Eco 529: Brunnermeier & Sannikov

Macro, Money and Finance

Problem Sets 2 – Solutions

Sebastian Merkel

Problem Set 2 – Problem 2 (KFE OU Process)

- General Kolmogorov Forward Equation (KFE)

$$\frac{\partial p}{\partial t}(x,t) = -\frac{\partial}{\partial x}\left(\mu(x,t)p(x,t)\right) + \frac{1}{2}\frac{\partial^2}{\partial x^2}\left(\sigma^2(x,t)p(x,t)\right)$$

- Describes density evolution of process X with

$$dX_t = \mu(X_t,t)dt + \sigma(X_t,t)dZ_t$$

- For this problem: $X =$ Ornstein-Uhlenbeck process

(continuous-time AR(1))

$$dX_t = \theta(\bar{x} - X_t)dt + \sigma dZ_t$$

- Get then special KFE

$$\frac{\partial p}{\partial t}(x,t) = \theta(x-\bar{x})\frac{\partial}{\partial x}p(x,t) + \theta p(x,t) + \frac{\sigma^2}{2}\frac{\partial^2}{\partial x^2}p(x,t)$$

Problem Set 2 – Problem 2 (KFE OU Process)

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- Tasks:
  - Solve equation numerically using different schemes and parameters
  - Compare with known closed-form solution
  - Identify problems with some schemes

Digression: a More General PDE

- OU- KFE PDE is

$$\frac{\partial p}{\partial t}(x,t) = \theta(x-\bar{x})\frac{\partial}{\partial x}p(x,t) + \theta p(x,t) + \frac{\sigma^2}{2}\frac{\partial^2}{\partial x^2}p(x,t)$$

- Instead, let's solve the generic linear PDE

$$\frac{\partial p}{\partial t}(x,t) = a(x,t)\frac{\partial^2}{\partial x^2}p(x,t) + b(x,t)\frac{\partial}{\partial x}p(x,t) + c(x,t)p(x,t) + d(x,t)$$

Digression: a More General PDE

- Let's solve the generic linear PDE

$$\frac{\partial p}{\partial t}(x,t) = a(x,t)\frac{\partial^2}{\partial x^2}p(x,t) + b(x,t)\frac{\partial}{\partial x}p(x,t) + c(x,t)p(x,t) + d(x,t)$$

- Let $x_0 < x_1 < \cdots < x_{N-1} < x_N$ be a grid in x-dimension
- A finite difference approximation transforms the PDE into a vector ODE

$$\frac{d\hat{p}}{dt}(t) = A(t)D^2\hat{p}(t) + B(t)D^1\hat{p}(t) + C(t)\hat{p}(t) + \hat{d}(t)$$

where

$$\hat{p}(t) := \begin{pmatrix} p(x_0,t) \\ p(x_1,t) \\ \vdots \\ p(x_N,t) \end{pmatrix}, \; A(t) := \begin{pmatrix} a(x_0,t) & 0 & \cdots & 0 \\ 0 & a(x_1,t) & & \vdots \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & \cdots & a(x_N,t) \end{pmatrix}, \; B(t) := \cdots, \; C(t) := \cdots, \; \hat{d}(t) := \begin{pmatrix} d(x_0,t) \\ d(x_1,t) \\ \vdots \\ d(x_N,t) \end{pmatrix}$$

and the matrices D^2, D^1 represent the finite difference approximation

Solving the Vector ODE – time step

- We can write this vector ODE more concisely as
$$\hat{p}'(t) = M(t)\hat{p}(t) + \hat{d}(t)$$
where
$$M(t) = A(t)D^2 + B(t)D^1 + C(t)$$
- We use two methods for the ODE (time step)
  1. Explicit Euler:
     - Evaluate right-hand side at the previous time grid point t
     - Equation for new vector $\hat{p}(t+\Delta t)$
$$\hat{p}(t+\Delta t) = \hat{p}(t) + \Delta t \cdot \big(M(t)\hat{p}(t) + \hat{d}(t)\big)$$
  2. Implicit Euler:
     - Evaluate right-hand side at the new time grid point $t + \Delta t$
     - Equation for new vector $\hat{p}(t+\Delta t)$
$$\big(I - \Delta t \cdot M(t+\Delta t)\big)\hat{p}(t+\Delta t) = \hat{p}(t) + \Delta t \cdot \hat{d}(t+\Delta t)$$
- *Remark*: on the two boundaries, we ignore the ODE and impose the boundary conditions instead

Solving the Vector ODE – the matrix $M(t)$

- Left to do: construct matrix $M(t)$
  - recall $M(t) = A(t)D^2 + B(t)D^1 + C(t)$
- The matrices $A(t)$, $B(t)$ and $C(t)$ are diagonal
- D^2 and D^1 are tridiagonal

$$D^k = \begin{pmatrix} * & * & & & & & \\ d^k_{1,-} & d^k_{1,0} & d^k_{1,+} & & & & \\ & d^k_{2,-} & d^k_{2,0} & d^k_{2,+} & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & \ddots & \ddots & \ddots & \\ & & & & d^k_{N-1,-} & d^k_{N-1,0} & d^k_{N-1,+} \\ & & & & & * & * \end{pmatrix}$$

- Conclusion: $M(t)$ is again tridiagonal, construct as sparse matrix

Example: D^2 for equally spaced grid

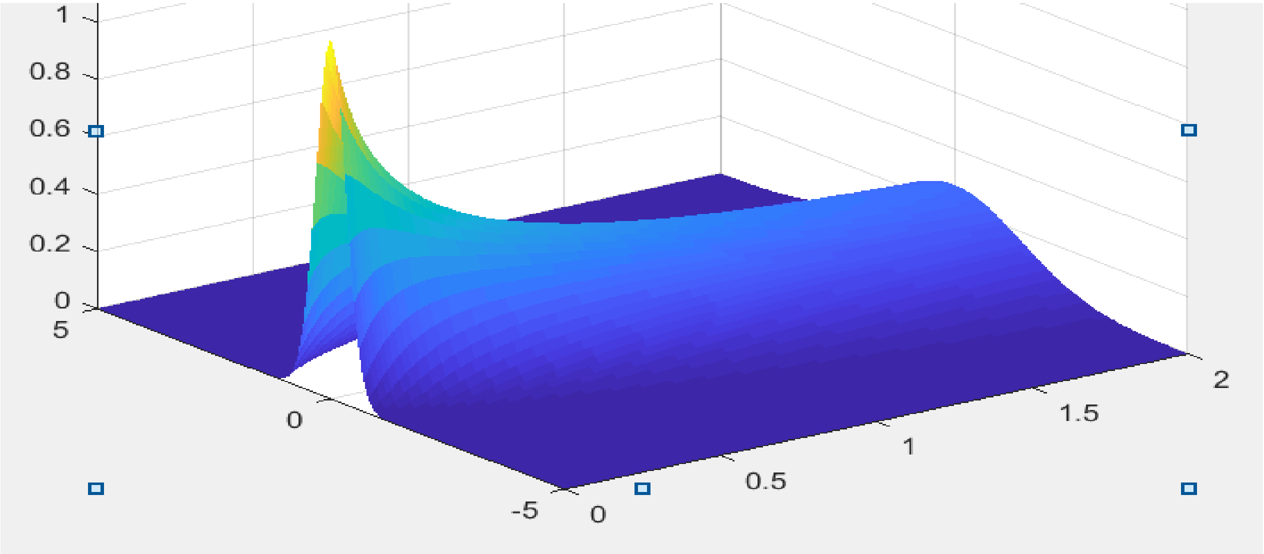
- Suppose we have an equally spaced grid,

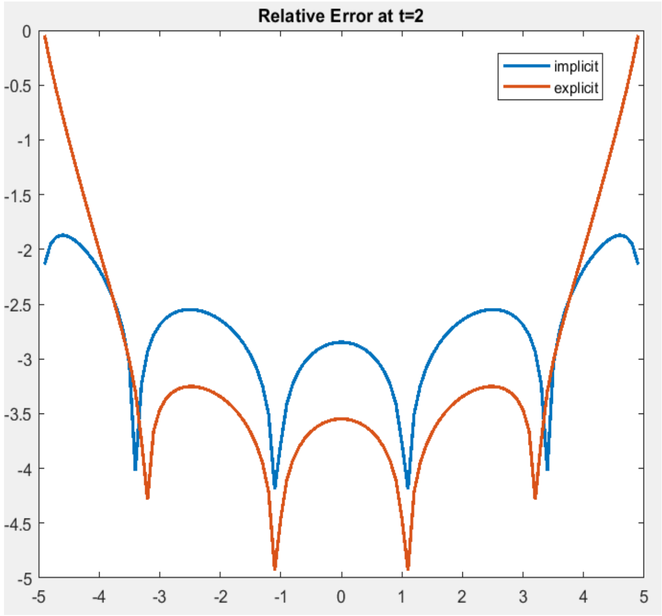
$$x_{i+1} = x_i + \Delta x$$

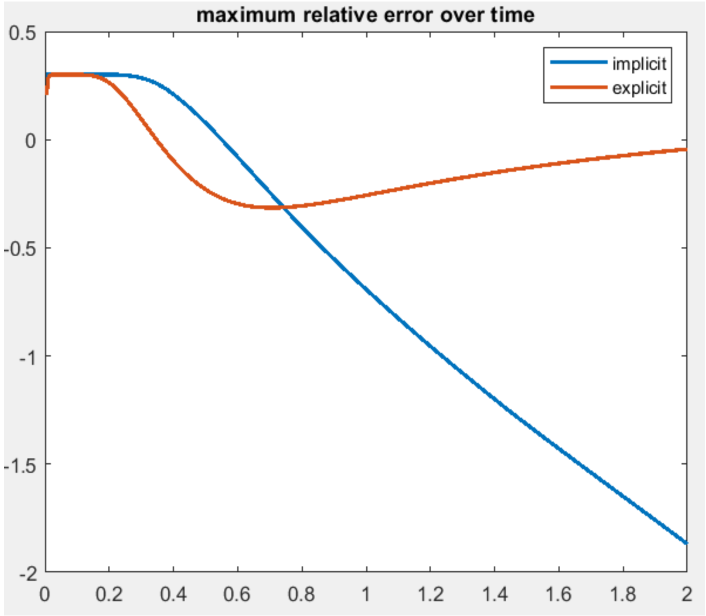
- Then D^2 is given by

$$D^2 = \begin{pmatrix} * & * & & & & & \\ \frac{1}{\Delta x^2} & -\frac{2}{\Delta x^2} & \frac{1}{\Delta x^2} & & & & \\ & \frac{1}{\Delta x^2} & -\frac{2}{\Delta x^2} & \frac{1}{\Delta x^2} & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & \ddots & \ddots & \ddots & \\ & & & & \frac{1}{\Delta x^2} & -\frac{2}{\Delta x^2} & \frac{1}{\Delta x^2} \\ & & & & & * & * \end{pmatrix}$$

Solution of KFE for $\theta = 0$



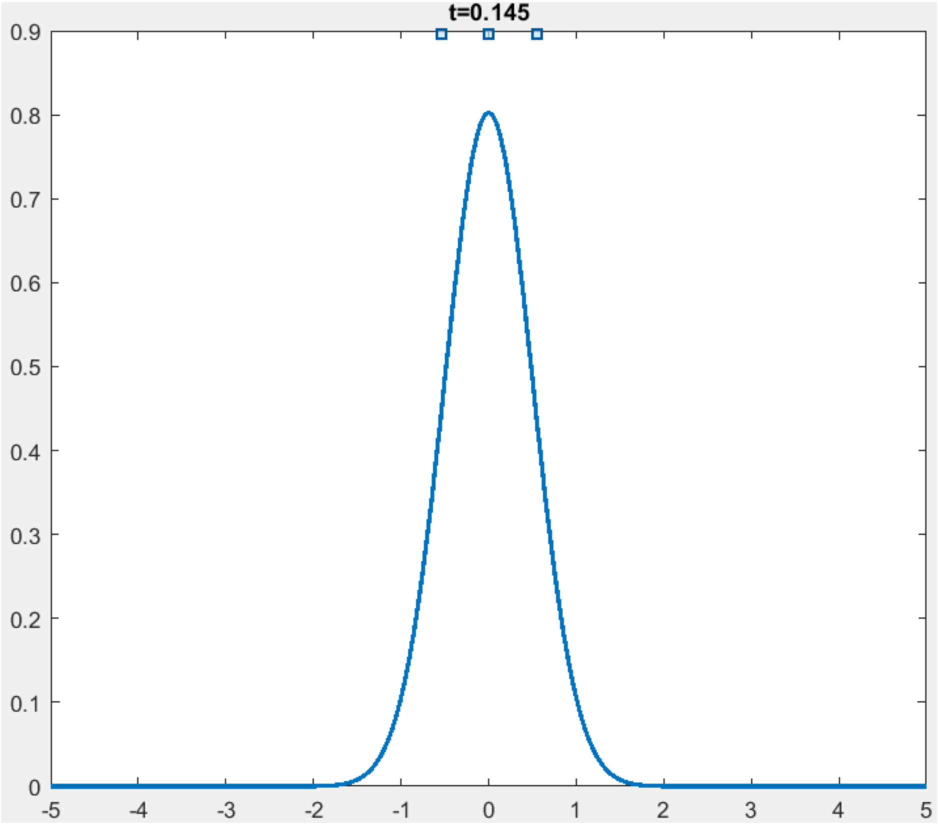




What Happens for Finer Space Grid?

- Implicit Method: errors slightly smaller

- Explicit Method:



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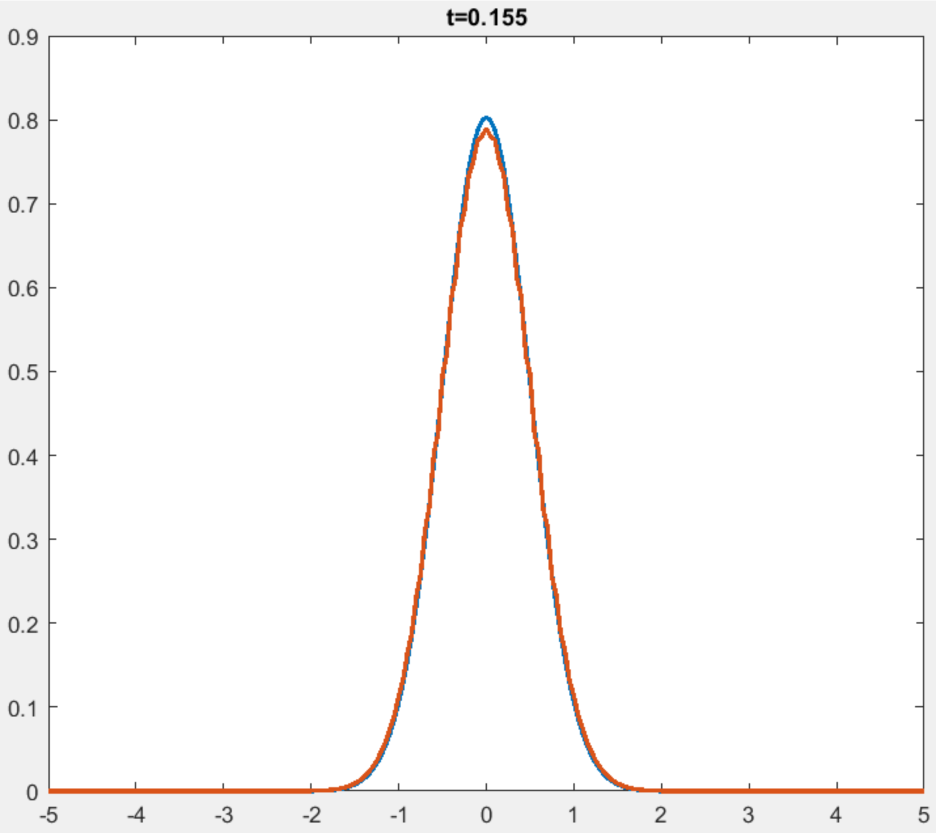

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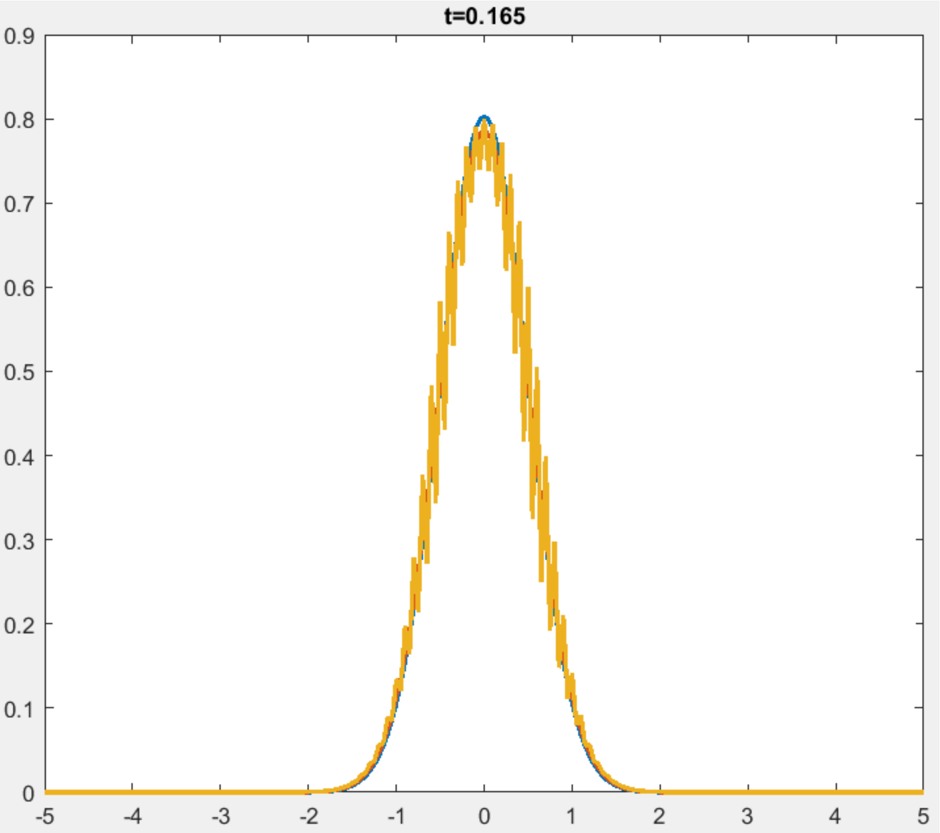

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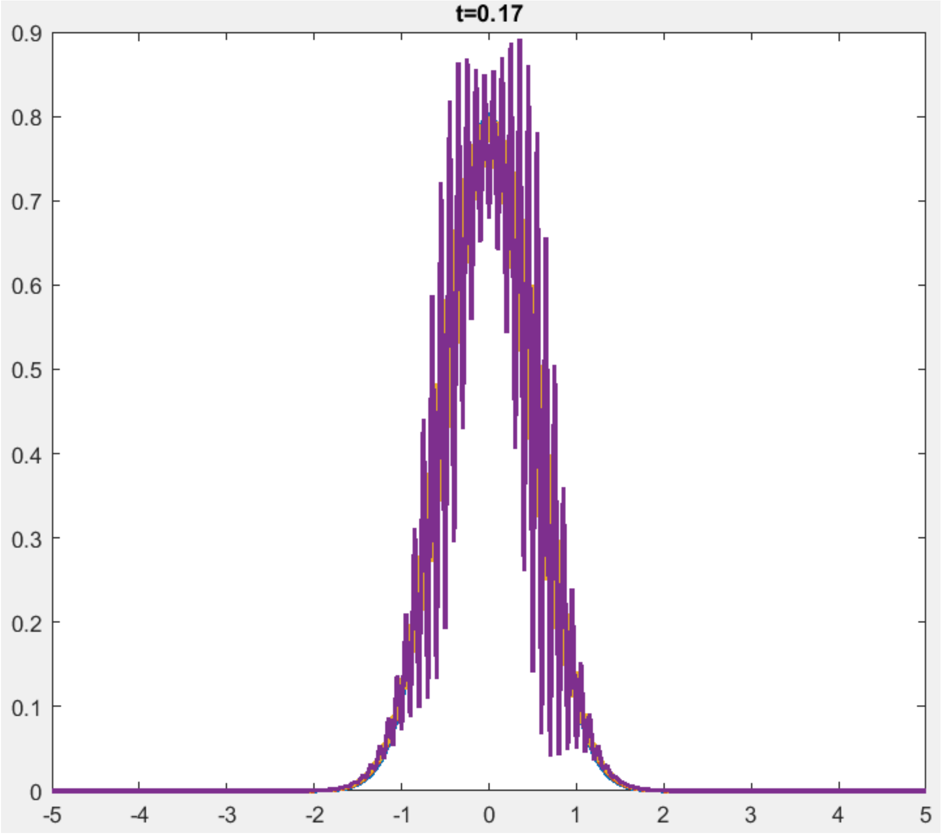

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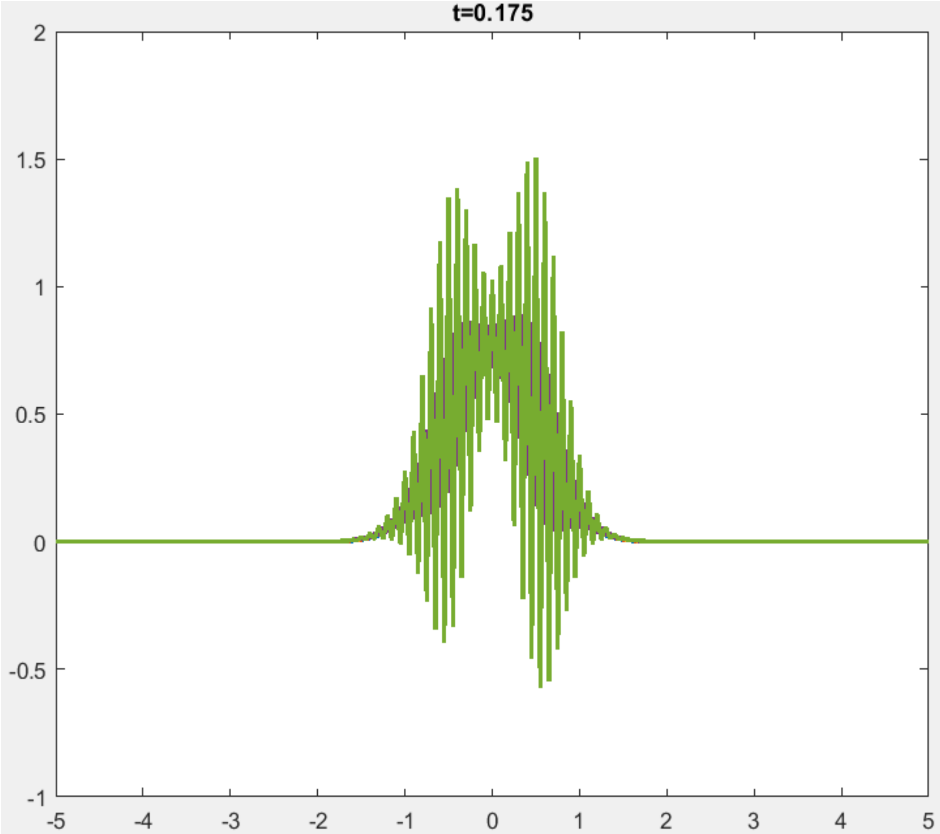
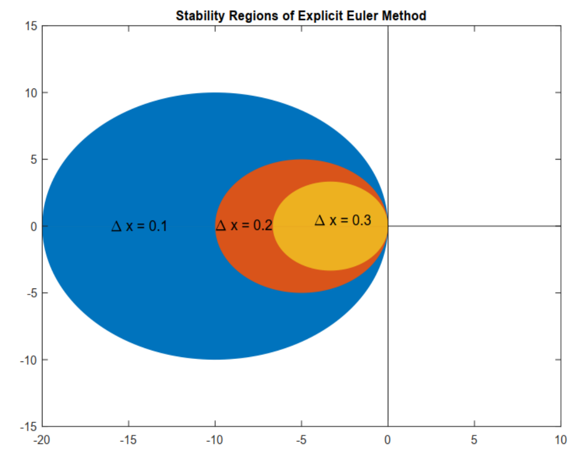
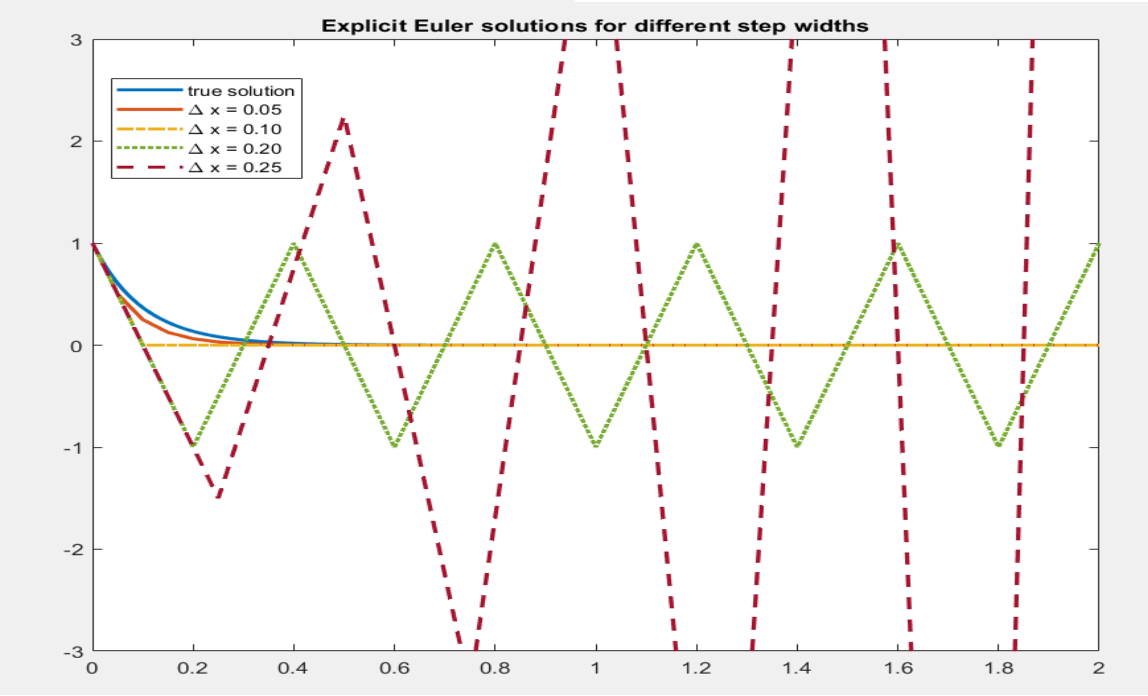


What Goes Wrong with the Explicit Method?

- Recall, Stability of ODEs:
  - Stable, if eigenvalue in stability region
  - Smallest eigenvalue of equation‘s space discretization $\lambda \approx -\frac{2\sigma^2}{\Delta x^2}$
  - If too small, explicit method becomes unstable:





Solution for $\theta = 3$