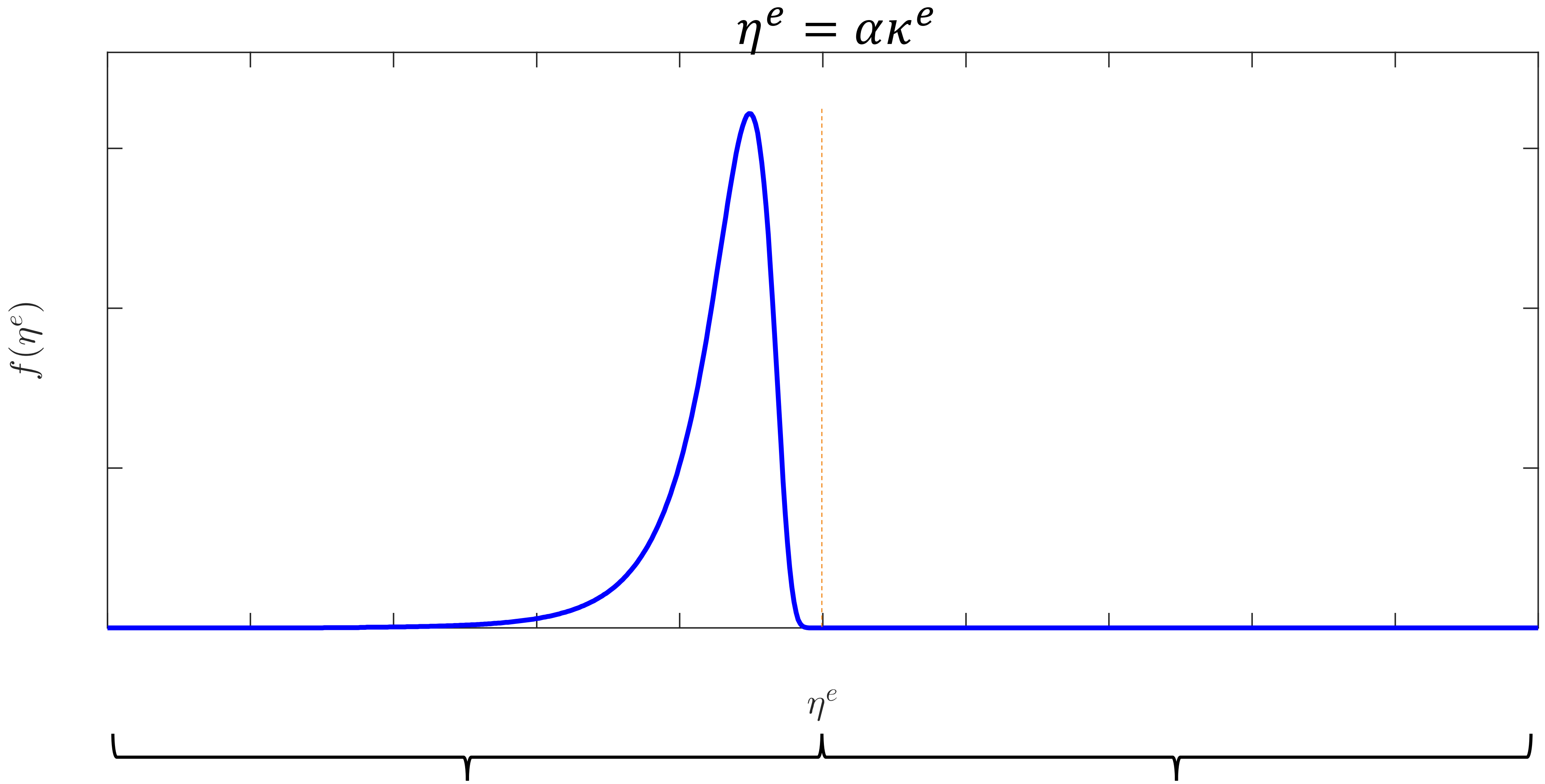
5. Stationary Distribution

- Stationary distribution of η^e



Experts' skin in the game constraint binds $\chi_t^e = \alpha\kappa_t^e$

Perfect risk-sharing region (infeasible)

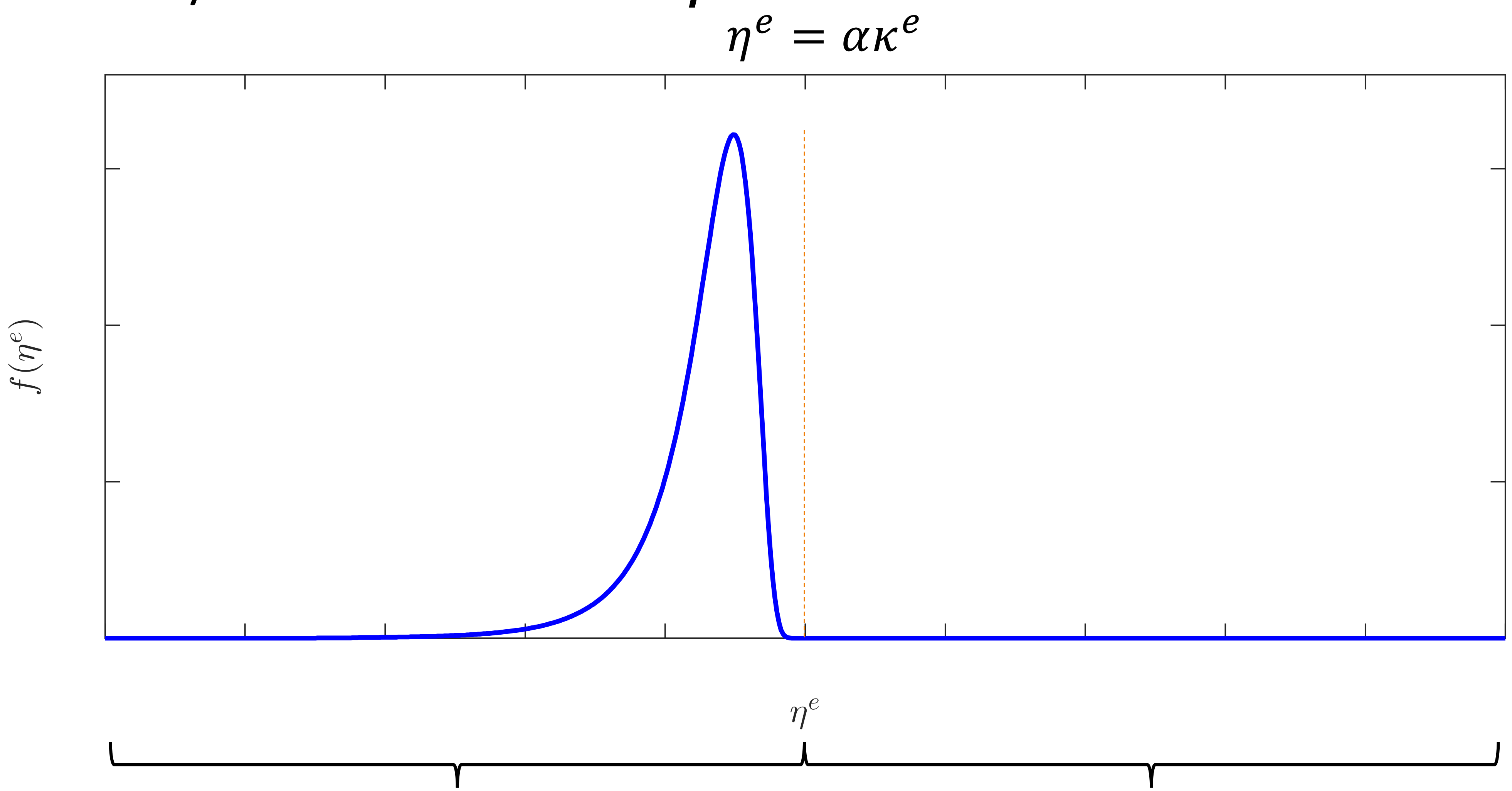
Poll 78: Is the constraint always (not just occasionally) binding

a) yes

b) no, only for some parameters $\rho^e > \rho^h$

5. Stationary Distribution

- Stationary distribution of η^e



Experts' skin in the game constraint binds $\chi_t^e = \alpha\kappa_t^e$

Perfect risk-sharing region (infeasible)

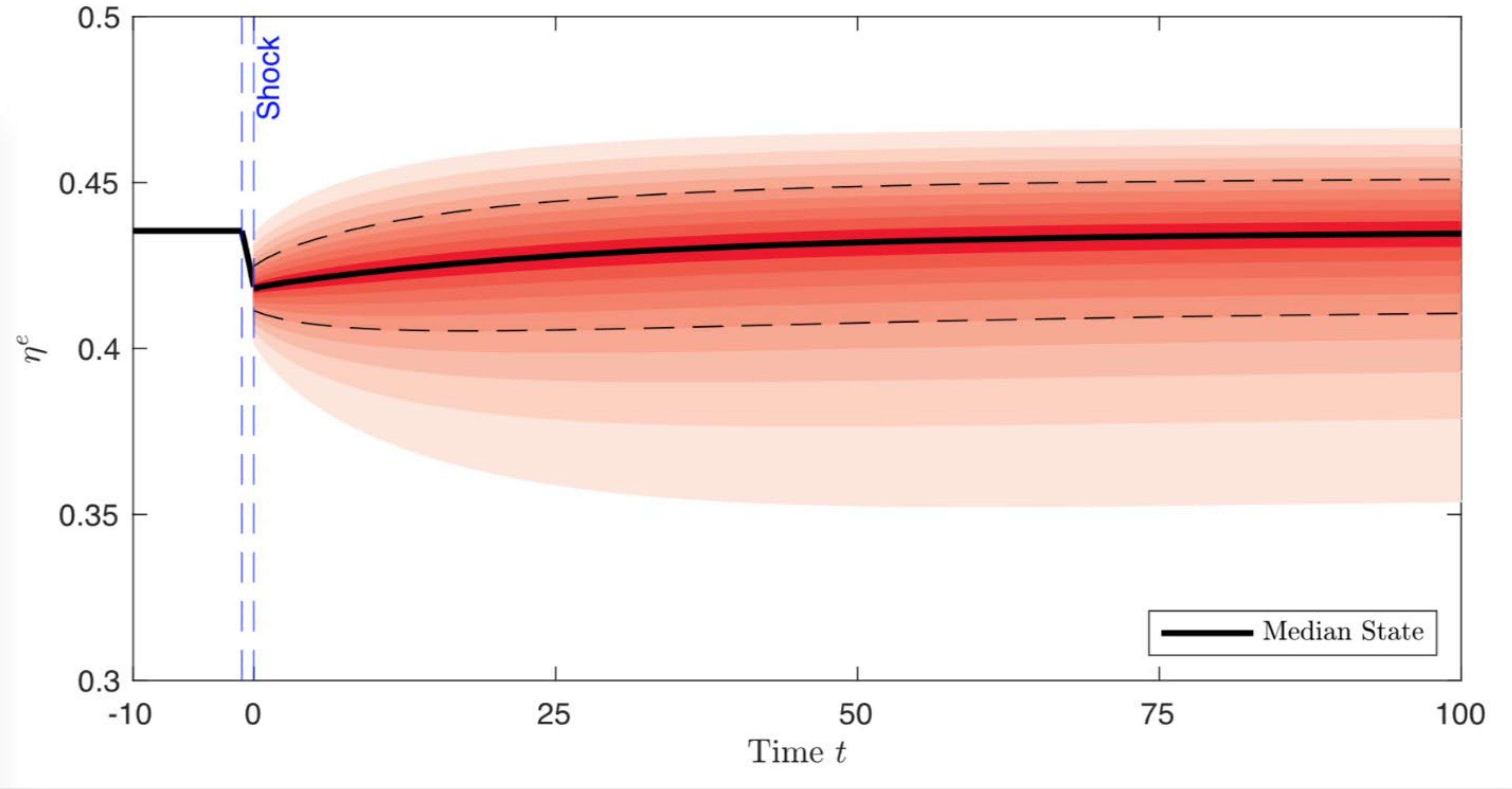
Poll 79: What happens for $\rho^e = \rho^h$

a) experts take over the economy, $\eta \to 1$

b) there is a steady state at $\eta = \alpha$

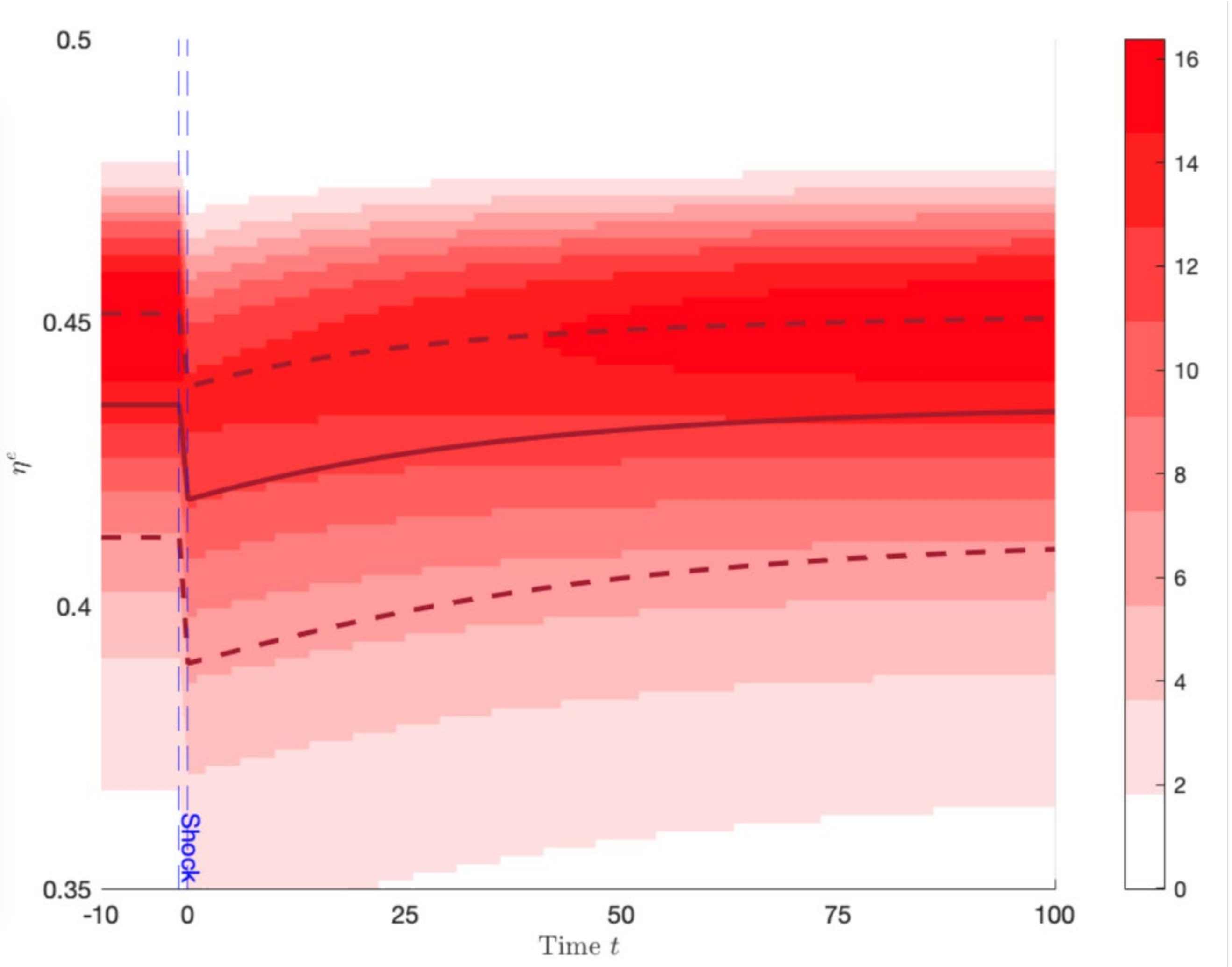
5. Fan chart and distributional impulse response

- … the theory to Bank of England’s empirical fan charts
- Starts at η_0, the median of stationary distribution
- Simulate a shock at 1% quantile of original Brownian shock ($dZ_t = -2.32\ dt$) for a period of $\Delta t = 1$.
- Converges back to stationary distribution



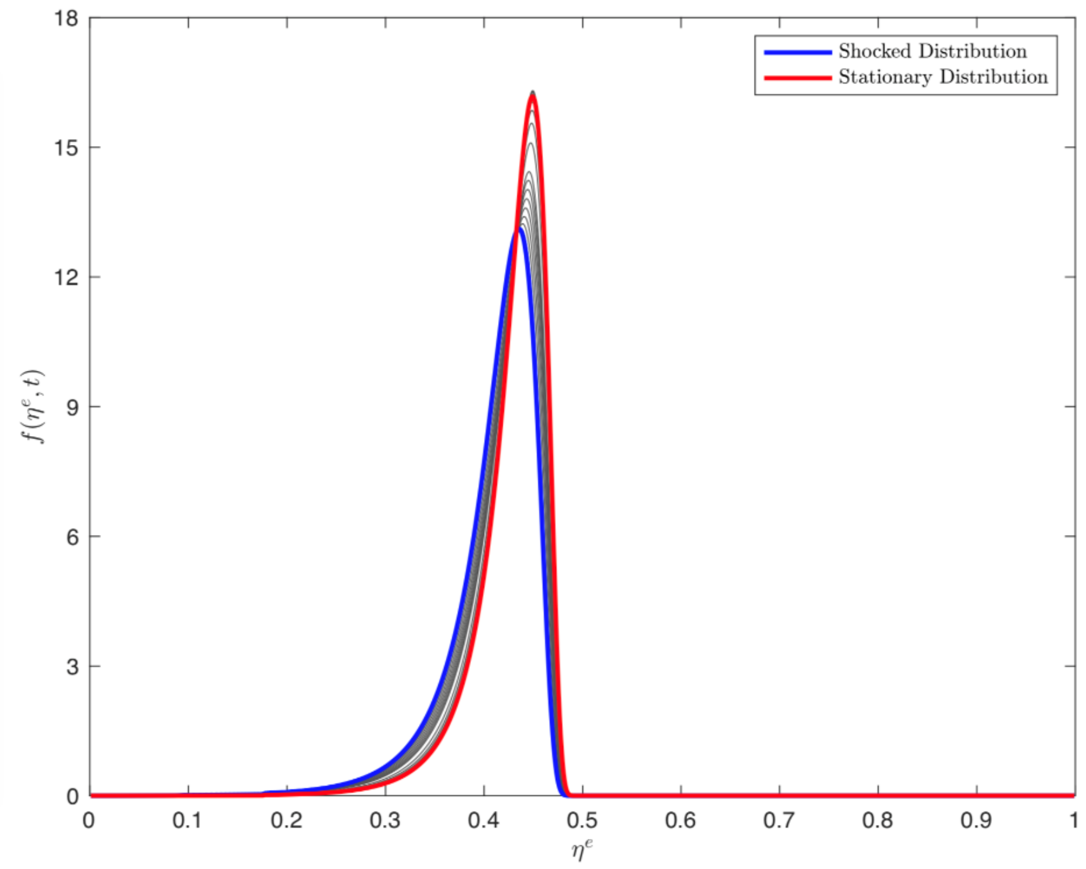
5. Fan chart and distributional impulse response

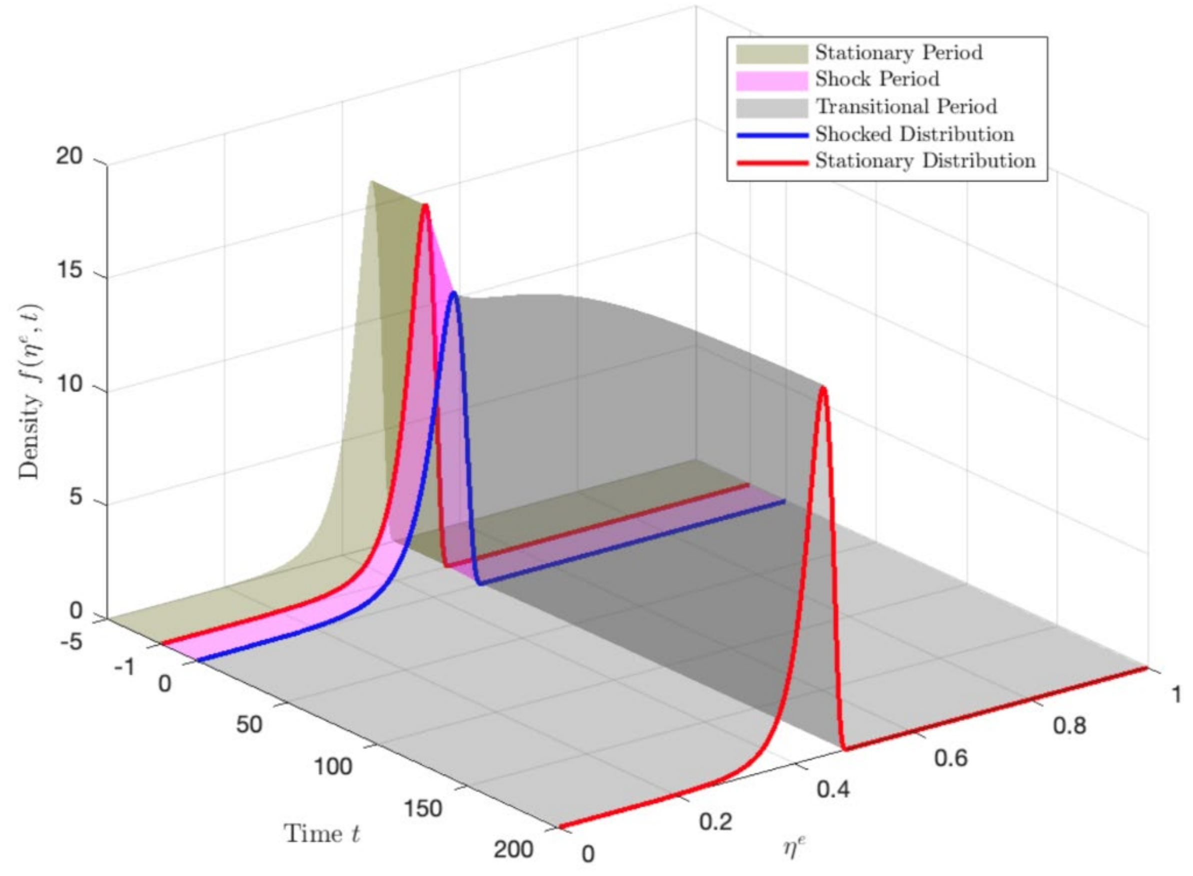
- Starts at stationary distribution
- Simulate a shock at 1% quantile of original Brownian shock ($dZ_t = -2.32\ dt$) for a period of $\Delta t = 1$.
- Converges back to stationary distribution



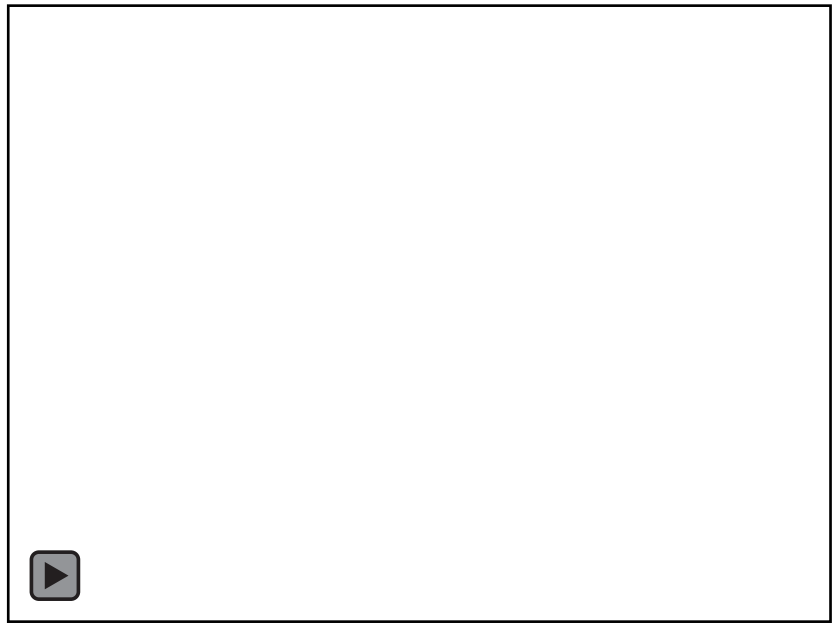
5. Density Diffusion

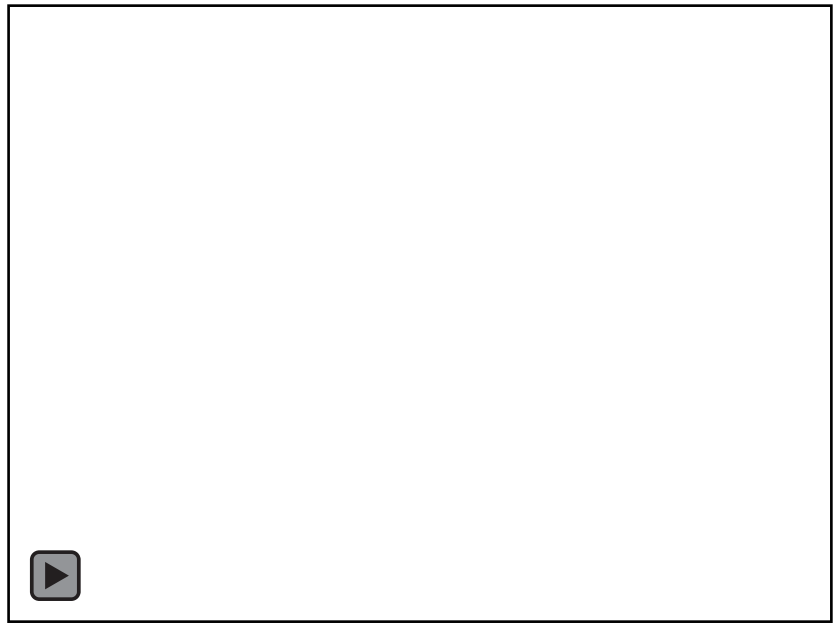
- Starts at stationary distribution
- Simulate a shock at 1% quantile of original Brownian shock ($dZ_t = -2.32\ dt$) for a period of $\Delta t = 1$.
- Converges back to stationary distribution

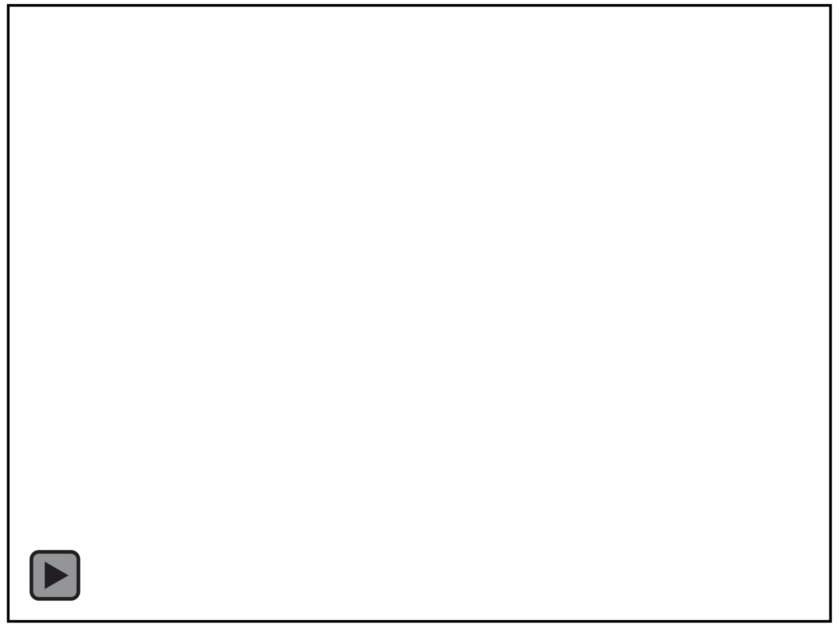


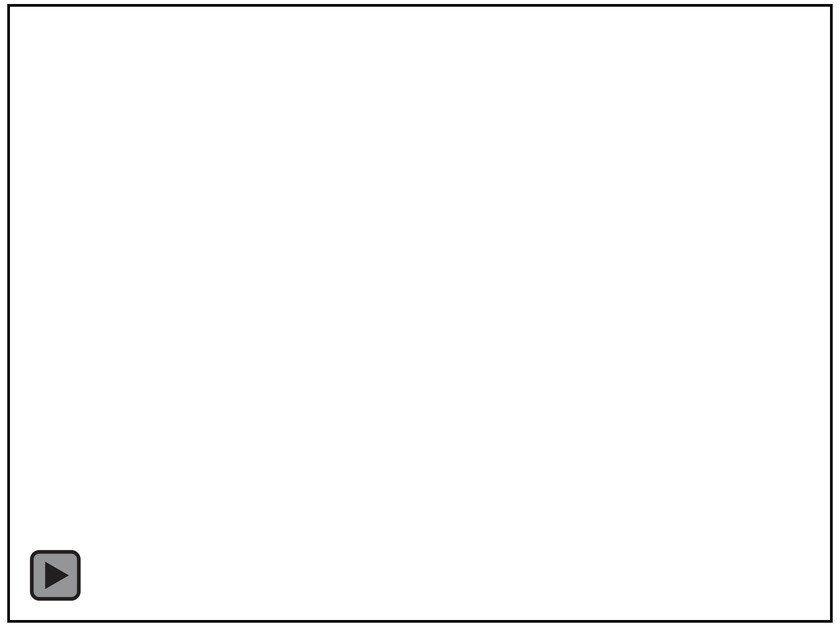


5.Density Diffusion Movies



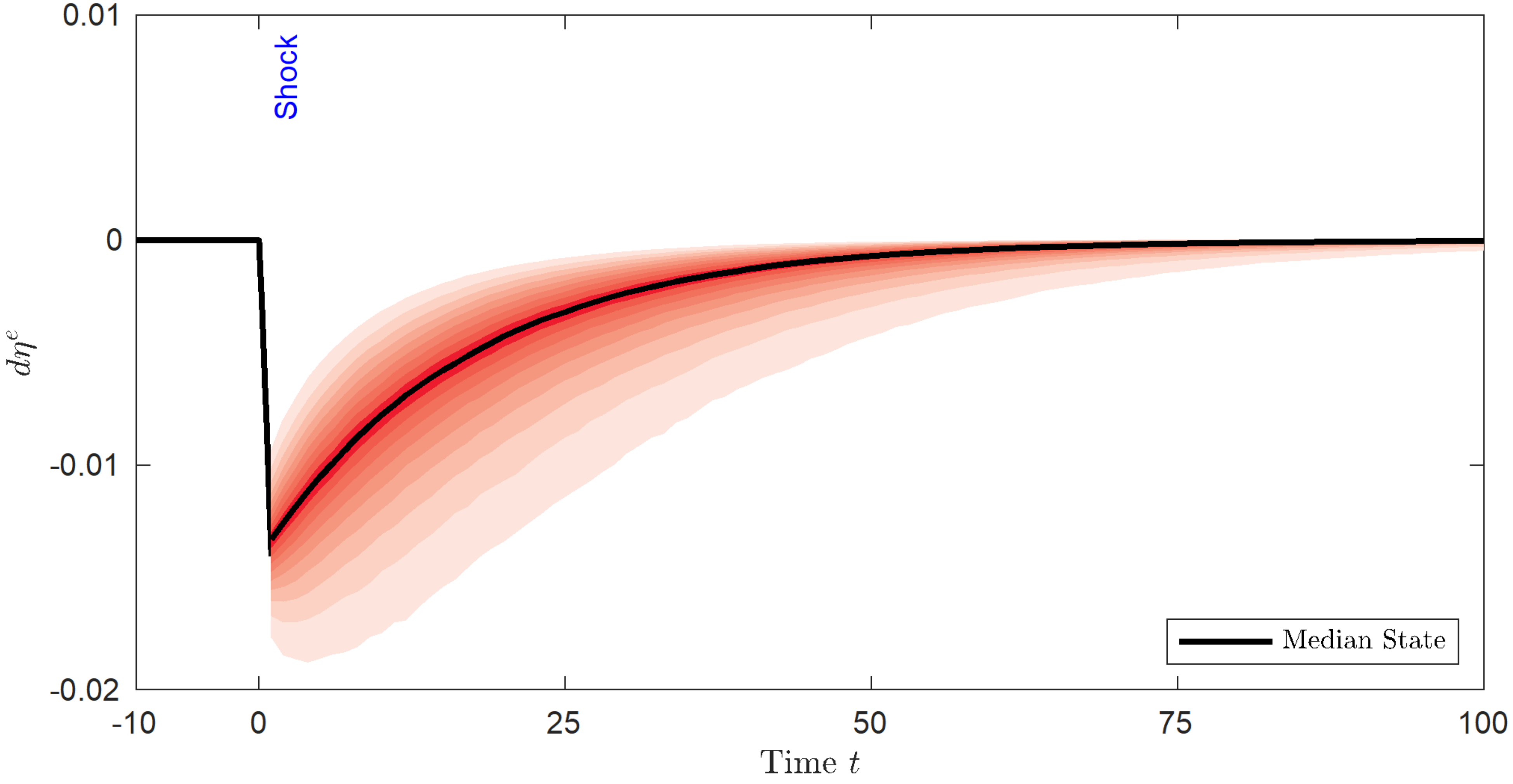






5. Distributional Impulse Response

- Difference between path with and without shock
- Difference converges to zero in the long-run



$\sigma = 0.15$