

# Macro, Money and (International) Finance – Problem Set 3

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**The submission deadline is Friday, October 2 (end of day Princeton time). Please submit your group's solution via email to Fernando Mendo (fmendolopez@gmail.com).**

## 1 Stochastic Lifetimes and Type Switching

We have discussed in Lecture 4 another common modeling device to generate a stationary wealth distribution in a model that would otherwise lead to the long-term dominance of one type of agent: stochastic lifetimes and type switching of the offsprings of dying agents. In this problem, you will work out some of the technical details involved. You may not have the mathematical training to make every single step in your argument fully rigorous. If that happens, try to argue intuitively why that step is logically correct (we are economists after all).

1. Let's start from a single-agent decision problem for an agent with a finite, but random, lifetime  $\tau \sim \text{Exp}(\lambda)$ <sup>1</sup> and no bequest motive. Expected utility at time  $t$  is

$$U_t = \begin{cases} \mathbb{E}_t \left[ \int_t^\tau e^{-\rho(s-t)} u(c_s) ds \right], & t \leq \tau \\ 0, & t > \tau \end{cases}.$$

The agent solves a consumption-portfolio problem with  $J$  assets  $1, \dots, J$ , the net worth evolution for  $t < \tau$  is given by

$$dn_t = -c_t dt + n_t \sum_{j=1}^J \theta_t^j dr_t^j.$$

Assume that the asset return processes  $dr_t^j$  satisfy

$$dr_t^j = \mu^j(X_t) dt + \sum_{m=1}^M \sigma_m^j(X_t) dZ_t^m,$$

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<sup>1</sup> $\text{Exp}(\lambda)$  refers to the exponential distribution with parameter  $\lambda > 0$ , its density is  $t \mapsto \lambda e^{-\lambda t}$  over  $[0, \infty)$ .

where  $Z^1, \dots, Z^m$  are independent Brownian motions,  $\mu^j, \sigma_m^j$  are some given functions and  $X_t$  is a Markov state that evolves according to

$$dX_t = \mu_X(X_t)dt + \sum_{m=1}^M \sigma_{X,m}(X_t) dZ_t^m.$$

Also assume that the Brownian motions  $Z^1, \dots, Z^m$  are independent of the idiosyncratic lifetime  $\tau_t$ . This setup implies in particular that there are no assets that allow the agent to hedge death/longevity risk.

We assume further that the agent's choices for  $\{c_t, \theta_t^1, \dots, \theta_t^J\}_{t < \tau}$  conditional on being alive at time  $t$  are restricted to Markov controls

$$c_t = c(t, X_t, n_t), \theta_t^1 = \theta^1(t, X_t, n_t), \dots, \theta_t^J = \theta^J(t, X_t, n_t) \quad (1)$$

that may only depend on the current time  $t$ , exogenous state  $X_t$  and endogenous state  $n_t$ .<sup>2</sup>

- (a) For each admissible control  $\{c_t, \theta_t^1, \dots, \theta_t^J\}_{t < \tau}$  of the agent that can be expressed in terms of policy functions according to equation (1), that equation actually defines a valid stochastic process  $\{c_t, \theta_t^1, \dots, \theta_t^J\}_{t=0}^\infty$  for all times  $t$ , not just  $t < \tau$ . Argue that this extension of the agent's control process to the time domain  $[0, \infty)$  must necessarily be independent of  $\tau$ .

For your answers of the following subquestion, you may even assume a stronger property than you have shown here: conditional on  $\tau > t$  and  $\{Z_s^1, \dots, Z_s^m\}_{s \leq t}$  (i.e. all time  $t$  information), the continuation process  $\{X_s, n_s, c_s, \theta_s^1, \dots, \theta_s^J\}_{s > t}$  is independent of the event  $\tau > T$  for all  $T > t$ .

- (b) Let  $V(t, X, n) := \sup_{\{c, \theta^1, \dots, \theta^J\}} \mathbb{E}[U_t(\{c, \theta^1, \dots, \theta^J\}) \mid X_t = X, n_t = n, \tau > t]$  be the agent's value function at time  $t$  conditional on still being alive (i.e. on  $\tau > t$ ).<sup>3</sup> Derive the HJB equation for  $V$ . Then show that policy functions for  $c, \theta^1, \dots, \theta^J$  that maximize the objective in the HJB equations do not directly depend on time  $t$  (just on the state vector  $(X_t, n_t)$ ). This means we can remove the explicit time-dependence in (1) without loss of generality.

*Hint:* you may use without proof that the exponential distribution is memoryless, that is  $\mathbb{P}(\tau \leq T \mid \tau > t) = \mathbb{P}(\tau \leq T - t)$  for all  $0 \leq t \leq T$ .<sup>4</sup>

- (c) For any admissible choice, define  $\{c_t\}_{t=0}^\infty$  for all  $t$  as in (a) and show that

$$U_0 = \mathbb{E} \left[ \int_0^\infty e^{-(\rho+\lambda)t} u(c_t) dt \right].$$

*Hint:* this can be solved without reference to any results from part (b).

- (d) Argue that solving the random lifetime problem over the interval  $[0, \tau]$  is equivalent to solving the infinite lifetime problem with a larger discount rate  $\rho + \lambda$ .

<sup>2</sup>Technical note: enlarging the decision problem to allow for arbitrary controls that are predictable with respect to the filtration generated by the Brownians and the process  $1_{\{\tau \leq t\}}$  would not improve the utility the agent could achieve. So this restriction is without loss of generality.

<sup>3</sup>While not explicit in the notation, the supremum is of course assumed to be taken over all admissible choices and the process  $\{n_s\}_{s=t}^\infty$  depends on the chosen control.

<sup>4</sup>This is the main reason why we use a stochastic lifetime with exponential distribution instead of a deterministic lifetime. With a deterministic lifetime, the time until death becomes a state variable whereas for an exponentially distributed lifetime, the distribution of the remaining lifetime is independent of how long the agent has already been around. This makes the problem very tractable, but also unsuitable for certain questions (e.g. life cycle behavior).

2. Part 1 shows us that from the perspective of individual choice problems, adding stochastic lifetimes simply increases the effective discount rate – provided there are no assets to hedge death risk. In this part we do some aggregate accounting.

Now assume that there are two types of agents, experts ( $i = e$ ) and households ( $i = h$ ). Both experts and households have some decision problems whose optimal solutions have the property that consumption-net-worth ratios and portfolio weights depend only on the agent’s type, but not on individual net worth. Consequently, individual net worth  $n^i(\tilde{i})$  of individual  $\tilde{i}$  of type  $i$  has some type-independent geometric drift  $\mu_t^{n,i}$  and volatility loading  $\sigma_t^{n,i,m}$  with respect to each of the Brownian shocks  $dZ_t^m$ . We are not concerned with the details of these individual decision problems but only with the behavior or aggregates.

Suppose all agents alive at any point in time  $t$  can be indexed by  $\tilde{i} \in [0, 1]$ . When agent  $\tilde{i}$  dies (at idiosyncratic rate  $\lambda$ ), then that agent is replaced by an offspring who inherits all the dying agent’s net worth, but whose type may differ: irrespective of the ancestor’s type, the newborn agent is an expert with probability  $\psi \in (0, 1)$  and a household with probability  $1 - \psi$ . We index the new agent again by the index  $\tilde{i}$  so that  $\tilde{i}$  no longer refers to an individual, but to a “dynastic chain” of agents. For  $i \in \{e, h\}$ , denote by  $n_t^i(\tilde{i})$  the net worth of the agent in that chain conditional on the current type being  $i$  and set  $n_t^i(\tilde{i}) = 0$  whenever the type of agent  $\tilde{i}$  is not  $i$ .

- (a) Argue that the the situation described above can be modeled by the stochastic evolution equations

$$\begin{aligned} dn_t^e(\tilde{i}) &= n_t^e(\tilde{i})\mu_t^{n,e} dt + \sum_{m=1}^M n_t^e(\tilde{i})\sigma_t^{n,e,m} dZ_t^m - n_{t-}^e(\tilde{i})d\tilde{J}_t^{e \rightarrow h}(\tilde{i}) + n_{t-}^h(\tilde{i})d\tilde{J}_t^{h \rightarrow e}(\tilde{i}) \\ dn_t^h(\tilde{i}) &= n_t^h(\tilde{i})\mu_t^{n,h} dt + \sum_{m=1}^M n_t^h(\tilde{i})\sigma_t^{n,h,m} dZ_t^m - n_{t-}^h(\tilde{i})d\tilde{J}_t^{h \rightarrow e}(\tilde{i}) + n_{t-}^e(\tilde{i})d\tilde{J}_t^{e \rightarrow h}(\tilde{i}) \end{aligned}$$

for all  $\tilde{i} \in [0, 1]$ , where  $\tilde{J}^{e \rightarrow h}(\tilde{i})$  and  $\tilde{J}^{h \rightarrow e}(\tilde{i})$  are independent (time-homogeneous) Poisson processes which are also independent across dynastic chains  $\tilde{i}$ . How does one have to choose the intensities  $\lambda^{e \rightarrow h}$  and  $\lambda^{h \rightarrow e}$ ?

- (b) Let  $N_t^i := \int_0^1 n_t^i(\tilde{i})d\tilde{i}$  be the aggregate net worth of type  $i \in \{e, h\}$  at time  $t$ .<sup>5,6</sup> Show that  $N^e$  and  $N^h$  are Ito processes and derive their laws of motion.

*Hint:* use the “law of large numbers”  $\int_0^1 d\tilde{J}_t^{i_1 \rightarrow i_2}(\tilde{i})d\tilde{i} = \mathbb{E}_t \left[ d\tilde{J}_t^{i_1 \rightarrow i_2} \right] = \lambda^{i_1 \rightarrow i_2} dt$ .

- (c) Derive the law of motion of  $\eta := \frac{N_t^e}{N_t^e + N_t^h}$ . How does it differ relative to a situation with infinitely-lived agents?

3. Reconsider the Basak-Cuoco model with log utility and identical time preference rates from Lecture 2. Instead of heterogeneous discount rates as in Problem Set 1, assume now that agents die at rate  $\lambda > 0$  and are replaced by newborn agents who inherit the wealth of the dying and become an expert with probability  $\psi$  and a household with probability  $1 - \psi$ . Solve this augmented model and compare your solution to both the one from the lecture and your solution to Problem 1 of Problem Set 1.

<sup>5</sup>This is a reasonable definition even though we integrate over the same set of  $\tilde{i}$ , because we have set  $n_t^i(\tilde{i})$  to zero whenever  $\tilde{i}$  is not of type  $i$ .

<sup>6</sup>If you have a mathematical background and wondered whether this integral exists: we ignore measurability issues here. Things get formally more involved, if you do not.

## 2 Money Model with Stochastic Volatility

Consider the model of Lecture 5 with log utility without government policy ( $\mu^B = i = \sigma^B = \mathbf{g} = 0$ ). In this problem, we add stochastic volatility to the model. Suppose idiosyncratic risk  $\tilde{\sigma}$  is no longer a constant, but evolves according to the exogenous stochastic process

$$d\tilde{\sigma}_t = b(\tilde{\sigma}^{ss} - \tilde{\sigma}_t)dt + \nu\sqrt{\tilde{\sigma}_t}dZ_t,$$

where  $\tilde{\sigma}^{ss}$ ,  $b$  and  $\nu$  are positive constants. You may assume that there are no aggregate capital shocks, i.e.  $\sigma = 0$ .

1. Characterize the equilibrium:

- (a) Use goods market clearing and optimal investment to express  $q^K$ ,  $q^B$  and  $\iota$  in terms of  $\vartheta := \frac{q^B}{q^K + q^B}$
- (b) Use agents' portfolio choice to derive an equation of the form  $\mu_t^\vartheta = f(\vartheta_t, \tilde{\sigma}_t)$ , where the function  $f$  only depends on model parameters (the “money valuation equation”)

2. Solve the model numerically. To do so, apply Ito's lemma to  $\vartheta_t = \vartheta(\tilde{\sigma}_t)$ , which allows you to transform the money valuation equation  $\mu_t^\vartheta = f(\vartheta_t, \tilde{\sigma}_t)$  into an ODE for the function  $\vartheta(\tilde{\sigma}_t)$ . Solve this equation with the methods from lecture 3 and plot  $\vartheta$ ,  $q^K$ ,  $q^B$ ,  $\iota$ ,  $r^f$  and  $\tilde{\zeta}$  as a function of  $\tilde{\sigma}$ .

For your plots, use the model parameters  $a = 0.2$ ,  $\phi = 1$ ,  $\rho = 0.01$ ,  $\tilde{\sigma}^{ss} = 0.2$ ,  $b = 0.05$ ,  $\nu = 0.02$ .

## 3 Money as a Medium of Exchange

In this problem, we add a medium-of-exchange role for the nominal asset in the model of Lecture 5. Take the baseline model from the lecture with log utility and assume a steady state. Now call the bonds “money” and denote them by  $M$  instead of  $B$ . We assume a constant money stock, so  $\mu^M = \sigma^M = 0$  and no interest is paid on money ( $i = 0$ ). We make money a medium of exchange by adding a simple cash-in-advance constraint to the model. Consider two possibilities:

A. Households face a cash-in-advance constraint for consumption and – potentially – investment expenditures:

For each household  $\tilde{i}$ ,

$$\alpha_c \tilde{c}_t^{\tilde{i}} + \alpha_\iota \tilde{l}_t^{\tilde{i}} \frac{\theta_t^{k,\tilde{i}} n_t^{\tilde{i}}}{q_t^K} \leq \theta_t^{m,\tilde{i}} n_t^{\tilde{i}},$$

where  $\alpha_c > 0$  and  $\alpha_\iota \geq 0$  are model parameters. Also, define velocity as

$$v_t := \frac{\tilde{c}_t^{\tilde{i}} + \frac{\alpha_\iota}{\alpha_c} \tilde{l}_t^{\tilde{i}}}{\theta_t^{m,\tilde{i}} n_t^{\tilde{i}}}.$$

B. Households face a cash-in-advance constraint in production:<sup>7</sup>

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<sup>7</sup>This can be interpreted as expenditures made in an unmodeled supply chain.

For each household  $\tilde{i}$ ,

$$\alpha \cdot a \frac{\theta_t^{k,\tilde{i}} n_t^{\tilde{i}}}{q_t^K} \leq \theta_t^{m,\tilde{i}} n_t^{\tilde{i}},$$

where  $\alpha \geq 0$  is a model parameter. Define velocity in this case as

$$v_t := \frac{a \frac{\theta_t^{k,\tilde{i}} n_t^{\tilde{i}}}{q_t^K}}{\theta_t^{m,\tilde{i}} n_t^{\tilde{i}}}.$$

Here,  $\theta_t^{k,\tilde{i}}$  denotes the households' portfolio share in capital and  $\theta_t^{m,\tilde{i}} = 1 - \theta_t^{k,\tilde{i}}$  the households' portfolio share in money. In both cases  $\theta_t^{m,\tilde{i}} n_t^{\tilde{i}}$  is the real value of  $\tilde{i}$ 's money balances and  $\frac{\theta_t^{k,\tilde{i}} n_t^{\tilde{i}}}{q_t^K} = k_t^{\tilde{i}}$  is the quantity of capital held by agent  $\tilde{i}$ .

1. Write down the HJB equation of agent  $\tilde{i}$ .
2. For both possibilities of the cash-in-advance constraint (consider one possibility at a time), derive the first-order conditions with respect to  $c_t$ ,  $\iota_t$ ,  $\theta_t^m$  and  $\theta_t^k = 1 - \theta_t^m$ . Show that in both cases the portfolio choice condition between capital and money can be written in the form

$$\frac{\mathbb{E}_t[dr_t^k]}{dt} - \frac{\mathbb{E}_t[dr_t^m]}{dt} = \tilde{\zeta}_t \tilde{\sigma} + \lambda_t v_t,$$

where  $\lambda_t$  is a re-scaled Lagrange multiplier on the constraint and  $v_t$  is the velocity of money holdings.

3. Fully characterize the steady-state equilibrium for the cash-in-advance constraint of the form A. by a set of equations. When is the constraint binding and how does it affect the equilibrium values of  $q^K$ ,  $q^M$ ,  $\iota$ ,  $\vartheta := \frac{q^M}{q^K + q^M}$  and  $c/n$ ? In the case  $\tilde{\sigma} = \alpha_\iota = 0$ , there is a simple closed-form solution. Derive it.
4. Characterize the equilibrium for the cash-in-advance constraint of the form B. by a set of equations and derive a closed-form solution for  $q^K$ ,  $q^M$ ,  $\iota$ ,  $\vartheta$  and  $c/n$ .
5. In the case  $\alpha_c = \alpha_\iota = \alpha$ , the constraints in A. and B. both say that for each unit of output agents must hold at least  $\alpha$  real units of money. Does this imply that the two specifications lead to the same equilibrium allocation? Why/why not?