Modern Macro, Money, and International Finance

Eco529

Lecture 07: Kolmogorov Forward Equation

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Toolboxes

- Stationary distribution

- Impulse Response Fan charts

- Evolution of cross-sectional distribution

Solving MacroModels Step-by-Step

0. Postulate aggregates, price processes & obtain return processes
1. For given C/N-ratio and SDF processes for each i *finance block*
   a. Real investment ι + Goods market clearing *(static)*
   - *Toolbox 1:* Martingale Approach, HJB vs. Stochastic Maximum Principle Approach

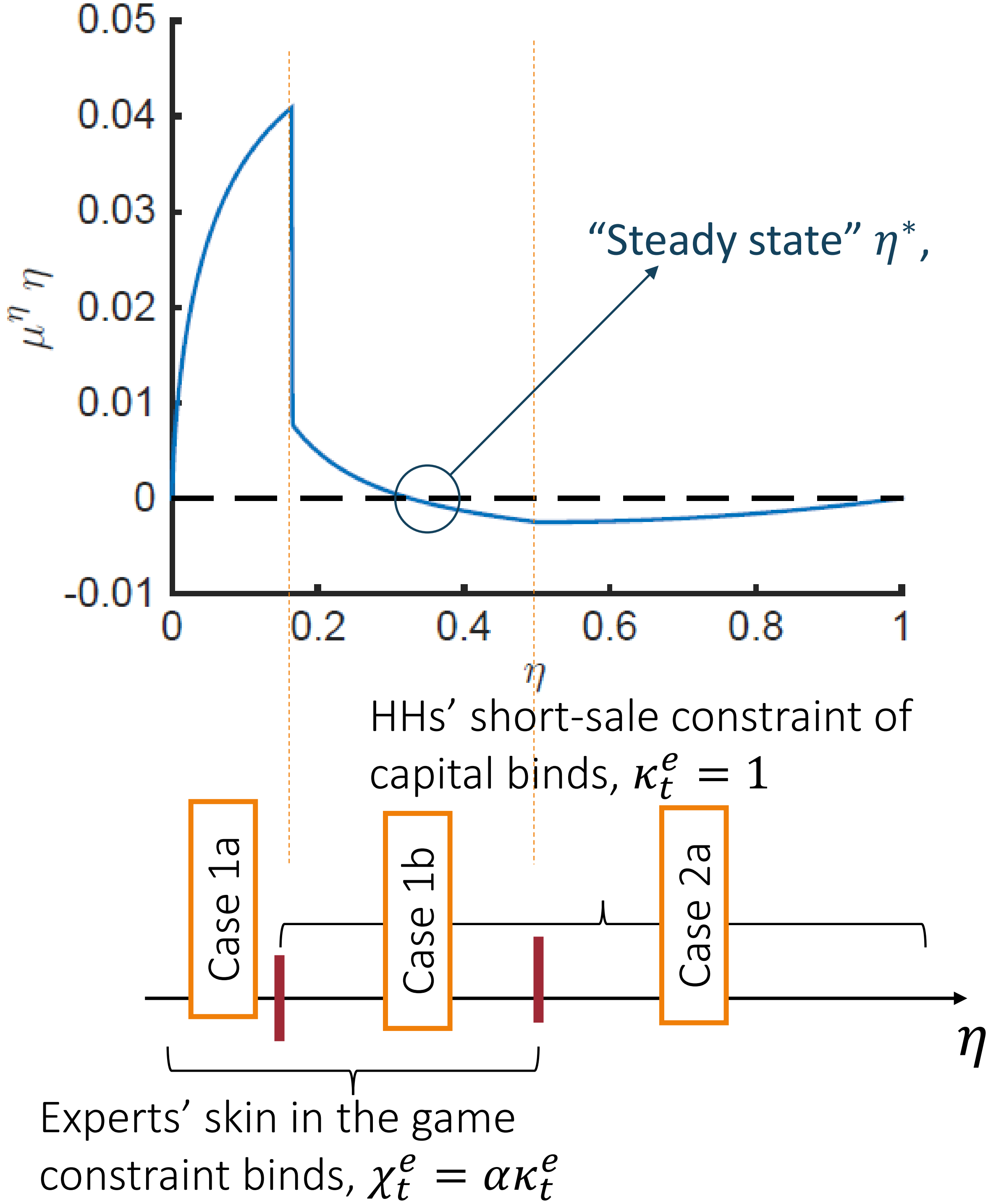
   b. Portfolio choice θ + Asset market clearing or
   Asset allocation κ & risk allocation χ
   - *Toolbox 2:* "price-taking social planner approach" – Fisher separation theorem

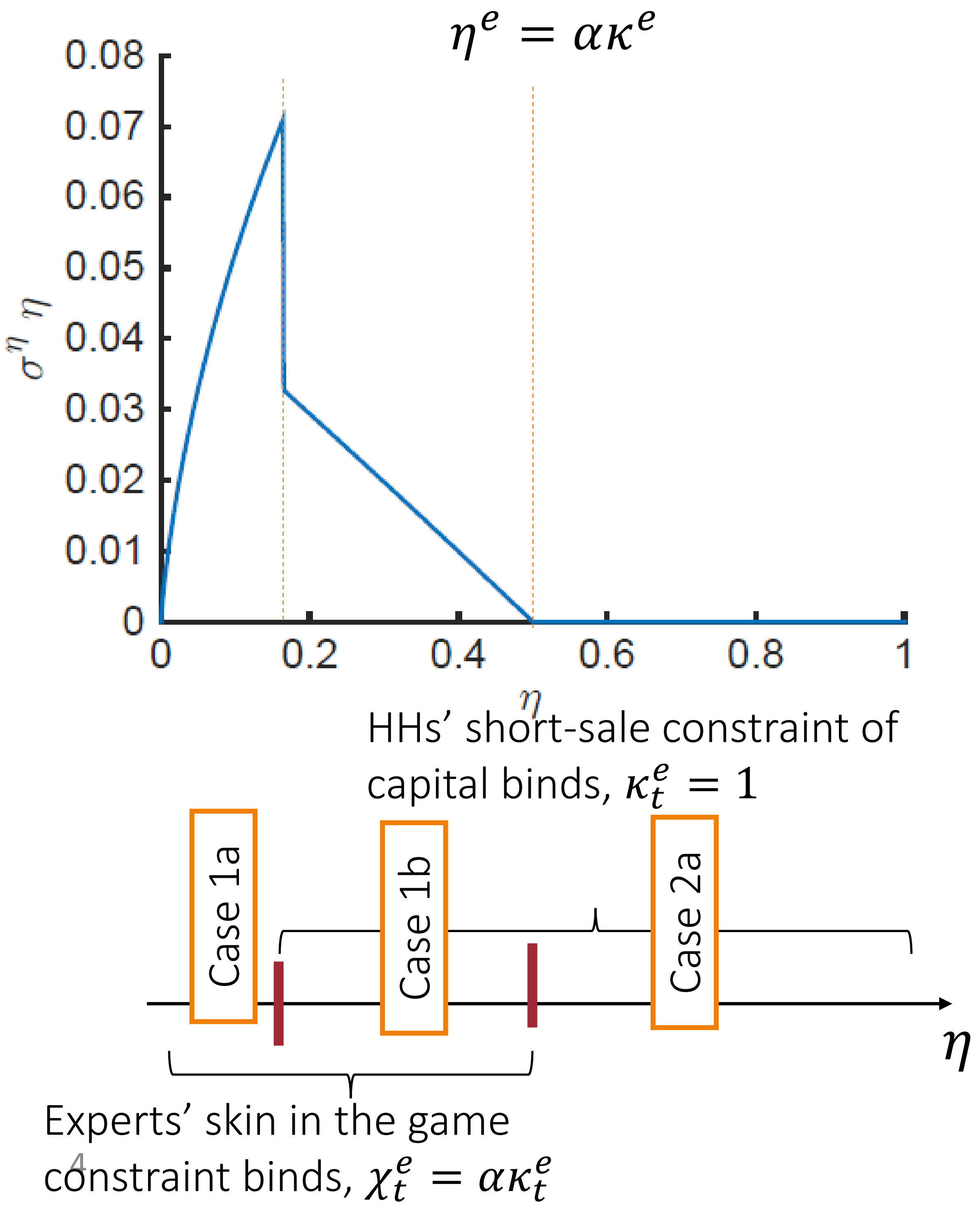
   c. "Money evaluation equation" ϑ
   - *Toolbox 3:* Change in numeraire to total wealth (including SDF)
2. Evolution of state variable η (and K) *forward equation*
3. Value functions *backward equation*
   a. Value fcn. as fcn. of individual investment opportunities ω
   - *Special cases: log-utility, constant investment opportunities*

   b. Separating value fcn. $V^i(n^{\tilde{i}};\eta,K)$ into $v^i(\eta)u(K)\left(n^{\tilde{i}}/n^i\right)^{1-\gamma}$
   c. Derive C/N-ratio and ς price of risk
4. Numerical model solution
   a. Transform BSDE for separated value fcn. $v^i(\eta)$ into PDE
   b. Solve PDE via value function iteration
5. KFE: Stationary distribution, Fan charts

5. From $\mu^{\eta^e}(\eta^e)$ & $\sigma^{\eta^e}(\eta^e)$ to Stationary Distribution

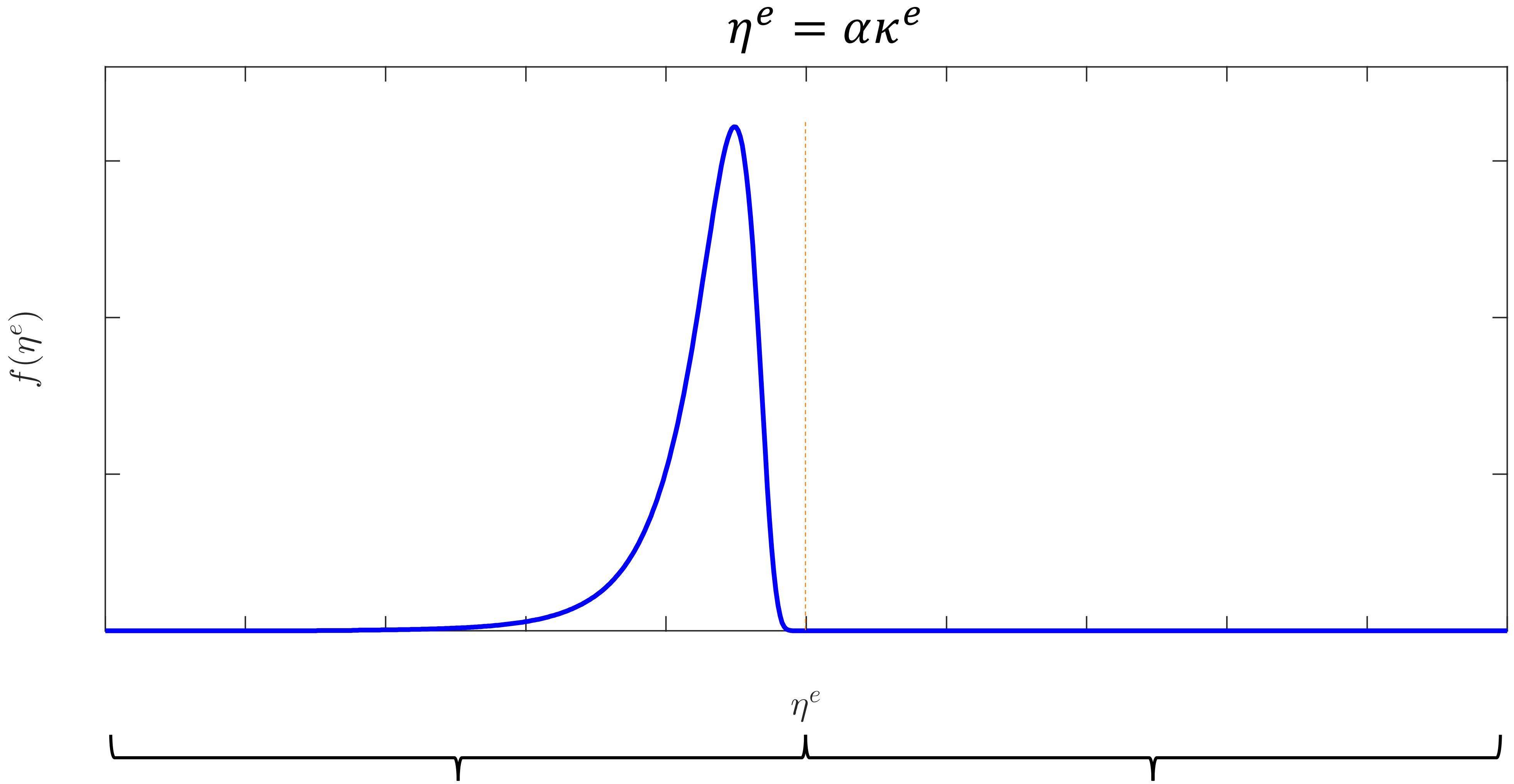
- Drift and Volatility of η^e





5. Preview: Stationary Distribution

- Stationary distribution of η^e



Experts' skin in the game constraint binds $\chi_t^e = \alpha\kappa_t^e$

Perfect risk-sharing region (infeasible)

5. Kolmogorov Forward Equation

- Given an initial distribution $f(\eta, 0) = f_0(\eta)$, the density diffusion follows parabolic PDE

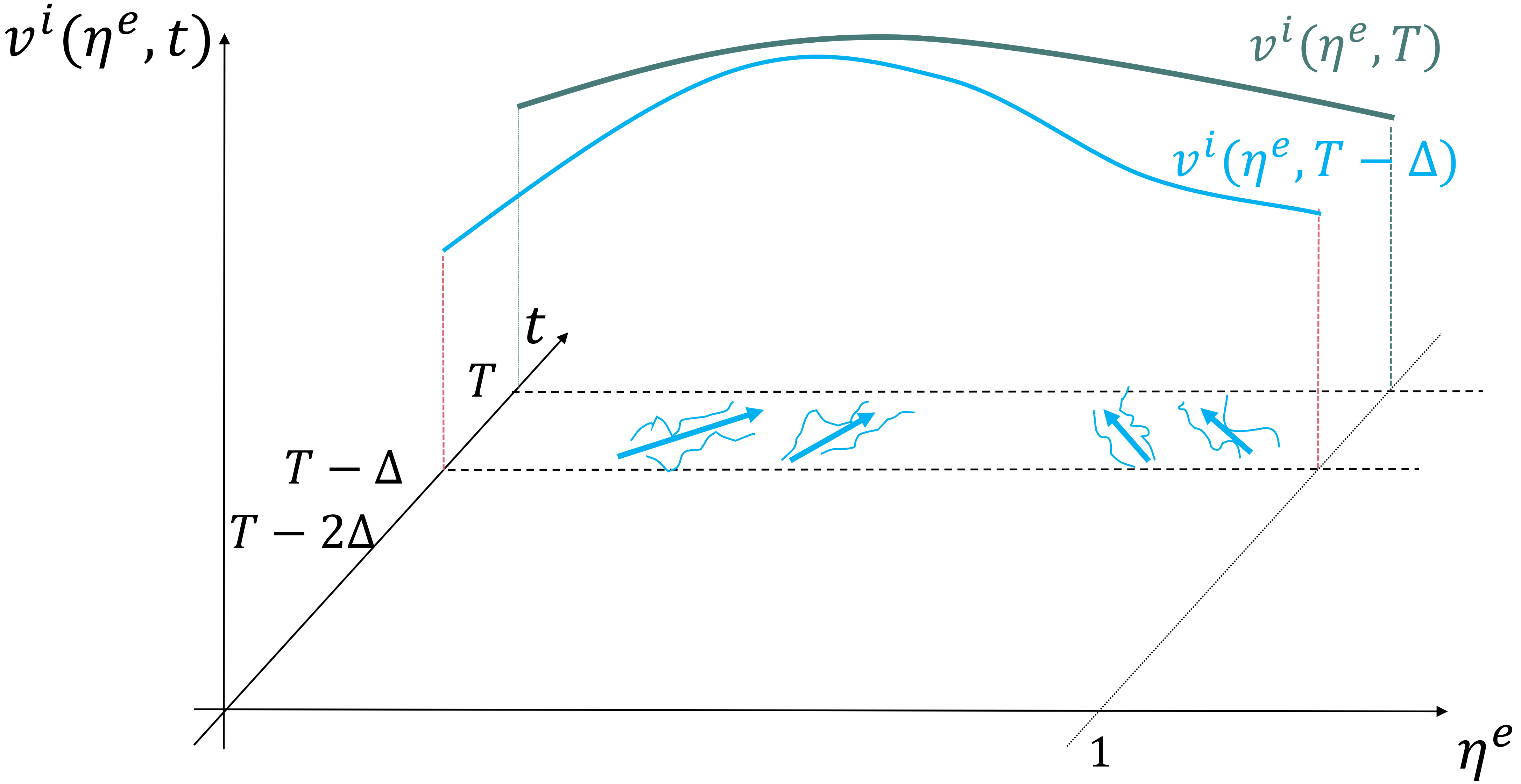
$$\frac{\partial f(\eta,t)}{\partial t} = -\frac{\partial[f(\eta,t)\mu(\eta)]}{\partial \eta} + \frac{1}{2}\frac{\partial^2[f(\eta,t)\sigma^2(\eta)]}{\partial \eta^2}$$

- "Kolmogorov Forward Equation" is in physics referred to as "Fokker-Planck Equation"

- Corollary: if stationary distribution $f(\eta)$ exists, it satisfies the ODE

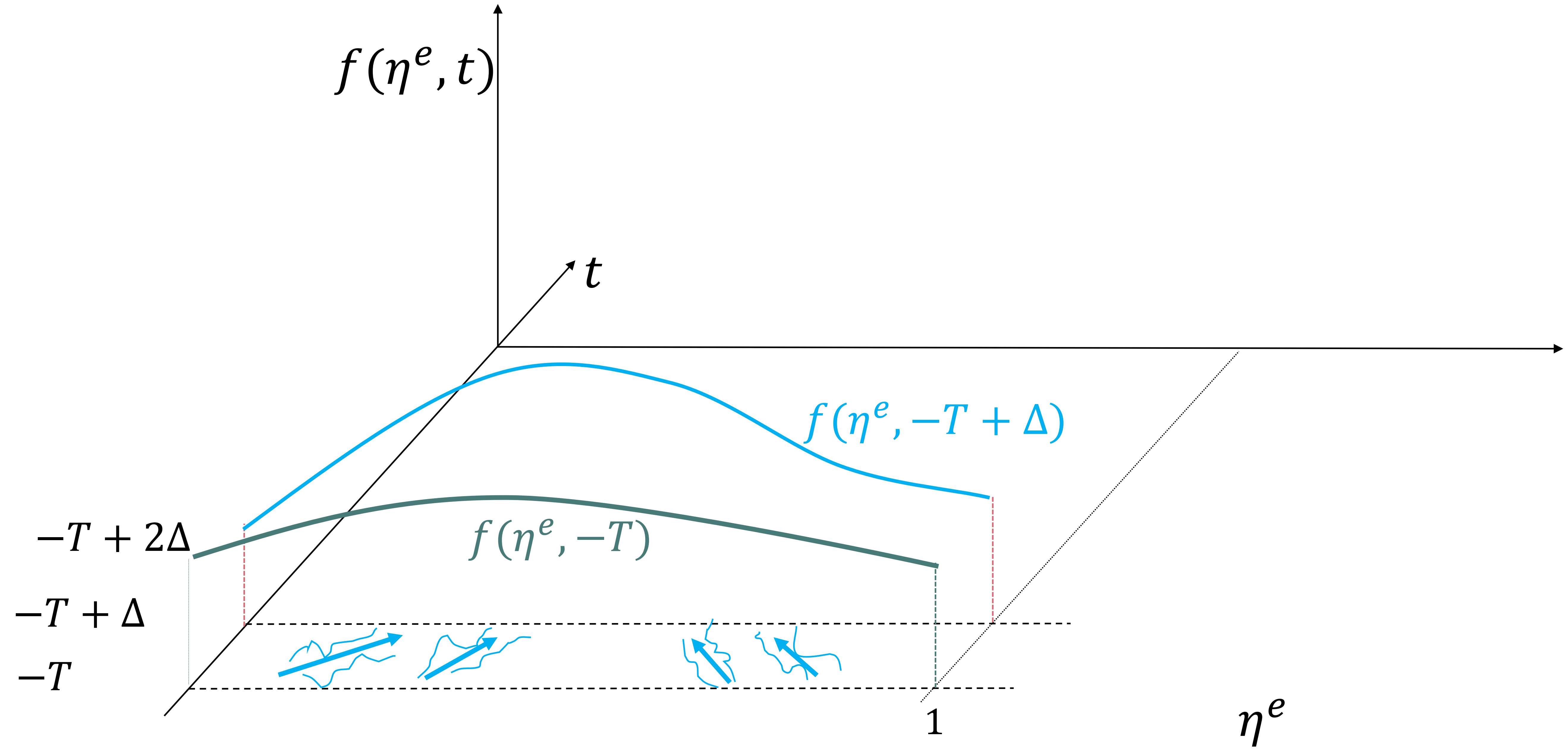
$$0 = -\frac{\partial[f(\eta,t)\mu(\eta)]}{\partial \eta} + \frac{1}{2}\frac{\partial^2[f(\eta,t)\sigma^2(\eta)]}{\partial \eta^2}$$

Recall: 4. Value Function Backwards Iteration



- Obtain descaled value function $v^i(\eta^e, T-\Delta)$
- Repeat previous steps

5. Forward Iteration … from past to the present



- Obtain descaled density function $f(\eta^e, -T+\Delta)$
- Repeat previous steps

5. Density Diffusion

- Given an initial distribution $f(\eta, 0) = f_0(\eta)$,
  the density diffusion follows parabolic PDE
  $$\frac{\partial f(\eta,t)}{\partial t} = -\frac{\partial[f(\eta,t)\mu(\eta)]}{\partial \eta} + \frac{1}{2}\frac{\partial^2[f(\eta,t)\sigma^2(\eta)]}{\partial \eta^2}$$

- Simpler than value function:
  - linear PDE, if $\mu(\eta)$ and $\sigma(\eta)$ are known functions that do not depend on $f(\cdot)$

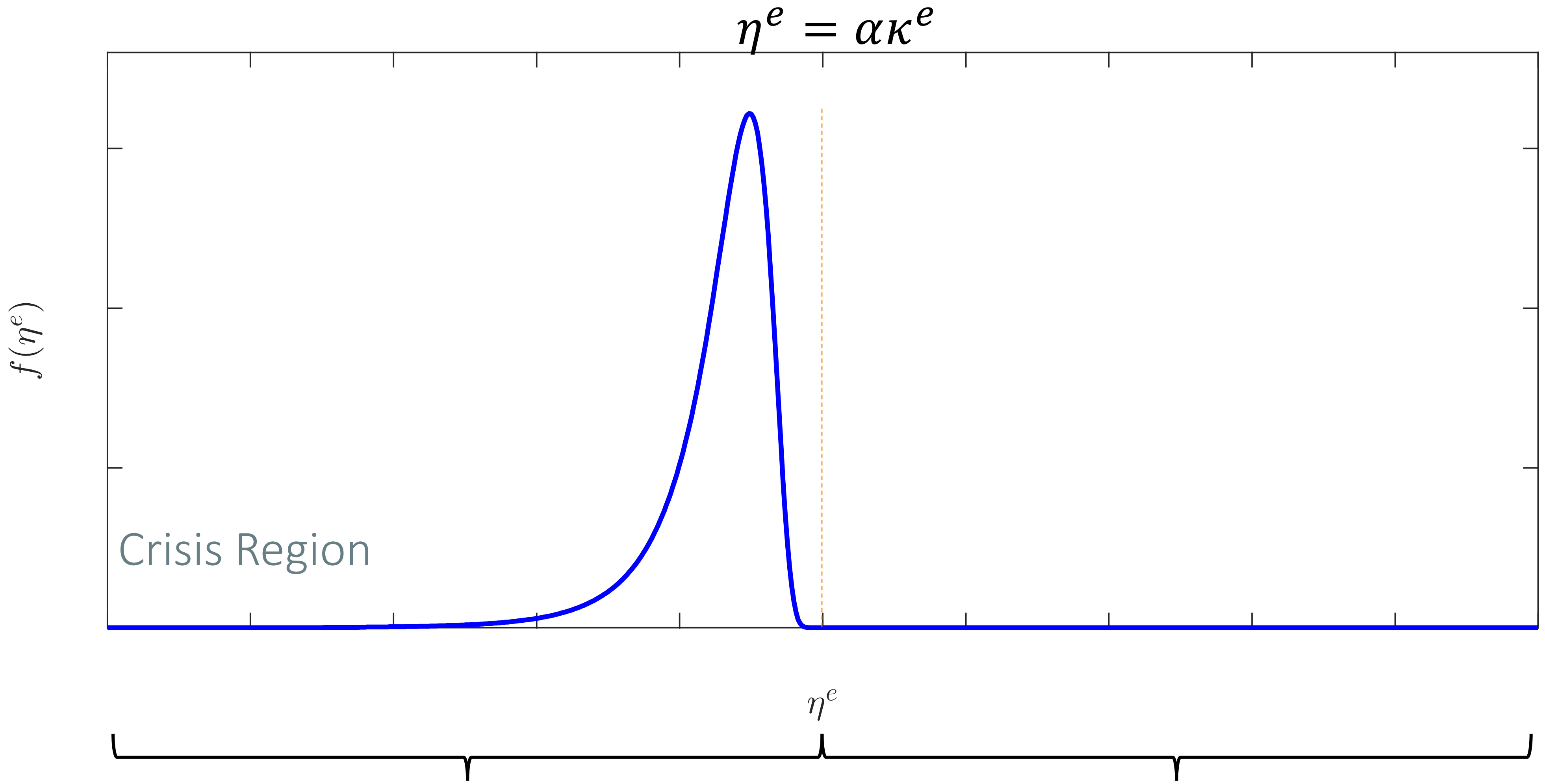
5. Stationary Distribution

- Iterate time-dependent KFE until convergence

- Simpler methods since linear
  $\mu(\eta)$ and $\sigma(\eta)$ are known functions that do not depend on $f(\cdot)$)
  - Discretize stationary KFE to obtain linear equation system
  - Complication: no unique solution (if f_∞ is a solution, so is αf_∞ for any $\alpha \in \mathbb{R}$)
    - Method 1:
      Determine the whole nullspace of the equation’s matrix and then find a vector in the nullspace that satisfies the normalization condition
    - Method 2:
      Add the normalization condition as a separate linear equation to the system.
      $\Rightarrow$ matrix is not longer square matrix
      make it a square matrix again
      - drop redundant equation (i.e. maintaining full rank of matrix)
        or regression

5. Stationary Distribution

- Stationary distribution of η^e



Experts' skin in the game constraint binds $\chi_t^e = \alpha\kappa_t^e$

Perfect risk-sharing region (infeasible)

- Crisis region is a tail event
  - Interest action happens there
  - Simulation method based on stationary distribution does not focus sufficiently on it

Poll 11: Is the constraint always (not just occasionally) binding

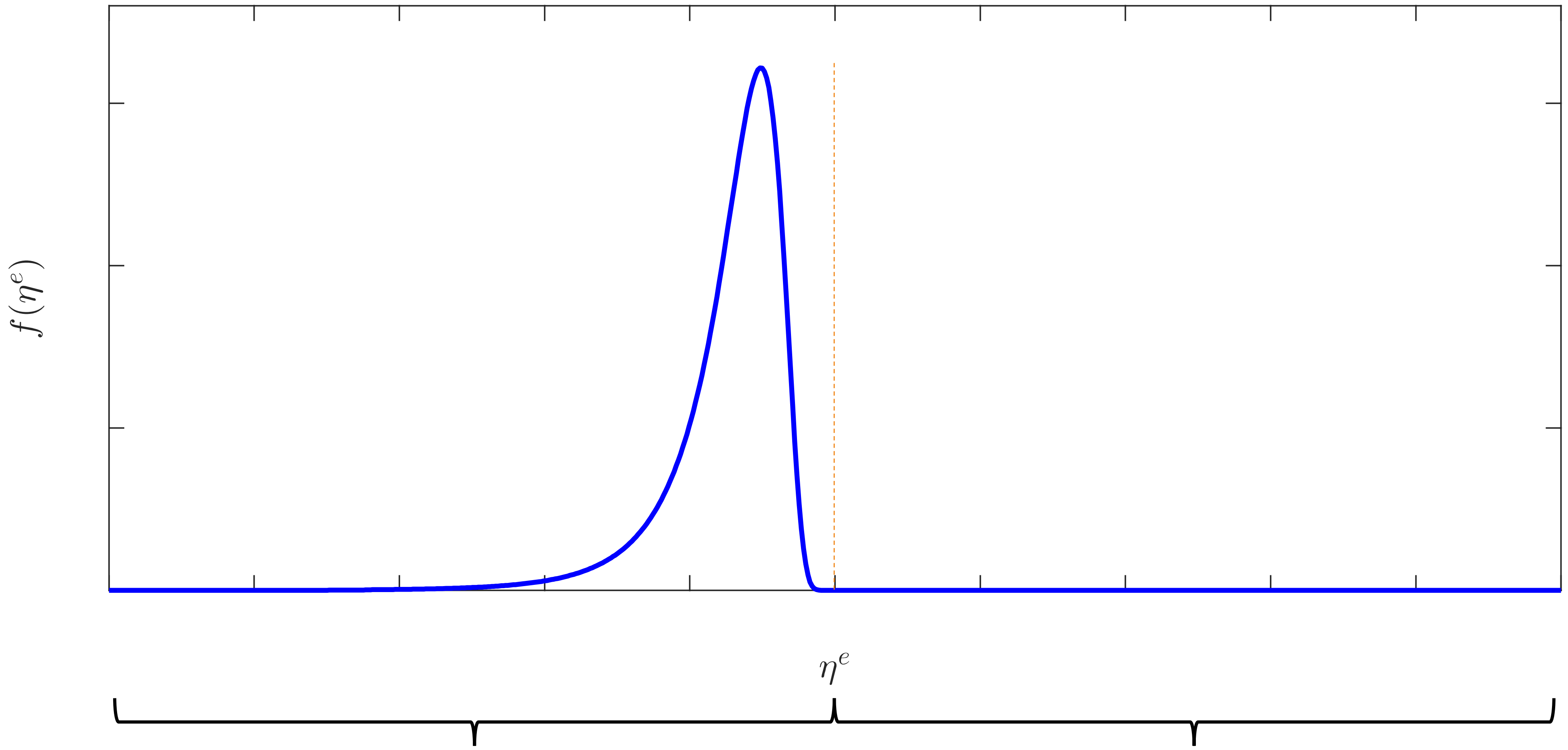
a) yes

b) no, only for some parameters $\rho^e > \rho^h$

5. Stationary Distribution

- Stationary distribution of η^e

$\eta^e = \alpha\kappa^e$



Experts' skin in the game constraint binds $\chi_t^e = \alpha\kappa_t^e$

Perfect risk-sharing region (infeasible)

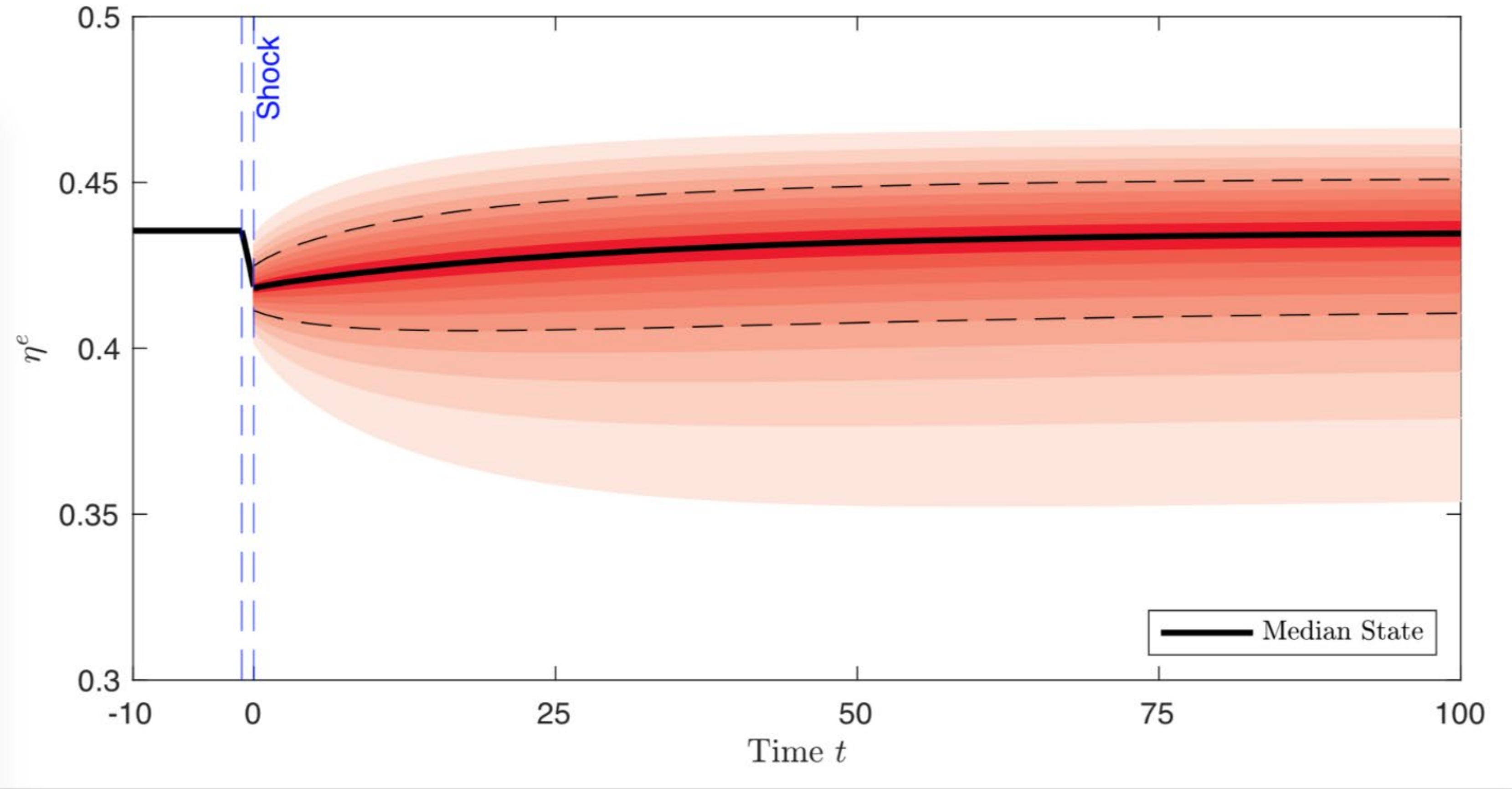
Poll 12: What happens for $\rho^e = \rho^h$

a) experts take over the economy, $\eta \to 1$

b) there is a steady state at $\eta = \alpha$

5. Fan chart and distributional impulse response

- … the theory to Bank of England's empirical fan charts
- Starts at η_0, the median of stationary distribution
- Simulate a shock at 1% quantile of original Brownian shock ($dZ_t = -2.32\ dt$) for a period of $\Delta t = 1$.
- Converges back to stationary distribution



5. Fan chart and distributional impulse response

- Starts at stationary distribution
- Simulate a shock at 1% quantile of original Brownian shock ($dZ_t = -2.32\ dt$) for a period of $\Delta t = 1$.
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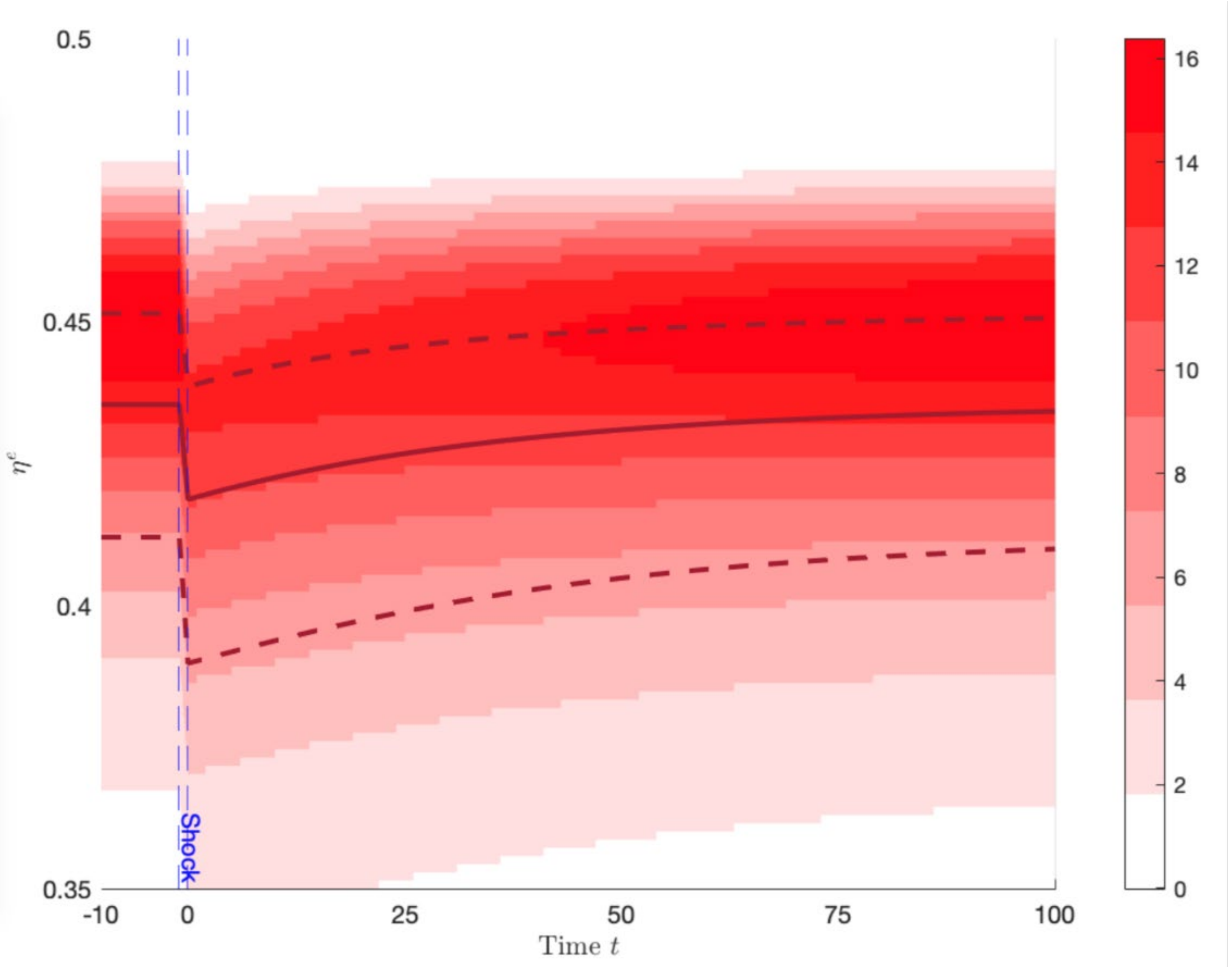
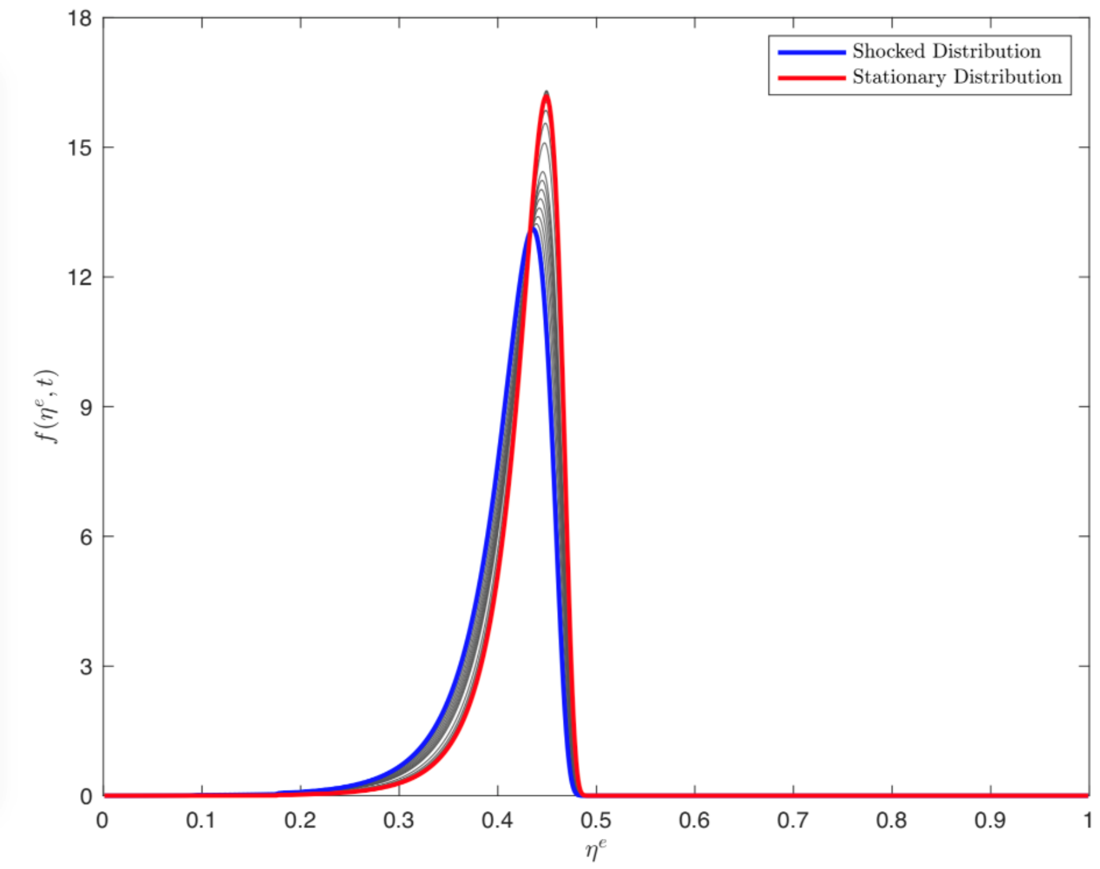
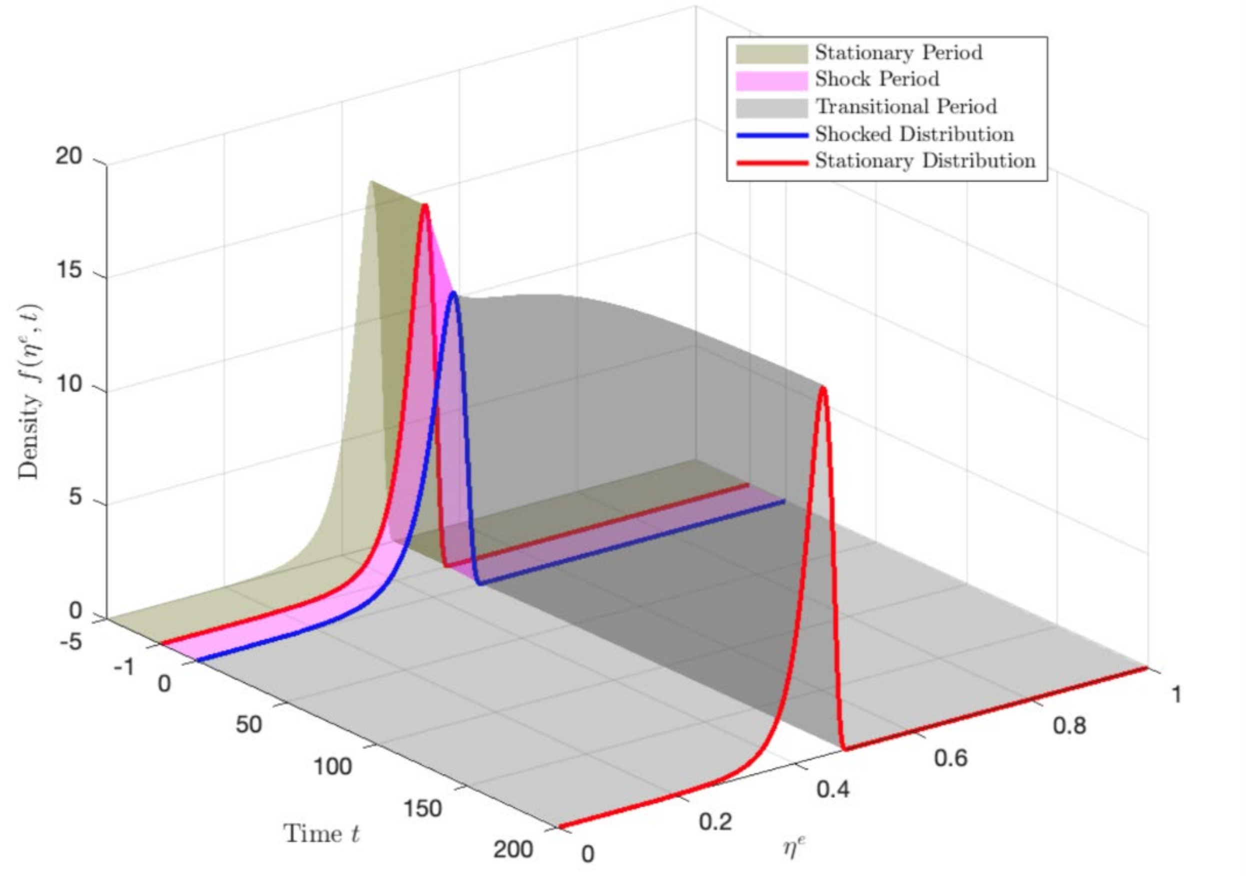


5. Density Diffusion

- Starts at stationary distribution
- Simulate a shock at 1% quantile of original Brownian shock ($dZ_t = -2.32\ dt$) for a period of $\Delta t = 1$.
- Converges back to stationary distribution

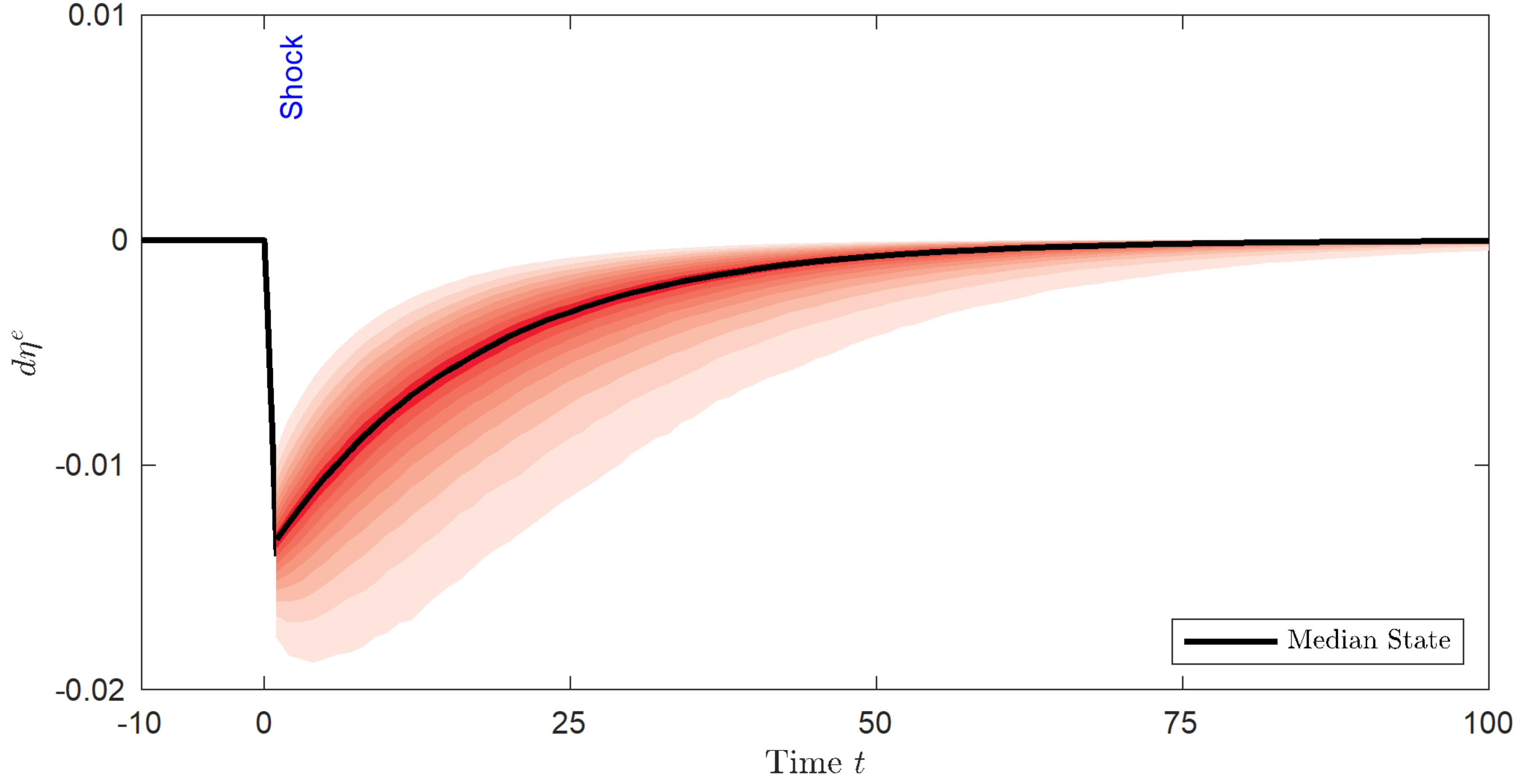




5.Density Diffusion Movies

5. Distributional Impulse Response

- Difference between path with and without shock
- Difference converges to zero in the long-run



$\sigma = 0.15$

The 3 Roles of KFE

- So far, KFE characterizes the
  1. Stationary probability distribution
  2. Density evolution of the system (distribution impulse ffan charts
     - Markov process maps probabilistic predictions
       for the initial state η_0 (i.e. density f_0)
       into probabilistic prediction for state η_t (i.e. density $f(\cdot, t)$)

- KFE as
  3. State equation (e.g. in Aiyagari-type models)
     describes the evolution of the cross-sectional distribution of net worth across a continuum of households (not the evolution of probability).
     - Mathematically identical (similar with jumps)

- In BruSan with 2 (finite) types: KFE takes on role 2. and 3.
  - With infinite types (like in Aiyagari/HANK models):
    infinite-dimensional object that summarizes cross-sectional wealth distribution = density evolution is governed by KFE

Toolboxes

- Stationary distribution
- Impulse Response Fan charts
- Evolution of cross-sectional distribution