Macro, Money and Finance

Problem Sets 4 – Solutions (selective)

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Problem 2 – Bank Money Creation

- Money model with CIA constraint in production (as Problem 3 last week)
- Now we add money-creating banks:
  - Bankers with wealth share η_t
  - Can invest in money and loans (return dr_t^ℓ)
  - Issue deposits (return dr_t^d)
  - Deposits relax household CIA constraint

$$\alpha \cdot a \frac{\theta_t^{k,h,\tilde{i}} n_t^{h,\tilde{i}}}{q_t^K} \leq \left(\theta_t^{m,h,\tilde{i}} + \theta_t^{d,h,\tilde{i}}\right) n_t^{h,\tilde{i}}$$

instead of

$$\alpha \cdot a \frac{\theta_t^{k,\tilde{i}} n_t^{\tilde{i}}}{q_t^K} \leq \theta_t^{m,\tilde{i}} n_t^{\tilde{i}}$$

Part 1 – Formal Model Description

- Return processes
  - Money

$$dr_t^m = \left(\Phi(\iota_t) - \delta + \mu_t^{q,M}\right) dt + \sigma dZ_t$$

  - Deposits and Loans

$$dr_t^d = i_t^d dt + dr_t^m, \qquad dr_t^{\ell,\tilde{i}} = i_t^\ell dt + dr_t^m + j_t^\ell d\tilde{J}_t^{\tilde{i}}$$

- Net worth evolutions
  - Bankers

$$\frac{dn_t^{b,\tilde{i}}}{n_t^{b,\tilde{i}}} = -\frac{c_t^{b,\tilde{i}}}{n_t^{b,\tilde{i}}} dt + \theta_t^{m,b,\tilde{i}} dr_t^m + \theta_t^{\ell,b,\tilde{i}} dr_t^{\ell,\tilde{i}} + \theta_t^{d,b,\tilde{i}} dr_t^d$$

  - Households

$$\frac{dn_t^{h,\tilde{i}}}{n_t^{h,\tilde{i}}} = -\frac{c_t^{h,\tilde{i}}}{n_t^{h,\tilde{i}}} dt + \theta_t^{m,h,\tilde{i}} dr_t^m + \theta_t^{d,h,\tilde{i}} dr_t^d + \theta_t^{k,h,\tilde{i}} dr_t^k + \theta_t^{\ell,h,\tilde{i}} dr_t^{\ell,\tilde{i}}$$

Part 1 – Agent Problems

- Bankers
  - Maximize utility $\mathbb{E}\left[\int_0^\infty e^{-\rho t} \log c_t^{b,\tilde{\imath}}\, dt\right]$
  - Choose $\{c_t^{b,\tilde{\imath}}\}, \{\theta_t^{m,b,\tilde{\imath}}\}, \{\theta_t^{d,b,\tilde{\imath}}\}, \{\theta_t^{\ell,b,\tilde{\imath}}\}$
  - s.t.
    - net worth evolution
    - $\theta_t^{m,b,\tilde{\imath}} + \theta_t^{d,b,\tilde{\imath}} + \theta_t^{\ell,b,\tilde{\imath}} = 1$
    - $\theta_t^{m,b,\tilde{\imath}} \geq 0$
- Households
  - Maximize utility $\mathbb{E}\left[\int_0^\infty e^{-\rho t} \log c_t^{h,\tilde{\imath}}\, dt\right]$
  - Choose $\{c_t^{h,\tilde{\imath}}\}, \{\iota_t^{\tilde{\imath}}\}, \{\theta_t^{m,h,\tilde{\imath}}\}, \{\theta_t^{d,h,\tilde{\imath}}\}, \{\theta_t^{\ell,h,\tilde{\imath}}\}, \{\theta_t^{k,h,\tilde{\imath}}\}$
  - s.t.
    - net worth evolution
    - $\theta_t^{m,h,\tilde{\imath}} + \theta_t^{d,h,\tilde{\imath}} + \theta_t^{\ell,h,\tilde{\imath}} + \theta_t^{k,h,\tilde{\imath}} = 1$
    - $\theta_t^{m,h,\tilde{\imath}}, \theta_t^{k,h,\tilde{\imath}}, \theta_t^{d,h,\tilde{\imath}} \geq 0$
    - cash-in-advance constraint

Part 2 – Frictionless Case (no CIA)

- Banker portfolio choice deposits vs loans

$$i_t^d = i_t^\ell$$

- Banker and Household choice deposits vs money

$$i_t^d \geq 0$$

… and with "=", if the agent holds money

  - Someone has to hold money in equilibrium $\Rightarrow i_t^d = 0$

- Conclusions from these preliminary considerations
  - $dr_t^m = dr_t^\ell = dr_t^d$
  - The three nominal assets are perfect substitutes for all agents

    $\Rightarrow$ gross quantities of deposits and loans are not determined by equilibrium conditions (can always add more deposits and back with additional loans)

Part 2 – Frictionless Case – Money Valuation

- Derivation of money valuation equation is standard
- Use loans as benchmark asset
- Portfolio choice of net worth relative to loans

$$\mu^{\eta} + \rho - \mu_t^{\vartheta} - i_t^{\ell} = 0$$

$$\mu^{1-\eta} + \rho - \mu_t^{\vartheta} - i_t^{\ell} = \tilde{\varsigma}_t^h \tilde{\sigma}_t^{n,h}$$

- Take η-weighted average, rearrange

here $i_t^{\ell} = 0$

$$\mu_t^{\vartheta} = \rho - \frac{1}{1-\eta_t} (1-\vartheta_t)^2 \tilde{\sigma}^2 - \cancel{i_t^{\ell}}$$

- Combine with first equation for η drift

$$\mu_t^{\eta} = -\frac{1}{1-\eta_t} (1-\vartheta_t)^2 \tilde{\sigma}^2$$

Part 2 – Frictionless Case – Steady State

- Recall:

$$\mu_t^\eta = -\frac{1}{1-\eta_t}\left(1-\vartheta_t\right)^2\tilde{\sigma}^2$$

- This is always negative!
  - $\Rightarrow$ steady state must be at $\eta = 0$
- Then steady-state money valuation equation is standard

$$\rho = (1-\vartheta)^2\tilde{\sigma}^2$$

$$\Rightarrow \vartheta = \frac{\tilde{\sigma}-\sqrt{\rho}}{\tilde{\sigma}}$$

- Get same equilibrium as in Lecture 5

Part 3 – Risk-free Loans

- Now add CIA constraint, but loans remain risk-free
- Banker portfolio choice still implies
$$i_t^d = i_t^\ell$$
- For households now
$$i_t^\ell - i_t^d = -\lambda_t^h (v_t^\ell - v_t^d) = \lambda_t^h \frac{1}{\alpha} \geq 0$$
  Where (compare last week)
  - λ^h: price of transaction services (for households)
  - $v^d = \frac{1}{\alpha}$: transaction services provided by deposits
  - $v^\ell = 0$: transaction services provided by loans
- Combining the two conditions: $\lambda_t^h = 0$
  - As long as $i_t^\ell - i_t^d > 0$, bankers issue deposits to make more loans and earn a spread
  - In equilibrium, bankers create so much deposit money that CIA is slack

Part 3 – Risk-free Loans – Balance Sheet Size

- Bank balance sheet minimal, if outside money held by households
- Cash-in-advance constraint requires

$$\left(\frac{\vartheta_t}{1-\eta_t}+\theta_t^{d,h,\tilde{i}}\right) \geq \alpha \cdot a \frac{\frac{1-\vartheta_t}{1-\eta_t}}{q_t^K} \Leftrightarrow (1-\eta_t)\,\theta_t^{d,h,\tilde{i}} \geq \alpha \cdot a \frac{1}{q_t^K+q_t^M}-\vartheta_t$$

- Multiply by net worth and integrate

$$(1-\eta_t)\overbrace{\int \theta_t^{d,h,\tilde{i}} n_t^{h,\tilde{i}} d\tilde{i}}^{=D_t} \geq \alpha \cdot a \frac{N_t^h}{q_t^K+q_t^M}-\vartheta_t N_t^h = \alpha \cdot a\,(1-\eta_t)\,K_t-(1-\eta_t)\,q_t^M K_t$$

- Thus (using known q^M expression)

$$N_t^b + D_t \geq \left(\alpha a + (\eta_t-\vartheta_t)\frac{1+\phi a}{1-\vartheta_t+\phi\rho}\right) K_t$$

Part 3 – Risk-free Loans – Allocation Impact of CIA Constraint

- Does the CIA constraint have any impact on the equilibrium allocation?
  - No, multiplier is zero, $\lambda_t^h = 0$
  - Agents would make same choices if constraint was absent

- Conclusions:
  - again same allocation as in Lecture 5 model
  - in presence of money-creating banks, medium-of-exchange role of money not relevant for determining value of *outside money*
  - from perspective of this model: BruSan are right to emphasize store of value role of money

Parts 4 and 5

- Why may banks not create arbitrary quantities of money in reality?
    1. Regulation, e.g.
        a) leverage constraints
        b) reserve requirements

    2. Limited competition
       (do not compete spread $i_t^\ell - i_t^d$ all the way to zero)

    3. Bank assets are risky

- Want to explore 1b) and 3 in parts 4 and 5

Part 4 – Reserve Requirements

- Change: bankers must hold $\psi \in [0,1]$ units of outside money (required reserves) for each unit of deposits they issue
- Adds a portfolio constraint

$$-\psi\theta_t^{d,b,\tilde{i}} \leq \theta_t^{m,b,\tilde{i}}$$

- Questions:
    1. When does the requirement matter? (parameter condition)
    2. Steady state values of η and ϑ?
    3. Show that this leads to the "money multiplier model"

Part 4 – When Does Requirement Matter?

- For $\psi < 1$, it is efficient for banks to hold all outside money
- Then aggregate version of reserve requirement is
$$\psi D_t \leq q_t^M K_t$$
- In equilibrium from part 3, D_t must be large to satisfy CIA
$$D_t \geq \alpha \cdot aK_t$$
- So requirement does not matter if $\left[\alpha a, \frac{q_t^M}{\psi}\right] \neq \emptyset$ (for q_t^M as in parts 2-3)
- Conversely: requirement matters if
$$\psi\alpha \cdot aK_t > q_t^M K_t \Leftrightarrow \psi\alpha a > \vartheta \frac{1 + \phi a}{1 - \vartheta + \phi\rho}$$

Part 4 – Steady State – Money Valuation

- Can still use loans as benchmark asset (does not enter CIA, so no additional terms relative to part 2)
- Then money valuation and η drift are as before

$$\mu_t^{\vartheta} = \rho - \frac{1}{1-\eta_t}\left(1-\vartheta_t\right)^2 \tilde{\sigma}^2 - i_t^{\ell}$$

$$\mu_t^{\eta} = -\frac{1}{1-\eta_t}\left(1-\vartheta_t\right)^2 \tilde{\sigma}^2$$

- Again, $\eta = 0$ in steady state (bankers do not need net worth to create money)

Part 4 – Steady State – Deposit Rate

- Steady-state version of money valuation equation

$$\rho = (1-\vartheta)^2 \tilde{\sigma}^2 + i^\ell$$

- Need to determine loan rate
  - Household portfolio choice (loans vs deposits)

$$i^\ell - i^d = -\lambda^h (v^\ell - v^d) = \lambda^h \frac{1}{\alpha}$$

  - Banker self-financing strategy:
    - 1 additional unit of deposits
    - invest in ψ units of money and $1-\psi$ units of loans

$$(1-\psi) i^\ell - i^d = 0$$

  - Combine the two:

$$i^\ell = \frac{1}{\psi}\left(i^\ell - i^d\right) = \lambda^h \frac{1}{\psi\alpha}$$

Part 4 – Steady State Solution

- Substitute into money valuation equation

$$\rho = (1-\vartheta)^2\,\tilde{\sigma}^2 + \lambda^h \frac{1}{\psi\alpha}$$

- This looks like equation from last week
  (with $\psi\alpha$ instead of α)

- We solve it the same way:
  - Either constraint slack, then valuation equation determines ϑ
  - Or constraint binds, then CIA

$$\alpha \cdot a \frac{1-\vartheta}{q^K} = \frac{D}{N^h} \Leftrightarrow \alpha \cdot a \frac{1}{q^M} = \frac{D}{q^M K} \Leftrightarrow \alpha \cdot \frac{a}{1+\phi a}\frac{1-\vartheta+\phi\rho}{\vartheta} = \frac{D}{q^M K_t}$$

$$\vartheta = \frac{\alpha a\,(1+\phi\rho)}{\frac{D}{q^M K}(1+\phi a)+\alpha a} \quad \Longrightarrow \quad \vartheta = \frac{\psi\alpha a\,(1+\phi\rho)}{1+a\,(\psi\alpha+\phi)}$$

Part 4 – Money Multiplier Model

- When constraint is binding,

$$i^d = (1-\psi)i^\ell = (1-\psi)\lambda^h \frac{1}{\psi\alpha}$$

- This is positive, so households strictly prefer to hold deposits ⇒ all outside money most be held by banks
- Reserve requirement must be binding for banks

  (otherwise they are unwilling to pay $i_t^d > 0$ while earning 0 on excess reserves)

- Integrating reserve requirement yields

$$\psi D_t = \int \psi \theta_t^{d,b,\tilde{i}} n_t^{b,\tilde{i}} d\tilde{i} = \int \theta_t^{m,b,\tilde{i}} n_t^{b,\tilde{i}} d\tilde{i} = q_t^M K_t$$

Part 5 – Risky Loans

- Changes:
  - No reserve requirement
  - But default risk in loans
  - At idiosyncratic jump times, $d\tilde{J}_t^{\tilde{\imath}}$, household $\tilde{\imath}$ can walk away from fraction $\hat{\beta}$ ($= 1 - \beta$ in problem set) of debt
  - Bankers cannot diversify (lose fraction $\hat{\beta}$ of loan value)
- New loan return

$$dr_t^\ell = i_t^\ell dt + dr_t^m - \hat{\beta} d\tilde{J}_t$$

- Questions:
  1. Money valuation equation and η evolution?
  2. Characterize steady state
  3. $\hat{\beta}$ comparative statics (numerically)

Part 5 – Money Valuation

- Use now deposits as benchmark asset

$$\mu_t^\eta + \rho - \mu_t^\vartheta - i_t^d + \tilde{\lambda} j_t^{n,b} = \tilde{\lambda}\tilde{\nu}_t^b \tilde{j}_t^{n,b},$$

$$\mu_t^{1-\eta} + \rho - \mu_t^\vartheta - i_t^d + \tilde{\lambda} j_t^{n,h} = \tilde{\varsigma}_t^h \tilde{\sigma}_t^{n,h} + \tilde{\lambda}\tilde{\nu}_t^h \tilde{j}_t^{n,h} + \lambda_t \frac{1}{\alpha}$$

- η-weighted average

$$\mu_t^\vartheta = \rho - \left((1-\eta_t)\,\tilde{\varsigma}_t^h \tilde{\sigma}_t^{n,h} + \eta_t \tilde{\lambda}\left(\tilde{\nu}_t^b - 1\right)\tilde{j}_t^{n,b} + (1-\eta_t)\,\tilde{\lambda}\left(\tilde{\nu}_t^h - 1\right)\tilde{j}_t^{n,h} + (1-\eta_t)\,\lambda_t \frac{1}{\alpha} \right)$$

- Substitue back into first equation

$$\mu_t^\eta = (1-\eta_t)\left(\tilde{\lambda}\left(\tilde{\nu}_t^b - 1\right)\tilde{j}_t^{n,b} - \tilde{\varsigma}_t^h \tilde{\sigma}_t^{n,h} - \tilde{\lambda}\left(\tilde{\nu}_t^h - 1\right)\tilde{j}_t^{n,h} - \lambda_t \frac{1}{\alpha} \right)$$

Part 5 – Money Premium

- Need to determine risk premia and money premium
- Start with money premium

- Banker portfolio choice loans vs deposits

$$i_t^\ell - \tilde{\lambda}\hat{\beta} - i_t^d = -\tilde{\lambda}\tilde{\nu}_t^b\hat{\beta}$$

- Household portfolio choice loans vs deposits

$$i_t^\ell - \tilde{\lambda}\hat{\beta} - i_t^d = -\tilde{\lambda}\tilde{\nu}_t^h\hat{\beta} + \lambda_t\frac{1}{\alpha}$$

- Combine

$$\lambda_t\frac{1}{\alpha} = \tilde{\lambda}\hat{\beta}\left(\tilde{\nu}_t^h - \tilde{\nu}_t^b\right)$$

- So money premium related to jump risk premium differential

Part 5 – Brownian risk premium

- Brownian risk premium is standard
- Log utility, so $\tilde{\varsigma}_t^h = \tilde{\sigma}_t^{n,h} = \theta_t^{k,h}\tilde{\sigma}$
- Capital market clearing: $\theta_t^{k,h} = \frac{1-\vartheta_t}{1-\eta_t}$
- Thus: $\tilde{\varsigma}_t^h \tilde{\sigma}_t^{n,h} = \left(\frac{1-\vartheta_t}{1-\eta_t}\tilde{\sigma}\right)^2$

Part 5 – Jump risk premia

- Price of jump risk (generic formula)

$$1-\tilde{\nu}_t^i = 1+\tilde{\jmath}_t^{\xi,i} = 1+\tilde{\jmath}_t^{1/n^i} = \frac{1}{1+\tilde{\jmath}_t^{n,i}} \Rightarrow \tilde{\nu}_t^i = \frac{\tilde{\jmath}_t^{n,i}}{1+\tilde{\jmath}_t^{n,i}}$$

- Net worth jumps

$$\tilde{\jmath}_t^{n,b} = -\theta_t^{\ell,b}\hat{\beta} \qquad \tilde{\jmath}_t^{n,h} = \frac{\eta_t\theta_t^{\ell,b}}{1-\eta_t}\hat{\beta}$$

- Prices of jump risk (here)

$$\tilde{\nu}_t^b = \frac{-\theta_t^{\ell,b}\hat{\beta}}{1-\theta_t^{\ell,b}\hat{\beta}} \qquad \tilde{\nu}_t^h = \frac{\eta_t\theta_t^{\ell,b}\hat{\beta}}{1-\eta_t+\eta_t\theta_t^{\ell,b}\hat{\beta}}$$

Part 5 – Money Valuation and η Drift

- Substitute everything in μ^{ϑ} and μ^{η} equations

$$\mu_t^{\vartheta} = \rho - \left(\frac{1}{(1-\eta_t)} (1-\vartheta_t)^2 \tilde{\sigma}^2 + \left(\frac{\eta_t + (1-\eta_t)\hat{\beta}}{1-\theta_t^{\ell,b}\hat{\beta}} - \frac{(1-\eta_t)\eta_t \left(1-\hat{\beta}\right)}{1-\eta_t+\eta_t\theta_t^{\ell,b}\hat{\beta}} \right) \tilde{\lambda}\theta_t^{\ell,b}\hat{\beta} \right)$$

$$\mu_t^{\eta} = (1-\eta_t) \left(\left(\frac{1}{1-\theta_t^{\ell,b}\hat{\beta}} + \frac{\eta_t}{1-\eta_t+\eta_t\theta_t^{\ell,b}\hat{\beta}} \right) \tilde{\lambda} \left(1-\hat{\beta}\right) \theta_t^{\ell,b}\hat{\beta} - \frac{(1-\vartheta_t)^2}{(1-\eta_t)^2}\tilde{\sigma}^2 \right)$$

Part 5 – Steady State

- Recall banker net worth pricing condition

$$\mu_t^\eta + \rho - \mu_t^\vartheta - i_t^d + \tilde{\lambda} \jmath_t^{n,b} = \tilde{\lambda} \tilde{\nu}_t^b \tilde{\jmath}_t^{n,b}$$

- In steady state

$$\rho = \tilde{\lambda} \left(\tilde{\nu}^b - 1 \right) \tilde{\jmath}^{n,b} = -\tilde{\lambda} \tilde{\nu}^b$$

- Thus: $$\tilde{\nu}^b = -\frac{\rho}{\tilde{\lambda}} \Rightarrow \tilde{\jmath}_t^{n,b} = -\frac{\rho}{\tilde{\lambda} + \rho}$$

- For this jump exposure, need $\theta_t^{\ell,b} = \frac{\rho}{\tilde{\lambda}+\rho} \cdot \frac{1}{\tilde{\beta}}$

Part 5 – Steady State

- Next: binding CIA

$$\alpha\cdot a\frac{\theta^{k,h}}{q^K}=\theta^{m,h}+\theta^{d,h}\Leftrightarrow\alpha\cdot a\frac{\frac{1-\vartheta}{1-\eta}}{q^K}=1-\frac{1-\vartheta}{1-\eta}+\frac{\eta\theta^{\ell,b}}{1-\eta}$$

$$\Leftrightarrow\frac{\alpha a}{1+\phi a}\left(1-\vartheta+\phi\rho\right)=\vartheta+\eta\frac{\rho}{\tilde{\lambda}+\rho}\frac{1}{\hat{\beta}}$$

- Solve for $1-\vartheta$

$$1-\vartheta=\frac{1+\phi a}{1+\left(\phi+\alpha\right)a}\left(1+\eta\frac{\rho}{\tilde{\lambda}+\rho}\frac{1}{\hat{\beta}}\right)-\frac{\alpha\phi a}{1+\left(\phi+\alpha\right)a}\rho$$

Part 5 – Steady State

- Final step: plug everything into steady-state money valuation equation

$$\rho = \frac{1}{(1-\eta)}\left(\frac{1+\phi a}{1+(\phi+\alpha)\,a}\left(1+\eta\frac{\rho}{\tilde{\lambda}+\rho}\frac{1}{\hat{\beta}}\right)-\frac{\alpha\phi a}{1+(\phi+\alpha)\,a}\rho\right)^2\tilde{\sigma}^2$$

$$+\left(\frac{\eta+(1-\eta)\,\hat{\beta}}{1-\theta^{\ell,b}\hat{\beta}}-\frac{(1-\eta)\,\eta\left(1-\hat{\beta}\right)}{1-\eta+\eta\theta^{\ell,b}\hat{\beta}}\right)\tilde{\lambda}\theta^{\ell,b}\hat{\beta}$$

- Can solve this numerically for η…
- Note: if there is no positive solution, then $\eta = 0$ (happens for large $\hat{\beta}$)

Part 5 – Numerical Results