

## ECO 529: Second Exercise on Numerical Methods

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The goal of this exercise is to solve the HJB equation for a particular optimization problem with two state variables. In problem 1, we'll address valuation equations from a certain general class, and produce code that can be used to solve these equations numerically. In problem 2, we'll use this code to solve the HJB equation for a particular application, via policy iteration.

**Problem 1.** Consider a two-dimensional state variable  $X = (X^1, X^2)$  which follows

$$dX_t^1 = \mu^1(X) dt + \sigma(X) dZ_t, \quad dX_t^2 = \mu^2(X) dt$$

on  $[0, \bar{X}^1] \times [0, \bar{X}^2]$ . Notice that only the first dimension  $X^1$  has volatility, which makes it fairly straightforward to discretize the differential operator for this process.

Assume that

$$\begin{aligned} \mu^1(0, X^2) \geq 0, \quad \mu^1(\bar{X}^1, X^2) \leq 0, \quad \mu^2(X^1, 0) \geq 0, \quad \mu^2(X^1, \bar{X}^2) \leq 0, \\ \text{and } \sigma(0, X^2) = \sigma(\bar{X}^1, X^2) = 0 \end{aligned}$$

to ensure that the process  $X$  stays within  $[0, \bar{X}^1] \times [0, \bar{X}^2]$ .

Consider the valuation equation for payoff flow  $g(X)$  and discount rate  $\rho$ .

Let us introduce a rectangular grid over the state space, by discretizing each dimension as follows:

$$0 = x_1^1 < x_2^1 \dots < x_{N_1}^1 = \bar{X}^1 \quad 0 = x_1^2 < x_2^2 \dots < x_{N_2}^2 = \bar{X}^2.$$

For simplicity, assume that each dimension is evenly spaced, with intervals between grid points  $\Delta x_1$  and  $\Delta x_2$ , respectively.

We can represent functions on the grid as  $N_1$  by  $N_2$  matrices or as column vectors of  $N = N_1 \times N_2$  components. In the later case, we take lexicographic order over the grid: the first coordinate of the long vector corresponds to  $(x_1^1, x_1^2)$ , the second corresponds to  $(x_2^1, x_1^2)$ , the  $N_1$ -th corresponds to  $(x_{N_1}^1, x_1^2)$ , the  $N_1 + 1$ -st corresponds to  $(x_1^1, x_2^2)$ , and so on.

Let  $g$  and  $f$  be the  $N$ -dimensional vectors that represent the values of the payoff flow and value function on the grid.

We seek to construct matrix  $M$  such that  $Mf$  represents the differential operator

$$\mu^1(X)f_1(X) + \mu^2(X)f_2(X) + \sigma(X)^2f_{11}(X)/2,$$

for the first partial derivatives  $f_1$  and  $f_2$ , and the second-order partial with respect to  $X_1$ ,  $f_{11}$ . We'll use the upwind method to evaluate the first derivative.

**(1)** If  $\mu^1(X) > 0$  at grid point  $n$ , then the term  $\mu^1(X)f_1(X)$  puts a nonzero value into which entries of matrix  $M$ ?

- (a)  $(n, n)$  and  $(n + 1, n)$     (b)  $(n, n)$  and  $(n - 1, n)$   
(c)  $(n, n)$  and  $(n, n + 1)$     (d)  $(n, n)$  and  $(n, n - 1)$

**(2)** If  $\mu^2(X) < 0$  at grid point  $n$ , then the term  $\mu^2(X)f_2(X)$  puts a nonzero value into which entries of matrix  $M$ ?

- (a)  $(n, n)$  and  $(n - N_1, n)$     (b)  $(n, n)$  and  $(n - N_2, n)$   
(c)  $(n, n)$  and  $(n, n - N_1)$     (d)  $(n, n)$  and  $(n, n - N_2)$

**(3)** If  $\sigma(X) > 0$  at grid point  $n$ , then the term  $\sigma(X)^2f_{11}(X)/2$  puts a nonzero value into which entries of matrix  $M$ ?

- (a)  $(n, n + 1)$  and  $(n, n - 1)$     (b)  $(n, n)$ ,  $(n, n + 1)$  and  $(n, n - 1)$   
(c)  $(n, n)$ ,  $(n, n + N_1)$  and  $(n, n - N_1)$   
(d)  $(n, n)$ ,  $(n + N_2, n)$  and  $(n - N_2, n)$     (e) None of the above

**(4)** What is the value of  $M_{n, n+1}$ ?

- (a)  $\mu^1(n)^+ + \sigma(n)^2/\Delta x_1/2$     (b)  $\mu^1(n)^+/\Delta x_1$   
(c)  $\mu^1(n)^+/\Delta x_1 + \sigma(n)^2/\Delta x_1^2$     (d)  $\mu^1(n)^+/\Delta x_1 + \sigma(n)^2/\Delta x_1^2/2$   
(e)  $\mu^1(n)^+/\Delta x_1 + \mu^2(n)^+/\Delta x_2 + \sigma(n)^2/\Delta x_1^2/2$

**(5)** What is the value of  $M_{n, n-1}$ ?

- (a)  $-\mu^1(n)^-/\Delta x_1 + \sigma(n)^2/\Delta x_1^2$     (b)  $-\mu^1(n)^-/\Delta x_1 + \sigma(n)^2/\Delta x_1^2/2$   
(c)  $\mu^1(n)^-/\Delta x_1 + \sigma(n)^2/\Delta x_1^2$     (d)  $\mu^1(n)^-/\Delta x_1 + \sigma(n)^2/\Delta x_1^2/2$

**(6)** What is the value of  $M_{n, n}$ ?

- (a)  $-\mu^1(n)/\Delta x_1 - \mu^2(n)/\Delta x_2 - \sigma(n)^2/\Delta x_1^2/2$
- (b)  $-\mu^1(n)/\Delta x_1 - \mu^2(n)/\Delta x_2 - \sigma(n)^2/\Delta x_1^2$
- (c)  $-|\mu^1(n)|/\Delta x_1 - |\mu^2(n)|/\Delta x_2 - \sigma(n)^2/\Delta x_1^2$
- (d)  $|\mu^1(n)|/\Delta x_1 + |\mu^2(n)|/\Delta x_2 - \sigma(n)^2/\Delta x_1^2/2$
- (e) None of the above.

(7) Assuming  $N_1 \neq N_2$ , what are the values at  $M_{n,n-N_1}$  and  $M_{n,n-N_2}$ ?

- (a) 0 and  $-\mu^2(n)^+/\Delta x_2$
- (b) 0 and  $-\mu^2(n)^-/\Delta x_2$
- (c)  $-\mu^2(n)^-/\Delta x_2$  and 0
- (d)  $\mu^2(n)^-/\Delta x_2$  and 0

(8) What is the maximal number of nonzero elements that  $M$  can have in a given row?

- (a) 2
- (b) 3
- (c) 4
- (d) 5
- (e) 6

The discretized HJB equation is  $\rho f = g + Mf$ . In class, we'll discuss the function to solve for  $f$  given  $g$ ,  $\mu$  and  $\sigma$ ,

$$f = \text{payoff\_policy\_2d}(dX1, dX2, \text{rho}, \text{MU1}, \text{MU2}, \text{S}, g),$$

which takes  $N_1 \times N_2$  matrices for the two components of the drift MU1 and MU2, volatility S and payoff flow G, as well as intervals between grid points for the two dimensions dX1 and dX2 and the discount rate rho.

**Problem 2.** Consider the following consumption-savings problem. The agent has income that follows the Cox-Ingersoll-Ross process

$$dx_t = a(\bar{x} - x_t) dt + \sigma\sqrt{x_t} dZ_t.$$

The agent can smooth income using risk-free savings technology with the risk-free rate of  $r$ . Savings  $s_t \geq 0$  follow

$$ds_t = (rs_t + x_t - c_t) dt,$$

where  $c_t$  is the agent's consumption. The agent has utility function  $u(c) = c^{1-\gamma}/(1-\gamma)$  and discount rate of  $\rho > r$ .

The goal of this problem is to use policy iteration to find the optimal consumption policy in this setting.

We'll use the Matlab function `payoff_policy_2d` from Problem 1. We'll take parameter values  $\rho = 0.1$ ,  $r = 0.05$ ,  $\gamma = 2$ ,  $a = 0.1$ ,  $\bar{x} = 1$  and  $\sigma = 0.3$ .

(1) What are good choices for the intervals for income and savings? What considerations should we take into account? What can we do to make sure the states stay within these intervals? Does it matter? What can we do to check whether it matters or not?

(2) To compute the value function via policy iteration, what is a good guess for a policy to start with?

(3) From the HJB equation, what is the first-order condition for optimal consumption?

- (a)  $f_2 = c^{-\gamma}$     (b)  $f_1 = c^{-\gamma}$   
 (c)  $f_2 = -c^{-\gamma}$     (d)  $f_1 = -c^{-\gamma}$

(4) Given the value function  $F$  for a candidate policy, shaped as an  $N_1 \times N_2$  matrix we can evaluate the first derivative of  $F$  with respect to savings as

$$F2 = \frac{F(:, 2 : N_2) - F(:, 1 : N_2 - 1)}{dX_2}.$$

What is a good way to determine numerically the optimal consumption given the value of this derivative? Here any matrix operation is to be interpreted in the element-by-element sense.

- (a)  $C1 = F2^{1/\gamma}$     (b)  $C1 = \max(F2, 100^{-\gamma})^{1/\gamma}$   
 (c)  $C1 = \min(F2, 100^{-\gamma})^{1/\gamma}$     (d)  $C1 = \min(F2, 0.0001^{-\gamma})^{1/\gamma}$

(5) Denote by  $C0 = r*S + X$  the consumption level that leads to zero drift of savings. At grid points  $[1 : N_1, 1 : N_2 - 1]$ , what is the optimal consumption subject to the constraint that the drift is nonnegative (so we use the right derivative)?

- (a)  $CR = \max(C1, C0(:, 1 : N_2 - 1))$     (b)  $CR = \max(C1, C0)$   
 (c)  $CR = \min(C1, C0(:, 1 : N_2 - 1))$     (d)  $CR = \min(C1, C0)$

(6) At grid points  $[1 : N_1, 2 : N_2]$ , what is the optimal consumption subject to the constraint that the drift is nonpositive (so we use the left derivative)?

- (a)  $CL = \max(C1, C0(:, 2 : N_2))$     (b)  $CL = \max(C1, C0)$   
(c)  $CL = \min(C1, C0(:, 2 : N_2))$     (d)  $CL = \min(C1, C0)$

**(7)** Which principles would you choose to update the policy for policy iteration?

- (a) At the lower boundary on savings, choose  $CR$ .  
(b) At the upper boundary on savings, choose  $CL$ .  
(c) In the middle, choose consumption by comparing

$$\frac{CR(:, 2 : N_2 - 1)^{1-\gamma}}{1 - \gamma} - F2(:, 2 : N_2 - 1) * CR(:, 2 : N_2 - 1) \quad \text{and}$$

$$\frac{CL(:, 1 : N_2 - 2)^{1-\gamma}}{1 - \gamma} - F2(:, 1 : N_2 - 2) * CL(:, 1 : N_2 - 2).$$

- (d) All of the above.