ECO 529:
Numerical Methods
In-class exercise #1

Goal: understand

- Why representing the differential operator through

$$M = \begin{bmatrix} - & + & 0 & 0 & 0 & 0 \\ + & - & + & 0 & 0 & 0 \\ 0 & + & - & + & 0 & 0 \\ 0 & 0 & + & - & + & 0 \\ 0 & 0 & 0 & + & - & + \\ 0 & 0 & 0 & 0 & + & - \end{bmatrix}$$

gives a monotone scheme

- Why this monotone scheme converges
- Study a stylized discrete-state setting that can give us intuition about equations in continuous state space

Recall from the recorded lectures

$$0 = f_t + u(x) + \mu(x) f_x + \frac{1}{2}\sigma(x)^2 f_{xx} - \rho f(x,t), \quad f(x,T) = U(x)$$

represent this as Mf

$$M = \begin{bmatrix} - & + & 0 & 0 & 0 & 0 \\ + & - & + & 0 & 0 & 0 \\ 0 & + & - & + & 0 & 0 \\ 0 & 0 & + & - & + & 0 \\ 0 & 0 & 0 & + & - & + \\ 0 & 0 & 0 & 0 & + & - \end{bmatrix}$$

rows
add to 0

- Explicit scheme

$$f(t-\Delta t,.) = \Delta t\, u(.) + ((1-\rho\Delta t)I + \Delta t\, M)\, f(t,.)$$

- Implicit scheme

$$f(t,.) = (I(1+\rho\,\Delta t) - \Delta t\, M)^{-1}(\Delta t\, u(.) + f(t+\Delta t,.))$$

monotone:
all entries ≥ 0

The Setting

Markov jump process on {1, 2, 3 … N}

Jump intensity from k to l is $\lambda_{kl} \geq 0$

Payoff flow in state k is g_k

Discount rate in state k is $\chi_k > 0$

Question 1

Write down the HJB equation for the value function f_k of this individual.

(a) $\chi_k f_k = g_k + \sum_{l \neq k} \lambda_{kl}(f_l - f_k)$

(b) $\chi_k f_k = g_k + \sum_{l \neq k} \lambda_{kl} f_l$

(c) $f_k = \chi_k g_k + \sum_{l \neq k} \lambda_{kl} f_l$

(d) $\chi_k f_k = g_k + \sum_{l \neq k} \lambda_{kl}(\chi_l f_l - \chi_k f_k)$

Question 2

χ, f, g are column vectors. Express the relationship between *f* and *g* in the form

$$\mathrm{diag}(\chi)\, f = g + M f$$

where diag(χ) is a diagonal matrix with coefficients χ_k. Then

(a) $M_{kk} = 0, \quad M_{kl} = \lambda_{kl}$

(b) $M_{kk} = \sum_{l \neq k} \lambda_{lk}, \quad M_{kl} = \lambda_{lk}$

(c) $M_{kk} = -\sum_{l \neq k} \lambda_{lk}, \quad M_{kl} = \lambda_{lk}$

(d) $M_{kk} = -\sum_{l \neq k} \lambda_{kl}, \quad M_{kl} = \lambda_{kl}$

$$M = \begin{bmatrix} M_{11} & M_{12} & M_{13} & \dots \\ M_{21} & M_{22} & M_{23} & \dots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

Question 3

$$M = \begin{bmatrix} -\sum_{l\neq 1}\lambda_{1l} & \lambda_{12} & \lambda_{13} & \cdots \\ \lambda_{21} & -\sum_{l\neq 2}\lambda_{2l} & \lambda_{23} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

Is matrix M invertible?

(a) Always (b) Never

(c) May be invertible or not, depending on parameters

Question 4

$$M = \begin{bmatrix} -\sum_{l\neq 1}\lambda_{1l} & \lambda_{12} & \lambda_{13} & \cdots \\ \lambda_{21} & -\sum_{l\neq 2}\lambda_{2l} & \lambda_{23} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

Is matrix $\mathrm{diag}(\chi) - M$ invertible?

(a) Always (b) Never

(c) May be invertible or not, depending on parameters

Question 5

Which is the strongest statement that can be made about the entries of the inverse of $\mathrm{diag}(\chi) - M$?

(a) All entries are positive (b) All entries are nonnegative
(c) The diagonal entries are nonnegative
(d) The diagonal entries are positive (e) Statements (b) and (d) hold

Question 6

Suppose at time 0, the state k is drawn from the distribution $h(0) \in \mathbb{R}^N$. Denote by $h(t)$ the probability distribution over states at time t. What differential equation does vector h satisfy?

(a) $h_k(t)' = -\sum_{l \neq k} \lambda_{kl} h_k(t)$

(b) $h_k(t)' = \sum_{l \neq k} \lambda_{kl} h_l(t) - \sum_{l \neq k} \lambda_{kl} h_k(t)$

(c) $h_k(t)' = \sum_{l \neq k} \lambda_{lk} h_l(t) - \sum_{l \neq k} \lambda_{kl} h_k(t)$

(d) $h_k(t)' = \sum_{l \neq k} \lambda_{kl} h_l(t) - \sum_{l \neq k} \lambda_{kl} h_k(t) - \chi_k h_k(t)$

Q7: In matrix form, write the differential equation that h satisfies.

$h_t = M^T h$, hence the operator from KFE is the adjoint of that of HJB

Question 8

Can matrix M have any eigenvalues with a positive real part? Can M have complex eigenvalues?

(a) Yes and yes (b) Yes and no (c) No and yes (d) No and no

Hint: M and M^T have the same eigenvalues. KFE is $h_t = M^T h$. What would happen if M^T had positive eigenvalues?

Question 8

Can matrix M have any eigenvalues with a positive real part? Can M have complex eigenvalues?

(a) Yes and yes (b) Yes and no (c) No and yes (d) No and no

Hint: M and M^T have the same eigenvalues. KFE is $h_t = M^T h$. What would happen if M^T had positive eigenvalues?

Distribution cannot explode, so can't have positive eigenvalues

$$M = \begin{bmatrix} -1 & 0 & 1 \\ 1 & -1 & 0 \\ 0 & 1 & -1 \end{bmatrix}$$

Eigenvalues 0, -1.5 $\pm$ 0.866 i

Question 9

(9) Does this Poisson process always have a stationary distribution? Assume that, for a given matrix M, this distribution exists and it is unique. Devise an algorithm, as simple as you can, to compute this stationary distribution.

Recap

- Matrix M has entries ≤ 0 on the diagonal, ≥ 0 elsewhere
- Sum of entries in any row is 0
- Can create any matrix with these properties by choosing $\lambda_{kl} \geq 0$

$$M = \begin{bmatrix} -\sum_{l \neq 1} \lambda_{1l} & \lambda_{12} & \lambda_{13} & \cdots \\ \lambda_{21} & -\sum_{l \neq 2} \lambda_{2l} & \lambda_{23} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

Then:

- Eigenvalues of M have real part ≤ 0
- Matrix $\mathrm{diag}(\chi) - M$ is invertible, has nonnegative entries

Let’s relate this to our setting…

- m-dimensional valuation equation (let’s even allow jumps)

$$\rho f(x) = g(x) + \underbrace{\mu(x) Df(x) + \frac{\operatorname{tr}\left[\sigma(x)\sigma(x)^T D^2 f(x)\right]}{2} + \int \phi(y \mid x)(f(y) - f(x))\, dy}_{\text{differential/jump operator}}$$

- Suppose, with N grid points, we can discretize this operator as Mf

Question 10

- m-dimensional valuation equation (let's even allow jumps)

$$\rho f(x) = g(x) + \underbrace{\mu(x) Df(x) + \frac{\mathrm{tr}\left[\sigma(x)\sigma(x)^T D^2 f(x)\right]}{2} + \int \phi(y \mid x)(f(y) - f(x))\, dy}_{\text{differential/jump operator}}$$

- Suppose, with N grid points, we can discretize this operator as Mf

- Which term poses the greatest challenges, in your opinion?

(a) The drift term (b) The volatility term (c) The jump term

Question 11

Now, suppose you are trying to solve the valuation equation via an implicit scheme. For the implicit scheme, the online lecture derives the equation

$$f(t,.) = (I(1+\rho\,\Delta t) - \Delta t\, M)^{-1}(\Delta t\, g(.) + f(t+\Delta t,.)).$$

In order to insure that solutions do not explode, we need the eigenvalues of $(I(1+\rho\,\Delta t) - \Delta t\, M)^{-1}$ to have real part less than 1 (why?).

(11) True or false: Matrices M and $I(1+\rho\,\Delta t) - \Delta t\, M$ always have the same **eigenvectors**. Matrix $I(1+\rho\,\Delta t) - \Delta t\, M$ is always invertible.

(a) True, false (b) False, true (c) True, true (d) False, false

Invertibility of the matrix for the implicit scheme

Matrix I (1 + ρ Δt) – Δt M is invertible because of question 4.

Is matrix $\mathrm{diag}(\chi) - M$ invertible?

(a) Always (b) Never

because (diag(χ) – M) f = g always has a solution f.

Moreover, the inverse has entries ≥ 0, hence implicit scheme is monotone (as long as M has the desired sign pattern)

Question 12

(12) Can you argue that the eigenvalues of $(I(1+\rho\,\Delta t) - \Delta t\, M)^{-1}$ have real part less than 1? Can you give an upper bound (on the largest real part)?

Question 12

(12) Can you argue that the eigenvalues of $(I(1+\rho\,\Delta t) - \Delta t\,M)^{-1}$ have real part less than 1? Can you give an upper bound (on the largest real part)?

1/(1 + ρ Δt) is an upper bound, because M and I (1 + ρ Δt) – Δt M have the same eigenvectors & M doesn't have eigenvalues with a positive real part

Hence, the implicit scheme

$$f(t,.) = (I(1+\rho\,\Delta t) - \Delta t\,M)^{-1}(\Delta t\,u(.) + f(t+\Delta t,.))$$

cannot explode as long as u remain bounded…

To wrap up…

- m-dimensional valuation equation (let's even allow jumps)

$$\rho f(x) = g(x) + \underbrace{\mu(x) Df(x) + \frac{\operatorname{tr}\left[\sigma(x)\sigma(x)^T D^2 f(x)\right]}{2} + \int \phi(y \mid x)(f(y) - f(x))\, dy}_{\text{differential/jump operator}}$$

- As long as we can discretize the operator as Mf, where M has desired properties, the resulting numerical scheme is monotone and stable