Modern Macro, Money, and International Finance

Eco529

Lecture 03: Optimization
Consumption and Portfolio Choice

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Cts.-time Macro: Macro-Finance vs HANK

Agents:	Heterogenous investor focus - Net worth distribution (often discrete)	Heterogenous consumer focus - Net worth distribution (often cts.)
Tradition:	Finance (Merton) *PORTFOLIO AND CONSUMPTION CHOICE*	DSGE (Woodford) *CONSUMPTION CHOICE*
	Full/global dynamical system - focused on non-linearities away from steady state (crisis …) - Length of recession is stochastic	Transition dynamics back to steady state - Zero probability shock - Length of recession is deterministic
Money due to:	Risk & financial frictions	Price stickiness
Risk:	Risk & financial frictions	No aggregate risk (in HANK paper)
Price of risk:	Idiosyncratic & aggregate risk	
Assets:	Capital, money, bonds with different risk profile - Risk-return trade-off - Liquidity-return trade-off - Flight to safety	All assets are risk-free - No risk-return trade-off - Liquidity-return trade-off

Notation: Ito Calculus

- Arithmetic Ito Process $dX_t = \mu_{X,t} dt + \sigma_{X,t} dZ_t$
  - X in the subscript of μ and σ
  - $\mu_{X,t}$ and $\sigma_{X,t}$ time varying

- Geometric Ito Process $dX_t = \mu_t^X X_t dt + \sigma_t^X X_t dZ_t$
  - X in the subscript of μ and σ
  - Stock goes up 32% or down 32% over a year.
    256 trading days $\frac{32\%}{\sqrt{256}} = 2\%$

- Note: This is not a general convention.

Basics of Ito Calculus

- Ito's Lemma: (geometric notation)

$$df(X_t) = f'(X_t)\mu_t^X X_t dt + \frac{1}{2} f''(X_t)(\sigma_t^X X_t)^2 dt + f'(X_t)\sigma_t^X X_t dZ_t$$

  - $u(c) = \frac{c^{1-\gamma}-1}{1-\gamma}, u'(c) = c^{-\gamma}$ volatility of process $\frac{dc_t^{-\gamma}}{c_t^{-\gamma}}$ is $-\gamma\sigma_t^c$

- Ito product rule: stock price × exchange rate

$$\frac{d(X_t Y_t)}{X_t Y_t} = (\mu_t^X + \mu_t^Y + \sigma_t^X \sigma_t^Y)dt + (\sigma_t^X + \sigma_t^Y)dZ_t$$

- Ito ratio rule:

$$\frac{d(X_t/Y_t)}{X_t/Y_t} = [\mu_t^X - \mu_t^Y + \sigma_t^Y(\sigma_t^Y - \sigma_t^X)]dt + (\sigma_t^X - \sigma_t^Y)dZ_t$$

Single-agent Consumption-Portfolio Choice

- Choose consumption $\{c_t\}_{t=0}^{\infty}$ and portfolio weights to $\{\theta_t\}_{t=0}^{\infty}$ maximize

$$\mathbb{E}\left[\int_0^\infty e^{-\rho t} u(c_t) dt\right], \text{ with } u(c) = \frac{c^{1-\gamma}-1}{1-\gamma}$$

- Subject to
  - Net worth evolution

$$\forall t > 0: dn_t = -c_t dt + n_t[\theta_t r_t dt + (1-\theta_t) dr_t^a]$$

  - A solvency constraint $\forall t > 0$: $n_t \geq 0$
    - (alternatively, a “no Ponzi condition” leading to identical solution)
  - Beliefs about
    - r_t risk-free rate
    - dr_t^a risky asset return process with risk premium of δ_t^a

$$dr_t^a = (r_t + \delta_t^a) dt + \sigma_t^a dZ_t$$

    - Agent takes prices/returns as given

Stochastic Control Methods in Continuous Time

- Hamilton-Jacobi-Bellman (HJB) Equation
  - Continuous-time version of Bellman equation
  - Requires Markovian formulation with explicit definition of state space
  - Postulate value function $V(n,\eta)$ as a function of state variable process $d\eta_t/\eta_t$
- Stochastic Maximum Principle
  - Conditions that characterize path of optimal solution (as opposed to whole value function)
  - Closer to discrete-time Euler equations than Bellman equation
  - Does not require Markovian problem structure
  - Postulate co-state variable ξ_t^i
- Martingale Method
  - (very general) shortcut for portfolio choice problems
  - Yields interpretable equations (effectively linear factor pricing equations)
  - But: tailored to specific problem class (portfolio choice), non-trivial to apply elsewhere
  - Postulate SDF process $d\xi_t^i/\xi_t^i$…

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State Space

- Suppose returns are a function of state variable η_t:

$$r_t = r(\eta_t), \;\; \delta_t^a = \delta^a(\eta_t), \;\; \sigma_t^a = \sigma^a(\eta_t)$$

- η_t evolves according to a Markov diffusion process

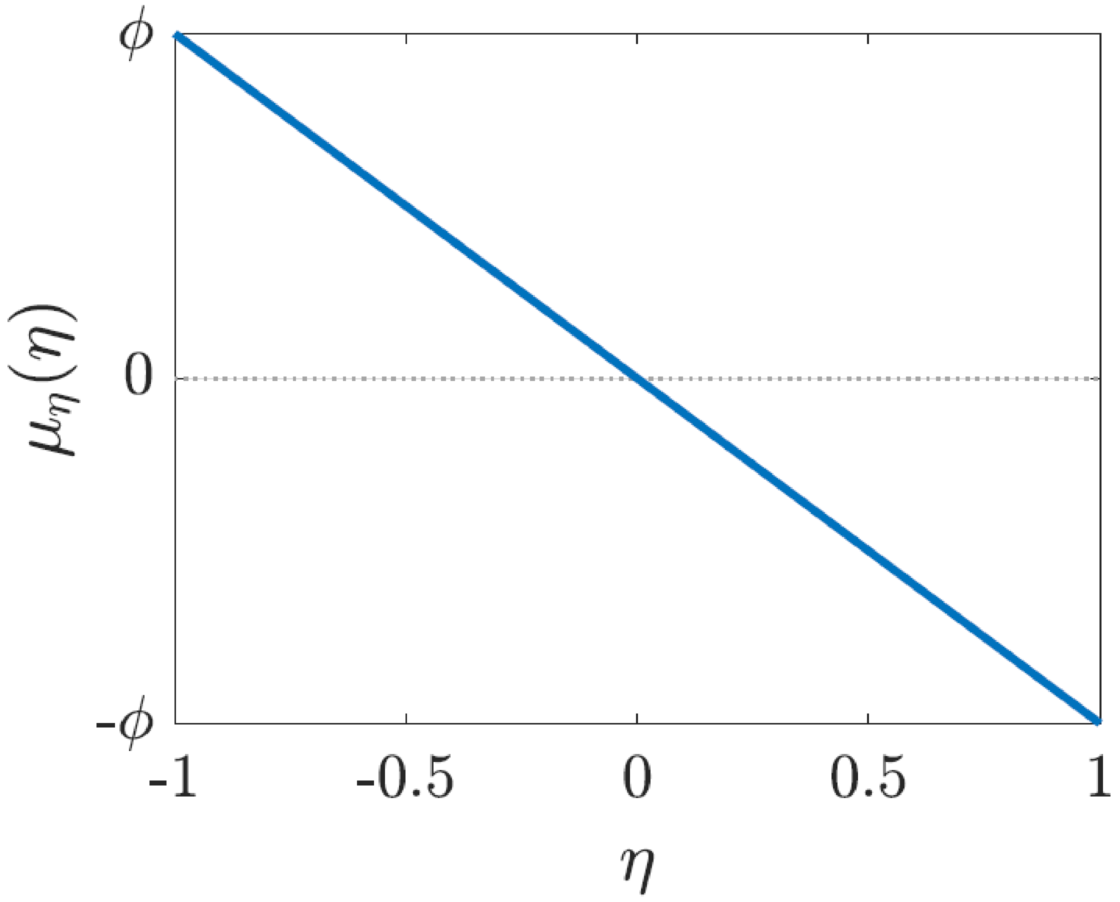
$$d\eta_t = \mu_t^\eta(\eta_t)\eta_t dt + \sigma_t^\eta(\eta_t)\eta_t dZ_t$$

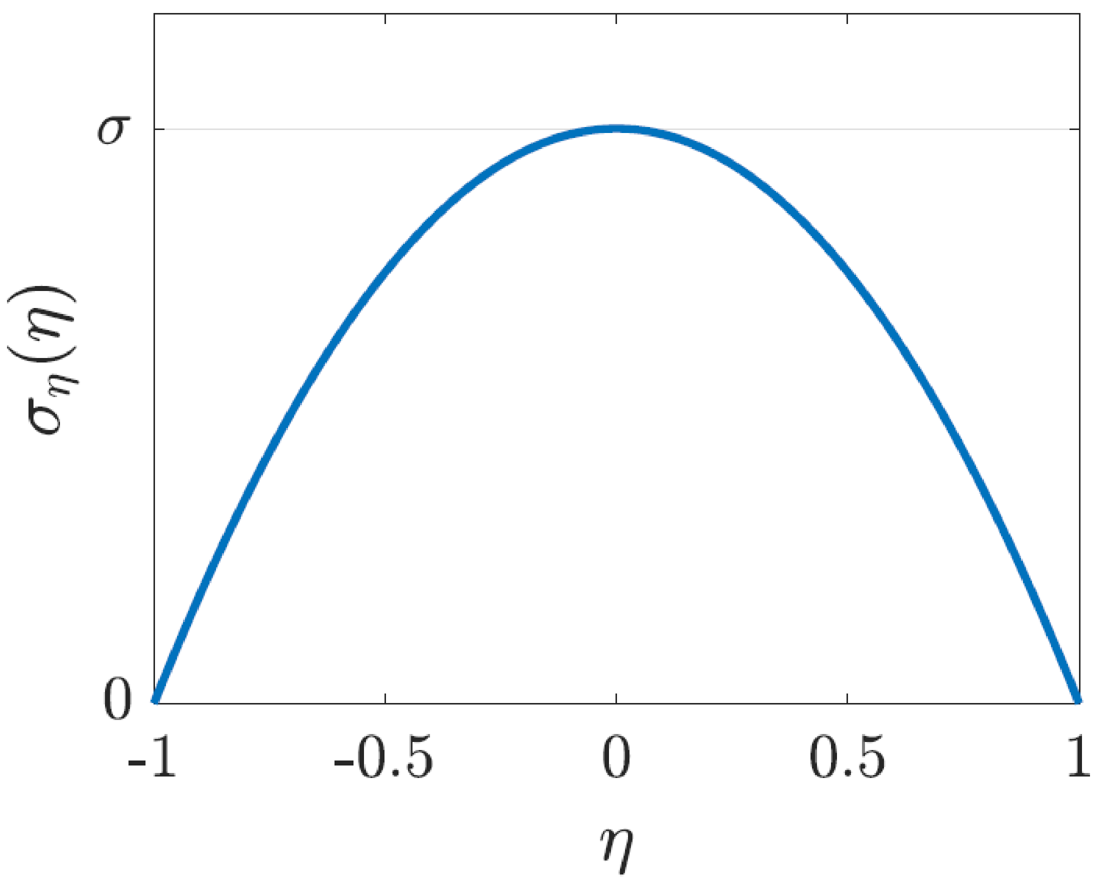
  - with given initial state η_0

- Then decision problem has two state variables:
  - n_t controlled state
  - η_t external state

  - For each initial state (n_0, η_0) we have a separate decision problem

Example: Functional Forms





- η-evolution (implies $\eta_t \in (-1,1)$)

$$\mu^\eta \eta = \mu_\eta = -\phi\eta, \qquad \sigma_\eta(\eta) = \sigma(1-\eta^2)$$

- Asset returns

$$r(\eta) = r^0 + r^1\eta,\ \delta^a(\eta) = \delta^0 - \delta^1\eta,\ \ \sigma^a(\eta) = \sigma^0 - \sigma^1\eta$$

- With parameters $r^0, r^1, \delta^0, \delta^1, \sigma^0, \sigma^1 \geq 0$

1. Hamilton-Jacobi-Bellman (HJB) Equation

- Stochastic Version of single-agent consumption-portfolio choice

- HJB Differential Equation

- Special Cases:
  - Constant Returns
  - Time-varying Returns

Value Function and Principle of Optimality

- Notation:
  - $\mathcal{A}(n,\eta)$: set of admissible choices $\{c_t,\theta_t\}_{t=0}^{\infty}$ given the initial conditions $n_0 = n$, $\eta_0 = \eta$
  - $\mathcal{A}_T(n,\eta)$: set of policies $\{c_t,\theta_t\}_{t=0}^{T}$ over $[0, T]$ that have admissible extensions to $[0,\infty)$, $\{c_t,\theta_t\}_{t=0}^{\infty} \in \mathcal{A}(n,\eta)$
- Define the value function of the decision problem

$$V(n,\eta) := \max_{\{c_t,\theta_t\}_{t=0}^{\infty}\in\mathcal{A}(n,\eta)} \mathbb{E}\left[\int_0^{\infty} e^{-\rho t}u(c_t)dt\right]$$

- It is easy to see that V satisfies the Bellman principle of optimality: for all $T > 0$

$$V(n,\eta) = \max_{\{c_t,\theta_t\}_{t=0}^{T}\in\mathcal{A}_T(n,\eta)} \mathbb{E}\left[\int_0^{T} e^{-\rho t}u(c_t)dt + e^{-\rho T}V(n_T,\eta_T)\right]$$

(where n_T depends on the choice $\{c_t,\theta_t\}_{t=0}^{T}$ over $[0, T]$).

A Stochastic Version of the HJB Equation: Derivation

- With $V_t := V(n_t, \eta_t)$, can write principle of optimality as

$$0 = \max_{\{c_t,\theta_t\}_{t=0}^T \in \mathcal{A}_T(n_0,\eta_0)} \mathbb{E}\left[\int_0^T e^{-\rho t} u(c_t)dt + e^{-\rho T} V_T - V_0\right]$$

- By the product rule

$$e^{-\rho T} V_T - V_0 = -\rho \int_0^T e^{-\rho t} V_t dt + \int_0^T e^{-\rho t} dV_t$$

- Combine with previous equation

$$0 = \max_{\{c_t,\theta_t\}_{t=0}^T \in \mathcal{A}_T(n_0,\eta_0)} \mathbb{E}\left[\int_0^T e^{-\rho t} \left(\left(u(c_t) - \rho V_t\right) dt + e^{-\rho t} dV_t\right)\right]$$

- Divide by T, take limit $T \searrow 0$:

(literally this yields the following equation only for $t = 0$, but we can shift time to arbitrary initial time due to Markovian nature of problem)

$$\boxed{\rho V_t dt = \max_{c_t,\theta_t} \left(u(c_t)dt + \mathbb{E}_t\left[dV_t\right]\right)}$$

A Stochastic Version of the HJB Equation: Interpretation

- Stochastic Version of HJB

$$\rho V_t dt = \max_{c_t,\theta_t} \{u(c_t)dt + \mathbb{E}[dV_t]\}$$

- This is an implicit backward stochastic differential equation (BSDE) for the value process V_t

- What does this mean?
  - Stochastic: equation for the stochastic process V_t not a deterministic function
  - Differential equation: relates time differential dV_t to process value V_t (& other variables)
  - Backward: forward-looking equation that must be solved backward in time, determines only expected time differential $\mathbb{E}[dV_t]$, volatility process is part of solution
  - Implicit: $\mathbb{E}[dV_t]$ is not explicitly solved for, instead part of non-linear expression on right-hand side (due to max operator)

Digression: Alternative Derivation: Time Approximation

- Usual way of writing discrete time Bellman equation ($\beta := e^{-\rho}$)

$$V(n_t, \eta_t) = \max_{c_t, \theta_t} \left(u(c_t) + \beta \mathbb{E}_t \left[V\left(n_{t+1}, \eta_{t+1} \right) \right] \right)$$

  More generally with generic period lentgh $\Delta t > 0$ ($\beta(\Delta t) := e^{-\rho \Delta t}$):

$$V(n_t, \eta_t) = \max_{c_t, \theta_t} \left(u(c_t) \Delta t + \beta(\Delta t) \mathbb{E}_t \left[V\left(n_{t+\Delta t}, \eta_{t+\Delta t} \right) \right] \right)$$

  Subtract $\beta(\Delta t) V(n_t, \eta_t)$ from both sides

$$\frac{1 - \beta(\Delta t)}{\Delta t} V(n_t, \eta_t) \Delta t = \max_{c_t, \theta_t} \left(u(c_t) \Delta t + \beta(\Delta t) \mathbb{E}_t \left[V\left(n_{t+\Delta t}, \eta_{t+\Delta t} \right) - V(n_t, \eta_t) \right] \right)$$

  Taking the limit $\Delta t \to 0$ yields again

$$\rho V(n_t, \eta_t) dt = \max_{c_t, \theta_t} \left(u(c_t) dt + \mathbb{E}_t \left[d\left(V\left(n_t, \eta_t \right) \right) \right] \right)$$

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- Special Cases:
  - Constant Returns
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The (Deterministic) HJB Equation

- Next step: transform stochastic version of HJB
  into a (non-stochastic) differential equation
- General idea: use Ito to express $\mathbb{E}[dV_t]$ in terms of derivatives of value function V_t

Poll: The (Deterministic) HJB Equation

- which of the following is the correct? [recall the definition $V_t = V(n_t, \eta_t)$]

(a) $$\mathbb{E}_t\left[dV_t\right] = \left(\partial_n V(n_t,\eta_t)\mu_{n,t} + \partial_\eta V(n_t,\eta_t)\mu_{\eta,t}\right) dt$$

(b) $$\mathbb{E}_t\left[dV_t\right] = \Bigg(\partial_n V(n_t,\eta_t)\mu_{n,t} + \partial_\eta V(n_t,\eta_t)\mu_{\eta,t} + \frac{1}{2}\left(\partial_{nn} V(n_t,\eta_t)\sigma_{n,t}^2 + \partial_{\eta\eta} V(n_t,\eta_t)\sigma_{\eta,t}^2\right)\Bigg) dt$$

(c) $$\mathbb{E}_t\left[dV_t\right] = \Bigg(\partial_n V(n_t,\eta_t)\mu_{n,t} + \partial_\eta V(n_t,\eta_t)\mu_{\eta,t} + \frac{1}{2}\left(\partial_{nn} V(n_t,\eta_t)\sigma_{n,t}^2 + \partial_{\eta\eta} V(n_t,\eta_t)\sigma_{\eta,t}^2 + \partial_{\eta n} V(n_t,\eta_t)\sigma_{\eta,t}\sigma_{n,t}\right)\Bigg) dt$$

(d) $$\mathbb{E}_t\left[dV_t\right] = \Bigg(\partial_n V(n_t,\eta_t)\mu_{n,t} + \partial_\eta V(n_t,\eta_t)\mu_{\eta,t} + \frac{1}{2}\left(\partial_{nn} V(n_t,\eta_t)\sigma_{n,t}^2 + \partial_{\eta\eta} V(n_t,\eta_t)\sigma_{\eta,t}^2\right) + \partial_{\eta n} V(n_t,\eta_t)\sigma_{\eta,t}\sigma_{n,t}\Bigg) dt$$

The (Deterministic) HJB Equation

- Next step: transform stochastic version of HJB into a (non-stochastic) differential equation
- General idea: use Ito to express $\mathbb{E}_t\left[dV_t\right]$ in terms of derivatives of value function V

  Here, $V_t = V(n_t, \eta_t)$, so we can write

$$\rho V_t dt = \max_{c_t,\theta_t}\Bigg(u(c_t) + \partial_n V(n_t,\eta_t)\mu_{n,t} + \partial_\eta V(n_t,\eta_t)\mu_{\eta,t} + \frac{1}{2}\left(\partial_{nn} V(n_t,\eta_t)\sigma_{n,t}^2 + \partial_{\eta\eta} V(n_t,\eta_t)\sigma_{\eta,t}^2\right) + \partial_{\eta n} V(n_t,\eta_t)\sigma_{\eta,t}\sigma_{n,t}\Bigg) dt$$

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- Next step: transform stochastic version of HJB into a (non-stochastic) differential equation
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  Here, $V_t = V(n_t, \eta_t)$, so we can write

$$\rho V_t dt = \max_{c_t,\theta_t}\Big(u(c_t) + \partial_n V(n_t,\eta_t)\mu_{n,t} + \partial_\eta V(n_t,\eta_t)\mu_{\eta,t} + \frac{1}{2}\left(\partial_{nn} V(n_t,\eta_t)\sigma_{n,t}^2 + \partial_{\eta\eta} V(n_t,\eta_t)\sigma_{\eta,t}^2\right) + \partial_{\eta n} V(n_t,\eta_t)\sigma_{\eta,t}\sigma_{n,t}\Big) dt$$

  For this problem, drifts and volatilities are

$$\mu_{n,t} = -c_t + n_t\left(r\left(\eta_t\right) + (1-\theta_t)\delta^a\left(\eta_t\right)\right) \qquad \mu_{\eta,t} = \mu_\eta\left(\eta_t\right)$$

$$\sigma_{n,t} = n_t(1-\theta_t)\sigma^a\left(\eta_t\right) \qquad \sigma_{\eta,t} = \sigma_\eta\left(\eta_t\right)$$

The (Deterministic) HJB Equation

- Combining the previous equation and dropping dt and time subscripts yields

$$
\begin{aligned}
\rho V(n,\eta) = \max_{c} \left(u(c) - \partial_n V(n,\eta) c\right) \\
+ \max_{\theta} \Bigg(\partial_n V(n,\eta) n \left(r(\eta) + (1-\theta)\delta^a(\eta)\right) \\
+ \left(\frac{1}{2}\partial_{nn} V(n,\eta) n (1-\theta)\sigma^a(\eta) + \partial_{\eta n} V(n,\eta)\sigma_\eta(\eta)\right) n(1-\theta)\sigma^a(\eta) \Bigg) \\
+ \partial_\eta V(n,\eta)\mu_\eta(\eta) + \frac{1}{2}\partial_{\eta\eta} V(n,\eta)\left(\sigma_\eta(\eta)\right)^2
\end{aligned}
$$

This is a nonlinear partial differential equation (PDE) for $V(n,\eta)$

Note: nonlinearity enters through the max operators

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Special Case: Constant Returns

Let's first assume that returns are constant: $r_t = r$, $\delta_t^a = \delta^a$, $\sigma_t^a = \sigma^a$

Can then drop η state from the problem and write the HJB as

$$\rho V(n) = \max_c \left(u(c) - V'(n)c\right) + \max_\theta \left(V'(n)n\left(r + (1-\theta)\delta^a\right) + \frac{1}{2}V''(n)n^2\left((1-\theta)\sigma^a\right)^2 \right)$$

To solve this equation, first solve the maximization problems:

- optimal consumption choice: marginal utility of consumption = marginal value of wealth

$$u'(c) = V'(n)$$

- optimal portfolio choice: Merton portfolio weight

$$1 - \theta = \left(-\frac{V''(n)n}{V'(n)}\right)^{-1} \frac{\delta^a}{(\sigma^a)^2}$$

Remarks:

- this has a flavor of mean-variance portfolio choice: $-\frac{V''(n)n}{V'(n)}$ is the relative risk aversion coefficient of V (locally at n), δ^a is the excess return and $(\sigma^a)^2$ is the risky asset's variance

Solving HJB for Constant Return Case

We could now plug optimal choices into HJB and solve the resulting ODE numerically

Instead for this problem: guess functional form and solve analytically

Guess: $V(n) = \frac{u(\omega n)}{\rho}$ with some constant $\omega > 0$

Plugging into HJB equation:

- $\gamma = 1$ (log utility):

$$\log \omega + \cancel{\log n} = \log \rho + \cancel{\log n} - 1 + \frac{1}{\rho}\left(r + \frac{1}{2\gamma}\left(\frac{\delta^a}{\sigma^a}\right)^2\right)$$

- $\gamma \neq 1$:

$$\rho \cancel{\frac{(\omega n)^{1-\gamma}}{\rho}} = \gamma \rho^{1/\gamma} \omega^{1-1/\gamma} \cancel{\frac{(\omega n)^{1-\gamma}}{\rho}} + (1-\gamma)\left(r + \frac{1}{2\gamma}\left(\frac{\delta^a}{\sigma^a}\right)^2\right) \cancel{\frac{(\omega n)^{1-\gamma}}{\rho}}$$

In both cases, n cancels out, thus verifying our guess (we can then solve for ω)

Full Solution for Constant Return Case

Value function:

$$V(n) = \frac{u(\omega n)}{\rho}$$

Optimal choices:

$$c(n) = \rho^{1/\gamma}\omega^{1-1/\gamma}n$$

$$1-\theta(n) = \frac{1}{\gamma}\frac{\delta^a}{(\sigma^a)^2}$$

Value function constant ω (for $\gamma \neq 1$):

$$\omega = \rho\left(1+\frac{\gamma-1}{\gamma}\frac{1}{\rho}\left(r-\rho+\frac{1}{2\gamma}\left(\frac{\delta^a}{\sigma^a}\right)^2\right)\right)^{\frac{\gamma}{\gamma-1}}$$

Discussion of Optimal Consumption Choice

$$c_t / n_t = \rho^{1/\gamma} \omega^{1-1/\gamma}$$

- Reaction of c/n to investment opportunities ω depends on EIS $\psi := 1/\gamma$:

$\psi < 1$ better investment opportunities $\Rightarrow$ consumption $\uparrow$, savings $\downarrow$

$\psi > 1$ better investment opportunities $\Rightarrow$ consumption $\downarrow$, savings $\uparrow$

$\psi = 1$ consumption-wealth ratio independent of investment opportunities

- Why this ambiguous relationship? Two effects:
  1. income effect:
     - improved investment opportunities ω make investor effectively richer
     - investor responds by increasing consumption in all periods
  2. substitution effect:
     - improved investment opportunities ω make savings more attractive
     - to benefit from them, investor reduces consumption now to get more consumption later

$\psi < 1$ substitution effect weak (consumption smoothing desire), income effect dominates

$\psi > 1$ investor less averse against temporal variation in comsumption, substitution effect dominates

General Case: Time-varying Returns

- Recall the HJB equation in the general case:

$$
\begin{aligned}
\rho V(n,\eta) = \max_c \left(u(c) - \partial_n V(n,\eta) c\right) \\
+ \max_\theta \Bigg(\partial_n V(n,\eta) n \left(r(\eta) + (1-\theta)\delta^a(\eta)\right) \\
+ \left(\frac{1}{2}\partial_{nn} V(n,\eta) n (1-\theta)\sigma^a(\eta) + \partial_{\eta n} V(n,\eta)\sigma_\eta(\eta)\right) n(1-\theta)\sigma^a(\eta)\Bigg) \\
+ \partial_\eta V(n,\eta)\mu_\eta(\eta) + \frac{1}{2}\partial_{\eta\eta} V(n,\eta)\left(\sigma_\eta(\eta)\right)^2
\end{aligned}
$$

Solution method 1: solve this two-dimensional PDE for V numerically

Solution method 2: guess $V(n,\eta) = \frac{u(\omega(\eta)n)}{\rho}$ and reduce to one-dimensional ODE for $\omega(\eta)$

Net worth multiplier/investment opportunity

Time-varying Returns: Optimal Consumption and Portfolio

Optimal consumption choice (after using guess from previous slide):

$$c(n,\eta) = \rho^{1/\gamma} \left(\omega(\eta)\right)^{1-1/\gamma} n$$

→ as for constant returns, but now investment opportunities $\omega(\eta)$ are state-dependent

Optimal portfolio choice (after using guess from previous slide):

$$1 - \theta\left(n,\eta\right) = \underbrace{\frac{1}{\gamma} \frac{\delta^a\left(\eta\right)}{\left(\sigma^a\left(\eta\right)\right)^2}}_{\text{myopic demand}} + \underbrace{\frac{1-\gamma}{\gamma} \frac{\frac{\omega'(\eta)}{\omega(\eta)} \sigma_\eta\left(\eta\right) \sigma^a\left(\eta\right)}{\left(\sigma^a\left(\eta\right)\right)^2}}_{\text{hedging demand}}$$

→ additional hedging demand term that depends on covariance $\sigma^\omega \sigma^a$ of investment opportunities with asset return

Time-varying Returns: Hedging Demand

$$1-\theta(n,\eta)=\underbrace{\frac{1}{\gamma}\frac{\delta^a(\eta)}{(\sigma^a(\eta))^2}}_{\text{myopic demand}}+\underbrace{\frac{1-\gamma}{\gamma}\frac{\frac{\omega'(\eta)}{\omega(\eta)}\sigma_\eta(\eta)\,\sigma^a(\eta)}{(\sigma^a(\eta))^2}}_{\text{hedging demand}}$$

- Why should variation in future investment opportunities be relevant for portfolio choice? Two opposing motives:
  1. if investment opportunities are good, it is valuable to have many resources available
     $\rightarrow$ invest in assets that pay off in states in which investment opportunities are good
  2. if investment opportunities are bad, that's a bad time for the investor and additional wealth is valuable
     $\rightarrow$ invest in assets that pay off in states in which investment opportunities are bad
- Which of the two dominates depends on γ:

$\gamma < 1$ investor not very risk averse, prefers to have resources available when it is profitable to invest

$\gamma > 1$ investor sufficiently risk averse to want to hedge against bad times

$\gamma = 1$ the two forces cancel out, investor acts myopically

- Remark: a very conservative investor ($\gamma \rightarrow \infty$) only cares about the hedging component

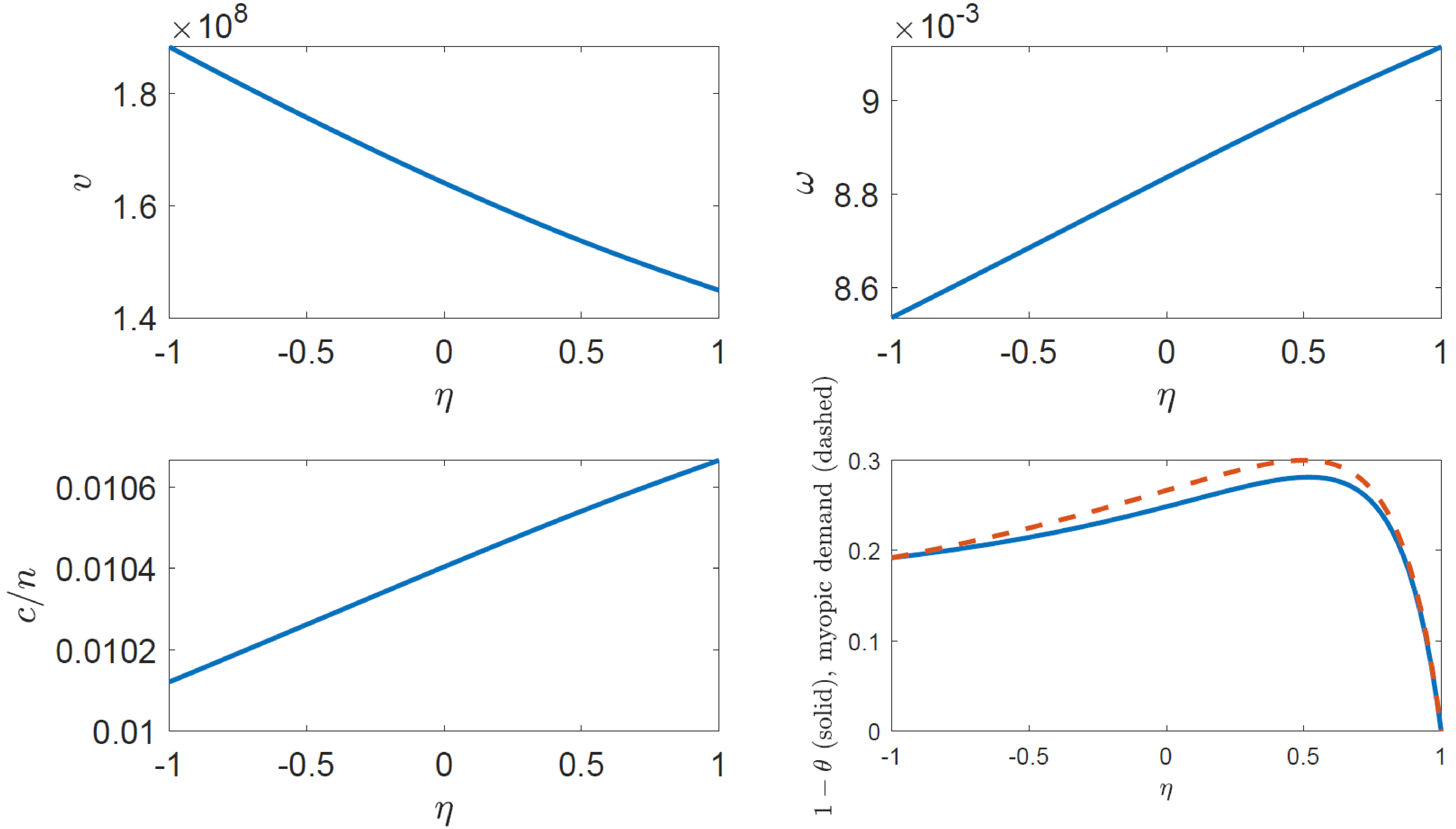
Determining Investment Opportunities

- When substituting optimal choices into HJB, n cancels out and we get ODE for $\omega(\eta)$

- One can solve this numerically for the function $\omega(\eta)$

- Details will be provided in Lecture 06 (later)
  - (E.g., solve equivalently for $v(\eta) \coloneqq \left(\omega(\eta)\right)^{1-\gamma}$ which is a “more linear” (less kinky) ODE.)

Example Solution



parameters: $\rho = 0.02, \gamma = 5, \phi = 0.2, \sigma = 0.1, r^0 = 0.02, r^1 = 0.01, \delta^0 = 0.3, \delta^1 = 0.03, \sigma^0 = 0.15, \sigma^1 = 0.1$

Stochastic Control Methods in Continuous Time

- Hamilton-Jacobi-Bellman (HJB) Equation
  - Continuous-time version of Bellman equation
  - Requires Markovian formulation with explicit definition of state space
  - Postulate value function $V(n, \eta)$ as a function of state variable process $d\eta_t/\eta_t$
- Stochastic Maximum Principle
  - conditions that characterize path of optimal solution
    (as opposed to whole value function)
  - closer to discrete-time Euler equations than Bellman equation
  - does not require Markovian problem structure
  - Postulate co-state variable ξ_t^i
- Martingale Method
  - (very general) shortcut for portfolio choice problems
  - yields interpretable equations (effectively linear factor pricing equations)
  - But: tailored to specific problem class (portfolio choice), non-trivial to apply elsewhere
  - Postulate SDF process $d\xi_t^i/\xi_t^i$…

Method 2: Stochastic maximum principle

- Consider a control problem
$$dX_t = \mu(X_t, A_t)dt + \sigma(X_t, A_t)dZ_t,$$
  - where A_t are the control and X_t are states.
- and finite-horizon problems with object function
$$E_0\left[\int_0^T g(t, X_t, A_t)dt + G(X_T)\right]$$
  - where $g(t, X_t, A_t)$ is payoff flow.
- Instead of solving such an optimization problem directly, one can work with p_t, the dynamic Lagrange multiplier on X_t
  - label p_t and its volatility q_t as *costates* of the system
  - then optimize the Hamiltonian
$$H_t = g(t, X_t, A_t) + \langle p_t, \mu(X_t, A_t)\rangle + tr[q_t^T \sigma(X_t, A_t)].$$
- The stochastic maximum principle: under necessary convexity condition, p_t must satisfy the BSDE
$$dp_t = -H_X(t, X_t, A_t, p_t, q_t)dt + q_t dZ_t$$
  with terminal condition $p_T = G'(X_T)$.

Method 2: Stochastic maximum principle

- Label co-state ξ_t^i and its volatility $-\varsigma_t^i \xi_t^i$
  - Link to HJB: co-state ξ_t^i acts like a Lagrange multiplier on the net worth evolution, marginal (time-$zero$) utility benefit of giving agent i an additional unit of (time t) wealth, $\xi_t^i = e^{-\rho t} V_t'(n_t)$
  - Link to Martingale Method: we will see later that co-state ξ_t^i will be the SDF, $-\varsigma_t^i \xi_t^i$ is the (arithmetic) volatility of ξ_t^i
- Hamiltonian

$$H^i = e^{-\rho t} \frac{(c_t^i)^{1-\gamma}}{1-\gamma} + \xi_t^i n_t^i \mu_t^{n^i} - \varsigma_t^i \xi_t^i n_t^i \sigma_t^{n^i}$$

$$= e^{-\rho t} \frac{(c_t^i)^{1-\gamma}}{1-\gamma} + \xi_t^i \left[-c_t^i + n_t^i \left(1-\theta_t^i\right)(r_t + \delta_t^a) + n_t^i \theta_t^i r_t - \varsigma_t^i n_t^i \left(1-\theta_t^i\right) \sigma_t^{r^a}\right]$$

Method 2: Stochastic maximum principle

- Label co-state ξ_t^i and its volatility $-\varsigma_t^i \xi_t^i$
  - **Link to HJB:** co-state ξ_t^i acts like a Lagrange multiplier on the net worth evolution, marginal (time-*zero*) utility benefit of giving agent i an additional unit of (time t) wealth, $\xi_t^i = e^{-\rho t} V_t'(n_t)$
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$$H^i = e^{-\rho t} \frac{(c_t^i)^{1-\gamma}}{1-\gamma} + \xi_t^i n_t^i \mu_t^{n^i} - \varsigma_t^i \xi_t^i n_t^i \sigma_t^{n^i}$$

$$= e^{-\rho t} \frac{(c_t^i)^{1-\gamma}}{1-\gamma} + \xi_t^i \left[-c_t^i + n_t^i (1-\theta_t^i)(r_t + \delta_t^a) + n_t^i \theta_t^i r_t - \varsigma_t^i n_t^i (1-\theta_t^i) \sigma_t^{r^a} \right]$$

- FOC w.r.t. θ_t^i, c_t^i :

$$e^{-\rho t} (c_t^i)^{-\gamma} = \xi_t^i,$$

$$\delta_t^a = \varsigma_t^i (\sigma + \sigma_t^q)$$

Method 2: Stochastic maximum principle

- Costate equation (additional FOC)

$$d\xi_t^i = -\frac{\partial H^i}{\partial n^i} dt \; - \varsigma_t^i \xi_t^i dZ_t$$

- The drift of ξ_t^i is given by

$$\mu_t^{\xi^i} \xi_t^i = -\frac{\partial H^i}{\partial n^i} = -\xi_t^i\left[\left(1-\theta_t^i\right)\left(r_t+\delta_t^a\right)+\theta_t^i r_t-\varsigma_t^i\left(1-\theta_t^i\right)\sigma_t^{r^a}\right] = -r_t \xi_t^i.$$

- Hence,

$$\frac{d\xi_t^i}{\xi_t^i} = -r_t dt - \varsigma_t^i dZ_t$$

  - $(\xi_t^i, -\varsigma_t^i)$ are indeed SDF and price of risk!
- Under log utility

$$\xi_t^i = \partial_n V_t^i = \frac{1}{\rho n_t^i}, \qquad \varsigma_t^i = \sigma_t^{n^i}$$

- Same result as HJB approach.

Stochastic Control Methods in Continuous Time

- Hamilton-Jacobi-Bellman (HJB) Equation
  - Continuous-time version of Bellman equation
  - Requires Markovian formulation with explicit definition of state space
  - Postulate value function $V(n, \eta)$ as a function of state variable process $d\eta_t / \eta_t$
- Stochastic Maximum Principle
  - conditions that characterize path of optimal solution (as opposed to whole value function)
  - closer to discrete-time Euler equations than Bellman equation
  - does not require Markovian problem structure
  - Postulate co-state variable ξ_t^i
- Martingale Method
  - (very general) shortcut for portfolio choice problems
  - yields interpretable equations (effectively linear factor pricing equations)
  - But: tailored to specific problem class (portfolio choice), non-trivial to apply elsewhere
  - Postulate SDF process $d\xi_t^i / \xi_t^i$…

Method 3: Martingale Approach – Discrete Time

$$\max_{\{c,\boldsymbol{\theta}\}} E_t\left[\sum_{\tau=t}^{T} \frac{1}{(1+\rho)^{\tau-t}} u(c_\tau)\right]$$

$$\text{s.t. } \boldsymbol{\theta}_t \boldsymbol{p}_t = \boldsymbol{\theta}_{t-1}(\boldsymbol{p}_t + \boldsymbol{d}_t) - c_t \text{ for all } t$$

- FOC w.r.t. θ_t: (deviate from optimal at t and $t+1$)

$$\xi_t p_t = E_t[\xi_{t+1}(p_{t+1} + d_{t+1})]$$

  - where $\xi_t = \frac{1}{(1+\rho)^t} \frac{u'(c_t)}{u'(c_0)}$ is the (multi-period) stochastic discount factor (SDF)
  - If projected on asset span, then pricing kernel ξ_t^*
  - Note: $\mathrm{MRS}_{t,\tau} = \xi_{t+\tau}/\xi_t$
- Consider portfolio, where one reinvests dividend d
  - Portfolio is a self-financing trading strategy, A, with price, p_t^A

$$\xi_t p_t^A = E_t[\xi_{t+1} p_{t+1}^A]$$

- Stochastic process, $\xi_t p_t^A$, is a martingale

Method 3: Martingale Approach – Cts. Time

$$\max_{\{\boldsymbol{\theta}_t, c_t\}_{t=0}^{\infty}} E\left[\int_0^\infty e^{-\rho t} u(c_t) dt\right]$$

s.t. $\frac{dn_t}{n_t} = -\frac{c_t}{n_t} dt + \sum_j \theta_t^j dr_t^j$ + labor income/endow/taxes

n_0 given

- Portfolio Choice: Martingale Approach
  - Let x_t^A be the value of a "self-financing trading strategy"(reinvest dividends)
- Theorem: $\xi_t x_t^A$ follows a Martingale, i.e., drift $= 0$.
  - Let $\frac{dx_t^A}{x_t^A} = \mu_t^A dt + \sigma_t^A dZ_t,$
  - Postulate $\frac{d\xi_t^i}{\xi_t^i} = \underbrace{\mu_t^{\xi^i}}_{-r_t^i} dt + \underbrace{\sigma_t^{\xi^i}}_{-\varsigma_t^i} dZ_t$
  - By Ito product rule

$$\frac{d(\xi_t^i x_t^A)}{\xi_t^i x_t^A} = \Big(\underbrace{-r_t^i + \mu_t^A - \varsigma_t^i \sigma_t^A}_{=0}\Big) dt + \text{volatility terms}$$

  - Expected return: $\boxed{\mu_t^A = r_t^i + \varsigma_t^i \sigma_t^A}$
    - For risk-free asset, i.e. $\sigma_t^A = 0$: $r_t^f = r_t^i$
    - Excess expected return to risky asset B: $\mu_t^A - \mu_t^B = \varsigma_t^i (\sigma_t^A - \sigma_t^B)$

Remark: What is ξ_t for CRRA utility

- ξ_t is $e^{-\rho t}u'(c_t) = e^{-\rho t}c_t^{-\gamma}$
  - $dc_t = \mu_t^c c_t dt + \sigma_t^c c_t dZ_t$

- Apply Ito's Lemma
  - Note: $u'' = -\gamma c^{-\gamma-1}$, $u''' = \gamma(\gamma+1)c^{-\gamma-2}$

- $$\frac{d\xi_t}{\xi_t} = -\underbrace{\left(\rho + \gamma\mu_t^c - \frac{1}{2}\gamma(\gamma+1)(\sigma_t^c)^2\right)}_{r_t^f}dt - \underbrace{\gamma\sigma_t^c}_{\varsigma_t}dZ_t$$

- Risk-free rate r_t^f
- Price of risk ς_t

- Aside: With Epstein-Zinn (-Duffie) preferences with EIS ψ
  - $r^r = \rho + \psi^{-1}\mu_t^c - \frac{1}{2}\gamma(\psi^{-1}+1)(\sigma_t^c)^2$

Method 3: Martingale Approach – Cts. Time

- Proof 1: Stochastic Maximum Principle (see Handbook chapter)
- Proof 2: Intuition (calculus of variation)

  remove from optimum Δ at t_1 and add back at t_2

  $$V(n, \omega, t) = \max_{\{\iota_s, \boldsymbol{\theta}_s, c_s\}_{s=t}^{\infty}} E_t\left[\int_0^{\infty} e^{-\rho(s-t)} u(c_s) ds | \omega_t = \omega\right]$$

  - s.t. $n_t = n$

  $$e^{-\rho t_1} \frac{\partial V}{\partial n}(n_{t_1}^*, x_{t_1}, t_1) x_{t_1}^A = E_{t_1}\left[e^{-\rho t_2} \frac{\partial V}{\partial n}(n_{t_2}^*, x_{t_2}, t_2) x_{t_2}^A\right]$$

  - See Lecture Notes and Merkel Handout