The Fiscal Theory of the Price Level with a Bubble*

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Abstract

This paper incorporates a bubble term in the standard Fiscal Theory of the Price Level equation to explain why countries with persistently negative primary surpluses can have a positively valued currency and low inflation. It also provides two illustrative models with closed-form solutions in which the return on government bonds is below the economy's growth rate. The government can "mine" the bubble by perpetually rolling over its debt. Despite the bubble, the price level remains determined provided government policy credibly promises primary surpluses off-equilibrium. Sufficient "fiscal space" ensures that the bubble term is attached to government bonds rather than other assets, like crypto assets. The analysis provides a new perspective on debt sustainability analysis.

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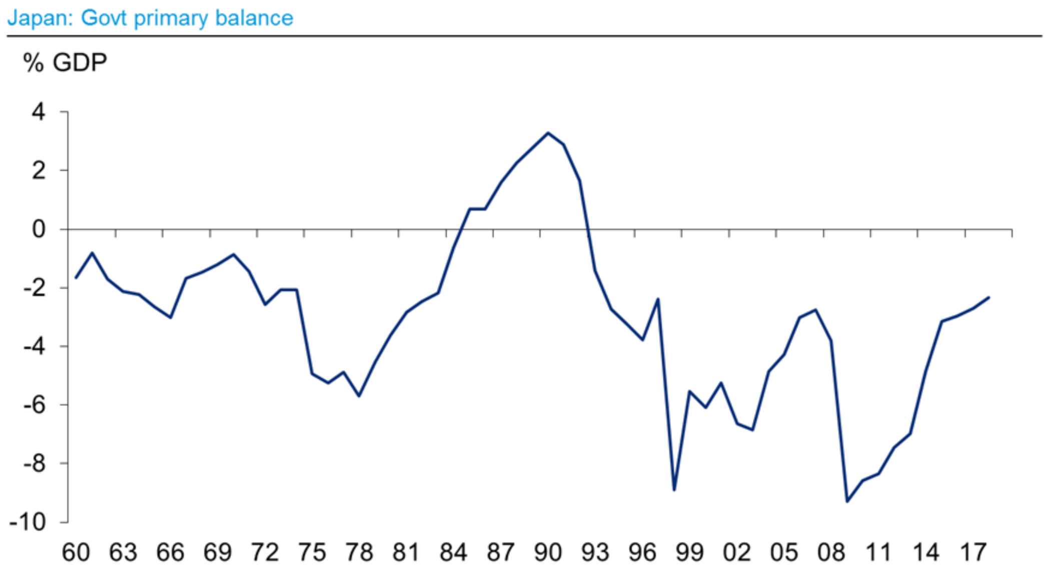

Figure 1: Japanese primary surplus 1960–2017

1 Introduction

What is the relationship between the market value and fundamental value of government debt? While most economic models imply equality of the two, in the data there seems to be a disconnect between these values for many countries. The fundamental value of government debt is the expected discounted stream of future primary surpluses – the difference between the government revenues and expenditures excluding interest payments. The Fiscal Theory of the Price Level (FTPL) links government debt valuation and sustainability to the determination of the price level and inflation. It argues that the price level is determined by equating the ratio of the nominal value of government debt and the price level to the fundamental value of debt. Absent government default, a decline in the expected future primary surpluses lowers the real value of debt by increasing the price level, i.e. higher inflation.

Empirically, for the United States, Jiang et al. (2019) suggest that the price of U.S. Treasury debt significantly exceeds the present value of primary surpluses. For Japan, the FTPL appears to be at odds with the data even more dramatically. The fact that since the 1960s Japan's primary surpluses were mostly negative (see Figure 1) and with no positive primary surplus in sight does not square with the decade-long depressed inflation. This is even more puzzling as the Bank of Japan has left no stone unturned to boost inflation closer to 2%.

In this paper, we revisit the key debt valuation equation underlying the FTPL and argue that including the typically ignored bubble term reconciles the theory with the experience of countries such as the U.S. and Japan. Indeed, we show that the transversality condition is often insufficient to rule out a bubble on the aggregate economy, refuting the usual justification to simply dismiss the bubble term. While most of our conclusions apply to government debt valuation more generally, we adopt a FTPL perspective since it avoids conceptual issues that arise when bubbles are possible[1] and it is well-suited to address the key issue of equilibrium multiplicity in the presence of bubbles.

Our analysis also provides a new perspective on debt sustainability analysis. On the one hand, the bubble creates additional, albeit limited, fiscal space, but on the other, it exposes government debt to a possible bursting of the bubble. Credible and scalable (backup) fiscal space is needed off-equilibrium to fend off such an event.

When can a bubble term emerge? An equilibrium with a bubble can emerge whenever the real interest rate is persistently below the growth rate of the economy, i.e., whenever $r \leq g$. It is well known that this can be the case in overlapping generations models (Samuelson 1958), models of perpetual youth (Blanchard 1985), and incomplete market models with uninsurable idiosyncratic risk à la Bewley (1980). In this paper we spell out the details of the FTPL with a bubble in two simple illustrative models. The first is a version of the Blanchard (1985) model. The second is a simple Bewley-type model based on Brunnermeier and Sannikov (2016a,b) in which $r \leq g$ arises naturally due to precautionary savings demand when agents can invest in both physical capital and government bonds. In the perpetual youth model, government debt allows a transfer of resources from future to current generations. In the uninsurable idiosyncratic risk model, government debt takes on the role of a safe asset which allows citizens to partially insure their idiosyncratic risk.

Even if a bubble can exist, why should we select an equilibrium with a public debt bubble? Why not the bubbleless equilibrium or equilibria with bubbles on other assets, say on corporate bonds or crypto assets? Are there policies that implement the public debt bubble as the unique equilibrium? To answer these questions we expand the FTPL uniqueness reasoning to a setting with a bubble and show how fiscal policy cannot only determine the price level but also select a unique bubble equilibrium. The required fiscal policy provides fiscal backing of the debt. But importantly, such backing is merely required off-equilibrium. If the off-equilibrium backing is credible, the government can simultaneously enjoy a bubble on its debt in equilibrium and rule out bubbles on other assets.

Our uniqueness argument suggests an approach to debt sustainability analysis that differs from the traditional one. A bubble on government debt permits the government to worry less about fiscal capacity *in equilibrium*, but raises the question how the government can prevent

[1] Conventional theories interpret the debt valuation equation as an "intertemporal government budget constraint". A bubble term raises the question how exactly this equation constrains government policy. Treating the equation instead as an equilibrium condition, as in the FTPL, sidesteps the issue.

fragility from multiple equilibria. Credible fiscal backing rests on "backup fiscal capacity", the political ability and willingness to adjust policy *off-equilibrium* and raise additional surpluses when markets lose faith in the government's liabilities. Debt sustainability analysis should include the evaluation of such "backup fiscal capacity" to fend off bubble bursts and jumps.

The issue of credibility is particularly pressing when there can be bubbles on other assets. Ruling out the no-bubble equilibrium merely requires a small amount of fiscal backing arbitrarily far in the future. But ruling out bubbles on other assets requires more aggressive off-equilibrium fiscal policies. In the context of our models, we formally analyze credibility of the off-equilibrium backing by relaxing the assumption of perfect commitment. That analysis yields a sharp prediction. When fiscal policy cannot commit to off-equilibrium tax hikes, it is still able to eliminate both the no bubble equilibrium and all bubbles on assets whose supply grows at a faster rate than the government's bond growth rate along the desired equilibrium path. But it is no longer able to eliminate bubbles on assets whose supply grows at a slower rate.

We also discuss alternative policies that are specifically directed at preventing certain bubbles on assets other than government debt. Such policies include insolvency laws that rule out private Ponzi schemes and holding restrictions or taxation of specific assets. When most or all assets in the economy are liabilities of entities subject to insolvency laws, these laws are generally effective in ruling out alternative bubble equilibria. Crypto assets such as bitcoin are not affected by insolvency laws, and hence additional regulation of these assets may be required to prevent bubbles on them and to preserve debt sustainability when backup fiscal capacity is in question. Our paper thus provides a new rationale for the regulation of crypto assets.

Besides raising the question of uniqueness and bubble fragility, a public debt bubble has also beneficial implications for debt sustainability: the debt bubble represents a fiscal resource. By "printing" bonds, the government imposes an inflation tax that reduces the return on the bonds. Since government bonds are a bubble, the government in a sense "mines a bubble" to generate seigniorage revenue. This seigniorage revenue can be used to finance government expenditures without ever having to raise extra taxes. There is a limit to bubble mining seigniorage, however. As more aggressive bubble mining reduces the attractiveness of government bonds as a store of value, private agents try to substitute away into other assets or reduce their total savings. As with traditional inflation taxes, bubble mining can erode the tax base. A Laffer curve emerges.

We also study optimal debt issuance policy. A positive rate of bubble mining with perpetually negative primary fiscal surpluses can be the optimal policy prescription, since bubble mining discourages bond holdings and boosts physical capital investments and thereby economic growth. Importantly, the optimal debt issuance policy only corrects for pecuniary externalities but it never reacts to the size of or need for public expenditures. The main takeaway is that welfare-maximizing policy should rely on taxes, not bubble mining, as the marginal funding source for (additional) public expenditures.

Literature. This paper contributes to the FTPL literature and its antecedent on the importance of fiscal arrangements in monetary economies (e.g. Sargent and Wallace, 1981). Classic references for the FTPL are Leeper (1991), Sims (1994), and Woodford (1995). For more comprehensive treatments see Leeper and Leith (2016) and a recent book draft by Cochrane (2021). That literature considers bubble-free environments. An exception is Bassetto and Cui (2018) who study the validity of the FTPL in low interest rate environments.[2] Our contribution differs to theirs in two ways. First, they focus exclusively on price determinacy and do not discuss the existence of a bubble and its implication for the government budget. Second, using a model that does feature the possibility of a bubble, a dynamically inefficient OLG setting, they conclude that the FTPL breaks down while we show in Section 6 how fiscal price level determination can succeed in the presence of a bubble.

Our paper is also related to an extensive literature on rational bubbles (e.g. Caballero and Krishnamurthy (2006), Farhi and Tirole (2012), Martin and Ventura (2012, 2016), Miao and Wang (2012, 2018), Asriyan et al. (2021)). Recent survey papers include Miao (2014) and Martin and Ventura (2018). A common theme in this literature is the existence of bubbles on assets issued by private agents and how these bubbles alleviate financial frictions. In contrast, our paper emphasizes bubbles on government debt and how such bubbles generate fiscal space. Closest to our discussion of equilibrium uniqueness is Asriyan et al. (2021) who show that monetary policy can select a unique equilibrium path for bubbly money and the price level out of a given pre-selected set of equilibria that are constrained by a "market psychology" that determines the evolution of private bubbles. By connecting the uniqueness question with the FTPL, we derive results that are considerably more far-reaching: using off-equilibrium fiscal backing, fiscal policy can select a unique equilibrium out of the set of all possible equilibria.[3]

Our uniqueness argument also relates to papers that seek to rule out hyperinflationary equilibria in models of fiat money. Wallace (1981), Obstfeld and Rogoff (1983) and Tirole (1985) show how policy interventions such as partial commodity backing or reserve requirements can ensure equilibrium uniqueness in models with money in the utility function or with bubbly money, respectively. Our off-equilibrium fiscal policy utilizes a similar idea, but we address the broader question how the policy can eliminate all alternative equilibria, including those with (station-

[2]Like Bassetto and Cui (2018), Farmer and Zabczyk (2020) also study the FTPL in an OLG model and conclude that the FTPL is unable to resolve equilibrium multiplicity. However, their result is based on indeterminancy in the underlying real model that is not directly related to either bubble multiplicity or indeterminacy of nominal valuations.

[3]Thus, our proposed policy effectively determines the "market psychology" that is pre-selected by Asriyan et al. (2021).

ary) bubbles on other assets than government debt.[4] In addition, we also investigate when such policies remain credible under imperfect commitment.

The possibility to run perpetual deficits through bubble mining relates our paper to a smaller literature on debt rollovers in overlapping generations economies that goes back to the classic contribution of Diamond (1965). The subsequent literature discusses primarily the relationship between debt rollovers and dynamic inefficiency (e.g. Ball et al., 1998; Blanchard and Weil, 2001). More recently, Blanchard (2019) uses model simulations to study the cost of public debt when safe interest rates are low but the economy is dynamically efficient, and concludes that public debt may have no fiscal cost.[5] Brumm et al. (2021) dispute Blanchard (2019)'s conclusion by presenting four settings in which public debt expansion is not the ideal policy to overcome the fundamental frictions causing the low interest rate. Methodologically, this literature does not established a link to bubbles,[6] but considers the dynamic stability properties of the debt-to-gdp ratio. While Diamond (1965)'s original example represents a case of bubble mining, the connection to bubbles and bubble mining is less clear-cut in the subsequent literature, particularly the dynamically efficient examples, as stochastic simulations often stipulate the imposition of a tax on paths in which debt rollovers fail. In contrast, our paper highlights that the possibility of perpetual deficits is neither related to intergenerational transfers nor to dynamic inefficiency, but to the possibility of public debt bubbles. In our analytically tractable (dynamically efficient) examples, we provide the precise conditions for the possibility of public debt bubbles, delineate the limits of bubble mining, and characterize optimal bubble mining policy. Our FTPL focus also allows us to make progress on the question of how policy can prevent coordination on adverse alternative equilibria, an issue that Blanchard (2019) recognizes as important, but ultimately ignores.

This paper abstracts from aggregate risk and then bubbles can exist if the risk-free rate r is below the economic growth rate g, consistent with the empirical evidence. However, $r < g$ does not necessarily imply the existence of bubbles in all models. It has been known at least since Bohn (1995) that aggregate risk can drive down the risk-free rate, yet the standard (bubble-free) debt valuation equation may continue to hold because the correct risk-adjusted rate for government debt differs from the risk-free rate. More recently, Barro (2021) and Mehrotra and Sergeyev (2021) provide examples in which disaster risk drives down r below g but markets are

[4]In Wallace (1981) and Obstfeld and Rogoff (1983), money is by assumption the only possible store of value and medium of exchange, respectively. Tirole (1985) does allow for bubbles on alternative assets, but his reserve requirement does not ensure uniqueness (it only eliminates asymptotically bubble-free equilibria).

[5]For empirical evidence on dynamic (in)efficiency, compare Abel et al. (1989) and Geerolf (2013). For additional evidence on the related question of the relationship between interest rates and growth rates, see also Lian et al. (2020) and Blanchard (2019)'s paper itself.

[6]In the OLG context, it has long been known (e.g. Tirole, 1985) that the allocational properties of Diamond's public debt policy can equivalently be achieved when there are bubbles (on any asset) in the economy. However, in our reading of the literature, the fact that the fiscal authority can perpetually roll over its debt precisely when there is a public debt bubble, has not been widely appreciated.

complete and thus bubbles cannot exist. In our view, the evidence is nevertheless in favor of the bubble interpretation. For one, in their empirical estimates, Jiang et al. (2019) find a large gap between the value of debt and the value of a claim to primary surpluses. That gap can be interpreted as an estimate of a positive bubble component. In addition, while aggregate risk affects discount rates, Brunnermeier et al. (2021a) point out that when government debt is a safe asset (negative β), as has been historically the case for advanced economies such as the U.S., it would command an even lower risk-adjusted interest rate which makes bubbles more likely.[7]

Since circulation of a previous draft, some recent papers have taken up and extended the core insights from our paper. Like this paper, Reis (2021) emphasizes the bubble as a fiscal resource that has implications for debt sustainability, but his focus is on the interaction with other policies while we focus on FTPL aspects and optimal bubble mining. Brunnermeier et al. (2021a) add aggregate risk to the example model presented in Section 3.2 to develop a safe asset theory of government debt. Kocherlakota (2021) studies a bubble on government debt caused by tail risk. Kocherlakota (2022) and Li and Merkel (2020) study monetary and fiscal policy in New Keynesian models with government debt bubbles and show that monetary policy may be inferior to fiscal policy in stabilizing inflation and the output gap. The uniqueness argument based on fiscal backing made in this paper serves as a theoretical underpinning for the (often implicit) equilibrium selection made in all of these papers.

The goverment debt valuation equation in models with transaction benefits of money can always be written without the usual flow seigniorage term by discounting at the appropriate money rate, see e.g. Cochrane (2021, Section 6.4.6). If the convenience yield on money is sufficiently large, government debt appears to have a bubble in the resulting equation. This paper is not about the transaction benefits of money, but about bubbles that arise even under marginal utility discounting.[8]

2 The Debt Valuation Equation with a Bubble

In this section, we derive the key valuation equation of the FTPL in a generic partial equilibrium setting. We then discuss under which conditions this equation can possibly have a bubble term that has previously been ignored in the literature. To conclude, we mention the possible sources of seigniorage consistent with the equation. In the remaining sections of the paper, we elaborate more on these points in the context of two fully worked out examples in general equilibrium.

[7] To the extent that rare disasters represent a Peso problem, the disaster stories of Barro (2021) and Mehrotra and Sergeyev (2021) can, however, also be made consistent with these facts.

[8] This difference is economically important, because the additional fiscal space a government gains from a public debt bubble is unrelated to empirically observable convenience yields as Brunnermeier et al. (2021a) point out.

2.1 Revisiting the Derivation of the Key FTPL Equation

The derivation of the debt valuation equation underlying the FTPL starts with the government flow budget constraint. In discrete time, this constraint is given by

$$\mathcal{B}_t + \mathcal{M}_t + \mathcal{P}_t T_t = (1 + i_{t-1})\,\mathcal{B}_{t-1} + \left(1 + i^m_{t-1}\right)\mathcal{M}_{t-1} + \mathcal{P}_t G_t,$$

where $\mathcal{B}_t$ is the nominal face value of outstanding government bonds, $\mathcal{M}_t$ is the nominal quantity of money in circulation, $\mathcal{P}_t$ is the price level, T_t are (real) taxes, G_t is (real) government spending, and i_t, i^m_t are the nominal interest rates paid on bonds and money, respectively. i^m_t can be smaller than i_t if money provides transaction services. If ξ_t is a real stochastic discount factor (SDF) process that prices government bonds, then $1 = \mathbb{E}_t\left[\xi_{t+1}/\xi_t \cdot \mathcal{P}_t/\mathcal{P}_{t+1}\,(1+i_t)\right]$. Using this property, dividing the government budget constraint by $\mathcal{P}_t$ and rearranging yields

$$\frac{\mathcal{B}_{t-1} + \mathcal{M}_{t-1}}{\mathcal{P}_t}\,(1 + i_{t-1}) = T_t - G_t + \overbrace{\left(i_{t-1} - i^m_{t-1}\right)}^{\Delta i_{t-1} :=}\frac{\mathcal{M}_{t-1}}{\mathcal{P}_t} + \mathbb{E}_t\left[\frac{\xi_{t+1}}{\xi_t}\,(1+i_t)\,\frac{\mathcal{B}_t + \mathcal{M}_t}{\mathcal{P}_{t+1}}\right].$$

Iterating this forward until period T implies

$$\frac{\mathcal{B}_{t-1} + \mathcal{M}_{t-1}}{\mathcal{P}_t}\,(1 + i_{t-1}) = \mathbb{E}_t\left[\sum_{s=t}^{T}\frac{\xi_s}{\xi_t}\,(T_s - G_s)\right] + \mathbb{E}_t\left[\sum_{s=t}^{T}\frac{\xi_s}{\xi_t}\Delta i_{s-1}\frac{\mathcal{M}_{s-1}}{\mathcal{P}_s}\right] + \mathbb{E}_t\left[\frac{\xi_T}{\xi_t}\frac{\mathcal{B}_T + \mathcal{M}_T}{\mathcal{P}_T}\right].$$

Up to this point, we have merely rearranged and iterated the government budget constraint and assumed that there is some SDF process ξ_t that prices government bonds in equilibrium. The literature now typically proceeds by invoking a private-sector transversality condition to eliminate the discounted terminal value of government debt when passing to the limit $T \to \infty$. In this paper, we focus on environments where the transversality condition does not eliminate the terminal value in the limit. When taking the limit $T \to \infty$, we therefore arrive at the more general equation[9]

$$\frac{\mathcal{B}_{t-1} + \mathcal{M}_{t-1}}{\mathcal{P}_t}\,(1 + i_{t-1}) = \underbrace{\mathbb{E}_t\left[\sum_{s=t}^{\infty}\frac{\xi_s}{\xi_t}\,(T_s - G_s)\right]}_{\text{PV of primary surpluses}} + \underbrace{\mathbb{E}_t\left[\sum_{s=t}^{\infty}\frac{\xi_s}{\xi_t}\Delta i_{s-1}\frac{\mathcal{M}_{s-1}}{\mathcal{P}_s}\right]}_{\text{PV of future transaction services}} + \underbrace{\lim_{T\to\infty}\mathbb{E}_t\left[\frac{\xi_T}{\xi_t}\frac{\mathcal{B}_T + \mathcal{M}_T}{\mathcal{P}_T}\right]}_{\text{bubble}}.$$

Relative to the standard valuation equation, this equation contains an additional bubble

[9] Mathematically, the sum of the three limits in the decomposition below may not be well-defined, even if the limit of the sum is. In this case, the right-hand side should be interpreted as the limit of the sum. For instance, this can happen if the bubble term is ∞, but the present value of surpluses is $-\infty$. While this may seem a pathological case, it can make sense economically because the bubble and surpluses are not separately tradeable, but necessarily bundled together to one asset: government debt. As long as the value of this asset is well-defined and finite, infinite subcomponents do not imply arbitrage opportunities or infinite utility.

term. That term does not vanish in the limit if government debt has a bubble component. More generally, for any asset, we say that the asset has a bubble component if its market value exceeds its fundamental value. We define the fundamental value as the discounted present value of the asset's cash flows where cash flows are discounted using the SDF ξ generated by the marginal utility of the marginal holder of the asset.[10]

From now on, we switch to continuous time in order to make our formal arguments more elegant. The continuous-time version of the last equation is given by[11]

$$\frac{\mathcal{B}_t + \mathcal{M}_t}{\mathcal{P}_t} = \mathbb{E}_t \left[\int_t^\infty \frac{\xi_s}{\xi_t} \left(T_s - G_s \right) ds \right] + \mathbb{E}_t \left[\int_t^\infty \frac{\xi_s}{\xi_t} \Delta i_s \frac{\mathcal{M}_s}{\mathcal{P}_s} ds \right] + \lim_{T \to \infty} \mathbb{E}_t \left[\frac{\xi_T}{\xi_t} \frac{\mathcal{B}_T + \mathcal{M}_T}{\mathcal{P}_T} \right]. \tag{1}$$

This equation for the real value of government debt holds in any monetary model. While most conventional monetary models treat this equation as an intertemporal government budget constraint that holds on- and off-equilibrium, in the FTPL it is an equilibrium condition that helps determining the price level.

2.2 When Can a Bubble Exist?

Equation (1) differs from the standard fiscal theory only by the presence of an additional bubble term. When can this bubble term be nonzero? Well-known examples are bubbles in OLG (Samuelson 1958) and perpetual youth (Blanchard 1985) models. In Section 3.1, we analyze a simple version of the latter model. In Section 3.2, we present another example with incomplete idiosyncratic risk sharing. Here, we make some generic points that apply across models.

For tractability, let us focus on environments with a stationary debt-to-GDP ratio and no aggregate risk. In this case, the real value of government debt is

$$\frac{\mathcal{B}_T + \mathcal{M}_T}{\mathcal{P}_T} = \frac{\mathcal{B}_t + \mathcal{M}_t}{\mathcal{P}_t} e^{g(T-t)},$$

where g is the growth rate of the economy, and $\xi_T / \xi_t = e^{-r^f (T-t)}$ with r^f denoting the real risk-free rate. By substituting these expressions into equation (1), we see that the bubble term does not vanish in the limit if $r^f \leq g$. More generally, the correct risk-adjusted discount rate compensating for the real risk inherent in $\frac{\mathcal{B}_T + \mathcal{M}_T}{\mathcal{P}_T}$ must be used instead of the risk-free rate to determine whether a bubble is possible.[12]

For any agent with recursive isoelastic utility (which includes CRRA utility) that is marginal

[10] Note that the primary surpluses are precisely the cash flows that a holder of the total public debt stock receives.
[11] A formal derivation can be found in Appendix A.1.
[12] Bohn (1995) provides an example of a stochastic economy in which $r^f < g$ but no bubbles can exist.

in the market for government debt, the risk-free rate is[13]

$$r^f = \rho + \psi^{-1}\mu^c - \frac{\gamma\left(1+\psi^{-1}\right)}{2}\|\sigma^c\|^2, \tag{2}$$

where $\rho > 0$ is the agent's time preference rate, γ is the relative risk aversion coefficient, ψ is the EIS, μ^c is the growth rate of agent-specific consumption, and σ^c is a vector or relative risk exposures of agent-specific consumption to Brownian risk factors.[14] $\|\cdot\|$ denotes the standard Euclidean norm. This equation is linked to the growth rate of the economy through individual consumption growth μ^c. For example, in a representative agent economy with a balanced growth path $\mu^c = g$.

Equation (2) suggests two reasons the growth rate may exceed the risk-free rate. First, expected consumption growth of individuals may be misaligned with aggregate growth, so that a higher growth rate g may not imply higher individual consumption growth μ^c. This is the case, for example, in OLG or perpetual youth models with population growth. We present a model of this type in Section 3.1. Second, large individual risk exposure (large $\|\sigma^c\|^2$) or risk aversion (large γ) may depress the risk-free rate through the last term in equation (2) and offset any positive effects of growth g on r^f through $\psi^{-1}\mu^c$.[15] We provide an example of this type in Section 3.2. Importantly, the main insights we derive from our two models do not depend on the precise channel through which a bubble can be sustained. These insights would equally apply to other environments in which a bubble term in equation (1) is possible.

The possibility of $r^f \leq g$ is not merely a theoretical curiosity. Historically, real interest rates on government bonds of advanced economies have mostly been below the growth rate. Even Abel et al. (1989), who are often cited as providing evidence against the existence of rational bubbles, report that the safe interest rate r^f is smaller than g. With the more recent decline in r^f, as stressed by Blanchard (2019), the evidence for $r^f < g$ has become more clear-cut. See also Geerolf (2013) and Lian et al. (2020).

In a large class of canonical general equilibrium models, transversality conditions prevent the possibility of $r^f \leq g$ that leads to a bubble. This is regularly true for complete-market models with long-lived agents. However, many interesting incomplete market models do permit bubbles. The transversality condition has less bite in these models, because *individual* bond holdings matter for transversality, while it is the *aggregate* bond value that enters equation (1).

[13]This equation assumes environments with non-stochastic investment opportunities.

[14]Here, we assume that all risk takes the form of Brownian risk. The intuition derived from the argument is unaltered if more general sources of consumption risk are permitted.

[15]In addition, bubble existence in this second case regularly requires idiosyncratic risk, i.e., the individual consumption growth volatility must differ from its aggregate counterpart. The reason is that in the presence of aggregate risk, the risk-free rate is not the correct comparison rate in the bubble existence condition. The correct rate adds a risk premium for aggregate risk on top of r^f that can offset the aggregate risk component in the last term of equation (2).

Individual bond holdings can have different properties than the aggregate bond stock because agents find it optimal to trade the bonds in order to partially complete the market. For example, individual bond holdings may grow at a lower rate or have higher risk than the aggregate bond stock. Then, individual transversality conditions can hold, yet the bubble term in (1) cannot be ruled out. Our two illustrative models presented in Section 3 cover these two cases. In the context of these models, we discuss the transversality condition in Section 4

2.3 Three Forms of Seigniorage

Equation (1) suggests three forms of seigniorage, which here we define simply as government spending that is not backed by offsetting future taxes. The first takes the form of a dilution of private claims to future primary surpluses through surprise devaluations of existing government debt or money.[16] Under rational expectations, this cannot be a regular source of revenue for governments. For the U.S., Hilscher et al. (2014) assess the possibility of future surprise devaluation based on option-implied (risk-neutral) probabilities and conclude that this form of seigniorage is perceived to be a negligible source of revenue. The likelihood of a devaluation exceeding 5% of GDP is less than 1%.

A second form of seigniorage comes from exploiting the liquidity benefits (convenience yield) of "narrow" money ($\mathcal{M}$ in equation (1)). This form of seigniorage can only be extracted from the portion of government debt that takes the form of "narrow" money and provides liquidity services. It depends on the interest rate differential $\Delta i = i - i^m$ between illiquid and liquid government debt. It is small if either that differential is small or if the stock of "narrow" money is only a small part of total government debt.[17] This form of seigniorage is not an important funding source for advanced economies. For example, in the U.S., Reis (2019) reports a flow revenue of approximately 0.36% of GDP and estimates a present value of $\approx$ 20% and, at most, 30% of GDP. Moreover, in the future the Δi term is likely to decline, because central banks pay interest on reserves and as money becomes more digitalized, its velocity rises.

Besides these standard forms of seigniorage, equation (1) suggests a third form of seigniorage that has remained unexplored in prior work and is the focus of this paper. The government can "mine" the bubble by using its outstanding government debt to run an ever-expanding Ponzi scheme: letting the stock of government debt grow generates a steady revenue flow that does not have to be paid for by future taxes as long as a bubble term is present in equation (1).

[16] Without long-term debt as in equation (1) such dilution must work through a sudden surprise inflation (an unexpected upward jump in $\mathcal{P}_t$). In a more realistic setting with long-term debt, news of higher inflation going forward would have similar effects and work through bond prices instead of the general price level.

[17] In reality, one has to distinguish between reserves, whose quantity is nonnegligible, but which pay interest and have therefore a small Δi_t, and cash, which has a much larger Δi_t, but whose quantity is almost negligible relative to the overall stock of government debt.

Unlike a surprise dilution through inflation, dilution of the bubble value is feasible even if it is fully expected by the private sector. This form of seigniorage is arguably larger than the officially measured seigniorage from growing narrow money $\mathcal{M}$ because all revenue from growing $\mathcal{B} + \mathcal{M}$ is relevant for bubble mining.

3 Two Models with a Bubble

There are several model structures in which rational bubbles can exist and thus the bubble term in equation (1) does not necessarily disappear. We illustrate this in two simple examples. Each contains one of the two mechanisms discussed in Section 2.2 through which the risk-free rate (2) can fall below the growth rate.

In the first model, a bubble can be sustained by the difference between the individual and the aggregate consumption growth rates. That model is a very simple version of the perpetual youth model (Blanchard, 1985). Government debt may circulate as a bubble because it facilitates trade between the current and not yet born future generations.

In the second model, a bubble can be generated through incomplete idiosyncratic risk sharing. That model is a streamlined version of Brunnermeier and Sannikov (2016a) without banks.[18] Government debt may circulate as a bubble because bond trading allows agents to self-insure against idiosyncratic shocks.

For simplicity, we abstract in both models from the presence of additional "narrow" money that yields transaction benefits.[19]

In this section, we set up both models and briefly outline their solution. We then discuss bubble existence, bubble mining policies, and price level determinations for both models simultaneously in the following Sections 4 to 6. A more formal derivation of the model solutions is presented in Appendix A.2.

3.1 Example I: Perpetual Youth

Environment. At each time t, there is a continuum of households indexed by $i \in [0, L_t]$, where L_t is the population mass at time t. We assume that population grows at a constant rate $g > 0$,

[18]The model version without banks has previously been analyzed in Brunnermeier and Sannikov (2016b) and Di Tella (2020). These papers frame the model as a model of money. Here, we add fiscal policy and reinterpret money as bonds. The bond interpretation is also adopted in the safe asset framework of Brunnermeier et al. (2021a).

[19]Other than adding an additional source of seigniorage, including transaction benefits into the analysis does not substantially alter our conclusions. For the FTPL with transaction benefits but no bubble, see, e.g., Sims (2019).

$dL_t = gL_t dt$. For simplicity, we abstract from death, so that all households are infinitely lived.[20] Households $i \in [0, L_t]$ alive at time t have logarithmic preferences

$$V_t^i := \mathbb{E}\left[\int_t^\infty e^{-\rho(s-t)} \log c_s^i ds\right]$$

with discount rate ρ.

Each household i born at a time $t_0 > 0$ is endowed at birth with one unit of human capital, $k_{t_0}^i = 1$. Human capital depreciates over time at a constant rate $\delta \geq 0$, $dk_t^i = -\delta k_t^i\, dt$, and k_t^i units of human capital produce an output flow of $ak_t^i dt$ ("labor income").

Denote by $K_t := \int_0^{L_t} k_t^i di$ the aggregate quantity of human capital in the economy. We choose initial conditions such that human capital per capita K_t/L_t is constant over time.[21] Then, K_t also grows at the constant rate g.

The key friction in the model is that agents are not able to trade with yet unborn generations. Instead, they can only enter financial contracts with other agents currently alive.

Besides households, there is a government that funds government spending, imposes taxes on labor income, and issues nominal bonds. The government has an exogenous need for real spending $\mathfrak{g}K_t dt$, where $\mathfrak{g}$ is a model parameter.[22] The government levies proportional labor income taxes (subsidies, if negative) τ_t on households. Outstanding government debt has a nominal face value of $\mathcal{B}_t$ and pays nominal interest i_t. $\mathcal{B}_t$ follows a continuous process $d\mathcal{B}_t = \mu_t^{\mathcal{B}} \mathcal{B}_t dt$, where the growth rate $\mu_t^{\mathcal{B}}$ is a policy choice of the government. In short, the government chooses the policy instruments τ_t, i_t, $\mu_t^{\mathcal{B}}$ as functions of histories of prices taking $\mathfrak{g}$ as given and subject to the nominal budget constraint[23,24]

$$\overbrace{\underbrace{\left(\mu_t^{\mathcal{B}} - i_t\right)}_{=:\breve{\mu}_t^{\mathcal{B}}} \mathcal{B}_t}^{\text{bond growth in excess of payouts}} \quad + \quad \overbrace{\mathcal{P}_t \underbrace{(\tau_t a - \mathfrak{g})}_{=:s_t} K_t}^{\text{(nominal) primary surpluses}} \quad = 0, \tag{3}$$

[20]At the expense of additional notation, one could easily assume a positive, but constant, death rate, provided there are annuity markets that insure against idiosyncratic death risk. See Blanchard (1985) for details. Other than simplifying notation, abstracting from death also stresses that an infinite lifespan does not preclude the existence of bubbles.

[21]Formally, the relevant condition is $K_0/L_0 = \frac{g}{g+\delta}$. This can be shown easily by computing the time derivative of K_t/L_t.

[22]Making spending proportional to total human capital K_t is equivalent to making spending proportional to population size L_t because K_t/L_t is constant.

[23]Letting policy depend on histories of endogenous price paths is common in the FTPL literature to discuss what happens off-equilibrium.

[24]At this point, we do not impose additional restrictions on government policy, including whether policy is characterized by monetary dominance or fiscal dominance, as the choice of the policy regime is irrelevant for most of our results. We do make more restrictive assumptions in Section 6 where we explain how to adjust the fiscal theory arguments for price level determination based on fiscal dominance if government debt has a bubble component.

where $\mathcal{P}_t$ denotes the price level as in Section 2.

The model is closed by the aggregate resource constraint

$$C_t + \mathfrak{g} K_t = a K_t, \tag{4}$$

where $C_t := \int_0^{L_t} c_t^i di$ is aggregate consumption.

Household Problem. Let b_t^i the (real) value of bond holdings of household i. Bond holdings satisfy the accumulation equation

$$db_t^i = \left(r_t^f b_t^i + (1 - \tau_t) a k_t^i - c_t^i \right) dt, \tag{5}$$

where, as in Section 2, r_t^f denotes the real risk-free rate, which equals the return on bonds in this model. The household chooses consumption $\{c_s^i\}_{s \geq t}$ to maximize utility V_t^i subject to the evolution of human capital and bond holdings (5) and subject to a standard no Ponzi condition.

In the appendix, we show that optimal consumption satisfies the familiar log-utility permanent-income consumption rule

$$c_t^i = \rho \left(b_t^i + q_t^K k_t^i \right), \tag{6}$$

where q_t^K is the shadow price of human capital. It is determined by the condition that the value of human capital must equal the present value of future after-tax labor income,

$$q_t^K k_t^i = \int_t^\infty e^{-\int_t^s r_{s'}^f ds'} (1 - \tau_s) a k_s^i ds. \tag{7}$$

Together with a standard transversality condition (to be discussed in Section 4), conditions (6) and (7) fully determine the optimal choices of each individual agent i.

Equilibrium. In this model, any equilibrium can be fully characterized by determining two "prices", the shadow value of human capital q_t^K and the nominal price level $\mathcal{P}_t$. Instead of working with the price level directly, it is more convenient to use the transformation $q_t^B := \frac{\mathcal{B}_t / \mathcal{P}_t}{K_t}$, which is the ratio of the real value of government debt to total human capital in the economy.[25] Total private net wealth consists of bond wealth, $q_t^B K_t$, and human capital wealth, $q_t^K K_t$.

The optimal consumption rule (6) implies that total wealth must be proportional to aggre-

[25] It is more convenient to work with this normalized version of the inverse price level $1/\mathcal{P}_t$ because the latter depends on the scale of the economy and the nominal quantity of outstanding bonds in equilibrium, whereas q_t^B does not.

gate consumption,[26]

$$q_t^B + q_t^K = \frac{a - \mathfrak{g}}{\rho}.$$

To characterize the equilibrium, we therefore need to determine the share of total wealth that is due to bond wealth. We denote this share by ϑ_t:

$$\vartheta_t := \frac{q_t^B}{q_t^B + q_t^K}.$$

The equilibrium behavior of ϑ_t itself can be derived by combining the Fisher equation for the bond return r_t^f with the government budget constraint (3) and the human capital valuation equation (7). The former equation is in this model

$$r_t^f = i_t - \pi_t = \underbrace{i_t - \mu_t^{\mathcal{B}}}_{=-\breve{\mu}_t^{\mathcal{B}}} + g + \mu_t^{q,B}, \tag{8}$$

where inflation $\pi_t := \dot{\mathcal{P}}_t / \mathcal{P}_t$ depends on the nominal bond growth rate, $\mu_t^{\mathcal{B}}$, the growth rate of the economy, g, and the appreciation of the normalized real bond price (q_t^B), $\mu_t^{q,B} := \dot{q}_t^B / q_t^B$. Notice that equation (8) defines the growth in debt in excess of the amount used to pay nominal interest, $\breve{\mu}_t^{\mathcal{B}}$. This quantity is related to seigniorage.

We combine these equilibrium conditions in the appendix and show that ϑ_t is characterized by the equation

$$\vartheta_t = \int_t^\infty e^{-\rho s} \left((1 - \vartheta_s) \left(\delta + g \right) - \breve{\mu}_s^{\mathcal{B}} \right) \vartheta_s ds. \tag{9}$$

This equation relates the current bond wealth share ϑ_t positively to the future flow $(1 - \vartheta_s)(\delta + g)$. Note that $g + \delta$ is the gap between aggregate and individual human capital growth. Bonds trading is more beneficial as this gap increases. The equation also relates ϑ_t negatively to the expected future path of $\breve{\mu}_t^{\mathcal{B}}$, which measures the dilution of the claims of existing bond holders through the issuance of new bonds.

Steady-State Equilibria. We now focus on government policies that hold $\breve{\mu}^{\mathcal{B}}$ and τ constant over time and consider steady-state equilibria with constant q^B and q^K – and thus constant ϑ. All such equilibria must solve equation (9) with constant ϑ. One can show that there is at most one steady-state equilibrium in which bonds have positive value. It is given by

$$\vartheta = 1 - \frac{\rho + \breve{\mu}^{\mathcal{B}}}{\delta + g}.$$

[26]The following equation also makes use of the aggregate resource constraint (4).

That formula describes a valid equilibrium if both the value of human capital and the value of bonds are positive. This is the case if and only if

$$\delta + g > \rho + \breve{\mu}^{\mathcal{B}}.$$

We make this parameter assumption from now on and focus exclusively on this "monetary steady state" with a positive value of government bonds. We also remark that there is always a second steady-state equilibrium, in which bonds have no value, $q^B = \vartheta = 0$,[27] and there are many additional non-stationary equilibria. However, we show in Section 6 that a simple off-equilibrium modification to the fiscal policy rule can select the monetary steady state as the unique equilibrium.

3.2 Example II: Uninsurable Idiosyncratic Risk

Environment. There is a continuum of households indexed by $i \in [0, 1]$. All households have identical logarithmic preferences

$$V_0^i := \mathbb{E}\left[\int_0^\infty e^{-\rho t} \log c_t^i dt\right]$$

with discount rate ρ.

Each agent operates one firm that produces an output flow $ak_t^i dt$, where k_t^i is the (physical) capital input chosen by the firm. Absent market transactions of capital, capital of firm i evolves according to

$$\frac{dk_t^i}{k_t^i} = \left(\Phi\left(\iota_t^i\right) - \delta\right) dt + \tilde{\sigma} d\tilde{Z}_t^i,$$

where $\iota_t^i k_t^i dt$ are physical investment expenditures of firm i (in output goods), Φ is a concave function that captures adjustment costs in capital accumulation, δ is the depreciation rate, and $\tilde{Z}^i$ is an agent-specific Brownian motion that is i.i.d. across agents i. $\tilde{Z}^i$ introduces firm-specific idiosyncratic risk. To obtain simple closed-form expressions, we choose the functional form $\Phi(\iota) = \frac{1}{\phi}\log\left(1 + \phi\iota\right)$ with adjustment cost parameter $\phi \geq 0$ for the investment technology.

The key friction in the model is that agents are not able to share idiosyncratic risk. While they are allowed to trade physical capital and risk-free assets, they cannot write financial contracts contingent on individual $\tilde{Z}^i$ histories. As a consequence, all agents have to bear the idiosyncratic risk inherent in their physical capital holdings.

As in Section 3.1, there is also a government that funds government spending, imposes taxes

[27] In that equilibrium, the price level is infinite, $\mathcal{P} = \infty$, and the government does not raise any primary surpluses, $\tau a = \mathfrak{g}$.

on firms, and issues nominal government bonds. With $K_t := \int k_t^i di$ denoting the aggregate capital stock, the government flow budget constraint is precisely as in the perpetual youth model (equation (3)).

The aggregate resource constraint in this model is

$$C_t + \mathfrak{g}K_t + I_t = aK_t, \tag{10}$$

where, compared to Section 3.1 (equation (4)), here output can also be used for physical investment, $I_t := \int \iota_t^i k_t^i di$.

Price Processes and Returns. We use notation in complete analogy to Section 3.1. Let q_t^K be the market price of a single unit of physical capital and let $q_t^B := \frac{\mathcal{B}_t/\mathcal{P}_t}{K_t}$ be the ratio of the real value of government debt to total capital in the economy. As before, we define $\vartheta_t := q_t^B/(q_t^B + q_t^K)$ as the share of total wealth that is due to bond wealth. We denote by $\mu_t^{q,B} := \dot{q}_t^B/q_t^B$ and $\mu_t^{q,K} := \dot{q}_t^K/q_t^K$ the instantaneous growth rates of q_t^B and q_t^K, respectively.

Households can trade two assets in positive net supply (if $q_t^B \neq 0$), bonds and capital. Assume that in equilibrium $\iota_t = \iota_t^i$ for all i (to be verified below) so that aggregate capital grows deterministically at rate $g_t = \Phi(\iota_t) - \delta$. Then, the return on bonds is

$$dr_t^{\mathcal{B}} = \left(-\check{\mu}_t^{\mathcal{B}} + \Phi(\iota_t) - \delta + \mu_t^{q,B}\right) dt \tag{11}$$

in full analogy to equation (8). The return on agent i's capital is

$$dr_t^{K,i}\left(\iota_t^i\right) = \left(\frac{(1-\tau_t)\,a - \iota_t^i}{q_t^K} + \Phi\left(\iota_t^i\right) - \delta + \mu_t^{q,K}\right) dt + \tilde{\sigma} d\tilde{Z}_t^i.$$

The expected capital return consists of the after-tax dividend yield, $\frac{(1-\tau_t)a-\iota_t^i}{q_t^K}$, and the capital gains rate, $\Phi\left(\iota_t^i\right) - \delta + \mu_t^{q,K}$. Capital returns are risky due to the presence of idiosyncratic risk $\tilde{\sigma} d\tilde{Z}_t^i$.

Household Problem and Equilibrium. The household problem is analogous to the one presented in Section 3.1 with the exception that agents now also choose the capital investment rate ι_t^i and the share of wealth θ_t^i invested in bonds as opposed to capital. We relegate the details of this problem to the appendix. The first-order conditions for the three choices are

$$q_t^K = \frac{1}{\Phi'\left(\iota_t^i\right)}, \qquad \text{Tobin's } q$$

$$c_t^i = \rho n_t^i, \qquad \text{permanent income consumption}$$

$$\frac{a - \mathfrak{g} - \iota_t}{q_t^K} - \frac{\mu_t^{\vartheta} - \breve{\mu}_t^{\mathcal{B}}}{1 - \vartheta_t} = \left(1 - \theta_t^i\right) \tilde{\sigma}^2, \qquad \text{Merton portfolio}$$

where n_t^i denotes the net worth of agent i, which consists of both capital and bond holdings, and $\mu_t^{\vartheta} := \dot{\vartheta}_t / \vartheta_t$ is the growth rate of the bond wealth share ϑ_t. Relative to Section 3.1, only the second condition is identical. The first condition is entirely new. It captures the optimal physical investment choice.[28] The third condition (portfolio choice) replaces condition (7), the valuation equation for nontraded human capital in Section 3.1. It equates the excess return on capital with the required risk premium $\left(1 - \theta_t^i\right) \tilde{\sigma}^2$ for bearing idiosyncratic risk.[29]

As in Section 3.1, the optimal consumption rule implies that total wealth is proportional to total consumption. The difference here is that total consumption is no longer a fixed proportion of aggregate capital but depends on the endogenous reinvestment choice ι_t. Nevertheless, one can show that asset prices q_t^B and q_t^K and the investment rate ι_t are still simple functions of ϑ_t,

$$\iota_t = \frac{(1 - \vartheta_t)(a - \mathfrak{g}) - \rho}{1 - \vartheta_t + \phi\rho},$$

$$q_t^B = \vartheta_t \frac{1 + \phi(a - \mathfrak{g})}{1 - \vartheta_t + \phi\rho},$$

$$q_t^K = (1 - \vartheta_t) \frac{1 + \phi(a - \mathfrak{g})}{1 - \vartheta_t + \phi\rho},$$

so that, again, the equilibrium is determined uniquely up to the dynamics of the bond wealth share ϑ_t. Similar to Section 3.1, ϑ_t must solve a valuation equation

$$\vartheta_t = \int_t^{\infty} e^{-\rho s} \left((1 - \vartheta_s)^2 \tilde{\sigma}^2 - \breve{\mu}_s^{\mathcal{B}} \right) \vartheta_s ds \tag{12}$$

that relates the current bond wealth share ϑ_t positively to expected future "services" $(1 - \vartheta_s)^2 \tilde{\sigma}^2$ and negatively to future bond dilution $\breve{\mu}_t^{\mathcal{B}}$. Equation (12) is structurally identical to equation (9) in the perpetual youth model. The only difference is that services are now derived from self-insurance against idiosyncratic risk and thus related to the magnitude of risk $\tilde{\sigma}$ the agents are exposed to.

Steady-State Equilibria. We consider again steady-state equilibria with constant policy choices $\breve{\mu}^{\mathcal{B}}$ and τ and constant (scaled) asset prices q^B and q^K. As in Section 3.1, there is at most one steady state equilibrium in which bonds have positive value.[30] In that "monetary steady state",

[28] In particular, because all agents face the same capital price q_t^K, they all choose the same investment rate ι_t^i, verifying our previous assumption.

[29] Government spending $\mathfrak{g}$ enters that third equation because we have used the government budget constraint (3) to substitute out taxes.

[30] There are again a non-monetary steady state with $q^B = 0$ and many nonstationary equilibria. As for the perpetual youth model, all these additional equilibria can be ruled out by a modification of the fiscal policy rule (see

if it exists, the bond wealth share is

$$\vartheta = \frac{\tilde{\sigma} - \sqrt{\rho + \breve{\mu}^{\mathcal{B}}}}{\tilde{\sigma}}$$

and the remaining equilibrium quantities are given by

$$\iota = \frac{\sqrt{\rho + \breve{\mu}^{\mathcal{B}}}\,(a - \mathfrak{g}) - \rho\tilde{\sigma}}{\sqrt{\rho + \breve{\mu}^{\mathcal{B}}} + \phi\rho\tilde{\sigma}}, \qquad q^B = \frac{\left(\tilde{\sigma} - \sqrt{\rho + \breve{\mu}^{\mathcal{B}}}\right)(1 + \phi\,(a - \mathfrak{g}))}{\sqrt{\rho + \breve{\mu}^{\mathcal{B}}} + \phi\rho\tilde{\sigma}}, \qquad q^K = \frac{\sqrt{\rho + \breve{\mu}^{\mathcal{B}}}\,(1 + \phi\,(a - \mathfrak{g}))}{\sqrt{\rho + \breve{\mu}^{\mathcal{B}}} + \phi\rho\tilde{\sigma}}.$$

These formulas describe a valid equilibrium only if idiosyncratic risk is sufficiently large,

$$\tilde{\sigma}^2 > \rho + \breve{\mu}^{\mathcal{B}}.$$

In what follows, we always make this assumption.

4 Transversality Condition and Existence of a Bubble

In both models, government debt can have value even in the absence of primary surpluses ($\breve{\mu}^{\mathcal{B}} \geq 0$) because it has a bubble component. In the perpetual youth model, it provides a store of value which allows agents to exchange some of their present labor income for a claim to the labor income of future generations. In the idiosyncratic risk model, bonds are the only store of value that is free of idiosyncratic risk and thus allow agents to self-insure against their risk exposures. In this section, we discuss why the private-sector transversality condition may not rule out the existence of a bubble despite the infinite lifespan of all agents. The key insight is that a bubble can exist because agents do not buy and hold government bonds, but optimally trade them. Such trading makes their individual bond portfolios look very different from the aggregate bond stock.

For each individual agent, a transversality condition for bond holdings,

$$\lim_{T\to\infty} \mathbb{E}\left[\xi_T^i b_T^i\right] = 0,$$

is necessary for an optimal choice. Here, as in Section 2, $\xi_t^i := e^{-\rho(t-t_0)} \frac{1}{c_t^i}$ denotes the SDF process of agent i.[31] The transversality condition appears to suggest that it should not be possible to have a nonzero bubble term in the debt valuation equation (1). However, this argument overlooks that individual bond wealth b_T^i that enters the transversality condition differs from the aggregate value of bonds $\mathcal{B}_T/\mathcal{P}_T$ that enters the valuation equation.

Section 6).

[31] t_0 denotes the birth time of agent i in the perpetual youth model and is simply set to zero in the idiosyncratic risk model.

In the perpetual youth model, this is the case because agents' optimal savings decision leads them to eventually decumulate their bond holdings to support additional consumption. Bond decumulation at the individual level is sustainable without a reduction in the aggregate value of bonds because agents can pass bonds on to newly born generations.

In the idiosyncratic risk model, the aggregate bond stock $B_T/\mathcal{P}_T$ evolves deterministically, yet individual bond wealth b_T^i is optimally chosen to be stochastic because agents constantly rebalance their portfolios in response to idiosyncratic shocks. Agents thus discount b_t^i at a risk-adjusted rate that takes into account their idiosyncratic risk. As idiosyncratic risk cancels out in the aggregate, when valuing a fixed fraction of the outstanding bond stock, as in the debt valuation equation (1), the relevant discount rate from the perspective of all agents is instead the risk-free rate.

Formally, we have in both models $c_t^i = \rho(b_t^i + q^K k_t^i)$ and $q^K k_t^i \geq 0$, so that

$$\mathbb{E}\left[\xi_T^i b_T^i\right] = e^{-\rho(T-t_0)}\frac{1}{\rho}\mathbb{E}\left[\frac{b_T^i}{b_T^i + q^K k_T^i}\right] \leq \frac{1}{\rho}e^{-\rho(T-t_0)} \to 0 \qquad (T\to\infty)$$

and thus the individual transversality condition is clearly satisfied in the equilibria determined in Sections 3.1 and 3.2. Yet, when determining agent i's time t_0 valuation of the entire government bond stock at time T, we obtain (up to a scaling constant)

$$\mathbb{E}\left[\xi_T^i \int b_T^j dj\right] = \mathbb{E}\left[\xi_T^i q^B K_T\right] = e^{-r^f(T-t_0)} q^B K_T = e^{\left(g-r^f\right)(T-t_0)} q^B K_{t_0}$$

and the latter expression does not converge to zero, if $r^f \leq g$.

The difference in the two equations is the presence of the dj-integral. In the perpetual youth model, that integral runs over an expanding interval $[0, L_t]$ of agents, so that aggregate bond wealth can grow at a rate g, even though each individual integrand b_T^j grows (asymptotically) only at the lower rate $g - \breve{\mu}^{\mathcal{B}} - \rho$. When the risk-free rate is in between the two growth rates, a bubble can exist, yet individual transversality conditions are satisfied.

In the idiosyncratic risk model, the integral is over a fixed interval $[0,1]$, so the (expected) growth rates of aggregate bond and individual bond wealth must be identical. Instead, the integral averages out idiosyncratic shocks and changes the risk characteristics relative to the individual integrands. All individual integrands b_T^i have idiosyncratic risk that is negatively correlated with agent i's SDF ξ_t^i. The effective discount rate in the individual transversality condition therefore contains a covariance term (risk premium) that raises the discount rate above r^f. For the total bond stock, idiosyncratic risk averages out and discounting happens at the risk-free rate.

Nothing in the model prevents the bubble existence condition $r^f \leq g$. Indeed, the growth rate of the economy equals the (human or physical) capital growth rate g and the risk-free rate equals the return on bonds. By equations (8) or (11), respectively

$$r^f = g - \breve{\mu}^{\mathcal{B}}. \tag{13}$$

Consequently, $r^f \leq g$, if and only if $\breve{\mu}^{\mathcal{B}} \geq 0$. A nonnegative value of $\breve{\mu}^{\mathcal{B}}$ is consistent with the existence condition of a monetary equilibrium if the specific reason that generates bond savings demand in the models is sufficiently strong relative to the time preference rate. In the perpetual youth model, this is the case if $\delta + g \geq \rho$, i.e. if population growth (g) and/or the decay rate in individual life-cycle labor income (δ) are sufficiently large. In the idiosyncratic risk model, this is the case if $\tilde{\sigma}^2 \geq \rho$, i.e. if idiosyncratic risk is sufficiently large.

5 Mining the Bubble

In this section, we show how the government can mine a bubble, i.e. finance government expenditures without ever raising taxes for them. We also discuss limits to bubble mining, under which circumstances bubble mining is inflationary, and optimal bubble mining policy.

5.1 The Bubble as a Fiscal Resource

Primary surplus is defined as $T_t - G_t = \tau a K_t - \mathfrak{g} K_t =: sK_t$. Due to our assumptions on fiscal policy, it grows at the same rate as K_t. From the government budget constraint (3), $s = -\breve{\mu}^{\mathcal{B}} q^B$. Hence, in our two models, the fiscal theory equation (1) reduces to[32]

$$q^B K_0 = \lim_{T\to\infty} \left(\underbrace{\int_0^T e^{-(r^f - g)t} s K_0 dt}_{=:PVS_{0,T}} + e^{-(r^f - g)T} q^B K_0 \right).$$

Provided $q^B > 0$, equation (13) implies precisely three cases:

1. $s > 0, \breve{\mu}^{\mathcal{B}} < 0$: then $r^f > g$, $PVS_{0,\infty} > 0$ and a bubble cannot exist. This is the "conventional" situation considered in the literature.

2. $s = \breve{\mu}^{\mathcal{B}} = 0$: then $r^f = g$, $PVS_{0,\infty} = 0$ and there is a finite positive bubble whose value exactly equals $q^B K_0$ and grows over time at the growth rate/risk-free rate.

3. $s < 0, \breve{\mu}^{\mathcal{B}} > 0$: then $r^f < g$ and thus the integral $PVS_{0,T}$ converges to $-\infty$ as $T \to \infty$. Yet, q^B is still positive, which is only possible if there is an offsetting infinite positive bubble.

[32] Since the models do not include "narrow" money, there is no Δi term.

These infinite values do not violate any no-arbitrage condition and are also not otherwise economically problematic, since the bubble cannot be traded separately from the claim to surpluses. Both are necessarily bundled in the form of government bonds. As long as $\frac{\mathcal{B}_t}{\mathcal{P}_t} = q^B K_t$ is determined and finite in equilibrium, the model remains economically and mathematically sensible despite the infinite values in the decomposition of the value of government bonds.

In all three cases, the (possible) presence of a bubble represents a fiscal resource that grants the government some extra leeway. Clearly in case 3, the government can run a perpetual deficit, "mine the bubble" and never has to raise taxes to fully fund all government expenditures. In case 2, the existence of the bubble is beneficial, because the value of government debt is positive – allowing agents to transfer resources across generations or self-insure against risk – despite the fact that the present value of primary surpluses is zero. Even in case 1, government debt is more sustainable since an unexpected drop of primary surpluses to zero results in a bubble instead of a total collapse of the value of debt.

5.2 The Bubble Mining Laffer Curve

In case 3 above, the bubble can become arbitrarily large. Does this mean that the government faces no budget constraint and can expand spending without limits? The answer must of course be no as real resources are still finite. Considering present value relationships can be misleading when $r^f \leq g$. Instead, it is instructive to look at flow quantities.

Specifically, primary deficits per unit of capital are given by

$$-s = \breve{\mu}^{\mathcal{B}} q^B.$$

The first factor, $\breve{\mu}_t^{\mathcal{B}}$, represents the rate of bubble mining: revenue raised by bond issuance that is not distributed to bond holders in the form of interest payments. If it is positive, the claim of old bond holders is diluted by the issuance of new bonds, i.e., a positive $\breve{\mu}^{\mathcal{B}}$ effectively represents a tax on existing bond holders.

The second factor, q^B, is the tax base, the real value of existing debt (per unit of capital). If this was unaffected by $\breve{\mu}^{\mathcal{B}}$, then the government could indeed generate arbitrarily large deficits by bubble mining. However, in both our examples, private agents react do the dilution of their claims by reducing bond demand. Thus, the tax base q^B reacts negatively to an increase in $\breve{\mu}^{\mathcal{B}}$. A standard Laffer curve intuition emerges.

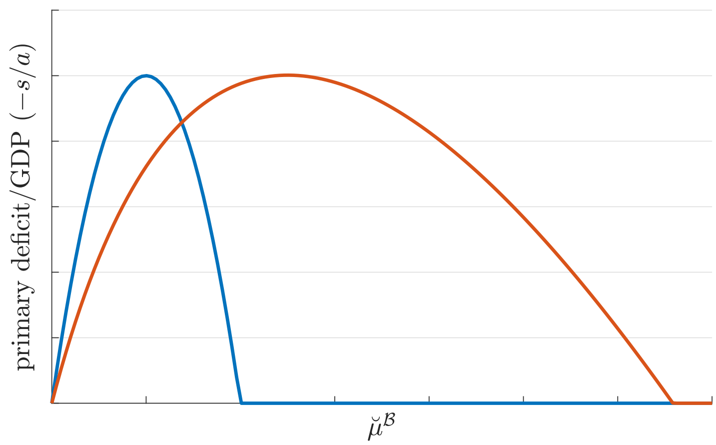

Figure 2: Typical shapes of Laffer curves for perpetual youth (blue) and idiosyncratic risk model (red). The figure has been created with parameters $\rho = g = \delta = 0.02$, $\mathfrak{g}/a = 0.2$, $\phi = 1$, and $\tilde{\sigma}$ to generate an identical peak for the idiosyncratic risk model ($\tilde{\sigma} \approx 0.29$).

We can see the reaction of q^B to $\breve{\mu}^{\mathcal{B}}$ explicitly from our closed-form solutions:

$$q^B = \begin{cases} \frac{\left(\delta+g-\rho-\breve{\mu}^{\mathcal{B}}\right)(a-\mathfrak{g})}{\rho\left(\delta+g\right)}, & \text{perpetual youth} \\ \frac{\left(\tilde{\sigma}-\sqrt{\rho+\breve{\mu}^{\mathcal{B}}}\right)\left(1+\phi(a-\mathfrak{g})\right)}{\sqrt{\rho+\breve{\mu}^{\mathcal{B}}}+\phi\rho\tilde{\sigma}}, & \text{idiosyncratic risk} \end{cases}.$$

This equation reveals two possible reasons how higher deficits may decrease q^B. First, there is a direct effect from increasing $\breve{\mu}^{\mathcal{B}}$. This emerges because higher debt growth makes bond savings less attractive, reduces bond demand, and thereby lowers the fraction ϑ of wealth that originates from bond wealth. If additional deficits are used to lower the output tax rate τ, this is the only effect. However, if additional deficits are used to fund government spending by raising $\mathfrak{g}$, q^B decreases further due to the presence of the term $a - \mathfrak{g}$. This second effect is a consequence of the resource constraint ((4) or (10)): when the government claims a larger share of output, consumption has to decline, which lowers all asset values symmetrically.[33]

Figure 2 provides an illustration of the typical shape of the Laffer curves in the two models. The assumption in Figure 2 is that $\mathfrak{g}$ remains unchanged, so that larger deficits imply smaller output taxes. For both models, the Laffer curve has a typical inverted-u shape. In the perpetual youth model, the Laffer curve is represented by a parabola, while, for identical ρ, the Laffer curve in the idiosyncratic risk model is typically flatter (or wider) because $\breve{\mu}^{\mathcal{B}}$ enters q^B through a square root term.

[33]This intuition breaks down for $\phi = 0$ in the idiosyncratic risk model, as then agents can convert existing capital goods freely into consumption goods and instead the growth rate is reduced.

Towards the right end of the Laffer curves, the mineable deficit becomes zero at finite rates of bubble mining $\breve{\mu}^{\mathcal{B}}$. At these values, the bubble existence inequalities determined in Section 3 no longer hold.[34] A bubble may disappear if it is mined too aggressively.

5.3 Bubble Mining and Inflation

Is bubble mining inflationary? Not necessarily. Among steady-state policies the answer depends on how the government mines the bubble, by issuing more debt or paying less interest, and on the impact of policy on economic growth.

Specifically, by the Fisher equation, inflation in our models is

$$\pi = i - r^f = i + \breve{\mu}^{\mathcal{B}} - g.$$

For a given nominal interest rate i, there is a direct inflationary effect from an increase in bubble mining $\breve{\mu}^{\mathcal{B}}$. Higher bubble mining at a given interest rate requires the government to grow its debt at a larger rate $\mu^{\mathcal{B}}$. When the growth rate is exogenously given, as in the perpetual youth example, then this is the only effect. But in general, there could be an additional indirect effect that operates through the growth rate g. This is the case in the idiosyncratic risk model: bubble mining decreases the attractiveness of bonds, making the agents want to hold more physical capital, which stimulates real investment and increases the steady-state growth rate g.[35] This latter effect tends to be deflationary.

When the growth rate is endogenous, an increase in $\breve{\mu}^{\mathcal{B}}$ may therefore in principle lower the $\breve{\mu}^{\mathcal{B}} - g$ term and thus inflation. However, this is unlikely to be the case for any realistic calibration of our idiosyncratic risk model: the effect on growth g is largest without capital adjustment costs ($\phi = 0$) and then

$$\frac{dg}{d\breve{\mu}^{\mathcal{B}}} = \frac{d\iota}{d\breve{\mu}^{\mathcal{B}}} = \frac{1}{2}\frac{\rho}{\rho + \breve{\mu}^{\mathcal{B}}}\frac{1}{1-\vartheta}.$$

For $\breve{\mu}^{\mathcal{B}} \geq 0$, this derivative can only be larger than 1 if $\vartheta > 1/2$, that is if the majority of private wealth is bond wealth. Despite the recent rise in the levels of public debt throughout advanced economies, this condition is unlikely to be satisfied in the foreseeable future. The most plausible situation is therefore the one in which the direct effect dominates the indirect growth effect.

[34] Reis (2021) derives expressions for the maximum deficity-debt ratio in the context of a related model and calls this the fiscal capacity of the economy. In our terminology, his "fiscal capacity" precisely corresponds to the rates of bubble mining $\breve{\mu}^{\mathcal{B}}$ at the right end point of the Laffer curve.

[35] This presumes that revenues from bubble mining are used to lower the output tax rate τ and government consumption $\mathfrak{g}K_t$ is kept constant. If instead bubble mining revenues are used to increase $\mathfrak{g}$, then the effect on the economic growth rate g is ambiguous as a larger $\mathfrak{g}$ tends to decrease both private consumption and private investment.

Thus, for a fixed nominal interest rate i, an increase in bubble mining is inflationary.

The government can also offset the inflationary effect of bubble mining further by lowering the interest rate i. This is possible whenever there is no binding lower bound on nominal interest rates. If i fully offsets the rise in $\breve{\mu}^{\mathcal{B}}$, so that $i + \breve{\mu}^{\mathcal{B}} = \mu^{\mathcal{B}}$ is unaffected, then only the indirect deflationary effect due to higher growth remains. By using the policy tools of debt growth and interest rate in the right proportion, the government can increase bubble mining in an inflation-neutral way.

Note, however, that the previous discussion solely centers on the steady-state inflation rate as a result of a steady-state level of bubble mining $\breve{\mu}^{\mathcal{B}}$. If the government was to announce more aggressive bubble mining going forward, government debt would become less attractive and its real value would have to fall, as we have seen in our Laffer curve discussion. This is brought about in equilibrium by an inflationary upward jump in the price level.[36]

5.4 Optimal Bubble Mining

Even if bubble mining is possible, is it ever socially optimal for the government to engage in mining? In this subsection, we characterize the optimal policy and draw two key conclusions:

First, a bubble in settings with frictions facilitates trade along certain dimensions – between generations or in response to idiosyncratic shocks – and mining the bubble inhibits these beneficial trades. Optimal policy therefore only calls for a positive rate of bubble mining, $\breve{\mu}^{\mathcal{B}} > 0$, if pecuniary externalities generate an equilibrium bubble that is "too large". Such a situation is never possible in the perpetual youth example, but can arise in the idiosyncratic risk example because the bubble crowds out real investment ι_t. The optimal policy there balances a trade-off between growth and risk sharing and may call for a positive rate of bubble mining, $\breve{\mu}^{\mathcal{B}} > 0$, if idiosyncratic risk is sufficiently large.

Second, the optimal degree of bubble mining is independent of the government spending need $\mathfrak{g}$. This implies that, under the optimal policy, any additional government spending is optimally funded by raising taxes, not by bubble mining. This conclusion holds in both models.

We limit our formal analysis here to the idiosyncratic risk model, as its welfare implications are richer and less well-known. A brief welfare analysis in the perpetual youth example can be found in Appendix A.4.

[36] If we were to add price stickiness to the model, this initial price level jump would translate into a transition period of larger inflation. See e.g. Li and Merkel (2020) for a closely related model with sticky prices.

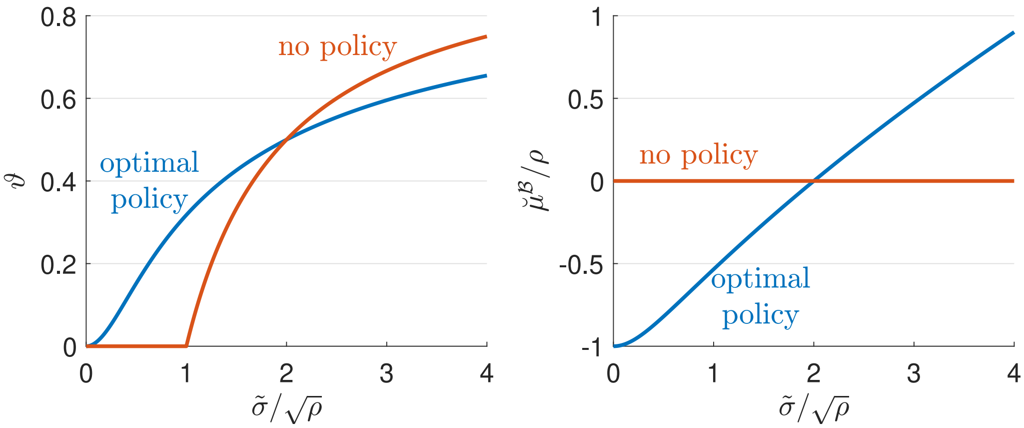

Figure 3: Optimal policy versus no policy ($\breve{\mu}^{\mathcal{B}} = 0$) in the idiosyncratic risk model for $\phi = 0$ as a function of $\tilde{\sigma}/\sqrt{\rho}$. The left panel depicts the nominal wealth share ϑ, the right panel the associated bubble mining policy $\breve{\mu}^{\mathcal{B}}$ normalized by the time-preference rate ρ.

Formally, expected utility of an agent with initial wealth share $\eta_0^i := n_0^i / \left((q_0^B + q_0^K) K_0 \right)$ is[37]

$$\mathbb{E}\left[\int_0^\infty e^{-\rho t} \log c_t^i dt\right] = \frac{\log \eta_0^i + \log K_0}{\rho}$$
$$+ \mathbb{E}\left[\int_0^\infty e^{-\rho t} \left(\underbrace{\log\left(\frac{\rho\left(1+\phi\left(a-\mathfrak{g}\right)\right)}{1-\vartheta_t+\phi\rho}\right)}_{=\log(a-\mathfrak{g}-\iota_t)} + \underbrace{\frac{1}{\phi\rho}\log\left(\frac{\left(1-\vartheta_t\right)\left(1+\phi\left(a-\mathfrak{g}\right)\right)}{1-\vartheta_t+\phi\rho}\right) - \frac{\delta}{\rho}}_{=\frac{(\Phi(\iota_t)-\delta)}{\rho}} - \underbrace{\frac{\left(1-\vartheta_t\right)^2\tilde{\sigma}^2}{2\rho}}_{=\frac{(1-\vartheta_t)^2\tilde{\sigma}^2}{2\rho}}\right) dt\right]. \tag{14}$$

For arbitrary Pareto weights, a social planner would like to manipulate ϑ_t period by period to maximize the integrand in the second line.[38] The first term in the integrand is utility from consumption $a - \mathfrak{g} - \iota_t$, which is increasing in ϑ_t because a higher ϑ_t depresses investment and leaves more resources for consumption. The second term is proportional to the endogenous component $\Phi(\iota_t)$ of the growth rate, which is decreasing in ϑ_t. The last term represents the reduction of utility due to idiosyncratic risk. Higher ϑ_t reduces residual consumption risk $(1-\vartheta_t)\tilde{\sigma}$ and thereby increases this term.[39]

While ϑ_t is not a policy instrument, the government can effectively choose ϑ_t directly by

[37] We provide a derivation of this equation in Appendix A.3.

[38] This is the case because the second line is the same for all agents i, whereas the first line depends only on initial conditions that cannot be affected by bubble mining policy.

[39] Representing the objective in this way highlights similarities to the classic analysis of the optimal quantity of debt by Aiyagari and McGrattan (1998). In their framework, a larger value of government debt increases liquidity by effectively relaxing borrowing constraints, but reduces the quantity of capital. Here, a larger debt wealth share directly improves risk sharing (even in the absence of borrowing constraints) but reduces the growth rate of capital and output.

adjusting $\breve{\mu}_t^{\mathcal{B}}$. We show in the appendix that there is a unique optimal solution ϑ^{opt} for ϑ_t, which is time-invariant, depends only on ρ, $\tilde{\sigma}$, and ϕ, and is strictly increasing in idiosyncratic risk $\tilde{\sigma}$. Figure 3 depicts this optimal nominal wealth share ϑ and the bubble mining rate $\breve{\mu}^{\mathcal{B}}$ required to implement it as a function of idiosyncratic risk.[40] It also compares the optimal policy to the competitive equilibrium without policy intervention ($\breve{\mu}^{\mathcal{B}} = 0$). Relative to that benchmark, optimal policy backs the value of government debt by primary surpluses (negative $\breve{\mu}^{\mathcal{B}}$) if risk is low. In these cases, the bubble created by market forces is too small (for $\tilde{\sigma} > \sqrt{\rho}$) or even absent (for $\tilde{\sigma} \leq \sqrt{\rho}$) and risk-sharing is suboptimal. If risk is high, market forces generate a bubble that is too large. Optimal policy then runs deficits (positive $\breve{\mu}^{\mathcal{B}}$) and funds government expenditures out of the bubble to encourage higher real investment and growth.

Market forces may fail to generate a bubble that achieves the optimal trade-off between growth and risk sharing. Inefficiencies are possible due to pecuniary externalities with respect to agents' portfolio choices because agents take returns as given when making these choices, yet their collective choice affects the risk-free rate and risk-premium on capital.[41] On one hand, a greater portfolio allocation to bonds discourages real investment ι in the economy, which in turn affects the real return on all assets through the growth term in the risk-free rate. This force tends to generate too much bond demand, a too high ϑ and thus under-investment in capital. On the other hand, a greater allocation to bonds increases the total value of bonds and thus reduces the residual (proportional) idiosyncratic consumption risk $(1 - \vartheta)\tilde{\sigma}$ that each agent has to bear.[42] This in turn affects asset returns through the precautionary motive in the risk-free rate and through the risk premium on capital. This second force tends to generate too little bond demand, a too low ϑ and thus over-investment in capital.

It is instructive, however, that the optimal value ϑ^{opt} for ϑ is independent of the government spending need $\mathfrak{g}$. Because $\mathfrak{g}$ does not appear in equation (12) either, then also the optimal degree of bubble mining $\breve{\mu}^{\mathcal{B}}$ to implement $\vartheta = \vartheta^{\text{opt}}$ must be independent of $\mathfrak{g}$. While the government could increase $\breve{\mu}^{\mathcal{B}}$ in response to an (unanticipated) increase in $\mathfrak{g}$ in order to fund the additional spending, this is never optimal.[43] The optimal policy should rely on taxes as the marginal funding source for additional government spending.

The reason for this result is that when government spending $\mathfrak{g}$ increases, the government must transfer a larger fraction of current output away from the private sector to itself. Taxing

[40] The figure assumes no capital adjustment cost, $\phi = 0$. It looks qualitatively identical for $\phi \in (0, \infty)$.

[41] These pecuniary externalities have been previously identified by Brunnermeier and Sannikov (2016b) and Di Tella (2020) in closely related frameworks.

[42] Specifically, higher bond prices benefit the bond-selling agents: those who suffered idiosyncratic losses and who have higher marginal utilities, on average.

[43] $\mathfrak{g}$ is here a model parameter and thus any change in $\mathfrak{g}$ must by construction be an unanticipated change in government spending. However, this fact is irrelevant for our conclusion. We would obtain the same result if we were to add government spending shocks to the model (including transitory changes that mean-revert in the long run).

current output is the most direct way of achieving this and does not distort the portfolio choice between capital and bonds.[44] In contrast, funding additional spending by increasing primary deficits and bubble mining dilutes the bubble at a faster rate and thereby distorts agents' portfolio choice in favor of larger capital holdings. Because the pecuniary externalities just discussed do not depend on either the level of government spending $\mathfrak{g}K_t$ or total output left for private uses $(a - \mathfrak{g})K_t$, the optimal portfolio distortion induced by $\breve{\mu}^{\mathcal{B}}$ is also independent of these quantities.[45]

In our models, government spending $\mathfrak{g}K_t$ is exogenously given and does not provide any utility to agents. We remark, that all the results in this subsection would equally apply if agents derived additively separable utility from public spending and also $\mathfrak{g}_t$ was a policy choice.

6 Price Level Determination and Equilibrium Uniqueness

A core theme in the FTPL literature is the question of price level determinacy, i.e. whether across the set of possible equilibria, there is a unique prediction for the price level. In many model environments, including ours, price level determinacy requires the equilibrium itself to be unique.

Uniqueness is not a purely technical issue but of economic importance. Price level determinacy has always been a concern in monetary economics, ultimately because an indeterminate price level casts doubt on the ability of government policy to control inflation. In the presence of a bubble, the uniqueness question becomes an even bigger concern. If there is multiplicity, then a bubble on government debt is an inherently fragile arrangement. Markets could coordinate any time on a different equilibrium, the government would lose the bubble on its debt and would either have to replace its value with a larger fundamental value by raising surpluses or accept an inflationary collapse of its currency (or default). Mining a bubble could then be a very risky proposition because it would expose the government's debt to sunspot revaluations that would have to be met with large and sudden fiscal corrections. In contrast, if there is a policy that can select a unique equilibrium with a bubble, then a bubble can in fact be a stable arrangement that is not threatened by shifts in market beliefs.

In Section 6.1, we show how fiscal policy can both determine the price level and ensure a

[44] Similarly, in the perpetual youth example, it does not distort the inter-generational resource transfer, which is ultimately about how *future* output left for private uses is split across individuals.

[45] The size of the pecuniary externalities does depends on the aggregate consumption-wealth ratio which equals the time preference rate ρ in our model with log utility. Admittedly, this is a somewhat knife-edge case that only holds for unit EIS. For general EIS, the aggregate consumption-wealth ratio depends on the growth rate of the economy, which in turn is increasing in output left for private uses $a - \mathfrak{g}$ per unit of capital. Nevertheless, our result represents an important benchmark case and the broader point that optimal bubble mining only adjusts to correct pecuniary externalities remains valid also for EIS $\neq 1$.

stable bubble value that remains attached to government debt. We first briefly revisit the conventional FTPL arguments and then discuss how the possible presence of a bubble complicates the situation. We ultimately conclude that fiscal price level determination as studied in the FTPL literature still works in the presence of a bubble, albeit the details of how to implement such a policy have to be adapted. In Sections 6.2–6.4, we analyze under which conditions fiscal price level determination remains credible in the absence of perfect commitment and how alternative policies can facilitate the equilibrium selection.

6.1 Fiscal Policy as an Equilibrium Selection Device

In the standard FTPL without a bubble, the story of price level determination is often told by starting from the key equation (1) (without a bubble) and interpreting it as an asset pricing equation. A holder of the total stock of government debt who absorbs all new issuances in the future receives as a cash flow in each period precisely the stream of primary surpluses. Like the value of a stock is determined by the present value of its future dividends, the value of government debt should thus be determined by the present value of its future cash flows, the primary surpluses. Because the nominal price level is the relative price between nominal bonds and consumption goods, this is the correct "asset price" that must adjust to clear the bond market.[46] If fiscal policy ensures that the present value of primary surpluses is unique, then precisely one value of debt and thus one price level is consistent with the (bubble-free) debt valuation equation (1), which is the main reason why this is the key equation of the theory. In most of the simpler models considered in the FTPL literature, a policy that commits to an exogenous sequence of primary surpluses can accomplish this task, but this specific feature is not a critical ingredient to the FTPL argument for determinacy.

In the presence of a bubble, the previous intuition breaks down because the size of the bubble is not determined by the present value identity itself. There is, however, an alternative intuition about the economic mechanism behind the FTPL that centers on goods market clearing and wealth effects and that remains fully operational when bubbles are possible.[47] A larger real value of government bonds, holding taxes constant, means bonds represent more net wealth for the private sector, which increases consumption demand through a wealth effect. The equilibrium price level is the price level at which consumption demand equals consumption supply.

However, this mechanism can generally not ensure a unique prediction for the price level in an environment with bubbles. Goods market clearing only determines the size of aggregate net wealth consisting of the pre-tax value of (human or physical) capital and the *aggregate bubble*,

[46] To be precise, the price level is the relative price between a maturing bond and consumption goods, while there are additional bond prices for longer-term bonds that depend on the term structure of nominal rates.

[47] This alternative intuition is in fact the original economic story told by Woodford (1995).

the sum of all bubble values in the economy. Even if the value of capital is given, there is no economic force that suggests that the residual, the aggregate bubble, must be attached to government debt. There could be bubble components on other assets, so that government debt has a smaller bubble component and a lower value. In addition, also the value of capital wealth is not given as the alternative nonmonetary equilibrium without a bubble shows. In this equilibrium, the value of capital wealth is larger because the absence of the trading benefits derived from the bubble depresses discount rates. Goods market clearing therefore at best imposes an upper bound on the value of government debt, but it does not pin it down uniquely.

Based on this reasoning, one may be tempted to conclude that in an environment with bubbles, fiscal policy is generally unable to select a unique equilibrium and thereby pin down the price level. This is the conclusion of Bassetto and Cui (2018) in the context of a closely related dynamically inefficient OLG model. They study constant tax policies that are not contingent on the price level and conclude that "the FTPL breaks down in [their] OLG economy" (p. 13).[48]

In contrast to their conclusion, we show in the remainder of this section how fiscal price level determination can succeed even in the presence of bubbles.[49] Our argument proceeds in three steps. We first consider positive surplus policies that eliminate all bubbles, including on government debt, and thus restore the standard fiscal theory intuition. By construction, such policies are unable to deliver uniqueness in the presence of an equilibrium bubble. However, we show in a second and third step that the elimination of all bubbles can be used as an "off-equilibrium threat" to select a unique bubble equilibrium. The second step focuses on eliminating equilibria that are asymptotically bubble-free whereas the third step discusses under which conditions the same policy specification can also eliminate bubbles on assets other than government debt. Relative to the conclusion of Bassetto and Cui (2018), our analysis highlights the importance of contingent policy that raises surpluses off-equilibrium to deliver fiscal price level determination in the presence of bubbles.

Positive Surplus Policies and Elimination of All Bubbles. We start by observing that wealth effects limit the total value of private-sector net wealth by an upper bound proportional to available resources. Any bubble (on any asset) represents net wealth, so that the aggregate bubble value cannot persistently outgrow the economy. As a consequence, the expected long-horizon growth rate of the bubble component on any asset cannot exceed the growth rate of the economy, g, and therefore no bubble on any asset can exist in an equilibrium that features $r^f > g$ in the long run.

[48] Bassetto and Cui (2018) do not discuss the possibility of bubbles explicitly. But it is our interpretation that their result is ultimately due to bubble multiplicity in their framework.

[49] Throughout, we limit attention to equilibria that are deterministic and feature absolutely continuous price paths. The same economic intuition also works to rule out non-time-continuous equilibria and equilibria driven by sunspot noise. However, in these cases, additional technical complications arise that make the mathematical arguments considerably more involved.

More precisely, a sufficient condition for the absence of bubbles is

$$\lim_{T \to \infty} \mathbb{E}\left[\xi_T Y_T\right] = 0, \tag{15}$$

where Y_T denotes aggregate output at time T. If output growth and the risk-free rate are constant, then this condition is equivalent to $r^f > g$.

We have seen in both of our models that any monetary steady state that features a constant and positive surplus-capital ratio $s > 0$ has the property $r^f > g$. We now argue that if the government commits to such a positive surplus policy by choosing[50]

$$\tau_t = \frac{s + \mathfrak{g}}{a}, \qquad \breve{\mu}_t^{\mathcal{B}} = -\frac{s}{q_t^B}, \tag{16}$$

any equilibrium consistent with this policy specification must satisfy condition (15) and, thus, no bubbles can exist.

The economic intuition for this result is very simple. If $r^f \leq g$, at least on average, then the present value of primary surpluses would be infinite and government debt would have infinite value.[51] Indeed, we have the inequality

$$\frac{\mathcal{B}_0}{\mathcal{P}_0} \geq \mathbb{E}\left[\int_0^\infty \xi_t s K_t dt\right] = \frac{s}{a}\mathbb{E}\left[\int_0^\infty \xi_t Y_t dt\right].$$

If condition (15) is violated, then the integral on the right does not converge and thus $\mathcal{B}_0/\mathcal{P}_0 = \infty$. However, then also total net wealth must be infinite in contradiction to our previous argument that wealth effects bound the ratio of total net wealth to output from above.[52] Therefore, condition (15) must be satisfied in any equilibrium under this policy.

The previous argument implies that a positive surplus policy is inconsistent with bubbles. The equilibrium is then indeed unique:

Proposition 1. *If government policy is specified by* (16) *with* $s \in (0, a - \mathfrak{g}]$,[53] *then there is a unique equilibrium. That equilibrium coincides with a monetary steady state and the initial price level* $\mathcal{P}_0$ *is uniquely determined.*

Some additional technical details are necessary to show that there is not just no scope for bubbles but the equilibrium is indeed unique. We relegate them to Appendix A.5.2.

[50] This is always consistent with the government's flow budget constraint (3) and thus a feasible policy specification after any price history.

[51] A possible bubble component can only increase the value of debt.

[52] Total net wealth at $t = 0$ is at least $\mathcal{B}_0/\mathcal{P}_\prime + q_t^K K_t$ (and possibly larger if other assets have bubble components). While higher taxes depress the second component, $q_t^K K_t$, that component always remains positive for $\tau < 1$, such that an infinitely large value of debt is not offset by an infinitely large present value of future taxes.

[53] We impose the additional upper bound $a - \mathfrak{g}$ to ensure that taxes τ_t do not exceed 1.

We remark that for the previous argument to work, it is crucial that the process of taxation to back government debt generates net wealth for the private sector. Otherwise, there would be no reason why the total value of debt has to be bounded because any positive wealth effect from higher debt would be offset by a negative wealth effect from the additional tax liability. This is not the case in our models because Ricardian equivalence does not apply to proportional output taxes in either model.[54] Backing of debt with surpluses would fail to select a unique bubble-free equilibrium if the government used taxes that are consistent with Ricardian equivalence and thus failed to generate net wealth.[55]

Uniqueness with an Equilibrium Bubble. We now discuss how government policy can select a unique equilibrium with a bubble *under the assumption that a bubble can only be on government debt*. We discuss bubbles on other assets below. Even under this assumption, the equilibrium is not necessarily unique as the bubble on government debt can take multiple values.

The steady state equilibria with a bubble derived in Section 3 are consistent, for example, with a policy that fixes $\breve{\mu}^{\mathcal{B}} \geq 0$ at a constant level and adjusts taxes τ_t such that the government budget constraint (3) holds after any price history.[56] We have seen that the same policy is also consistent with a second steady-state equilibrium in which government bonds have no value, $q^B = \vartheta = 0$. In addition, there is a continuum of nonstationary equilibria. We show in Appendix A.5.1 that there is a one-to-one relationship between model equilibria and paths for the bond wealth share ϑ_t that are contained in $[0, 1]$ and solve the ordinary differential equation (ODE)

$$\dot{\vartheta}_t = \left(\rho + \breve{\mu}_t^{\mathcal{B}} - f\left(1 - \vartheta_t\right)\right)\vartheta_t \tag{17}$$

where f is a strictly increasing function (for positive arguments) given by

$$f(x) = \begin{cases} x\left(\delta + g\right), & \text{perpetual youth model} \\ x^2\tilde{\sigma}^2, & \text{idiosyncratic risk model} \end{cases}.$$

Equation (17) can be obtained by taking the time derivative in equation (9) for model I and in equation (12) for model II. We can therefore study the set of all equilibria by analyzing ODE (17).

[54]In the perpetual youth model, taxation of future generations implies that the value of debt exceeds the present value of taxes for the currently alive. In the idiosyncratic risk model, individual tax liabilities are contingent on individual idiosyncratic shock histories, so that individuals discount their future tax liability at a higher rate than the risk-free rate.

[55]In both models, this would be the case if surpluses were raised exclusively by imposing lump-sum taxes on the currently alive agents.

[56]We have opted not to choose the opposite specification where τ is constant and $\breve{\mu}_t^{\mathcal{B}}$ adjusts to make the government budget constraint hold because this is only a valid policy specification if $\tau \geq \mathfrak{g}/a$. For $\tau < \mathfrak{g}/a$, there are histories of prices in which no value of $\breve{\mu}_t^{\mathcal{B}}$ is consistent with equation (3) (e.g. $\mathcal{P} = \infty$). Also, the Laffer curves discussed in Section 5 implies that even if fixing constant deficits was a feasible choice, it would not uniquely determine the equilibrium.

With constant $\breve{\mu}^{\mathcal{B}} \geq 0$, there is a continuum of solution paths for ϑ that remain inside the interval $[0, 1]$. This continuum can be indexed by the initial value $\vartheta_0 \in [0, \vartheta^*]$, where ϑ^* denotes the monetary steady-state level of ϑ. For any initial value but the right endpoint, ϑ asymptotically converges to 0.[57] Conversely, if for some reason agents expected that the equilibrium value of ϑ could never fall below a positive threshold $\underline{\vartheta} > 0$, then all equilibria but the monetary steady state $\vartheta_0 = \vartheta^*$ could be ruled out.

These considerations suggest a simple off-equilibrium modification of the fiscal policy rule to achieve equilibrium uniqueness: fix an arbitrary threshold $0 < \underline{\vartheta} \leq \vartheta^*$ and, whenever ϑ_t falls below $\underline{\vartheta}$, switch from a constant debt growth rule (constant $\breve{\mu}^{\mathcal{B}}$) to a positive surplus rule as discussed previously for as long as $\vartheta_t < \underline{\vartheta}$.[58] This works because, as we have shown previously, the positive surplus rule implies a unique equilibrium with a positive value of debt. Furthermore, an expectation of the regime change cannot become self-fulfilling: the value of debt under the positive surplus rule is so large that $\vartheta_t > \vartheta^* \geq \underline{\vartheta}$ and thus the threshold criterion would be violated if the positive surplus policy was permanently in place.

By analyzing an appropriately modified version of ODE (17), we show in Appendix A.5.3 that this threshold policy indeed succeeds in selecting a unique equilibrium. Any mathematical solution to this ODE with $\vartheta_0 < \vartheta^*$ would eventually lead to $\vartheta_t < 0$, which is inconsistent with an equilibrium solution. This is the case because a low bond valuation can only be consistent with the positive surpluses expected for $\vartheta < \underline{\vartheta}$ if agents expect bond valuations to fall even further at a speed that would imply $\vartheta_t < 0$ at a finite time t.

Proposition 2. *Under the threshold policy discussed in this section, there is a unique equilibrium for any threshold $\underline{\vartheta} \in (0, \vartheta^*]$. The unique equilibrium does not depend on $\underline{\vartheta}$ and satisfies $\vartheta_t = \vartheta^*$ for all t.*

We emphasize again that our policy rule modifies fiscal policy only off-equilibrium. Along the equilibrium path, the government is free to choose any rate of bubble mining $\breve{\mu}^{\mathcal{B}}$, effectively selecting the monetary steady state we have analyzed so far.[59]

Bubbles on Other Assets. We now remove the assumption that only government debt can have a bubble. Instead, we allow for any intrinsically worthless asset in limited supply to have a bubble component.[60] Our main result is that the same threshold policy as studied previously

[57]This is implied by the fact that the right-hand side of equation (17) is negative for all $\vartheta_t \in (0, \vartheta^*)$.

[58]The proposed modification is reminiscent of Obstfeld and Rogoff (1983), who show how an off-equilibrium commodity backing of money can rule out hyperinflationary equilibria in models of money as a medium of exchange. However, there are some differences here that become apparent below, see footnote 59.

[59] This is an important difference to the Obstfeld and Rogoff (1983) policy that requires the government to hold the stock of commodity backing in equilibrium. While the required backing can be arbitrarily small if only a bubble on government debt is permitted, we will show next that in the presence of other possible bubbles, the backing may have to be "large". The equivalent Obstfeld and Rogoff (1983) policy would reduce to full commodity backing in that case and eliminate all bubbles *in equilibirum*.

[60]We limit the formal discussion to bubbles on intrinsically worthless assets, but the economic points apply more broadly, see Section 6.3.

can rule out bubbles on other assets, too, if either the supply of these assets grows sufficiently fast or the taxation threshold $\underline{\vartheta}$ is tight, i.e. $\underline{\vartheta} = \vartheta^*$.

To see this, we extend our models by adding an intrinsically worthless additional asset that can potentially also have a bubble component[61] – for concreteness called cryptocoin. To keep matters simple, we assume that the nominal supply of cryptocoins grows at a constant exogenous rate $\breve{\mu}^{\mathcal{C}} \geq 0$ and that, if the inequality is strict, newly generated cryptocoins are produced by capital. Under this assumption, seigniorage from cryptocoin growth accrues to capital owners symmetrically to how bubble mining by the government leads to lower capital taxes (for given $\mathfrak{g}$). We denote by $q_t^C K_t$ the real value of the stock of cryptocoins and define in analogy to before the shares $\vartheta_t^B := q_t^B / \left(q_t^B + q_t^C + q_t^K\right)$, $\vartheta_t^C := q_t^C / \left(q_t^B + q_t^C + q_t^K\right)$, and $\hat{\vartheta}_t := \vartheta_t^B + \vartheta_t^C$ of total wealth that is due to bond wealth, cryptocoin wealth, and the combination of the two, respectively. With this notation, clearly $\vartheta_t^B \leq \hat{\vartheta}_t$. We can solve these augmented models as before.[62] The key result is that there is a one-to-one relationship between model equilibria and time paths $t \mapsto (\vartheta_t^B, \hat{\vartheta}_t)$ satisfying $0 \leq \vartheta_t^B \leq \hat{\vartheta}_t \leq 1$ and the two ODEs

$$\dot{\hat{\vartheta}}_t = \left(\rho - f\left(1 - \hat{\vartheta}_t\right)\right)\hat{\vartheta}_t + \breve{\mu}_t^{\mathcal{B}}\vartheta_t^B + \breve{\mu}^{\mathcal{C}}\vartheta_t^C, \tag{18}$$

$$\dot{\vartheta}_t^B = \left(\rho - f\left(1 - \hat{\vartheta}_t\right)\right)\vartheta_t^B + \breve{\mu}_t^{\mathcal{B}}\vartheta_t^B, \tag{19}$$

where the function f is the same as in equation (17) above.

The structure of equation (19) is sufficiently similar to that of equation (17) that we can rule out any solution paths for ϑ^B that ever fall below the threshold $\underline{\vartheta}$ by the same argument as previously: if ϑ^B ever fell below $\underline{\vartheta}$, then this low valuation of bonds would only be consistent with the positive surplus policy if agents expected ϑ^B to fall below 0 in finite time. However, such an expectation is not consistent with any equilibrium.

Thus, all equilibrium solution paths must satisfy $\vartheta_t^B \geq \underline{\vartheta}$ at all times. Whether this is sufficient to also rule out cryptocoin bubbles, i.e. equilibria with $\vartheta_t^C > 0$, depends on the threshold $\underline{\vartheta}$ and the growth rate of cryptocoins:

First, if the threshold is tight, $\underline{\vartheta} = \vartheta^*$, then there is no "space" for a bubble on cryptocoins: if $\hat{\vartheta}_t > \vartheta_t^B \geq \vartheta^*$, one can show that equation (18) implies that $\hat{\vartheta}_t$ must grow without bounds and can therefore not be a valid equilibrium path.

Second, even if $\underline{\vartheta} < \vartheta^*$, the policy still succeeds in eliminating cryptocoin bubbles if the supply growth rate of the latter is larger than the desired bubble mining rate in equilibrium, $\breve{\mu}^{\mathcal{C}} > \breve{\mu}^{\mathcal{B}}$. The reason is that the dilution rate of cryptocoins then exceeds that of government

[61]It is without loss of generality to consider only a single additional asset as we could always combine the portfolio of all potential bubble assets in the economy into a single asset.

[62]We provide details in Appendix A.6.

bonds. By no arbitrage, households are only willing to hold cryptocoins if they expect them to appreciate in value relative to government bonds over time. Because the aggregate bubble is bounded (relative to the size of the economy), this is only possible if ϑ_t^B shrinks over time, precisely as in the equilibria that lead to the no bubble equilibrium. As above, any positive lower bound $\underline{\vartheta}$ on ϑ_t^B makes such price paths inconsistent with a perfect foresight equilibrium.

In the remaining case, $\underline{\vartheta} < \vartheta^*$ and $\breve{\mu}^{\mathcal{C}} \leq \breve{\mu}^{\mathcal{B}}$, one can show that there are multiple solution pairs $(\vartheta, \hat{\vartheta})$ to equations (18) and (19) that satisfy $0 \leq \vartheta_t \leq \hat{\vartheta}_t \leq 1$. In this case, the policy fails to select a unique equilibrium. In total, we obtain the following proposition.

Proposition 3. *Under the threshold policy discussed in this section, all equilibria have the property that $\vartheta_t^B \geq \underline{\vartheta}$ for all t. If $\underline{\vartheta} = \vartheta^*$ or $\breve{\mu}^{\mathcal{C}} > \breve{\mu}^{\mathcal{B}}$, the equilibrium is unique and satisfies $\vartheta_t^B = \hat{\vartheta}_t = \vartheta^*$ for all t.*

We remark that the distinction between $\breve{\mu}^{\mathcal{C}} > \breve{\mu}^{\mathcal{B}}$ and $\breve{\mu}^{\mathcal{C}} \leq \breve{\mu}^{\mathcal{B}}$ affects the nature of the off-equilibrium backing considerably. In the former case, any threshold $\underline{\vartheta}$ works. Any promise of arbitrarily small surpluses arbitrarily far in the future is sufficient to rule out all equilibria but the one featuring a stationary public debt bubble. In the latter case, only $\underline{\vartheta} = \vartheta^*$ works. The fiscal authority must start raising surpluses immediately in response to any drop of the value of debt below its equilibrium value.

6.2 Imperfect Commitment

The previous results assume that the government can perfectly commit to any policy that is fully contingent on realized price histories. We now ask when off-equilibrium fiscal backing is credible if the government has a social welfare objective and imperfect commitment power. Because a positive rate of bubble mining can only be optimal in the idiosyncratic risk model (compare Section 5.4), we restrict attention here to that model.

We make the following modification to the setup of our augmented model with two potential bubbles. We replace the single infinite-horizon government by a sequence of governments, each with a finite term of office $T > 0$. Government $j \in \{0, 1, \dots\}$ chooses the policy variables $i_t, \mu_t^{\mathcal{B}}, \tau_t$ subject to the budget constraint (3) for periods $t \in [jT, (j+1)T)$, but takes the policy choices at other times (made by other governments $j' \neq j$) as given and cannot commit to react to policies chosen by previous governments. All governments maximize the same social welfare function which is given by some weighted average of individual agent utilities as stated in equation (14).

We describe the details of the modified setup in Appendix A.8. Effectively, each government j takes prices $q^B_{(j+1)T}, q^C_{(j+1)T}, q^K_{(j+1)T}$ at the end of its tenure as given and chooses an optimal (Ramsey) policy path over $[jT, (j+1)T)$ that implies price paths for q_t^B, q_t^C, q_t^K in the competitive equilibrium of our model over that time interval.[63] An *equilibrium in this policy game* consists of a set of price paths $\{q^B_t, q^C_t, q^K_t\}^\infty_{t=0}$ and policies $\{i_t, \mu^B_t, \tau_t\}^\infty_{t=0}$ such that (1) the sequences correspond to a competitive equilibrium of the augmented model with two bubbles and (2) for each j, the policy sequence restricted to the interval $[jT,(j+1)T)$ is optimal for government j conditional on terminal prices $q^B_{(j+1)T}, q^C_{(j+1)T}, q^K_{(j+1)T}$.

We assume that model parameters are such that in the setting of Section 5.4, a positive rate of bubble mining is optimal and denote that rate by $\breve{\mu}^*$. We denote by ϑ^* the optimal value of ϑ corresponding to that policy ($\vartheta^* = \vartheta^{\text{opt}}$ in the notation of Section 5.4). With these assumptions and definitions, we can formulate our main result, which characterizes the possible values of government bonds in the equilibria of the policy game:[64]

Proposition 4. *Any equilibrium of the policy game features a positive aggregate bubble with $\hat{\vartheta}_t \geq \vartheta^*$. In addition:*

(i) If $\breve{\mu}^{\mathcal{C}} > \breve{\mu}^*$, the equilibrium is unique and features $\hat{\vartheta}_t = \vartheta^B_t = \vartheta^*$.

(ii) If $\breve{\mu}^{\mathcal{C}} \leq \breve{\mu}^*$, there exists an equilibrium for any initial value $\vartheta^B_0 \in [0, \vartheta^*]$.

We prove this result in Appendix A.8. There are two key takeaways from this proposition. First, even under imperfect commitment, fiscal policy always eliminates the no bubble equilibrium. Second, equilibria with bubbles on cryptocoins can only be ruled out in case (i), i.e. if the growth rate of cryptocoins exceeds the optimal bubble mining rate $\breve{\mu}^*$ in the equilibrium with a stationary public debt bubble (and no other bubbles). Otherwise, equilibria with arbitarily small public debt bubbles cannot be ruled out. Selecting a unique equilibrium under imperfect commitment fails in precisely the cases in which the switching threshold $\underline{\vartheta}$ has to be tight in the results under perfect commitment presented in the previous subsection.

To gain intuition for the result, we first explain why commitment is not required to eliminate the no bubble equilibrium. In many standard models, there is a basic time-inconsistency problem of nominal debt: a government would prefer to inflate away pre-existing nominal liabilities in order to avoid current and future distortionary taxes necessary to repay the debt. Key to this argument is that the taxes required to support additional debt are welfare-reducing. But this is not the problem here. Even though debt is funded by distortionary capital taxes that reduce growth, a larger value of government debt can nevertheless be beneficial because it improves risk sharing (compare Section 5.4). If the total value of safe assets such as government bonds is sufficiently small, each finite-horizon government has incentives to raise taxes during

[63] Once terminal prices over the finite time interval have been fixed, there is no scope for multiplicity anymore. Each policy path over the interval is associated with a unique equilibrium.

[64] As in the previous subsection, we continue to restrict attention to deterministic equilibria with absolutely continuous asset price paths.

its own term of office to create more safe assets and improve risk sharing. Commitment power is thus not needed to eliminate equilibria in which the total value of safe assets is too small. This explains why the no bubble equilibrium can be ruled out even under limited commitment.

Importantly, however, it is the total value of safe assets ($(q_t^B + q_t^C)K_t$) not the value of public debt ($q_t^B K_t$) that is relevant for risk sharing. For a sufficiently large cryptocoin bubble, safe assets are abundant and the marginal welfare impact of additional (tax-funded) debt becomes negative. In this case, the standard intuition is restored: a government would prefer not to honor the full outstanding real value of its nominal liabilities.

The question is thus whether in equilibrium, a sufficiently large cryptocoin bubble can provide a substitute for government debt as a safe asset. The answer is yes if the growth rate of cryptocoins is sufficiently low, but it is no otherwise. If $\breve{\mu}^{\mathcal{C}} > \breve{\mu}^*$, cryptocoins are diluted at a faster rate than the optimal bubble mining rate, so that, unless cryptocoins are worthless, the aggregate bubble must be suboptimally small, at least in some future period. In such a future period, the government in charge would have incentives to raise taxes in order to create additional safe assets (through government debt). These incentives to tax by some future government implement a form of off-equilibrium fiscal backing that eliminates cryptocoin bubbles in the first place.

6.3 Remark: Bubbles on Private Assets and Ponzi Schemes

We have logically attached the bubble on other assets to a separate intrinsically worthless asset in exogenous supply. However, there is no economic difference if instead we considered one of the following alternative arrangements.

First, bubbles could be attached to long-lived liabilities issued by households in the model. These liabilities neither need to be intrinsically worthless, nor in exogenous supply. If, for example, a private household issued a perpetuity with a positive fundamental value, there could be an additional bubble component attached to it and the household may be able to create additional bubbles when issuing more perpetuities. Whether this is possible is ultimately a matter of coordination of market beliefs and thus depends on the equilibrium selection. If other agents are only willing to buy such a bond at a price that does not exceed the present value of future coupon payments, then bubble creation fails and the agent has to pay back in present value exactly what she has borrowed. However, when rational bubbles are possible, then other agents could coordinate on an equilibrium in which they are willing to pay more for the bond than the present value of coupon payments in the expectation that they can pass it on to others at a high price in the future.

However, we can in such cases always logically split the assets into a bubble-free asset and an intrinsically worthless asset and simply select an equilibrium in which the two are always effectively held as a bundle. Therefore, any policy that rules out intrinsically worthless bubbles ultimately also rules out bubbles on assets issued by private agents.

Second, the benefits of issuing a bubbly asset can be generated in an alternative way without issuing long-lived liabilities. We have defined a bubble as a situation in which the market value of an asset exceeds its fundamental value and this works only for long-lived assets. However, the economic equivalent of issuing a bubbly asset can also be achieved through a Ponzi scheme, a chain of debt issuance that is perpetually grown and rolled over such that the present value of time-T debt liabilities does not converge to zero as the horizon T approaches infinity. Unlike issuing a long-lived asset with a bubble, each individual debt claim in this chain can have finite maturity and be priced according to its fundamental value and thus not have a bubble component. Yet, when the totality of all debt claims is considered as a bundle, the Ponzi scheme represents a bubble because the present value of payouts to debt holders falls short of the total value of debt issued. An agent able to run a Ponzi schemes can effectively mine this bubble by growing such "Ponzi debt" at a faster rate.

Formally, the ability of private households to run Ponzi schemes would require that markets do not enforce a strict no Ponzi condition on individual agents as we have assumed so far. If the market does not impose a strict no Ponzi condition on agent i, agent i's transversality condition becomes $\mathbb{E}_0 \xi_T^i b_T^i \to -b_0^{p,i} < 0$, where $b_0^{p,i}$ is the present value of bubble mining ("Ponzi wealth") that the market permits the agent in a given equilibrium. The equilibrium allocation is then equivalent to the one of a model in which the agent issues a long-lived bubble asset of value $b_0^{p,i}$ at time 0, so that $b_0^{p,i}$ is included in the agent's measured (bond) wealth b_0^i and the agent faces a strict no Ponzi condition $\liminf_{T\to\infty} \mathbb{E}_0 \xi_T^i b_T^i \geq 0$.

Because equilibria with private Ponzi schemes and private bubble issuance are equivalent, policies that rule out bubbles on private assets also eliminate equilibria featuring private Ponzi schemes. This is in particular true for the off-equilibrium fiscal policy analyzed in Sections 6.1 and 6.2.

Importantly, private agents could not resort to a similar equilibrium selection policy as the government to force the bubble onto their liabilities. While the government can use taxation to raise surpluses that grow proportionally with the economy, private households in our models are unable to generate resources that grow in lockstep with the economy and thus provide off-equilibrium backing to their liabilities in the same way as the government. In the perpetual youth example, the output share of each individual agent decays over time as more and more other agents are born. In the idiosyncratic risk example, there are some states in which an agent has experienced many negative shocks and become arbitrarily poor relative to the size of the economy. Beyond these models, both channels are arguably present in reality and restrict, for example, the ability of private corporations to implement an off-equilibrium selection policy in their favor.

6.4 Alternative Policies

Fiscal policy is not the only policy that can rule out bubbles on other assets. We briefly discuss here how other policies, insolvency law, restrictions on certain assets, and financial repression, can facilitate the equilibrium selection.

Insolvency Law. Institutional rules such as insolvency laws can effectively impose no Ponzi conditions on private agents through the legal system. If effective, such rules can rule out all Ponzi scheme run by private agents, so that at most the government is able to run a Ponzi scheme.

While effective to eliminate Ponzi schemes, such rules do not directly rule out bubbles on long-lived assets. However, because most assets in reality are not long-lived, eliminating Ponzi schemes goes a long way in narrowing down the possibilities for alternative bubbles.

In addition, insolvency law can indirectly affect the viability of bubbles on long-lived assets that are liabilities of firms subject to insolvency law such as stocks. If the claims of holders of these assets are wiped out in bankruptcy, then the institutional environment triggers a forced bubble burst in any bankruptcy event.[65] To the extent that all firms face some background bankruptcy risk, bubbles on their long-lived liabilities are harder to sustain because their required return conditional on no bankruptcy must rise to compensate investors for the bubble burst in bankruptcy.[66]

Restrictions on Specific Assets. Specific asset classes (of long-lived assets) can also be targeted directly by legal restrictions to prevent bubbles on them. This is particularly relevant for cryptoassets that are not liabilities of entities affected by insolvency law and thus not affected by considerations in the previous paragraph.

For example, the government could impose a tax on holding such asset. In our model, a tax on cryptocoins whose tax revenues are used to lower output taxes would have the same effect as a higher exogenous growth rate $\breve{\mu}^{\mathcal{C}}$. A sufficiently high tax can therefore drive the effective $\breve{\mu}^{\mathcal{C}}$ above $\breve{\mu}^{*}$ and thereby ensure that off-equilibrium fiscal backing remains credible.

Alternatively, a government could also impose trading restrictions on long-lived assets that could support a bubble. Because the ultimate economic value of a bubble results from enabling

[65] Specifically, it is important that a bankruptcy also inhibits the continued trading of the fundamentally worthless claims post-bankruptcy. This is, e.g., the case if the claims only exist in digital accounts and are deleted in bankruptcy.

[66] This is effectively very similar to a larger dilution rate due to supply growth of the asset. Thus off-equilibrium fiscal backing is more credible against such bubbles.

beneficial trades (compare Section 4), bubbles on illiquid assets are less likely or even outright impossible.

Financial Repression. Instead of making bubbles on other specific assets more difficult, financial repression tools such as reserve and liquidity requirements seek to support the demand for government liabilities relative to other assets. Formally, they make $\mathcal{B}$ more like $\mathcal{M}$ in equation (1) and open up a spread Δi between bonds and cryptocoins. If conducted on a sufficiently large scale, such policies can even drive r^f (the illiquid rate) above g and eliminate bubbles, yet keep the government's funding costs $r^f - \Delta i$ low, so that the government still enjoys the same advantages as with a bubble.[67]

7 Conclusion and Lessons for Debt Sustainability Analysis

This paper integrates the typically ignored bubble term in the FTPL, which is necessary to explain low inflation in countries with persistently negative primary surpluses. We conclude with some lessons for debt sustainability analysis. Applying these lessons to assess debt sustainability of specific countries appears an interesting avenue for future research.

The traditional concern of debt sustainability analysis is the ability of a government to generate the future primary surpluses necessary to back its outstanding debt obligations (in equilibrium). A public debt bubble opens up the possibility that debt may be sustainable even in the absence of such future surpluses. Whether it is, requires an analysis of the bubble. There are two aspects to be considered.

First, debt sustainability analysis should attempt to quantify the fiscal space created by the bubble (in equilibrium). This can be done, for example, by determining the maximum debt level that can be supported in the absence of any primary surpluses or by mapping out the Laffer curve. To do so, one has to determine not only the size of the bubble for the policy regime in place, but also how the bubble reacts to changes in policy. An example of such an exercise is carried out by Brunnermeier et al. (2021a) who extend the idiosyncratic risk example provided in this paper, calibrate it to U.S. data, and quantify the Laffer curve.

Second, debt sustainability analysis should assess stability of the bubble by considering the defenses in place to prevent coordination to other equilibria. The primary defense discussed in this paper is (contingent) fiscal policy. Just like traditional debt sustainability analysis, assessing the stability of the bubble thus starts with an assessment of the government's capacity to generate future primary surpluses. However, the emphasis shifts from the capacity to raise actual future surpluses in equilibrium to backup capacity, the mere ability and credibility to do

[67] See e.g. Di Tella (2020) and Merkel (2020) for frameworks similar to our idiosyncratic risk model in which monetary frictions may crowd out bubbles.

so off-equilibrium. Backup capacity may by limited, for example, by actual limits to taxation, by imperfect commitment, or by political frictions. In addition to analyzing backup fiscal capacity, the stability of the public debt bubble can be assessed by identifying other potential assets to which the bubble could jump and the policies in place to prevent such bubble jumps (e.g. insolvency law, regulation of crypto assets). Going beyond the closed economy framework of this paper, this analysis should also consider foreign assets such as foreign government debt as competing stores of value that may carry a bubble. Some important policy considerations in the international case are explored in Brunnermeier et al. (2021b).

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A Appendix

A.1 Derivation of the Continuous-time Fiscal Theory Equation (Equation (1))

As in the discrete-time case, the derivation of the fiscal theory equation starts with the government flow budget constraint, which here is

$$\left(\mu_t^{\mathcal{B}}\mathcal{B}_t + \mu_t^{\mathcal{M}}\mathcal{M}_t + \mathcal{P}_t T_t\right) dt = \left(i_t \mathcal{B}_t + i_t^m \mathcal{M}_t + \mathcal{P}_t G_t\right) dt,$$

where $\mathcal{B}_t$, $\mathcal{M}_t$, T_t, G_t, i_t and i_t^m have the same meaning as in the main text and $\mu_t^{\mathcal{B}}$, $\mu_t^{\mathcal{M}}$ are the growth rates of nominal bonds and money, respectively.[68] Multiplying the budget constraint by the nominal SDF $\xi_t/\mathcal{P}_t$ and rearranging yields

$$\left(\left(\mu_t^{\mathcal{B}} - i_t\right)\frac{\xi_t}{\mathcal{P}_t}\mathcal{B}_t + \left(\mu_t^{\mathcal{M}} - i_t\right)\frac{\xi_t}{\mathcal{P}_t}\mathcal{M}_t\right) dt = -\xi_t\left((T_t - G_t) + \Delta i_t \frac{\mathcal{M}_t}{\mathcal{P}_t}\right) dt. \tag{20}$$

Next, Ito's product rule implies

$$\begin{aligned}
d\left(\frac{\xi_t}{\mathcal{P}_t}\mathcal{B}_t\right) &= \left(\mu_t^{\mathcal{B}} - i_t\right)\frac{\xi_t}{\mathcal{P}_t}\mathcal{B}_t dt + \frac{\xi_t}{\mathcal{P}_t}\mathcal{B}_t\left(\frac{d\left(\xi_t/\mathcal{P}_t\right)}{\xi_t/\mathcal{P}_t} + i_t dt\right),\\
d\left(\frac{\xi_t}{\mathcal{P}_t}\mathcal{M}_t\right) &= \left(\mu_t^{\mathcal{M}} - i_t\right)\frac{\xi_t}{\mathcal{P}_t}\mathcal{M}_t dt + \frac{\xi_t}{\mathcal{P}_t}\mathcal{M}_t\left(\frac{d\left(\xi_t/\mathcal{P}_t\right)}{\xi_t/\mathcal{P}_t} + i_t dt\right).
\end{aligned}$$

Solving these last two equations for $\left(\mu_t^{\mathcal{B}} - i_t\right)\frac{\xi_t}{\mathcal{P}}{}_t\mathcal{B}_t dt$ and $\left(\mu_t^{\mathcal{M}} - i_t\right)\frac{\xi_t}{\mathcal{P}_t}\mathcal{M}_t dt$, respectively, and substituting the results into equation (20) yields (after rearranging)

$$d\left(\frac{\xi_t}{\mathcal{P}_t}\left(\mathcal{B}_t + \mathcal{M}_t\right)\right) = -\xi_t\left(\mathcal{P}_t\left(T_t - G_t\right) + \Delta i_t \mathcal{M}_t\right) dt + \xi_t \frac{\mathcal{B}_t + \mathcal{M}_t}{\mathcal{P}_t}\left(\frac{d\left(\xi_t/\mathcal{P}_t\right)}{\xi_t/\mathcal{P}_t} + i_t dt\right),$$

or in integral form

$$\xi_T \frac{\mathcal{B}_T + \mathcal{M}_T}{\mathcal{P}_T} - \xi_t \frac{\mathcal{B}_t + \mathcal{M}_t}{\mathcal{P}_t} = -\int_t^T \xi_s\left(T_s - G_s\right) ds - \int_t^T \xi_s \Delta i_s \frac{\mathcal{M}_s}{\mathcal{P}_s} ds + \int_t^T \xi_s \frac{\mathcal{B}_s + \mathcal{M}_s}{\mathcal{P}_s}\left(\frac{d\left(\xi_s/\mathcal{P}_s\right)}{\xi_s/\mathcal{P}_s} + i_s dt\right).$$

Up to this point, we have merely rearranged and integrated the government budget constraint. To derive the fiscal theory equation, the literature proceeds by using two equilibrium conditions. First, if the nominal SDF $\xi/\mathcal{P}$ prices the government bonds, then its expected rate of change must be the negative of the nominal interest rate. Then, the last stochastic integral on

[68] Here we abstract from long-term bonds and the possibility of taxes, spending, and adjustments in $\mathcal{B}$ and $\mathcal{M}$ that are not absolutely continuous over time (e.g., lumpy adjustments in response to a Poisson shock). Such elements could be easily added, but require more complicated notation without generating additional insights for our purposes.

the right must be a martingale and disappears when taking conditional time-t expectations $E_t[\cdot]$. Second, a private-sector transversality condition is invoked to eliminate a terminal value of government debt when passing to the limit $T \to \infty$. We perform the first operation, but do not want to restrict attention to environments where transversality can rule out a nonzero discounted terminal value. When taking the limit $T \to \infty$, we therefore arrive at the more general equation (1).

A.2 Omitted Details in Section 3

In this section, we present some additional formal details about the two example models and their solution.

A.2.1 Example I: Perpetual Youth

Details on the Household Problem. The HJB equation for household i's problem is

$$\rho V_t\left(b_t^i, k_t^i\right) - \partial_t V_t\left(b_t^i, k_t^i\right) = \max_{c^i}\left\{\log c^i + \partial_b V_t\left(b_t^i, k_t^i\right)\left(r_t^f b_t^i + (1-\tau_t)\, a k_t^i - c^i\right) + \partial_k V_t\left(b_t^i, k_t^i\right)\left(-\delta k_t^i\right)\right\}.$$

This is a standard consumption-savings problem, so we conjecture a functional form $V_t\left(b^i, k^i\right) = \alpha_t + \frac{1}{\rho}\log\left(b^i + q_t^K k^i\right)$ for the value function, where α_t and q_t^K depend on (aggregate) investment opportunities, but not on individual state variables b^i and k^i. We verify this conjecture below.

Substituting the functional form guess into the HJB equation yields

$$\rho\alpha_t + \log\left(b^i + q_t^K k^i\right) - \dot{\alpha}_t - \frac{1}{\rho}\frac{\dot{q}_t^K k^i}{b^i + q_t^K k^i} = \max_{c^i}\left\{\log c^i - \frac{1}{\rho}\frac{1}{b^i + q_t^K k^i} c^i\right\} + \frac{1}{\rho}\frac{1}{b^i + q_t^K k^i}\left(r_t^f b^i + (1-\tau_t)\, a k^i - \delta q_t^K k^i\right). \quad (21)$$

The first-order condition for the maximization with respect to c^i is

$$0 = \frac{1}{c^i} - \frac{1}{\rho\left(b^i + q_t^K k^i\right)},$$

which immediately implies the consumption rule (6) stated in the main text.

Substituting the optimal consumption choice into equation (21) and canceling and rearranging terms implies

$$\left(\rho\alpha_t - \dot{\alpha}_t + 1 - \log\rho - \frac{r_t^f}{\rho}\right) b^i + \left(\left(\rho\alpha_t - \dot{\alpha}_t + 1 - \log\rho\right) q_t^K - \frac{(1-\tau_t)\, a - \delta q_t^K + \dot{q}_t^K}{\rho}\right) k^i = 0. \tag{22}$$

Clearly, the functional form guess can only be correct if equation (22) is satisfied for all states (b^i, k^i). But that means in particular that α_t and q_t^K must satisfy the two equations

$$\rho\alpha_t - \dot{\alpha}_t + 1 - \log\rho = \frac{r_t^f}{\rho}, \tag{23}$$

$$\rho\alpha_t - \dot{\alpha}_t + 1 - \log\rho = \frac{(1-\tau_t)\, a - \delta q_t^K + \dot{q}_t^K}{\rho q_t^K}, \tag{24}$$

that are obtained by setting either b^i or k^i to zero in equation (22). Conversely, if there are functions α_t and q_t^K that satisfy equations (23) and (24), then clearly also (22) holds for all (b^i, k^i) (and all times t) and because (22) is equivalent to the original HJB equation (21), also that equation must then hold for all (b^i, k^i) (and all times t). To verify the conjectured functional form for V, it is thus sufficient to show that α_t, q_t^K satisfying (23) and (24) do indeed exist.

Equation (23) is a linear ordinary differential equation (ODE) in α_t. The general solution for this ODE is

$$\alpha_t = \beta e^{\rho t} + \int_t^\infty e^{-\rho(s-t)} \left(1 - \log\rho - \frac{r_s^f}{\rho}\right) ds,$$

where $\beta \in \mathbb{R}$ is an integration constant. This solution is well-defined as long as the given risk-free rate path r_t^f is sufficiently regular (e.g., continuous and bounded). Thus, a solution to (23) does indeed exist and with the specific choice $\beta = 0$, the resulting HJB solution function V_t also satisfies a transversality condition (which is necessary for optimality).

Next, combine equations (23) and (24) to substitute out the α_t-dependent terms, which yields the ODE

$$\frac{r_t^f}{\rho} = \frac{(1-\tau_t)\, a - \delta q_t^K + \dot{q}_t^K}{\rho q_t^K}$$

for q_t^K. Rearranging implies the equation

$$\dot{q}_t^K = -(1-\tau_t)\, a + \left(r_t^f + \delta\right) q_t^K, \tag{25}$$

which is a differential version of equation (7) in the main text. To derive the latter equation, use $k_s^i = k_t^i e^{-\delta(s-t)}$ and thus

$$e^{-\int_t^s r_{s'}^f ds'} q_s^K k_s^i = e^{-\int_t^s \left(r_{s'}^f + \delta\right) ds'} q_s^K k_t^i.$$

Taking the time-derivative of the product on the right yields

$$d\left(e^{-\int_t^s\left(r^f_{s'}+\delta\right)ds'}q^K_s\right)=-\left(r^f_s+\delta\right)e^{-\int_t^s\left(r^f_{s'}+\delta\right)ds'}q^K_s ds+e^{-\int_t^s\left(r^f_{s'}+\delta\right)ds'}\dot{q}^K_s ds,$$

so that

$$\frac{d\left(e^{-\int_t^s\left(r^f_{s'}+\delta\right)ds'}q^K_s k^i_t\right)}{ds}=e^{-\int_t^s\left(r^f_{s'}+\delta\right)ds'}\underbrace{\left(\dot{q}^K_s-\left(r^f_s+\delta\right)q^K_s\right)}_{=-(1-\tau_s)a\text{ by (25)}}k^i_t.$$

Integrating both sides over $s\in[t,T]$ yields

$$\begin{aligned}e^{-\int_t^s\left(r^f_{s'}+\delta\right)ds'}q^K_T k^i_t-q^K_t k^i_t&=-\int_t^T e^{-\int_t^s\left(r^f_{s'}+\delta\right)ds'}(1-\tau_s)ak^i_t ds\\&=-\int_t^T e^{-\int_t^s r^f_{s'}ds'}(1-\tau_s)ak^i_s ds\end{aligned}$$

Rearranging and taking the limit $T\to\infty$ implies equation (7).

Derivation of Equation (9). We use the notation

$$\mu^{q,B}_t:=\frac{\dot{q}^B_t}{q^B_t},\qquad \mu^{q,K}_t:=\frac{\dot{q}^K_t}{q^K_t},\qquad \mu^{\vartheta}_t:=\frac{\dot{\vartheta}_t}{\vartheta_t}.$$

These definitions are unproblematic as long as the denominator expressions are different from zero, that is as long as both bonds and human capital have positive value in equilibrium. By definition of ϑ_t, we have $\mu^{\vartheta}_t=(1-\vartheta_t)\left(\mu^{q,B}_t-\mu^{q,K}_t\right)$.

We first divide equation (25) by q^K_t and then plug in the risk-free rate expression from equation (8) in the main text. The resulting equation is

$$\mu^{q,K}_t=-(1-\tau_t)\frac{a}{q^K_t}-\breve{\mu}^{\mathcal{B}}_t+g+\mu^{q,B}_t+\delta.$$

Next, the government budget constraint (3) implies $\tau_t a=\mathfrak{g}-\breve{\mu}^{\mathcal{B}}_t q^B_t$ and we have

$$q^B_t=\vartheta_t\frac{a-\mathfrak{g}}{\rho},\qquad q^K_t=(1-\vartheta_t)\frac{a-\mathfrak{g}}{\rho}.$$

Substituting these expressions into the previous equation and rearranging yields

$$\mu^{q,K}_t-\mu^{q,B}_t=-\frac{\rho}{1-\vartheta_t}-\frac{\breve{\mu}^{\mathcal{B}}_t}{1-\vartheta_t}+g+\delta$$

and thus

$$\mu^{\vartheta}_t=\rho+\breve{\mu}^{\mathcal{B}}_t-(1-\vartheta_t)\left(g+\delta\right).$$

This is a backward equation for ϑ_t that has been derived under the assumption that bonds have a positive value ($q_t^{\mathcal{B}} > 0 \Leftrightarrow \vartheta_t > 0$). In particular, in these cases multiplying the equation by ϑ_t represents an equivalence transformation. Furthermore, if $\vartheta_t = 0$, then no arbitrage requires also $\dot{\vartheta}_t = 0$; otherwise, agents could earn an infinite risk-free return from investing into bonds. Consequently, the ODE

$$\dot{\vartheta}_t = \Big(\rho + \breve{\mu}_t^{\mathcal{B}} - (1 - \vartheta_t)\,(\delta + g)\Big)\,\vartheta_t \tag{26}$$

must hold along any equilibrium path, regardless of whether bonds have positive value or not.

Equation (26) is the differential version of equation (9) stated in the main text. The latter equation can be derived from (26) by appropriate time integration along the same lines as the FTPL equation from the flow budget constraint (see Appendix A.1) and equation (7) from equation (25) (see household problem above). For this reason, we omit the proof here.

Steady-State Equilibria. In steady state, equation (26) simplifies to

$$\Big(\rho + \breve{\mu}^{\mathcal{B}} - (1 - \vartheta)\,(\delta + g)\Big)\,\vartheta = 0.$$

This is a second-order polynomial and has precisely two solutions, $\vartheta = 0$ and $\vartheta = 1 - \frac{\rho + \breve{\mu}^{\mathcal{B}}}{\delta + g}$. Only the latter solution can be associated with a positive value of government bonds, $q_t^{\mathcal{B}} > 0$, and it is if and only if

$$1 > \frac{\rho + \breve{\mu}^{\mathcal{B}}}{\delta + g}.$$

A.2.2 Example II: Uninsurable Idiosyncratic Risk

Details on the Household Problem. Because agents can freely adjust portfolios at each time instant in this model, the household problem can be written in terms of a single state variable, net worth $n_t^i = q_t^K k_t^i + b_t^i$. As in the main text, let $\theta_t^i := b_t^i / n_t^i$ denote the fraction of net worth invested into bonds. Then net worth evolves according to

$$\frac{dn_t^i}{n_t^i} = -\frac{c_t^i}{n_t^i}dt + dr_t^{\mathcal{B}} + \left(1 - \theta_t^i\right)\left(dr_t^{K,i}\left(\iota_t^i\right) - dr_t^{\mathcal{B}}\right), \tag{27}$$

where returns $dr_t^{\mathcal{B}}$ and $dr_t^{K,i}\left(\iota_t^i\right)$ are as stated in the main text.

The household chooses consumption c_t^i, real investment ι_t^i, and the portfolio share θ_t^i to maximize utility V_0^i subject to (27). The HJB equation for this problem is[69]

$$\rho V_t\left(n^i\right) - \partial_t V_t\left(n^i\right) = \max_{c^i,\theta^i,\iota^i}\Bigg\{\log c^i + V_t'\left(n^i\right)\left[-c^i + n^i\left(\frac{dr_t^{\mathcal{B}}}{dt} + \left(1-\theta^i\right)\overbrace{\left(\frac{a-\mathfrak{g}-\iota^i}{q_t^K} + \Phi\left(\iota^i\right) - \Phi(\iota_t) - \frac{\mu_t^{\vartheta} - \breve{\mu}_t^{\mathcal{B}}}{1-\vartheta_t}\right)}^{=\frac{\mathbb{E}_t\left[dr_t^{K,i}\left(\iota_t^i\right)\right]}{dt} - \frac{dr_t^{\mathcal{B}}}{dt}}\right)\right] + \frac{1}{2}V_t''\left(n^i\right)\left(n^i\right)^2\left(1-\theta^i\right)^2\tilde{\sigma}^2\Bigg\}.$$

This is a standard consumption-portfolio-choice problem, so we conjecture again a functional form $V_t\left(n^i\right) = \alpha_t + \frac{1}{\rho}\log n_t^i$ for the value function, where as for the previous model, α_t depends on (aggregate) investment opportunities, but not on individual net worth n^i.

Substituting this guess into the HJB equation yields

$$\begin{aligned}\rho\alpha_t + \log\left(n^i\right) - \dot{\alpha}_t = {} & \max_{c^i}\left(\log c^i - \frac{c^i}{\rho n^i}\right) \\ & + \frac{1}{\rho}\max_{\theta^i,\iota^i}\left(\left(1-\theta^i\right)\left(\frac{a-\mathfrak{g}-\iota^i}{q_t^K} + \Phi\left(\iota^i\right) - \Phi(\iota_t) - \frac{\mu_t^{\vartheta} - \breve{\mu}_t^{\mathcal{B}}}{1-\vartheta_t}\right) - \frac{\left(1-\theta^i\right)^2\tilde{\sigma}^2}{2}\right) \\ & + \frac{1}{\rho}\frac{dr_t^{\mathcal{B}}}{dt}. \end{aligned} \tag{28}$$

The first-order conditions for the maximization with respect to c^i, ι^i and $1-\theta^i$ are

$$\begin{aligned} 0 &= \frac{1}{c^i} - \frac{1}{\rho n^i}, \\ 0 &= \frac{1-\theta^i}{\rho}\left(\Phi'(\iota^i) - \frac{1}{q_t^K}\right) \\ 0 &= \left(\frac{a-\mathfrak{g}-\iota^i}{q_t^K} + \Phi\left(\iota^i\right) - \Phi(\iota_t) - \frac{\mu_t^{\vartheta} - \breve{\mu}_t^{\mathcal{B}}}{1-\vartheta_t}\right) - \left(1-\theta^i\right)\tilde{\sigma}^2 \end{aligned}$$

These three equations immediately imply the three conditions stated in the main text.

Expressing q^B, q^K, ι in Terms of ϑ. Combining the aggregate resource constraint (10) with the optimal consumption rule (aggregated over all agents) relates total wealth to total consumption in each period,

$$q_t^B + q_t^K = \frac{1}{\rho}C_t/K_t = \frac{a-\mathfrak{g}-\iota_t}{\rho}.$$

[69] Here, we have used the government budget constraint (3) to eliminate $\tau_t a$ in the return on capital.

Divide both equations by q_t^K, use $1-\vartheta_t = \frac{q_t^K}{q_t^B+q_t^k}$ on the left-hand side and $q_t^K = \frac{1}{\Phi'(\iota_t)} = 1+\phi\iota_t$ on the right hand side to obtain an equation that relates ϑ_t to the investment rate ι_t:

$$\frac{1}{1-\vartheta_t} = \frac{a-\mathfrak{g}-\iota_t}{1+\phi\iota_t}.$$

Solving for ι_t yields

$$\iota_t = \frac{(1-\vartheta_t)(a-\mathfrak{g})-\rho}{1-\vartheta_t+\phi\rho}$$

and substituting that equation into $q_t^K = 1+\phi\iota_t$ and $q_t^B = \frac{\vartheta_t}{1-\vartheta_t} q_t^K$ implies

$$q_t^B = \vartheta_t \frac{1+\phi(a-\mathfrak{g})}{1-\vartheta_t+\phi\rho},$$
$$q_t^K = (1-\vartheta_t)\frac{1+\phi(a-\mathfrak{g})}{1-\vartheta_t+\phi\rho}.$$

Derivation of Equation (12). Bond market clearing and the fact that all households choose the same θ_t^i imply $\theta_t^i = \vartheta_t$ and substituting this and into the first-order condition for θ^i yields

$$\frac{a-\mathfrak{g}-\iota_t}{q_t^K} - \frac{\mu_t^\vartheta - \check{\mu}_t^\mathcal{B}}{1-\vartheta_t} = (1-\vartheta_t)\tilde{\sigma}^2.$$

Now use $\frac{a-\mathfrak{g}-\iota_t}{q_t^K} = \frac{\rho}{1-\vartheta_t}$, multiply by $1-\vartheta_t$ and solve for μ_t^ϑ:

$$\mu_t^\vartheta = \rho + \check{\mu}_t^\mathcal{B} - (1-\vartheta_t)^2\tilde{\sigma}^2.$$

As in the perpetual youth model, this is a backward equation for ϑ_t that has been derived under the assumption that bonds have a positive value ($\vartheta_t > 0$). By the same arguments as there, the equivalent ODE

$$\dot{\vartheta}_t = \left(\rho + \check{\mu}_t^\mathcal{B} - (1-\vartheta_t)^2\tilde{\sigma}^2\right)\vartheta_t \tag{29}$$

remains valid on all equilibrium paths, even if $\vartheta_t = 0$. As before, this is the differential version of the integral equation (12) stated in the main text and the latter can be easily derived from the former by time integration.

Steady-State Equilibria. In steady state, equation (29) simplifies to

$$\left(\rho + \check{\mu}_t^\mathcal{B} - (1-\vartheta_t)^2\tilde{\sigma}^2\right)\vartheta_t = 0.$$

This is a third-order polynomial and has precisely three solutions, $\vartheta = 0$ and $\vartheta = 1 \pm \frac{\sqrt{\rho+\check{\mu}^\mathcal{B}}}{\tilde{\sigma}}$. The solution with $\vartheta = 0$ is associated with worthless government bonds, $q^B = 0$. The solution with "+" has always the property $\vartheta > 1$ and would therefore imply either a negative capital price (if $q^B > 0$) or a negative value of government bonds. Both cases violate free disposal and therefore this solution cannot be a valid equilibrium.

The remaining solution $\vartheta = 1 - \frac{\sqrt{\rho+\breve{\mu}^{\mathcal{B}}}}{\tilde{\sigma}}$ can be a valid equilibrium solution if it is nonnegative. If it is even positive, then the associated equilibrium features a positive value of bonds. This is the case if and only if

$$\tilde{\sigma} > \sqrt{\rho + \breve{\mu}^{\mathcal{B}}}.$$

A.3 Omitted Details in Section 5.4

Derivation of Equation (14). Because all agents consume the same constant fraction ρ of their wealth, the consumption share c_t^i/C_t of agent i at time t must equal the agent's wealth share η_t^i. We can therefore write, using the aggregate resource constraint 10,

$$c_t^i = \eta_t^i C_t = \eta_t^i \left(a - \mathfrak{g} - \iota_t\right) K_t.$$

Thus, expected utility of agent i is given by

$$\mathbb{E}\left[\int_0^\infty e^{-\rho t} \log c_t^i dt\right] = \mathbb{E}\left[\int_0^\infty e^{-\rho t} \left(\log \eta_t^i + \log\left(a - \mathfrak{g} - \iota_t\right) + \log K_t\right) dt\right]. \tag{30}$$

To compute the integrals in equation (30), note that if

$$\frac{dx_t}{x_t} = \mu_t^x dt + \tilde{\sigma}_t^x d\tilde{Z}_t,$$

then

$$\mathbb{E}\left[\int_0^\infty e^{-\rho t} \log x_t dt\right] = \frac{\log x_0}{\rho} + \mathbb{E}\left[\int_0^\infty e^{-\rho t} \frac{\mu_t^x - \left(\tilde{\sigma}_t^x\right)^2/2}{\rho} dt\right]. \tag{31}$$

This follows from a simple calculation:

$$\begin{aligned}
\int_0^\infty e^{-\rho t}\left(\log x_t - \log x_0\right) dt &= \int_0^\infty e^{-\rho t} \int_0^t d\log x_s dt \\
&= \int_0^\infty e^{-\rho t} \left(\int_0^t \mu_s^x ds + \int_0^t \tilde{\sigma}_t^x d\tilde{Z}_s - \frac{1}{2}\int_0^t \left(\tilde{\sigma}_t^x\right)^2 ds\right) dt \\
&= \int_0^\infty \int_s^\infty e^{-\rho t} dt \left(\mu_s^x - \frac{1}{2}\left(\tilde{\sigma}_t^x\right)^2\right) ds + \int_0^\infty e^{-\rho t} \int_0^t \tilde{\sigma}_t^x d\tilde{Z}_s dt \\
&= \int_0^\infty e^{-\rho s} \frac{\mu_s^x - \left(\tilde{\sigma}_t^x\right)^2/2}{\rho} ds + \int_0^\infty e^{-\rho t} \int_0^t \tilde{\sigma}_t^x d\tilde{Z}_s dt.
\end{aligned}$$

When taking expectations, the second term disappears because it is a martingale. Thus, we obtain formula (31).

To apply formula (31), we need to determine $\frac{dK_t}{K_t}$ and $\frac{d\eta_t^i}{\eta_t^i}$. We know that

$$\frac{dK_t}{K_t} = \big(\Phi(\iota_t) - \delta\big)\, dt. \tag{32}$$

For η_t^i, we have

$$\begin{aligned}
\frac{d\eta_t^i}{\eta_t^i} &= \frac{dn_t^i}{n_t^i} - \frac{d\bar{q}_t}{\bar{q}_t} - \frac{dK_t}{K_t} \\
&= \left(-\rho dt + dr_t^{\mathcal{B}} + (1-\vartheta_t)\left(dr_t^{K,i}(\iota_t) - dr_t^{\mathcal{B}}\right)\right) - \mu_t^{\bar{q}} dt - \big(\Phi(\iota_t) - \delta\big)\, dt \\
&= \left(-\rho - \breve{\mu}_t^{\mathcal{B}} + \mu_t^{\vartheta}\right) dt + (1-\vartheta_t)\left(\frac{a - \mathfrak{g} - \iota_t}{q_t^K} + \frac{\breve{\mu}_t^{\mathcal{B}} - \mu_t^{\vartheta}}{1-\vartheta_t}\right) dt + (1-\vartheta_t)\,\tilde{\sigma} d\tilde{Z}_t^i \\
&= \left(-\rho + (1-\vartheta_t)\frac{\rho}{1-\vartheta_t}\right) dt + (1-\vartheta_t)\,\tilde{\sigma} d\tilde{Z}_t^i \\
&= (1-\vartheta_t)\,\tilde{\sigma} d\tilde{Z}_t^i
\end{aligned} \tag{33}$$

where $\bar{q}_t := q_t^B + q_t^K$ and $\mu_t^{\bar{q}} := \frac{\dot{\bar{q}}_t}{\bar{q}_t}$. Here, the third line uses the return expressions and the government budget constraint (3) and the fourth line the aggregate resource constraint (10).

Equations (32) and (33) together with formula (31) allow us to compute the integrals in (30):

$$\begin{aligned}
\mathbb{E}\left[\int_0^\infty e^{-\rho t} \log \eta_t^i dt\right] &= \frac{\log \eta_0^i}{\rho} - \frac{1}{2\rho}\mathbb{E}\left[\int_0^\infty e^{-\rho t}(1-\vartheta_t)^2 \tilde{\sigma}^2 dt\right], \\
\mathbb{E}\left[\int_{t_0}^\infty e^{-\rho t} \log K_t dt\right] &= \frac{\log K_0}{\rho} + \mathbb{E}\left[\int_0^\infty e^{-\rho t}\frac{\big(\Phi(\iota_t) - \delta\big)}{\rho} dt\right].
\end{aligned}$$

Consequently,

$$\begin{aligned}
\mathbb{E}\left[\int_{t_0}^\infty e^{-\rho t} \log c_t^i dt\right] &= \frac{\log \eta_0^i + \log K_0}{\rho} \\
&+ \mathbb{E}\left[\int_0^\infty e^{-\rho t}\left(\log(a - \mathfrak{g} - \iota_t) + \frac{\big(\Phi(\iota_t) - \delta\big)}{\rho} - \frac{(1-\vartheta_t)^2\tilde{\sigma}^2}{2\rho}\right) dt\right]
\end{aligned}$$

After substituting ι_t as a function of ϑ_t (as stated in Section 3.2) and the functional form $\Phi(\iota) = \frac{1}{\phi}\log\big(1 + \phi\iota\big)$, we obtain equation (14).

Existence, Uniqueness and Properties of ϑ^{opt}. Taking first order conditions for maximizing the time-t integrand in equation (14) with respect to ϑ_t implies

$$(1-\vartheta_t)^3\,\tilde{\sigma}^2 + \phi\rho\,(1-\vartheta_t)^2\,\tilde{\sigma}^2 + \rho\,(1-\vartheta_t) - \rho = 0. \tag{34}$$

This is a third-order polynomial equation in $1-\vartheta_t$ and has thus precisely three complex solutions. Because the coefficients on all monomials of positive order are nonnegative and the constant coefficient is negative, standard results on polynomial roots imply that precisely one of these complex solutions is real and that solution must be positive. Consequently, there is a unique real number $\vartheta^{\text{opt}} < 1$ such that $1-\vartheta^{\text{opt}}$ satisfies the first-order condition.[70] It is also easy to see that $\vartheta^{\text{opt}} > 0$ as otherwise the positive-sign terms in equation (34) would exceed the negative-sign term in absolute value. Therefore, there is a unique optimal $\vartheta^{\text{opt}} \in (0,1)$ that maximizes the time-t integrand in equation (14) with respect to ϑ_t. Because the coefficients in equation (34) just depend on the parameters $\tilde{\sigma}$, ρ and ϕ, so does ϑ^{opt}. By the implicit function theorem, ϑ^{opt} must be strictly increasing in $\tilde{\sigma}$.

A.4 Welfare and Optimal Bubble Mining in the Perpetual Youth Example

In this appendix, we briefly outline the welfare analysis for the perpetual youth model. We start by computing the expected utility of a single agent i born at time $t_0(i)$:

$$\mathbb{E}\left[\int_{t_0(i)}^{\infty} e^{-\rho t}\log c_t^i dt\right] = e^{-\rho t_0(i)}\,\frac{X^i_{t_0(i)} + \log\left(a-\mathfrak{g}\right) + \left(\vartheta g - (1-\vartheta)\,\delta\right)/\rho}{\rho} \tag{35}$$

with

$$X_t^i = \begin{cases} \log\left(1-\vartheta\right), & t>0 \\ \log\eta_0^i + \log K_0, & t=0 \end{cases}$$

where $\eta_0^i := \left(b_0^i + q_0^K k_0^i\right) / \left((q_0^B + q_0^K)K_0\right)$ is the initial wealth share at time $t=0$ of an agent i that is already alive at that time.

The derivation of equation (35) proceeds in analogy to the derivation of equation (14) for the idiosyncratic risk model: we use the consumption rule and the aggregate resource constraint to write

$$c_t^i = \eta_t^i\left(a-\mathfrak{g}\right)K_t,$$

[70] The objective is not generally concave, but this first-order condition nevertheless always corresponds to a global maximum as can be readily verified by studying its asymptotic properties as $1-\vartheta \to \infty$ and $1-\vartheta \to 0$.

then decompose

$$\mathbb{E}\left[\int_{t_0(i)}^{\infty} e^{-\rho t}\log c_t^i dt\right] = \int_{t_0(i)}^{\infty} e^{-\rho t}\left(\log\eta_t^i + \log\left(a-\mathfrak{g}\right) + \log K_t\right) dt$$

and then compute the three integrals separately using a version of formula (31):[71]

$$\begin{aligned}
\int_{t_0(i)}^{\infty} e^{-\rho t}\log\eta_t^i dt &= e^{-\rho t_0(i)}\left(\frac{\log\eta_{t_0(i)} - \int_{t_0(i)}^{\infty} e^{-\rho(t-t_0(i))}\left(1-\vartheta_t\right) dt\cdot\left(\delta+g\right)}{\rho}\right),\\
\int_{t_0(i)}^{\infty} e^{-\rho t}\log\left(a-\mathfrak{g}\right) dt &= e^{-\rho t_0(i)}\frac{\log\left(a-\mathfrak{g}\right)}{\rho},\\
\int_{t_0(i)}^{\infty} e^{-\rho t}\log K_t dt &= e^{-\rho t_0(i)}\frac{\log K_{t_0(i)} + g/\rho}{\rho}.
\end{aligned}$$

Combining terms, imposing a steady state, and using the definition $X_{t_0(i)}^i := \log\left(\eta_{t_0(i)}^i K_{t_0(i)}\right)$ and the fact that for $t_0(i) > 0$, $b_{t_0(i)}^i = 0$ and thus

$$\eta_{t_0(i)}^i K_{t_0(i)} = \frac{b_{t_0(i)}^i + q_{t_0(i)}^K k_{t_0(i)}}{q_{t_0(i)}^B + q_{t_0(i)}^K} = 1-\vartheta_{t_0},$$

we obtain equation (35).

Next, let $\lambda(i)$ denote some weighting function with the properties $\int_0^\infty \lambda(i)di < \infty$ and $\lambda(i) \geq 0$ for all i and consider the social welfare function

$$\mathbb{W} := \int_0^\infty \lambda(i)\mathbb{E}\left[\int_{t_0(i)}^{\infty} e^{-\rho t}\log c_t^i dt\right] di$$

Using equation (35), we can write this as

$$\begin{aligned}
\mathbb{W} &= \int_0^\infty \lambda(i)e^{-\rho t_0(i)}\frac{X_{t_0(i)}^i + \log\left(a-\mathfrak{g}\right) + \left(\vartheta g - \left(1-\vartheta\right)\delta\right)/\rho}{\rho} di\\
&= \frac{1}{\rho}\int_0^\infty \lambda(i)e^{-\rho t_0(i)} X_{t_0(i)}^i di + \int_0^\infty \lambda(i)e^{-\rho t_0(i)} di\cdot\frac{\log\left(a-\mathfrak{g}\right) + \left(\vartheta g - \left(1-\vartheta\right)\delta\right)/\rho}{\rho}\\
&= \frac{1}{\rho}\int_0^{L_0} \lambda(i)e^{-\rho t_0(i)}\left(\log\eta_0^i + \log K_0\right) di + \frac{1}{\rho}\int_{L_0}^\infty \lambda(i)e^{-\rho t_0(i)}\log\left(1-\vartheta\right) di\\
&\qquad + \overline{\Lambda}\cdot\frac{\log\left(a-\mathfrak{g}\right) + \left(\vartheta g - \left(1-\vartheta\right)\delta\right)/\rho}{\rho}
\end{aligned}$$

[71] Note that $d\eta_t^i/\eta_t^i = -\left(1-\vartheta_t\right)\left(\delta+g\right)dt$, as a simple calculation shows.

$$= \underbrace{\frac{1}{\rho}\int_0^{L_0} \lambda(i) e^{-\rho t_0(i)} \left(\log \eta_0^i + \log K_0\right) di}_{=:\mathbb{W}_0} + \Lambda \cdot \frac{\log(1-\vartheta)}{\rho} + \overline{\Lambda} \cdot \frac{\log(a - \mathfrak{g}) + \left(\vartheta g - (1-\vartheta)\delta\right)/\rho}{\rho},$$

where in the last line, the constant $\mathbb{W}_0$ does not depend on ϑ and we define

$$\Lambda := \int_{L_0}^{\infty} \lambda(i) e^{-\rho t_0(i)} di$$
$$\overline{\Lambda} := \int_0^{\infty} \lambda(i) e^{-\rho t_0(i)} di$$

which are the total discounted welfare weights attached to all future generations and to all generations, respectively.

A planner with this welfare objective would therefore like to manipulate ϑ to maximize the sum of the last two terms in the expression for $\mathbb{W}$. The first of these terms captures the share of total output added by newborns that they consume themselves. This is decreasing in ϑ as a larger value of bonds means that newborns without financial wealth transfer a larger share of their labor income to previously existing generations as they gradually redeem those generations' bond holdings. This first term is multiplied by Λ because only newborn agents after the initial time are negatively affected. The second term is increasing in ϑ as a larger value of bonds allows for more inter-generational wealth transfer and thereby increases the consumption growth rate of all agents. This second term is multiplied by $\overline{\Lambda}$ because consumption growth of all generations is affected equally.

The first-order condition for maximizing $\mathbb{W}$ with respect to ϑ is

$$\Lambda \frac{1}{1-\vartheta^{\text{opt}}} = \overline{\Lambda} \frac{g+\delta}{\rho} \Rightarrow \vartheta^{\text{opt}} = 1 - \frac{\Lambda}{\overline{\Lambda}} \frac{\rho}{g+\delta}.$$

Thus, there is a unique optimal solution ϑ^{opt} that is strictly decreasing in ρ and strictly increasing in $g+\delta$.

The interpretation of this optimal policy prescription is straightforward when comparing ϑ^{opt} to the competitive equilibrium value of ϑ in the monetary steady state, $\vartheta = 1 - \frac{\rho+\breve{\mu}^{\mathcal{B}}}{\delta+g}$. In the special case that the planner does not care about existing generations, $\overline{\Lambda} = \Lambda$, the intergenerational resource transfer generated by the bubble in the competitive equilibrium without policy intervention is optimal. The size of the bubble is thus constrained efficient. It optimally trades off the lower initial value of consumption for newborns with the higher consumption growth rate that results from a larger bubble value. Therefore, zero primary surpluses and no bubble mining, $\breve{\mu}^{\mathcal{B}} = 0$, implement the optimal policy in this case.

Whenever the planner does care about the generation initially alive, $\overline{\Lambda} > \Lambda$, then she desires to transfer additional resources from future generations to initial generations than in the benchmark case where $\overline{\Lambda} = \Lambda$. This requires increasing the value of debt (lowering ϑ) and taxing future generations. Therefore, the government optimally runs positive primary surpluses and uses these surpluses to back its debt, $\breve{\mu}^{\mathcal{B}} < 0$, instead of mining the bubble. Consequently, a positive rate of bubble mining ($\breve{\mu}^{\mathcal{B}} > 0$) is never optimal in this model.

As in the idiosyncratic risk model, the optimal value of ϑ^{opt} is independent of the government spending need $\mathfrak{g}$ and $\mathfrak{g}$ does also not appear in equation (26), so that also the degree of bubble mining $\breve{\mu}^{\mathcal{B}}$ required to implement $\vartheta = \vartheta^{\text{opt}}$ must be independent of $\mathfrak{g}$. The reason for this result is similar to the idiosyncratic risk model: when government spending $\mathfrak{g}$ increases, the government must transfer a larger fraction of current output away from the private sector to itself and taxing current output is the most direct way that does not distort the inter-generational resource transfers. In contrast, funding additional spending by increasing primary deficits (and bubble mining) dilutes the bubble at a faster rate and lowers the interest rate. This increases not just the resources available to the government today, but also increases the initial consumption level of future newborn agents at the expense of consumption growth for all agents. As the optimal trade-off between initial consumption and consumption growth is independent of the aggregate level of private consumption $C_t = (a - \mathfrak{g})K_t$, the optimal distortion induced by $\breve{\mu}^{\mathcal{B}}$ should also be independent of the government spending need.

A.5 Uniqueness in the Model without Alternative Bubbly Assets

In this appendix, we present the left-out technical steps and proofs necessary to establish the uniqueness results stated in Section 6.1 in the absence of bubbles on assets other than government bonds. We present the extension with such bubbles in Appendix A.6 and defer the proof of uniqueness results in the presence of such bubbles (in particular, Proposition 3) to Appendix A.7.

As stated in footnote 49, we restrict attention throughout to equilibria that are deterministic in aggregate variables and feature absolutely continuous price paths. In addition, we impose as a (purely technical) regularity condition on government policy that the ratio $\bar{s}_t := \frac{s_t}{q_t^B + q_t^K}$ of primary surpluses to total wealth must be a bounded and measurable function $t \mapsto \bar{s}_t$ along any equilibrium path. The precise nature of this additional assumption is of no relevance for any of the proofs below. But some regularity condition is required for all mathematical objects to be well-defined.

A.5.1 A Technical Lemma

We start with a technical lemma that reduces the study of model equilibria to the study of solution paths $t \mapsto \vartheta_t$ to the ODE (17) stated in the main text. The basic idea behind that lemma is that knowing the relative price ϑ_t is sufficient to recover the remaining equilibrium quantities whereas any equilibrium implies a relative price ϑ_t that must solve the ODE.

For technical reasons that concern the boundary case $\vartheta_t = 0$ (when bonds have no value), we write ODE (17) in a slightly different, but equivalent form in the following lemma.[72]

Lemma 1. *An absolutely continuous function* $[0, \infty) \to \mathbb{R}, t \mapsto \vartheta_t$ *corresponds to a (unique) model equilibrium for some government policy, if and only if*

(i) $0 \leq \vartheta_t \leq 1$ *for all* $t \geq 0$*;*

(ii) there is a (bounded and measurable) function $[0, \infty) \to \mathbb{R}, t \mapsto \bar{s}_t$ *satisfying* $\bar{s}_t \geq 0$ *whenever* $\vartheta_t = 0$ *such that* ϑ *solves the ODE*

$$\dot{\vartheta}_t = \big(\rho - f\left(1 - \vartheta_t\right)\big)\, \vartheta_t - \bar{s}_t \tag{36}$$

where the function f *is as defined in Section 6.1 of the main text.*

We remark that the quantity $\bar{s}_t$ corresponds to the ratio of primary surpluses to total wealth in equilibrium. Whenever $\vartheta_t > 0$, the identity $\bar{s}_t = \breve{\mu}_t^{\mathcal{B}} \vartheta_t$ holds.

Proof of Lemma 1. The proof that any equilibrium path for ϑ must satisfy conditions (i) and (ii) is straightforward. Condition (i) follows from the definition $\vartheta_t = q_t^B / (q_t^B + q_t^K)$ and the fact that $q_t^B, q_t^K \geq 0$. The latter fact is an implication of free disposal of both capital and bonds in the idiosyncratic risk model. In the perpetual youth model, free disposal of bonds only implies $q_t^B \geq 0$, while $q_t^K \geq 0$ follows directly from equation (7). Condition (ii) has already been shown for both models in Appendix A.2: as shown there, equation (26) is necessary for an equilibrium in the perpetual youth model and equation (29) is necessary for an equilibrium in the idiosyncratic risk model. Both equations are equivalent to ODE (36). The additional restriction on policy, $\bar{s}_t \geq 0$ whenever $\vartheta_t = 0$ follows from our requirement that any government policy must be consistent with the government budget constraint.[73]

The proof of the opposite direction is conceptually also straightforward, but quite lengthy as one needs to construct all equilibrium quantities from the ϑ path alone. We provide this

[72] That the two are equivalent is a direct consequence of the government flow budget constraint (3).

[73] Economically, when the market is unwilling to absorb any bond issuances at a positive price, the government lacks the means to finance primary deficits.

proof explicitly only for the idiosyncratic risk model. The proof for the perpetual youth model is largely identical, but simpler as there are fewer conditions to be verified.

It is to show that any solution $\vartheta : [0, \infty) \to [0, 1]$ to (36) corresponds to a unique equilibrium of the model. For any such function, define ι, q^B, and q^K consistent with the expressions given in the main text, i.e.,

$$\begin{aligned}
\iota_t &= \frac{(1-\vartheta_t)(a-\mathfrak{g})-\rho}{1-\vartheta_t+\phi\rho}, \\
q_t^B &= \vartheta_t \frac{1+\phi(a-\mathfrak{g})}{1-\vartheta_t+\phi\rho}, \\
q_t^K &= (1-\vartheta_t)\frac{1+\phi(a-\mathfrak{g})}{1-\vartheta_t+\phi\rho}.
\end{aligned}$$

Because $\vartheta_t \in [0, 1]$ at all times, $q_t^B, q_t^K \geq 0$, so these expressions are consistent with free disposal of both bonds and capital. We now verify that ι_t, q_t^B, q_t^K and $\theta_t := \vartheta_t$ satisfy all household choice conditions and the aggregate resource constraint.

One immediately verifies that ι_t and q_t^K satisfy households' optimal investment choice condition, $q_t^K = \frac{1}{\Phi'(\iota_t)} = 1 + \phi\iota_t$. In addition, total wealth of all households is $\left(q_t^B + q_t^K\right) K_t$ and because individual consumption demand $c_t^i = \rho n_t^i$ implies an aggregate consumption demand of $C_t = \rho\left(q_t^B + q_t^K\right) K_t$, we obtain

$$\begin{aligned}
C_t + \mathfrak{g}K_t + \iota_t K_t &= \left(\rho\left(q_t^B + q_t^K\right) + \mathfrak{g} + \iota_t\right) K_t \\
&= \left(\rho\frac{1+\phi(a-\mathfrak{g})}{1-\vartheta_t+\phi\rho} + \mathfrak{g} + \frac{(1-\vartheta_t)(a-\mathfrak{g})-\rho}{1-\vartheta_t+\phi\rho}\right) K_t \\
&= \left(\left(\frac{\phi\rho}{1-\vartheta_t+\phi\rho} + \frac{(1-\vartheta_t)}{1-\vartheta_t+\phi\rho}\right)(a-\mathfrak{g}) + \mathfrak{g}\right) K_t \\
&= aK_t,
\end{aligned}$$

so this equilibrium candidate satisfies the aggregate resource constraint (10).

It is left to show that for the asset prices q^B and q^K, agents' bond portfolio share $\theta_t = \vartheta_t$ is consistent with their optimal choice condition for θ_t. We consider two cases:

1. If $\vartheta_t > 0$, then equation (36) (that ϑ satisfies by assumption) is equivalent to $\mu_t^\vartheta = \rho + \breve{\mu}_t^{\mathcal{B}} - (1-\vartheta_t)^2 \tilde{\sigma}^2$. This is because then $\bar{s}_t = \breve{\mu}_t^{\mathcal{B}}\vartheta_t$ by the government budget constraint (3). Rearranging the latter equation and using $\theta_t = \vartheta_t$ yields

$$1 - \theta_t = \frac{1}{\tilde{\sigma}^2}\frac{\rho + \breve{\mu}_t^{\mathcal{B}} - \mu_t^\vartheta}{1-\vartheta_t}. \tag{37}$$

Next, by definition of ι_t and q_t^K

$$\frac{a-\mathfrak{g}-\iota_t}{q_t^K} = \frac{(a-\mathfrak{g})\left(1-\vartheta_t+\phi\rho\right)-(1-\vartheta_t)\,(a-\mathfrak{g})+\rho}{(1-\vartheta_t)\left(1+\phi\,(a-\mathfrak{g})\right)}$$
$$= \frac{\left(1+\phi\,(a-\mathfrak{g})\right)\rho}{(1-\vartheta_t)\left(1+\phi\,(a-\mathfrak{g})\right)} = \frac{\rho}{1-\vartheta_t},$$

and substituting this into equation (37) yields

$$1-\theta_t = \frac{1}{\tilde{\sigma}^2}\left(\frac{a-\mathfrak{g}-\iota_t}{q_t^K} - \frac{\mu^\vartheta-\breve{\mu}_t^{\mathcal{B}}}{1-\vartheta_t}\right),$$

which is precisely households' first-order condition with respect to θ_t as stated in the main text.

2. If $\vartheta_t = 0$, then $q_t^B = 0$, hence bonds have no value and the return on bonds is not well-defined. Consequently, the household portfolio choice condition as stated in the main text is not directly applicable. Instead, households demand a finite quantity of bonds (which is consistent with equilibrium and $\theta_t = 0$), if and only if $\dot{q}_t^B \leq 0$, i.e., the value of bonds is expected to remain nonpositive in the infinitesimal future. Because $q_s^B \geq 0$ for all s, this condition reduces here to $\dot{q}_t^B = 0 \Leftrightarrow \dot{\vartheta}_t = 0$. We therefore have to show that $\vartheta_t = 0$ implies $\dot{\vartheta}_t = 0$.

   The latter follows immediately from conditions (i) and (ii) in the lemma. $\vartheta_t = 0$ means $\bar{s}_t \geq 0$ and so by ODE (36), $\dot{\vartheta}_t \leq 0$. If this was holding with strict inequality, ϑ would have to be negative at some times and violate condition (i). Consequently, it must be that $\dot{\vartheta}_t = 0$.

□

A.5.2 Omitted Steps in Proof of Proposition 1

We have already shown in the main text that the policy (16) with $s > 0$ is inconsistent with equilibrium bubbles on any asset. It is therefore without loss of generality (w.l.o.g.) to complete the proof of Proposition 1 within the context of this baseline model that allows for bubbles at most on government bonds, but not on any other asset.

By Lemma 1, it remains to be shown that there is a unique solution path ϑ_t that is consistent with both the policy specification (16) and ODE (36) and remains within the interval $[0, 1]$ at all times.

Under this specific policy, ODE (36) can be written as

$$\dot{\vartheta}_t = \big(\rho - f(1-\vartheta_t)\big)\,\vartheta_t - sh(1-\vartheta_t) \tag{38}$$

where h is a (weakly) increasing function given by

$$h(x) = \begin{cases} \frac{\rho}{a-\mathfrak{g}}, & \text{perpetual youth model} \\ \frac{x+\phi\rho}{1+\phi(a-\mathfrak{g})}, & \text{idiosyncratic risk model} \end{cases}. \tag{39}$$

In addition, the values of h (for positive arguments) are always positive and bounded away from zero.

It is sufficient to show that the right-hand side of equation (38) crosses zero at precisely one value for $\vartheta_t \in [0,1]$, is negative below that value and positive above it.[74] If $\rho - f(1-\vartheta_t) < 0$ and $\vartheta_t \geq 0$, then the right-hand side of (38) is negative. In addition, monotonicity of f implies that also $\rho - f(1-\vartheta)$ for any other $\vartheta \in [0,\vartheta_t]$, so that the right-hand side remains negative for lower values of ϑ_t. Instead, if $\rho - f(1-\vartheta_t) \geq 0$, then the monotonicity properties of f and h imply that the right-hand side of (38) is strictly increasing (and so $\rho - f(1-\vartheta) \geq 0$ also for values $\vartheta \geq \vartheta_t$). Finally, the right-hand side of (38) is continuous, negative for $\vartheta_t = 0$ (because h is positive) and nonnegative for $\vartheta_t = 1$,[75] such that, by the intermediate value theorem, it must cross 0 at some value $\vartheta_t \in (0,1]$.

A.5.3 Proof of Proposition 2

By Lemma 1, we need to show that $\vartheta_t = \vartheta^*$ is the only solution to ODE (36) that is consistent with the specified threshold policy and contained in the interval $[0,1]$. The threshold policy described in the text is formally given by

$$\tau_t = \begin{cases} \frac{-\breve{\mu}^{\mathcal{B}} q_t^{B} + \mathfrak{g}}{a}, & \vartheta_t \geq \underline{\vartheta} \\ \frac{s+\mathfrak{g}}{a}, & \vartheta_t < \underline{\vartheta} \end{cases}, \qquad \breve{\mu}_t^{\mathcal{B}} = \begin{cases} \breve{\mu}^{\mathcal{B}}, & \vartheta_t \geq \underline{\vartheta} \\ -\frac{s}{q_t^{B}}, & \vartheta_t < \underline{\vartheta} \end{cases},$$

where $\breve{\mu}^{\mathcal{B}} \geq 0$ and $s > 0$ are given constants.[76]

[74] Then, all mathematical solutions that start at a different value than that steady state value drift off to values below 0 or above 1 in finite time and can thus not correspond to valid equilibria.

[75] If $\vartheta_t = 1$, then $f(1-\vartheta_t) = 0$ and

$$sh(1-\vartheta_t) \leq (a-\mathfrak{g})h(0) \leq \rho,$$

where the first inequality follows from the assumption $s \leq a - \mathfrak{g}$. Combining these results implies that the right-hand side of (38) must be nonnegative.

[76] We remark that the proof below also works if we do not assume $\breve{\mu}^{\mathcal{B}} \geq 0$. However, in this case s must be larger than the (positive) equilibrium surplus generated by the policy if $\vartheta_t = \vartheta^*$ at all times.

Under this policy, ODE (36) can be written as[77]

$$\dot{\vartheta}_t = \begin{cases} \left(\rho - f\left(1-\vartheta_t\right) + \breve{\mu}^{\mathcal{B}}\right)\vartheta_t, & \vartheta_t \geq \underline{\vartheta} \\ \left(\rho - f\left(1-\vartheta_t\right)\right)\vartheta_t - sh\left(1-\vartheta_t\right), & \vartheta_t < \underline{\vartheta} \end{cases},$$

where h is a strictly positive function that has been defined in in the proof of Proposition 1 (equation (39)).

By definition, ϑ^* is the unique solution to the equation

$$\rho - f(1-\vartheta) + \breve{\mu}^{\mathcal{B}} = 0.$$

Because the left-hand side of this equation is strictly increasing in ϑ, it must be negative for any $\vartheta < \vartheta^*$ and positive for any $\vartheta > \vartheta^*$. In addition, because $\breve{\mu}^{\mathcal{B}} \geq 0$ by assumption, we can also conclude that $\rho - f(1-\vartheta) < 0$ if $\vartheta < \vartheta^*$.

The previous considerations together with $sh(1-\vartheta_t) > 0$ allow us to make the following conclusions about the ODE stated previously: (1) $\vartheta_t = \vartheta^* \Rightarrow \dot{\vartheta}_t = 0$, (2) $\vartheta_t \in (\vartheta^*, 1] \Rightarrow \dot{\vartheta}_t > 0$, (3) $\vartheta_t \in [0, \vartheta^*) \Rightarrow \dot{\vartheta}_t < 0$. Conclusion (1) implies that $\vartheta_t = \vartheta^*$ is a solution that always remains inside the interval $[0,1]$. Conclusion (2) implies that any solution that is ever above ϑ^* must be larger than 1 within a finite time and can thus not be contained in $[0,1]$. Conclusion (3) implies, symmetrically, that any solution that is ever below ϑ^* must turn negative within a finite time and can thus also not be contained in $[0,1]$. Consequently, $\vartheta_t = \vartheta^*$ is the unique solution that satisfies $0 \leq \vartheta_t \leq 1$ for all t.

A.6 Model Extension with Bubbles on Other Assets

The setup is as outlined in Section 6. We start by imposing on households the additional constraint that they have to invest a fraction x_t of their (bond) savings b_t^i into government bonds and a fraction $1 - x_t$ into the other bubble asset, where x_t is exogenously given. It is then easy to see that the household problem is precisely as in Section 3.1 for the perpetual youth model and as in Section 3.2 for the idiosyncratic risk model, except that b_t^i denotes savings in the bubble portfolio (whereas before we set $x_t \equiv 1$).

In addition, also the model solution is precisely as before with two exceptions:

First, we have to replace everywhere q_t^B with $q_t^B + q_t^C$ and ϑ_t with $\hat{\vartheta}_t$ as only the total bubble

[77] Recall $\bar{s}_t = s_t\vartheta_t/q_t^B = (\tau_t a - \mathfrak{g})\vartheta_t/q_t^B$.

portfolio matters, not its individual components. In particular, we now obtain the equations

$$q_t^B + q_t^C = \hat{\vartheta}_t \frac{a - \mathfrak{g}}{\rho},$$
$$q_t^K = \left(1 - \hat{\vartheta}_t\right) \frac{a - \mathfrak{g}}{\rho}$$

for $q_t^B + q_t^C$ and q_t^K in the perpetual youth model and the equations

$$\iota_t = \frac{(1 - \hat{\vartheta}_t)(a - \mathfrak{g}) - \rho}{1 - \hat{\vartheta}_t + \phi\rho},$$
$$q_t^B + q_t^C = \hat{\vartheta}_t \frac{1 + \phi(a - \mathfrak{g})}{1 - \hat{\vartheta}_t + \phi\rho},$$
$$q_t^K = \left(1 - \hat{\vartheta}_t\right) \frac{1 + \phi(a - \mathfrak{g})}{1 - \hat{\vartheta}_t + \phi\rho}.$$

for ι_t, $q_t^B + q_t^C$, and q_t^K in the idiosyncratic risk model. In both cases, the individual components q_t^B and q_t^C can be recovered from the relationship

$$q_t^B = x_t(q_t^B + q_t^C)$$

that is determined by the exogenous portfolio split x_t.

Second, the solution procedure that resulted in the valuation equation for ϑ_t in the main text has to take into account that the effective dilution rate of the bubble depends no longer just on $\breve{\mu}_t^{\mathcal{B}}$, but on both growth rates, $\breve{\mu}_t^{\mathcal{B}}$ and $\breve{\mu}^{\mathcal{C}}$, as well as on the exogenous portfolio split x_t. It is easy to see that the correct return $d\hat{r}_t^{\mathcal{B}}$ on the bubble portfolio that replaces $dr_t^{\mathcal{B}}$ in the baseline models must be

$$d\hat{r}_t^{\mathcal{B}} = \left(g_t + x_t\mu_t^{q,B} + (1 - x_t)\mu_t^{q,C} - \left(x_t\breve{\mu}_t^{\mathcal{B}} + (1 - x_t)\breve{\mu}^{\mathcal{C}}\right)\right) dt,$$

Effectively, $\breve{\mu}^{\mathcal{B}}$ in the baseline models is replaced with $x_t\breve{\mu}_t^{\mathcal{B}} + (1 - x_t)\breve{\mu}^{\mathcal{C}}$ and $\mu_t^{q,B}$ in the baseline model is replaced with $x_t\mu_t^{q,B} + (1 - x_t)\mu_t^{q,C}$, the geometric drift of $q_t^B + q_t^C$. As a consequence, the derivation of the ODE for $\hat{\vartheta}_t$ changes relative to the baseline models:

1. For the perpetual youth model, when equating the return of the bubble portfolio with the risk-free rate, $d\hat{r}_t^{\mathcal{B}} = r_t^f dt$, we obtain

$$r_t^f = g + x_t\mu_t^{q,B} + (1 - x_t)\mu_t^{q,C} - \left(x_t\breve{\mu}_t^{\mathcal{B}} + (1 - x_t)\breve{\mu}^{\mathcal{C}}\right)$$

   instead of equation (8).

2. For the idiosyncratic risk model, the portfolio choice first-order condition becomes

$$\frac{a - \mathfrak{g} - \iota_t}{q_t^K} - \frac{\mu_t^{\hat{\vartheta}} - \left(x_t \breve{\mu}_t^{\mathcal{B}} + (1 - x_t)\breve{\mu}^{\mathcal{C}}\right)}{1 - \vartheta_t} = \left(1 - \theta_t^i\right)\tilde{\sigma}^2$$

instead of the condition stated in Section 3.2.

In both cases, the only difference is that the drift of q_t^B or ϑ_t is replaced with the one of $q_t^B + q_t^C$ or $\hat{\vartheta}_t$, respectively and that $\breve{\mu}_t^{\mathcal{B}}$ is replaced with $x_t \breve{\mu}_t^{\mathcal{B}} + (1 - x_t)\breve{\mu}^{\mathcal{C}}$. Otherwise, the derivation of $\mu_t^{\hat{\vartheta}}$ proceeds precisely as in the baseline models (there for μ_t^{ϑ}) and thus the result must be

$$\mu_t^{\hat{\vartheta}} = \rho + x_t \breve{\mu}_t^{\mathcal{B}} + (1 - x_t)\breve{\mu}^{\mathcal{C}} - f\,(1 - \vartheta_t)\,.$$

Using $q_t^B = x_t(q_t^B + q_t^C)$, $q_t^C = (1 - x_t)(q_t^B + q_t^C)$, this implies equation (18) stated in the main text.

In sum, the restricted model with the additional constraint on household portfolios has precisely the same equations as the baseline model, except that ϑ_t is replaced with $\hat{\vartheta}_t$, q_t^B with $q_t^B + q_t^C$, and $\breve{\mu}_t^{\mathcal{B}}$ with $x_t \breve{\mu}_t^{\mathcal{B}} + (1 - x_t)\breve{\mu}^{\mathcal{C}}$. Therefore, any theoretical proposition about the baseline model holds equally also for the restricted model for any exogenous $x_t \in (0, 1]$. This includes Lemma 1, so that a time path $\hat{\vartheta}_t$ corresponds to a (unique) equilibrium of the restricted model if and only if it solves equation (18) (for a suitable policy) and satisfies $0 \leq \hat{\vartheta}_t \leq 1$ for all t.

The results in the previous paragraph remain true in particular if we choose for x_t the process $q_t^B/(q_t^B + q_t^C)$ that arises endogenously in the actual model where the portfolio choice between the two bubble assets is not constrained.[78] Then, there is an additional no-arbitrage condition between the two bubble assets because both assets must earn the risk-free rate for the household to be indifferent between the two.[79] One way to write this condition is as $dr_t^{\mathcal{B}} = d\hat{r}_t^{\mathcal{B}}$, which is equivalent to

$$\begin{aligned} g + \frac{q_t^B}{q_t^B + q_t^C}\mu_t^{q,B} + \frac{q_t^C}{q_t^B + q_t^C}\mu_t^{q,C} - \frac{q_t^B}{q_t^B + q_t^C}\breve{\mu}_t^{\mathcal{B}} + \frac{q_t^C}{q_t^B + q_t^C}\breve{\mu}^{\mathcal{C}} &= g + \mu_t^{q,B} - \breve{\mu}_t^{\mathcal{B}} \\ \Leftrightarrow \qquad \underbrace{\frac{q_t^B}{q_t^B + q_t^C}\mu_t^{q,B} + \frac{q_t^C}{q_t^B + q_t^C}\mu_t^{q,C} - \mu_t^{\bar{q}}}_{=\mu_t^{\hat{\vartheta}}} + \frac{\vartheta_t^C}{\hat{\vartheta}_t}\left(\breve{\mu}_t^{\mathcal{B}} - \breve{\mu}^{\mathcal{C}}\right) &= \underbrace{\mu_t^{q,B} - \mu_t^{\bar{q}}}_{=\mu_t^{\vartheta,B}} \end{aligned}$$

[78] This is simply, because if x_t happens to be the process $q_t^B/(q_t^B + q_t^C)$ from the actual model, then the constrained portfolio choice is already unconstrained optimal and imposing the constraint becomes redundant.

[79] This statement remains true even if we allow for stochastic "bubble fluctuations" between the two assets. The reason is that the aggregate bubble portfolio has a return $d\hat{r}_t^{\mathcal{B}}$ that is free of any risk, thus agents are perfectly hedged against such "bubble fluctuations" and, at the margin, require a zero risk premium for holding these risks.

where $\bar{q}_t := q_t^B + q_t^C + q_t^K$ and $\mu_t^{\bar{q}} := \frac{\dot{\bar{q}}_t}{\bar{q}_t}$. Solving for $\mu_t^{\vartheta,B}$ and multiplying by ϑ_t^B yields

$$\dot{\vartheta}_t^B = \frac{\vartheta_t^B}{\hat{\vartheta}_t}\left(\dot{\hat{\vartheta}}_t + \vartheta_t^C\left(\breve{\mu}_t^{\mathcal{B}} - \breve{\mu}^{\mathcal{C}}\right)\right),$$

which, after using equation (18) to eliminate $\dot{\hat{\vartheta}}_t$, is equivalent to equation (19) in the main text. This implies that equation (19) is necessary for any model equilibrium in the actual (unrestricted) model. It is also clearly the case that $0 \leq \vartheta_t^B \leq \hat{\vartheta}_t \leq 1$ for all t.

Conversely, if we have time paths $\vartheta_t^B, \hat{\vartheta}_t$ with $0 \leq \vartheta_t^B \leq \hat{\vartheta}_t \leq 1$ for all t that satisfy equations (18) and (19), then the previous arguments show that (a) there is an (unique) equilibrium of the restricted model with $x_t := \vartheta_t/\hat{\vartheta}_t$ such that $\hat{\vartheta}_t = (q_t^B + q_t^C)/(q_t^B + q_t^C + q_t^K)$ in equilibrium, (b) for $q_t^B := \vartheta_t/\hat{\vartheta}_t(q_t^B + q_t^C)$, the return on government bonds $dr_t^{\mathcal{B}}$ as defined by equation (11)[80] satisfies the no-arbitrage condition $dr_t^{\mathcal{B}} = d\hat{r}_t^{\mathcal{B}}$,[81] and (c) both government bonds and the other bubble asset have a nonnegative value at all times (due to $0 \leq q_t^B \leq q_t^B + q_t^C$). Consequently, the equilibrium of the restricted model for $x_t = \vartheta_t/\hat{\vartheta}_t$ is even an equilibrium of the actual, unrestricted, model, as the no arbitrage condition ensures that agents are willing to hold a fraction x_t of the bubble portfolio in government debt and due to nonnegative valuations, there is no deviation strategy that relies on disposing certain assets (or accumulating them indefinitely, which is equivalent).

To sum up, the following version of Lemma 1 holds in the extended model. Like in Lemma 1, we use a slight reformulation of the ODEs (18) and (19) and replace the policy variable $\breve{\mu}_t^{\mathcal{B}}$ with $\tilde{s}_t := s_t/\bar{q}_t - \breve{\mu}^{\mathcal{C}}\vartheta_t^B$. One immediately checks with the help of the government flow budget constraint (3) that the two formulations are equivalent.[82]

Lemma 2. *Absolutely continuous functions $\vartheta, \hat{\vartheta} : [0, \infty) \to \mathbb{R}$ correspond to a (unique) model equilibrium of the extended model with bubbles on cryptocoins for some government policy, if and only if*

(i) $0 \leq \vartheta_t^B \leq \hat{\vartheta}_t \leq 1$ for all $t \geq 0$;

(ii) there is a (bounded and measurable) function $[0, \infty) \to \mathbb{R}, t \mapsto \tilde{s}_t$ satisfying $\tilde{s}_t \geq 0$ whenever $\vartheta_t = 0$ such that ϑ and $\hat{\vartheta}$ solve the two ODEs

$$\dot{\hat{\vartheta}}_t = \left(\rho + \breve{\mu}^{\mathcal{C}} - f\left(1 - \hat{\vartheta}_t\right)\right)\hat{\vartheta}_t - \tilde{s}_t \tag{40}$$

$$\dot{\vartheta}_t^B = \left(\rho + \breve{\mu}^{\mathcal{C}} - f\left(1 - \hat{\vartheta}_t\right)\right)\vartheta_t^B - \tilde{s}_t \tag{41}$$

[80] This is the equation for the idiosyncratic risk model. For the perpetual youth model, one simply replaces $\Phi(\iota_t) - \delta$ with g.

[81] This follows immediately from the derivation of (19) above, as all transformations have been equivalence transformations.

[82] Specifically, the government budget constraint implies $\tilde{s}_t = \left(\breve{\mu}_t^{\mathcal{B}} - \breve{\mu}^{\mathcal{C}}\right)\vartheta_t^B$.

where the function f is as defined in Section 6.1 of the main text.

A.7 Proof of Proposition 3

We start by remarking that for the threshold policy discussed in the main text, the ODEs (40) and (41) can be written as

$$\dot{\hat{\vartheta}}_t = \begin{cases} \left(\rho - f\left(1-\hat{\vartheta}_t\right)\right)\hat{\vartheta}_t + \breve{\mu}^{\mathcal{C}}\left(\hat{\vartheta}_t - \vartheta_t^B\right) + \breve{\mu}^{\mathcal{B}}\vartheta_t^B, & \vartheta_t \geq \underline{\vartheta} \\ \left(\rho - f\left(1-\hat{\vartheta}_t\right)\right)\hat{\vartheta}_t + \breve{\mu}^{\mathcal{C}}\left(\hat{\vartheta}_t - \vartheta_t^B\right) - sh\left(1-\hat{\vartheta}_t\right), & \vartheta_t < \underline{\vartheta} \end{cases}, \tag{42}$$

$$\dot{\vartheta}_t^B = \begin{cases} \left(\rho - f\left(1-\hat{\vartheta}_t\right)\right)\vartheta_t^B + \breve{\mu}^{\mathcal{B}}\vartheta_t^B, & \vartheta_t \geq \underline{\vartheta} \\ \left(\rho - f\left(1-\hat{\vartheta}_t\right)\right)\vartheta_t^B - sh\left(1-\hat{\vartheta}_t\right), & \vartheta_t < \underline{\vartheta} \end{cases}. \tag{43}$$

where the function h is as defined in the proof of Proposition 1 (equation (39)). Important here is only that h is positive, bounded away from zero, and weakly increasing.

The proof of the first part of the proposition proceeds in two lemmas that exclude certain paths for $\hat{\vartheta}_t$ and ϑ_t^B as valid equilibrium paths. Throughout the proof, we repeatedly make us of the following simple fact about ODEs: if the right-hand side of an ODE for a function x_t is continuous in x_t, strictly negative for all $x_t \in [0, \underline{x})$, and strictly positive for all $x_t \in (\overline{x}, 1]$, then all solution paths with $x_t < \underline{x}$ for some t reach 0 in finite time and all solution paths with $x_t > \bar{x}$ for some t reach 1 in finite time.[83]

We start with a lemma that excludes a special case as a possible equilibrium candidate and thereby helps simplify the remaining proof. In the following, let ϑ^0 denote the unique solution to the equation $f(1-\vartheta) = \rho$.

Lemma 3. *Any solution $(\hat{\vartheta}, \vartheta^B)$ to ODEs (42) and (43) such that $0 \leq \vartheta_t^B \leq \hat{\vartheta}_t \leq 1$ for all t cannot simultaneously satisfy the following three inequalities for any time t:*

$$\vartheta_t^B < \underline{\vartheta}, \qquad \hat{\vartheta}_t \geq \vartheta^0, \qquad \dot{\hat{\vartheta}}_t > 0.$$

Proof. The proof of this result relies on showing that if all three inequalities hold at some time t_0 then necessarily also $\dot{\hat{\vartheta}}_t \geq \dot{\hat{\vartheta}}_{t_0} > 0$ for all $t \geq t_0$ and thus $\hat{\vartheta}_t$ must exceed 1 in finite time.

[83] The proof of this simple fact is omitted, but the basic idea is that close to the boundaries, $\dot{x}_t$ is bounded away from zero and thus boundaries must be reached in finite time as opposed to asymptotically.

By equation (42), for all t

$$\dot{\hat{\vartheta}}_t \geq \underbrace{\left(\rho - f\left(1-\hat{\vartheta}_t\right)\right)\hat{\vartheta}_t - sh\left(1-\hat{\vartheta}_t\right)}_{=:F(\hat{\vartheta}_t)} + \breve{\mu}^C \vartheta_t^C$$

where $\vartheta_t^C := \hat{\vartheta}_t - \vartheta_t^B$. Whenever $\vartheta_t^B < \underline{\vartheta}$, then the inequality even holds with equality. This is in particular the case for $t = t_0$.

Note that F is a strictly increasing function on the interval $[\vartheta^0, 1]$, so if $\hat{\vartheta}_t \geq \hat{\vartheta}_{t_0}$, then $F(\hat{\vartheta}_t) \geq F(\hat{\vartheta}_{t_0})$. In addition, if $\hat{\vartheta}_t \geq \hat{\vartheta}_{t_0} \geq \vartheta^0$, then $\rho + \breve{\mu}^C - f\left(1-\hat{\vartheta}_t\right) \geq 0$ and thus ϑ_t^C is nondecreasing over time.[84]

From these considerations, it is straightforward to establish for all $t \geq t_0$

$$\dot{\hat{\vartheta}}_t \geq F(\hat{\vartheta}_t) + \breve{\mu}^C \vartheta_t^C \geq F(\hat{\vartheta}_{t_0}) + \breve{\mu}^C \vartheta_{t_0}^C = \dot{\vartheta}_{t_0} > 0.$$

This completes the proof of the lemma. □

Lemma 4. *Any solution* $(\hat{\vartheta}, \vartheta^B)$ *to ODEs* (42) *and* (43) *such that* $0 \leq \vartheta_t^B \leq \hat{\vartheta}_t \leq 1$ *for all* t *satisfies* $\vartheta_t^B \in [\underline{\vartheta}, \vartheta^*]$ *for all* t.

Proof. We first show that if $\vartheta_t^B < \underline{\vartheta}$ for any time t, then the solution cannot always be contained inside the interval $[0, 1]$. Specifically, we restrict attention w.l.o.g. to solutions that never satisfy all three inequalities in Lemma 3 and show that $0 \leq \vartheta_t^B < \underline{\vartheta}$ implies $\dot{\vartheta}_t^B < 0$. Consequently, ϑ_t^B must then reach 0 in finite time and cannot be nonnegative for all t.

To show this, we assume $\vartheta_t^B < \underline{\vartheta}$ and distinguish two cases:

1. If $\hat{\vartheta}_t \leq \vartheta^0$, then $\left(\rho - f\left(1-\hat{\vartheta}_t\right)\right)\vartheta_t^B \leq 0$ and hence

$$\dot{\vartheta}_t^B = \left(\rho - f\left(1-\hat{\vartheta}_t\right)\right)\vartheta_t^B - sh\left(1-\hat{\vartheta}_t\right) \leq -sh\left(1-\hat{\vartheta}_t\right) < 0$$

2. If $\hat{\vartheta}_t > \vartheta^0$, then in particular $\hat{\vartheta}_t > \vartheta_t^B$ because, by assumption, $\vartheta_t^B < \underline{\vartheta}$ but $\underline{\vartheta} \leq \vartheta^* \leq \vartheta^0$. In addition, $\hat{\vartheta}_t > \vartheta^0$ also implies $\left(\rho - f\left(1-\hat{\vartheta}_t\right)\right)\vartheta_t^B > 0$. Consequently,

$$\dot{\vartheta}_t^B = \left(\rho - f\left(1-\hat{\vartheta}_t\right)\right)\vartheta_t^B - sh\left(1-\hat{\vartheta}_t\right)$$

[84] $\dot{\vartheta}_t^C = \left(\rho + \breve{\mu}^C - f\left(1-\hat{\vartheta}_t\right)\right)\vartheta_t^C$.

$$< \left(\rho - f\left(1 - \hat{\vartheta}_t\right)\right)\hat{\vartheta}_t + \breve{\mu}^{\mathcal{C}}\left(\hat{\vartheta}_t - \vartheta_t^B\right) - sh\left(1 - \hat{\vartheta}_t\right) = \dot{\hat{\vartheta}}_t$$

Now, by assumption not all inequalities in Lemma 3 are simultaneously satisfied, so that necessarily $\dot{\hat{\vartheta}} \leq 0$. Combining this with the previous inequality yields $\dot{\vartheta}_t^B < 0$.

Thus, in any case $\vartheta_t^B < \underline{\vartheta}$ implies $\dot{\vartheta}_t^B < 0$. This concludes the proof of $\vartheta_t^B \geq \underline{\vartheta}$.

For the proof of $\vartheta_t^B \leq \vartheta^*$, suppose otherwise that $\vartheta^* < \vartheta_t^B$. Then also $\hat{\vartheta}_t \geq \vartheta_t^B > \vartheta^*$ and thus $\dot{\vartheta}_t^B = \left(\rho + \breve{\mu}^{\mathcal{B}} - f\left(1 - \hat{\vartheta}_t\right)\right)\vartheta_t^B > 0$. Such a solution would thus have to exceed 1 in finite time. □

Together with the result from Lemma 2, the previous lemma completes the proof of the first part of Proposition 3. For the remaining part, we treat the cases $\underline{\vartheta} = \vartheta^*$ and $\breve{\mu}^{\mathcal{C}} > \breve{\mu}^{\mathcal{B}}$ separately:

The result for the former case follows easily from Lemma 4. If $\underline{\vartheta} = \vartheta^*$, then that lemma implies the inequality chain

$$\vartheta^* = \underline{\vartheta} \leq \vartheta_t^B \leq \vartheta^*$$

for any solution path ϑ^B to ODE (43) that remains contained in $[0, 1]$ at all times. By Lemma 2, any equilibrium path for ϑ^B must have this property. Consequently, $\vartheta_t^B = \vartheta^*$ is the only possible equilibrium path. Then, in particular $\dot{\vartheta}_t^B = 0$ and ODE (43) also implies $\hat{\vartheta}_t = \vartheta^*$. It is also clear that $\vartheta_t^B = \hat{\vartheta}_t = \vartheta^*$ is indeed a valid equilibrium.

We now consider the case $\breve{\mu}^{\mathcal{C}} > \breve{\mu}^{\mathcal{B}}$. For the remainder of this appendix, we restrict attention to this case and show that also then necessarily $\vartheta_t^B = \hat{\vartheta}_t = \vartheta^*$ in any equilibrium.

We first note that under the assumptions $\vartheta_t^B \geq \underline{\vartheta}$ (w.l.o.g. by Lemma 4) and $\breve{\mu}^{\mathcal{C}} > \breve{\mu}^{\mathcal{B}}$, any equilibrium path for $\hat{\vartheta}$ must satisfy the inequality $\hat{\vartheta}_t \leq \vartheta^*$ for all t and with strict inequality whenever $\vartheta_t^B < \hat{\vartheta}_t$. To see this, note that

$$\begin{aligned}\dot{\hat{\vartheta}}_t &= \left(\rho - f\left(1 - \hat{\vartheta}_t\right)\right)\hat{\vartheta}_t + \breve{\mu}^{\mathcal{C}}\left(\hat{\vartheta}_t - \vartheta_t^B\right) + \breve{\mu}^{\mathcal{B}}\vartheta_t^B \\ &= \left(\rho + \breve{\mu}^{\mathcal{B}} - f\left(1 - \hat{\vartheta}_t\right)\right)\hat{\vartheta}_t + \left(\breve{\mu}^{\mathcal{C}} - \breve{\mu}^{\mathcal{B}}\right)\left(\hat{\vartheta}_t - \vartheta_t^B\right).\end{aligned}$$

If $\hat{\vartheta}_t \geq \vartheta^*$ and either this holds even with strict inequality or the additional condition $\vartheta_t^B < \hat{\vartheta}_t$ holds, then clearly $\dot{\hat{\vartheta}}_t > 0$. Consequently, $\hat{\vartheta}_t$ would grow larger than 1 in finite time, contradicting Lemma 2.

Now, $\hat{\vartheta}_t \leq \vartheta^*$ implies (still assuming w.l.o.g. $\vartheta_t^B \geq \underline{\vartheta} > 0$)

$$\dot{\vartheta}_t^B = \left(\rho + \breve{\mu}^{\mathcal{B}} - f\left(1 - \hat{\vartheta}_t\right)\right)\vartheta_t^B \leq 0$$

and with strict inequality whenever $\hat{\vartheta}_t < \vartheta^*$.

But that means that only $\vartheta_t^B = \hat{\vartheta}_t = \vartheta^*$ is a possible equilibrium solution. If the second equation holds only with (strict) inequality, then by the previous results $\dot{\vartheta}_t^B < 0$ and thus ϑ_t^B decays below $\underline{\vartheta}$ in finite time, contradicting to Lemma 4. If instead the first equation holds only with (strict) inequality, then by the previous results also the second must hold with strict inequality and thus the same argument applies.

This completes the proof of Proposition 3.

A.8 Uniqueness under Imperfect Commitment

In this Appendix, we provide a more detailed formal description of the policy game outlined in Section 6.2 and prove Proposition 4.

A.8.1 The Policy Game

We have described asset prices in the main text by the three variables q_t^B, q_t^C, q_t^K and policies by the three variables $i_t, \mu_t^{\mathcal{B}}, \tau_t$. However, Lemma 2 reveals that it is sufficient to consider the two relative price paths ϑ_t^B and $\hat{\vartheta}_t$ and the policy choice is effectively the one-dimensional choice of $\tilde{s}_t$ with the constraint $\tilde{s}_t \geq 0$ if $\vartheta_t^B = 0$. As this saves on notation, we formulate the policy game in terms of these transformed variables instead of working with the original variables as in the main text.

Problem of a Government. Our assumption is that governments do not react directly to the policy choices of previous governments.[85] As a consequence, each government j takes the terminal asset values $\vartheta_{(j+1)T}^B, \hat{\vartheta}_{(j+1)T}$ as given because actions of future governments (that will ultimately determine these values) are not directly impacted by government j's actions.

It is sufficient to describe the problem of government $j = 0$ in more detail. The problem of governments $j > 0$ is identical with the only difference that time has to be shifted to the right by jT time units.

Government $j = 0$ takes terminal asset values $0 \leq \vartheta_T^B \leq \hat{\vartheta}_T \leq 1$ as given and makes a policy choice for $\{\tilde{s}_t\}_{t\in[0,T)}$ in order to control the price dynamics of ϑ^B and $\hat{\vartheta}$ over $[0, T)$ in line with ODEs (40) and (41). As in Lemma 2, the policy $\{\tilde{s}_t\}_{t\in[0,T)}$ must satisfy the constraint $\tilde{s}_t \geq 0$ whenever $\vartheta_t^B = 0$. Also in line with Lemma 2, we only admit policies that lead to valuation

[85]This assumption rules out "strange" equilibria in which each government does something seemingly suboptimal out of the fear of punishment by the subsequent government. The subsequent government in turn is compelled to punish by a threat of even fiercer punishment by governments following it. Such additional "strange" equilibria may exist for sufficiently small T by folk theorem logic, but they are of no interest for our purposes.

paths satisfying $\hat{\vartheta}_t \leq 1$ for all $t \in [0, T]$, as otherwise the resulting solution to ODE (40) would not correspond to a valid competitive equilibrium of the underlying model.[86]

Because each path for $\bar{s}$ (that is bounded and measurable) implies a unique solution for ϑ^B and $\hat{\vartheta}$ (given terminal values), there is no gain from requiring that a policy must be specified as a full function from observed histories of prices as we have previously done. Instead, it is sufficient to think about the government's problem as a Ramsey problem, i.e. choosing a time path of the policy variable that is actually followed conditional on terminal asset values being ϑ^B_T and $\hat{\vartheta}_T$.

A strategy of government $j = 0$ is therefore a function that maps pairs of terminal asset values $(\vartheta^B_T, \hat{\vartheta}_T)$ satisfying $0 \leq \vartheta^B_T \leq \hat{\vartheta}_T \leq 1$ into a time path $\{\tilde{s}_t\}_{t\in[0,T)}$ for the policy variable. Given a strategy, ODEs (40) and (41) then provide a mapping from pairs $(\vartheta^B_T, \hat{\vartheta}_T)$ into time paths $\{\vartheta^B_t, \hat{\vartheta}_t\}_{t\in[0,T)}$ of asset values of the interval $[0, T)$.

The government desires to maximize a social welfare function as in Section 5.4 and Appendix A.4:

$$\mathbb{W} := \int_0^1 \lambda(i)\mathbb{E}\left[\int_0^\infty e^{-\rho t} \log c^i_t dt\right] di,$$

where $\{\lambda(i)\}_{i\in[0,1]}$ is a given distribution of (nonnegative) welfare weights. As we have argued in Section 5.4, individual utility can be written in the form (14) that separates individual and aggregate variables, so that the former (and the welfare weights) are not relevant for a planner's optimal choices. In addition, the government in question here has no way of influencing prices after period T. Consequently, we can assume that government j maximizes the simpler objective

$$\mathcal{W}\left(\{\hat{\vartheta}_t\}_{t\in[0,T]}\right) := \int_0^T e^{-\rho T} \underbrace{\left(\log\left(\frac{\rho\left(1+\phi\left(a-\mathfrak{g}\right)\right)}{1-\hat{\vartheta}_t+\phi\rho}\right) + \frac{1}{\phi\rho}\log\left(\frac{\left(1-\hat{\vartheta}_t\right)\left(1+\phi\left(a-\mathfrak{g}\right)\right)}{1-\hat{\vartheta}_t+\phi\rho}\right) - \frac{\delta}{\rho} - \frac{\left(1-\hat{\vartheta}_t\right)^2\tilde{\sigma}^2}{2\rho}\right)}_{=:\Psi(\hat{\vartheta}_t)} dt.$$

We remark that Ψ is a strictly concave function with a unique global maximizer. This maximizer has been denoted ϑ^{opt} in Section 5.4 and we write $\vartheta^* := \vartheta^{\text{opt}}$ in Section A.8 and the proof of Proposition 4 below.

Equilibrium. An equilibrium in the policy game consists of absolutely continuous valuation paths $\{\vartheta^B_t, \hat{\vartheta}_t\}_{t\in[0,\infty)}$ and a bounded and measurable government policy path $\{\tilde{s}_t\}_{t\in[0,\infty)}$ such that for all $j = 0, 1, \ldots,$ $\{\tilde{s}_t\}_{t\in[jT,(j+1)T)}$ is the optimal policy chosen by government j given terminal valuations $(\vartheta^B_{(j+1)T}, \hat{\vartheta}_{(j+1)T})$ and $\{\vartheta^B_t, \hat{\vartheta}_t\}_{t\in[jT,(j+1)T}$ is the valuation path over $[jT, (j+1)T)$ implied by that optimal policy.

[86] We do not need to impose the inequality $\vartheta^B_t \leq \hat{\vartheta}_t$ explicitly because the structure of the ODEs ensures that this condition is automatically satisfied at all times if it is satisfied at the terminal time $t = T$.

Note that, due to Lemma 2, this definition automatically implies that $\{\vartheta_t^B, \hat{\vartheta}_t\}_{t\in[0,\infty)}$ corresponds to a competitive equilibrium of the idiosyncratic risk model for the policy choice $\{\tilde{s}_t\}_{t\in[0,\infty)}$.

A.8.2 Proof of Proposition 4

We first show that any equilibrium in the policy game must feature $\hat{\vartheta}_t \geq \vartheta^*$ at all times. The proof of this part is organized into a sequence of technical lemmas that provide key results about the associated ODEs (40) and (41) and the optimal choices of any individual government.

Lemma 5. *Let $(\vartheta_T^{B,i}, \hat{\vartheta}_T^i)$, $i = 1, 2$ be two sets of terminal conditions (with $\vartheta_T^{B,i} \leq \hat{\vartheta}_T^i$) and $\{\tilde{s}_t^i\}_{t\in[0,T)}$, $i = 1, 2$ two policy paths over $[0, T)$. Let $\{\vartheta_t^{B,i}, \hat{\vartheta}_t^i\}_{t\in[0,T]}$ be the implied solution paths to ODEs (40) and (41). Define $\vartheta_t^{C,i} := \hat{\vartheta}_t^i - \vartheta_t^{B,i}$. If $\hat{\vartheta}_t^1 \leq \hat{\vartheta}_t^2$ for all $t \in [0, T]$ and $\hat{\vartheta}_T^2 - \hat{\vartheta}_T^1 \leq \vartheta_T^{B,2} - \vartheta_T^{B,1}$, then $\vartheta_t^{C,1} \geq \vartheta_t^{C,2}$.*

Proof. Let $\alpha(\hat{\vartheta}) := \rho + \breve{\mu}^{\mathcal{C}} - \left(1 - \hat{\vartheta}\right)^2 \tilde{\sigma}^2$, note that $\alpha'(\hat{\vartheta}) = 2\left(1 - \hat{\vartheta}\right)\tilde{\sigma}^2 > 0$ for $\hat{\vartheta} \in (0, 1)$. ODEs (40) and (41) and the definition of $\vartheta_t^{C,i}$ imply

$$\dot{\vartheta}_t^{C,i} = \left(\rho + \breve{\mu}^{\mathcal{C}} - \left(1 - \hat{\vartheta}_t^i\right)^2 \tilde{\sigma}^2\right)\vartheta_t^{C,i} = \alpha(\hat{\vartheta}_t^i)\vartheta_t^{C,i}.$$

This has the solution

$$\vartheta_t^{C,i} = \vartheta_T^{C,i} \exp\left(-\int_t^T \alpha(\hat{\vartheta}_s^i) ds\right). \tag{44}$$

If $\hat{\vartheta}_t^1 \leq \hat{\vartheta}_t^2$ for all t, then $\alpha(\hat{\vartheta}_t^1) \leq \alpha(\hat{\vartheta}_t^2)$ for all t, and so

$$\exp\left(-\int_t^T \alpha(\hat{\vartheta}_s^1) ds\right) \geq \exp\left(-\int_t^T \alpha(\hat{\vartheta}_s^2) ds\right). \tag{45}$$

In addition, if $\hat{\vartheta}_T^2 - \hat{\vartheta}_T^1 \leq \vartheta_T^{B,2} - \vartheta_T^{B,1}$, then $\hat{\vartheta}_T^1 - \vartheta_T^{B,1} \geq \hat{\vartheta}_T^2 - \vartheta_T^{B,2}$ and so

$$\vartheta_T^{C,1} = \hat{\vartheta}_T^1 - \vartheta_T^{B,1} \geq \hat{\vartheta}_T^2 - \vartheta_T^{B,2} = \vartheta_T^{C,2}. \tag{46}$$

Combining inequalities (45) and (46) with equation (44) implies $\vartheta_t^{C,1} \geq \vartheta_t^{C,2}$. □

Lemma 6. *Consider the problem of government $j = 0$ with terminal condition $(\vartheta_T^B, \hat{\vartheta}_T)$ and let $\{\hat{\vartheta}_t\}_{t\in[0,T]}$, $\{\hat{\vartheta}'_t\}_{t\in[0,T]}$ be two absolutely continuous time paths satisfying the terminal condition for $\hat{\vartheta}$. Suppose there is a feasible policy that implements $\{\hat{\vartheta}_t\}_{t\in[0,T]}$ as a time path for $\hat{\vartheta}$, $\{\hat{\vartheta}'_t\}_{t\in[0,T]}$ has bounded derivative, and $\hat{\vartheta}'_t \geq \hat{\vartheta}_t$ for all $t \in [0, T]$. Then there is also a feasible policy that implements $\{\hat{\vartheta}'_t\}_{t\in[0,T]}$ as a time path for $\hat{\vartheta}$.*

Proof. First, use ODE (40) to back out the associated policy $\tilde{s}'_t$ such that $\hat{\vartheta}'_t$ satisfies the ODE with terminal condition $\hat{\vartheta}_T$.[87] We need to show that $\tilde{s}'$ is a feasible policy choice.

Given the time paths $\tilde{s}'_t$ for policy and $\hat{\vartheta}'_t$ for $\hat{\vartheta}$, there is a unique solution $\vartheta^{B\prime}_t$ to ODE (41) that satisfies the terminal condition $\vartheta^{B\prime}_T = \vartheta^B_T$.

The policy $\tilde{s}'$ is feasible if it satisfies the condition $\tilde{s}'_t \geq 0$ whenever $\vartheta^{B\prime}_t = 0$. Inspecting ODE (41) reveals that this property is equivalent to $\vartheta^{B\prime}_t \geq 0$ for all t and we choose to prove this equivalent property.

Let ϑ^B_t denote the solution for ϑ^B associated with the feasible path $\hat{\vartheta}_t$. Then $\vartheta^B_t \geq 0$ for all t. By assumption, also $\hat{\vartheta}_t \leq \hat{\vartheta}'_t$ and thus by Lemma 5, $\vartheta^C_t \geq \vartheta^{C\prime}_t$. Combining these facts yields for all $t \in [0, T]$ the inequality chain

$$0 \leq \vartheta^B_t = \hat{\vartheta}_t - \vartheta^C_t \leq \hat{\vartheta}'_t - \vartheta^{C\prime}_t = \vartheta^{B\prime}_t.$$

Thus, policy $\tilde{s}'$ is indeed feasible. □

Lemma 7. *Consider the problem of government $j = 0$ for a given terminal condition. Suppose $\{\hat{\vartheta}^\circ_t\}_{t\in[0,T]}$ is an arbitrary absolutely continuous time path with bounded derivative that satisfies $\hat{\vartheta}^\circ_T \leq \hat{\vartheta}_T$ and $\hat{\vartheta}^\circ_t \leq \vartheta^*$ for all t. Then the optimal policy choice implies $\hat{\vartheta}_t \geq \hat{\vartheta}^\circ_t$ for all $t \in [0, T]$.*

Proof. Let $\hat{\vartheta}_t$ be the time path for $\hat{\vartheta}$ implied by the optimal solution. Define $\hat{\vartheta}'_t := \max\{\hat{\vartheta}_t, \hat{\vartheta}^\circ_t\}$. Because both $\hat{\vartheta}$ and $\hat{\vartheta}^\circ$ are absolutely continuous with bounded derivative, so is also the pointwise maximum $\hat{\vartheta}'$. In addition, by construction $\hat{\vartheta}'_t \geq \hat{\vartheta}_t$ for all t and by assumption on the terminal values, $\hat{\vartheta}'_T = \hat{\vartheta}_T$. Thus, all assumptions of Lemma 6 are satisfied and we can conclude that there is a feasible policy that implements the time path $\hat{\vartheta}'$.

We show next that this alternative policy generates at least as high welfare as the original plan leading to $\hat{\vartheta}$ and strictly larger welfare if $\hat{\vartheta}$ and $\hat{\vartheta}'$ are different. Because $\hat{\vartheta}$ is optimal by assumption, the latter cannot be the case and so it must be that $\hat{\vartheta}_t = \hat{\vartheta}'_t \geq \hat{\vartheta}^\circ_t$ for all t.

To do so, we use that the relevant part of the government's objective can be written as

$$\mathcal{W}\left(\{\hat{\vartheta}\}_{t\in[0,T]}\right) = \int_0^T e^{-\rho t}\Psi(\hat{\vartheta}_t)dt$$

and Ψ is a strictly concave function with global maximizer ϑ^*. For each $t \in [0, T]$ we have two cases

(i) if $\hat{\vartheta}_t \geq \hat{\vartheta}^\circ_t$, then $\hat{\vartheta}'_t = \hat{\vartheta}_t$ and thus $\Psi(\hat{\vartheta}'_t) = \Psi(\hat{\vartheta}_t)$;

[87] $\tilde{s}'_t$ is always uniquely defined, except possibly on a set of measure zero. It can be backed out by taking the time derivative of $\hat{\vartheta}'_t$ and comparing terms with equation (40). It is also easy to see that $\tilde{s}'_t$ must be bounded because $\hat{\vartheta}'_t$ has bounded derivative by assumption and all other terms in equation (40) are also bounded.

(ii) if $\hat{\vartheta}_t < \hat{\vartheta}_t^\circ$, then $\vartheta^* \geq \hat{\vartheta}_t' > \hat{\vartheta}_t$ and thus $\Psi(\hat{\vartheta}_t') > \Psi(\hat{\vartheta}_t)$.

So in any case, $\Psi(\hat{\vartheta}_t') \geq \Psi(\hat{\vartheta}_t)$ for all $t \in [0, T]$. Thus $\hat{\vartheta}'$ must yield at least as high welfare as $\hat{\vartheta}$. In addition, if case (ii) ever occurs, then $\Psi(\hat{\vartheta}_t') > \Psi(\hat{\vartheta}_t)$ must hold on a set of positive measure for t because all expressions are continuous in t. But that would imply $\mathcal{W}\left(\{\hat{\vartheta}'\}_{t\in[0,T]}\right) > \mathcal{W}\left(\{\hat{\vartheta}\}_{t\in[0,T]}\right)$. Therefore, $\hat{\vartheta}'$ yields strictly higher welfare unless the two paths are identical at all times. □

The previous lemma implies the following corollary that is key to the proof that $\hat{\vartheta}_t \geq \vartheta^*$ in all equilibria in the policy game.

Corollary 1. *Consider the problem of any government j for any terminal condition at time $(j+1)T$. Then the optimal solution to the government problem features $\hat{\vartheta}_{jT} \geq \vartheta^*$.*
In addition, if the terminal condition satisfies $\hat{\vartheta}_{(j+1)T} \geq \vartheta^*$, then the optimal solution even satisfies $\hat{\vartheta}_t \geq \vartheta^*$ for all $t \in [jT, (j+1)T]$.

Proof. We show the assertion for $j = 0$. Because the problem of government $j > 0$ is identical to the problem of government $j = 0$ except for a time shift, the result for $j = 0$ immediately carries over to any $j \geq 0$.

We start by proving the additional statement in the case $\hat{\vartheta}_{(j+1)T} \geq \vartheta^*$. This also proves the main assertion in this case. To see that this statement must hold, simply define $\hat{\vartheta}_t^\circ := \vartheta^*$ and apply Lemma 7 to conclude $\hat{\vartheta}_t \geq \hat{\vartheta}_t^\circ = \vartheta^*$ for all $t \in [0, T]$.

Next, suppose $\hat{\vartheta}_{(j+1)T} < \vartheta^*$. In this case, the time path $\hat{\vartheta}^\circ$ defined by

$$\hat{\vartheta}_t^\circ := \vartheta^* + \frac{t}{T}\hat{\vartheta}_{(j+1)T}$$

satisfies the terminal condition and $\hat{\vartheta}_t^\circ \leq \vartheta^*$ for all $t \in [0, T]$. We can thus again apply Lemma 7 and conclude $\hat{\vartheta}_t \geq \hat{\vartheta}_t^\circ$. In particular, $\hat{\vartheta}_0 \geq \hat{\vartheta}_0^\circ = \vartheta^*$. □

Corollary 2. *Any equilibrium in the policy game features $\hat{\vartheta}_t \geq \vartheta^*$ for all t.*

Proof. By the previous corollary, any government j always chooses a policy that implies $\hat{\vartheta}_{jT} \geq \vartheta^*$. But because this holds for all j, this means that any government $j-1$ for $j \geq 1$ faces a terminal condition with $\hat{\vartheta}_{jT} \geq \vartheta^*$. Applying again the previous corollary, we can even conclude $\vartheta_t \geq \vartheta^*$ for all $t \in [(j-1)T, jT]$. As this must hold for all $j \geq 1$, we obtain $\vartheta_t \geq \vartheta^*$ for all $t \in [0, \infty)$. □

The previous result completes the proof of the first part of Proposition 4 that $\hat{\vartheta}_t \geq \vartheta^*$.

We next characterize all equilibria in the policy game in which the inequality always holds with equality, $\hat{\vartheta}_t = \vartheta^*$ for all t. In this case, we only need to check under which conditions there is a feasible infinite-horizon policy path $\{\tilde{s}_t\}_{t\in[0,\infty)}$ and there are valuation paths $\{\hat{\vartheta}_t, \vartheta_t^B\}_{t\in[0,\infty)}$ for this policy consistent with ODEs (40) and (41) and $\hat{\vartheta}_t = \vartheta^*$ for all t. Whenever this is the case, this automatically represents an equilibrium of the policy game because this particular path for $\hat{\vartheta}_t$ is the global maximizer of the objective of any government $j = 0, 1, \ldots$ and so no government can have an incentive to deviate.

Inspecting ODE (40) reveals that $\hat{\vartheta}_t = \vartheta^*$ for all t requires the policy

$$\tilde{s}_t = \left(\rho + \breve{\mu}^C - (1-\vartheta^*)^2\tilde{\sigma}^2\right)\vartheta^* =: \alpha\vartheta^*.$$

Under this candidate policy, ODE (41) becomes

$$\dot{\vartheta}_t^B = \alpha\left(\vartheta_t^B - \vartheta^*\right)$$

and thus $\vartheta_t^C = \hat{\vartheta}_t - \vartheta_t^B = \vartheta^* - \vartheta_t^B$ must satisfy the ODE

$$\dot{\vartheta}_t^C = \alpha\vartheta_t^C.$$

This is a standard linear ODE with the general solution

$$\vartheta_t^C = \vartheta_0^C e^{\alpha t},$$

where $\vartheta_0^C \in \mathbb{R}$ parameterizes the possible solutions. All solutions with $\vartheta_0^C < 0$ imply $\vartheta_t^B = \hat{\vartheta}_t - \vartheta_t^C > \hat{\vartheta}_t$ and can thus not correspond to model equilibria by Lemma 2. We now consider two cases

(i) If $\breve{\mu}^C > \breve{\mu}^*$, then $\alpha > 0$ and thus any solution with $\vartheta_0^C > 0$ generates a ϑ^C path that grows without bounds. Eventually, such a solution implies $\vartheta_t^C > \vartheta^* = \hat{\vartheta}_t$ and thus $\vartheta_t^B < 0$. By Lemma 2 such a solution does therefore not correspond to a model equilibrium. Consequently, only the solution with $\vartheta_0^C = 0$, i.e. $\vartheta_t^C = 0$ for all $t \geq 0$ remains. This solution implies $\vartheta_t^B = \hat{\vartheta}_t = \vartheta^*$ and this corresponds indeed to a model equilibrium (again, by Lemma 2).

(ii) If $\breve{\mu}^C \leq \breve{\mu}^*$, then $\alpha \leq 0$. For any initial condition $\vartheta_0^C \in [0, \vartheta^*]$, the implied path satisfies $\vartheta_t^C \in [0, \vartheta^*]$ for all $t \geq 0$. This implies $\vartheta_t^B = \vartheta^* - \vartheta_t^C \in [0, \vartheta^*]$ for all $t \geq 0$. By Lemma 2, any such solution corresponds to a model equilibrium.

To complete the proof of Proposition 4, we only need to show that in the case $\breve{\mu}^C > \breve{\mu}^*$ (case (i) above), there can be no other equilibria in the policy game. Such other equilibria must necessarily feature $\hat{\vartheta}_t > \vartheta^*$ for some time t by the previous discussion. But it is easy to see that this cannot be possible:

First, we show that there is no equilibrium with $\vartheta_t^C > 0$ for any $t \geq 0$. The proof is largely identical to the one in case (i) above. We know already that $\hat{\vartheta}_t \geq \vartheta^*$ in any equilibrium and thus with α defined as previously,

$$\dot{\vartheta}_t^C \geq \alpha \vartheta_t^C.$$

Therefore, any solution must satisfy $\vartheta_{t_1}^C \geq \vartheta_{t_0}^C e^{\alpha(t_1 - t_0)}$ for any $t_0 < t_1$. If $\vartheta_{t_0}^C > 0$, then, because $\alpha > 0$, there is some t_1 such that $\vartheta_{t_1}^C > 1$ and thus $\vartheta_{t_1}^B < 0$, so that this cannot be an equilibrium solution by Lemma 2. Consequently, $\vartheta_{t_0}^C = 0$ is the only possibility. Because t_0 was arbitrary, any equilibrium must feature $\vartheta_t^C = 0$ for all $t \geq 0$. From now on, we can therefore identify $\hat{\vartheta}_t$ and ϑ_t^B and simply write ϑ_t. Note that ODEs (40) and (41) coincide in this case.

Second, we show that $\vartheta_t > \vartheta^*$ at any time t is impossible in any equilibrium of the policy game. We proceed in two steps:

1. If any government j faces a terminal condition $\vartheta_{(j+1)T} = \vartheta^*$, then it is feasible and (strictly) optimal to implement $\vartheta_t = \vartheta^*$ for all $t \in [jT, (j+1)T]$. Feasibility follows from the previous discussion, optimality from the structure of the government's objective. Therefore, the observation $\vartheta_t > \vartheta^*$ for some t cannot result from the policy of any government that faces the terminal condition $\vartheta_{(j+1)T} = \vartheta^*$.

2. We show next that if any government j was to face a terminal condition $\vartheta_{(j+1)T} > \vartheta^*$, its optimal policy would still imply $\vartheta_{jT} = \vartheta^*$ at the beginning of its period of office, so that the previous government $j-1$ would face the terminal condition $\vartheta_{jT} = \vartheta^*$. But if this holds for all j, then a terminal condition $\vartheta_{(j+1)T} > \vartheta^*$ cannot actually be the outcome of any equilibrium.

   We prove this last result by contradiction and assume w.l.o.g. that $j = 0$. Suppose the optimal response of the government to a terminal condition $\vartheta_T > \vartheta^*$ was leading to a price path $\{\vartheta_t\}_{t\in[0,T]}$ such that $\vartheta_0 > \vartheta^*$. We construct a feasible alternative price path $\{\vartheta'_t\}_{t\in[0,T]}$ that generates higher welfare than $\{\vartheta_t\}_{t\in[0,T]}$ contradicting the assumption that the latter is the result of an optimal policy choice by the government.

   For the construction of $\{\vartheta'_t\}_{t\in[0,T]}$, let $\tilde{s}_t$ denote the optimal policy chosen by the government. For $x > 0$, define $\tilde{s}''_t := \tilde{s}_t - x$. Consider the solution ϑ''_t to ODE (40) under this policy with the terminal condition $\vartheta''_T = \vartheta_T$. Clearly, that solution satisfies $\vartheta''_t < \vartheta_t$ for all $t < T$.[88] Also, if x is sufficiently large, ϑ''_t must cross ϑ^* in the interval $[0, T]$. We define the

[88] This is because the ODE's right-hand side for policy $\tilde{s}''$ is strictly larger than that for $\tilde{s}$ if evaluated at any given $\hat{\vartheta}$ path. By standard ODE comparison logic, ϑ'' must thus fall faster than ϑ when moving backward in time.

last crossing time,

$$t_0 := \sup\{t \in [0,T] \mid \vartheta''_t \leq \vartheta^*\}.$$

By continuity of ϑ'' and $\vartheta'' = \vartheta_T$, $\vartheta''_{t_0} = \vartheta^*$ whenever $t_0 > 0$. Clearly t_0 is increasing in x and $t_0 \to T$ as $x \to \infty$. Consequently, for any given $\varepsilon > 0$, we can choose $x > 0$ sufficiently large such that $t_0 > T - \varepsilon$. Choose such x appropriately and define next

$$\tilde{s}'_t := \begin{cases} \alpha\vartheta^*, & t < t_0 \\ \tilde{s}''_t, & t \geq t_0 \end{cases},$$

where α is defined as previously. Now let ϑ'_t be the solution to ODE (40) with terminal condition $\vartheta'_T = \vartheta_T$ and the policy $\tilde{s}'$. Because the ODE coincides with the one for ϑ'' on $[t_0, T]$ and because the terminal conditions are identical, it must be that $\vartheta'_t = \vartheta''_t$ for all $t \in [t_0, T]$. Furthermore, $\vartheta'_{t_0} = \vartheta''_{t_0} = \vartheta^*$ and we have seen above that the policy $\tilde{s} = \alpha\vartheta^*$ makes the right-hand side of ODE (40) vanish whenever $\hat{\vartheta}_t = \vartheta^*$. Consequently, the solution ϑ' must satisfy $\vartheta'_t = \vartheta^*$ for all $t \leq t_0$.

The path ϑ' just constructed has the property $\vartheta'_t = \vartheta^*$ for all $t \leq T - \varepsilon$ and $\vartheta'_t \leq \bar{\vartheta} := \max_{t\in[0,T]} \vartheta_t \leq 1$ for all $t \in [T - \varepsilon, T]$. Consequently, the relevant part of the government's welfare objective under this path is

$$\begin{aligned}
\mathcal{W}\left(\{\vartheta'_t\}_{t\in[0,T]}\right) &= \int_0^T e^{-\rho t}\Psi\left(\vartheta'_t\right) dt \\
&= \int_0^{T-\varepsilon} e^{-\rho t}\Psi\left(\vartheta^*\right) dt + \int_{T-\varepsilon}^T e^{-\rho t}\Psi\left(\vartheta'_t\right) dt \\
&\geq \int_0^T e^{-\rho t}\Psi\left(\vartheta^*\right) dt - \int_{T-\varepsilon}^T e^{-\rho t}\left(\Psi\left(\bar{\vartheta}\right) - \Psi\left(\vartheta^*\right)\right) dt \\
&\geq \mathcal{W}^* - \varepsilon\left(\Psi\left(\bar{\vartheta}\right) - \Psi\left(\vartheta^*\right)\right),
\end{aligned}$$

where $\mathcal{W}^*$ denotes the global maximum of the government's objective. As we have constructed such an alternative path ϑ' for any $\varepsilon > 0$, we can move the achievable welfare arbitrarily close to the global maximum $\mathcal{W}^*$ by choosing $\varepsilon > 0$ sufficiently small. Conversely, $\vartheta_0 > \vartheta^*$ and by continuity there must be numbers $\delta_0, \delta_1 > 0$ such that $\vartheta_t \geq \vartheta^* + \delta_0$ for all $t \leq \delta_1$. This is sufficient to conclude that the welfare attained under the optimal policy, $\mathcal{W}\left(\{\vartheta_t\}_{t\in[0,T]}\right)$ must fall short of $\mathcal{W}^*$, a contradiction.