Financial and Monetary Economics

Eco529 Fall 2020

Lecture 05: One Sector Money Model with Idiosyncratic Risk

Markus K. Brunnermeier

Princeton University

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Key Takeaways

- Money as a bubble
  - As store of value – link to FTPL
  - Medium of exchange

- Technical Takeaways
  - Recall: Change to total wealth numeraire to derive “money evaluation” equation
  - Idiosyncratic risk
    - Isolating it from value function

Roadmap

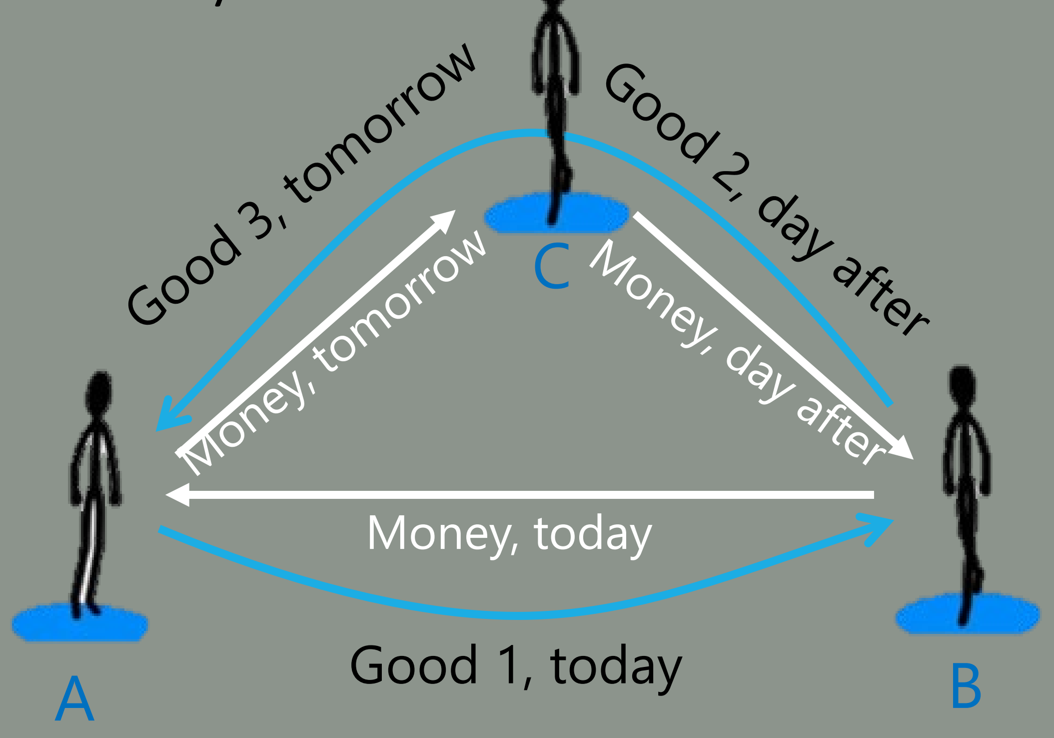
- Solve one-sector money model
  - Different ways to derive money evaluation equation
  - Value function with idiosyncratic risk
  - Bubble/Ponzi scheme, r vs. g vs. ς and transfersality condition
  - Mining the bubble and MMT

- Fiscal Theory of the Price Level (with a bubble)
  - FTPL equation
  - Price Level Determination
  - Monetary vs. fiscal authority

- Medium of Exchange Role of Money

The 4 Roles of Money

- Unit of account
  - Intratemporal: Numeraire — bounded rationality/price stickiness
  - Intertemporal: Debt contract — incomplete markets
- Store of value
  - "I Theory of Money without I"
    Less risky than other "capital" – no idiosyncratic risk
  - Fiscal theory of the price level
- Medium of exchange
  - Overcome double-coincidence of wants problem



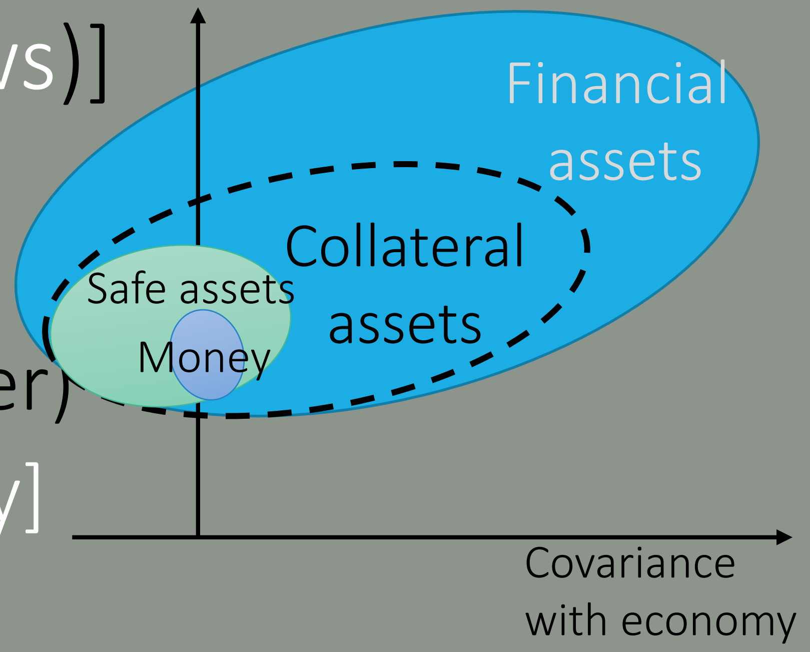
- Record keeping device – money is memory
  - Virtual ledger

Safe Assets ⊇ (Narrow) Money

- Asset Price = E[PV(cash flows)] + E[PV(service flows)]

  dividends/interest

  - Service flows/convenience yield
    1. Collateral: relax constraints (Lagrange multiplier)
    2. Safe asset: [good friend analogy]
       - When one needs funds, one can sell at stable price … since others buy
       - Partial insurance through **retrading** - market liquidity!
    3. Money (narrow): relax double-coincidence of wants
  - Higher Asset Price = lower expected return



- Problem: safe asset + money status might burst like a bubble
  - Multiple equilibria: [safe asset tautology]

Models on Money as Store of Value

\Friction	OLG	Incomplete Markets + idiosyncratic risk	
Risk	deterministic	**endowment risk** **borrowing constraint**	**return risk** Risk tied up with Individual capital
Only money	Samuelson	Bewley	“I Theory without I” Brunnermeier-Sannikov (AER PP 2016)
With capital	Diamond	Aiyagari	

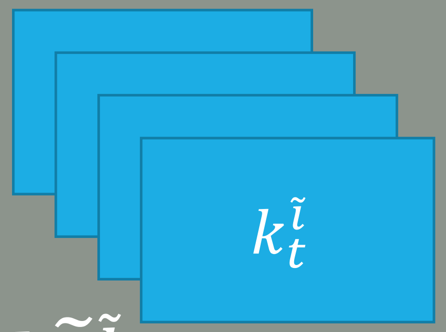
One Sector Model with Gov. Bonds/Broad Money as Safe Asset

- Agent $\tilde{\imath}$'s preferences

$$E\left[\int_0^\infty e^{-\rho t}\left(\frac{\left(c_t^{\tilde{\imath}}\right)^{1-\gamma}}{1-\gamma} + f(g_t K_t)\right)dt\right]$$

- Each agent operates one firm
  - Output $\qquad y_t^{\tilde{\imath}} = a k_t^{\tilde{\imath}}$
  - Physical capital k

$$\frac{dk_t^{\tilde{\imath}}}{k_t^{\tilde{\imath}}} = \left(\Phi\left(\iota_t^{\tilde{\imath}}\right) - \delta\right)dt + \sigma dZ_t + \tilde{\sigma} d\tilde{Z}_t^{\tilde{\imath}}$$



- Financial Friction: Incomplete markets:
  Agents cannot share $d\tilde{Z}_t^{\tilde{\imath}}$

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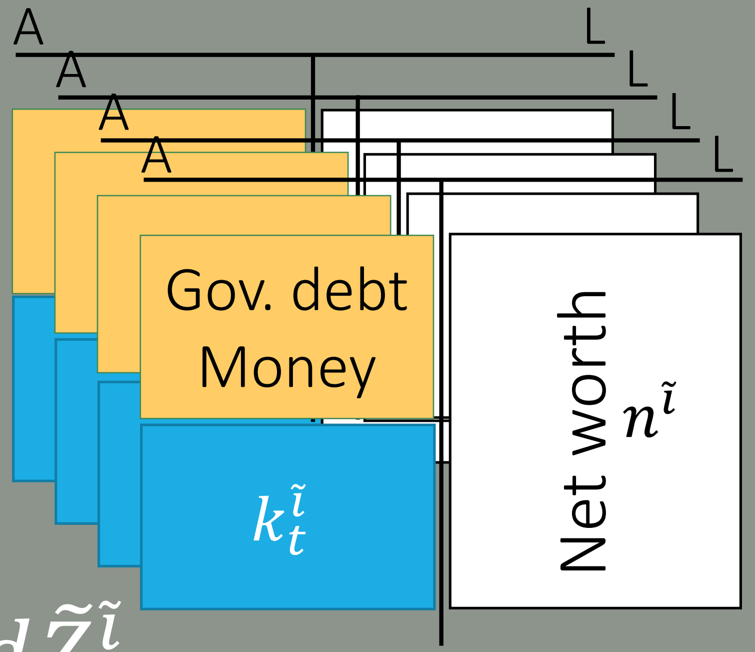


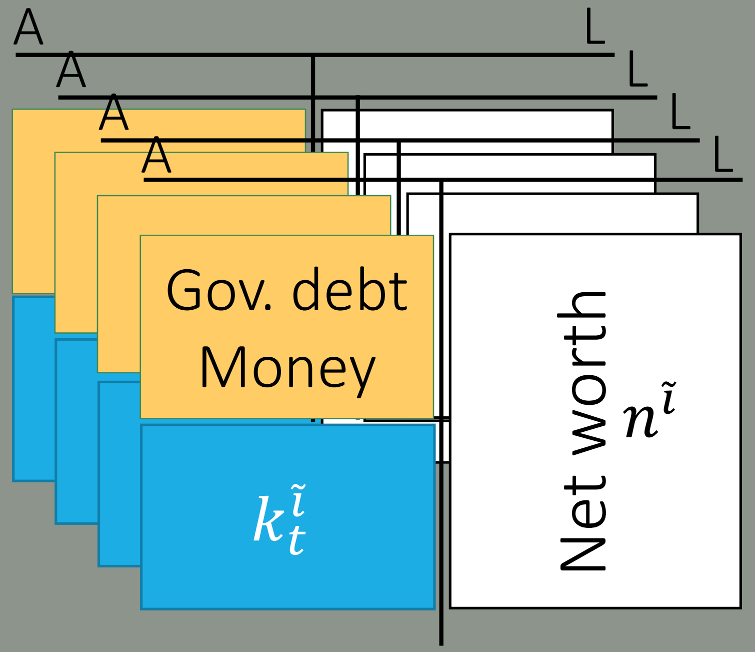
- Financial Friction: Incomplete markets:
  Agents cannot share $d\tilde{Z}_t^{\tilde{\iota}}$

- Goods market clearing:

$$C_t + g_t K_t = (a - \iota_t)K_t$$

Taxes, Money/Bond Supply, Gov. Budget

- Policy Instruments
  - Government spending $K_t d\mathcal{g}_t$
    - $K_t \mu_t^{\mathcal{g}} dt + K_t \sigma_t^{\mathcal{g}} dZ_t$
  - government debt supply
    $$\frac{d\mathcal{B}_t}{\mathcal{B}_t} = \mu_t^{\mathcal{B}} dt + \sigma_t^{\mathcal{B}} dZ_t$$
  - nominal interest rate i_t
  - proportional tax $K_t d\tau_t$ on capital
  - lump-sum tax τ_t^{ls} ($= 0$ for this talk)
- Government budget constraint (BC)
  $$i_t \mathcal{B}_t + \wp_t K_t d\mathcal{g}_t = \mu_t^{\mathcal{B}} \mathcal{B}_t + \wp_t K_t d\tau_t + \wp_t \tau_t^{ls} + seigniorage$$
- Assume here:
  - No lump-sum taxes
  - Gov. chooses $\mu^{\mathcal{B}}$, i; while τ_t adjusts to satisfy (BC)



One Sector Model with Money/Gov. debt

- Seigniorage is distributed
  1. Proportionally to bond/money holdings
     - No real effects, only nominal
  2. Proportionally to capital holdings
     - Bond/Money return decreases with dB_t (change in debt level/money supply)
     - Capital return increases
     - Pushes citizens to hold more capital
  3. Proportionally to net worth
     - Fraction of seigniorage goes to capital - same as 2.
     - Rest of seigniorage goes to money holders - same as 1.
  4. Per capita
     - No real effects: people simply borrow against the transfers they expect to receive

Solving MacroModels Step-by-Step

0. Postulate aggregates, price processes & obtain return processes
1. For given C/N-ratio and SDF processes for each i *finance block*
   a. Real investment ι + Goods market clearing *(static)*
   - *Toolbox 1:* Martingale Approach, HJB vs. Stochastic Maximum Principle Approach

   b. Portfolio choice θ + Asset market clearing or
   Asset allocation κ & risk allocation χ
   - *Toolbox 2:* "price-taking social planner approach" – Fisher separation theorem

   c. "Money evaluation equation" ϑ
   - *Toolbox 3:* Change in numeraire to total wealth (including SDF)
2. Evolution of state variable ~~η (and~~ K) *forward equation*
3. Value functions *backward equation*
   a. Value fcn. as fcn. of individual investment opportunities ω
   - *Special cases: log-utility, constant investment opportunities*

   b. Separating value fcn. $V^i(n^{\tilde{i}};\eta,K)$ into $v^i(\eta)\left(\tilde{\eta}^{\tilde{i}}\right)^{1-\gamma}u(K)$
   c. Derive $\breve{\rho} = C/N$-ratio and ς price of risk
4. Numerical model solution
   a. Transform BSDE for separated value fcn. $v^i(\eta)$ into PDE
   b. Solve PDE via value function iteration
5. KFE: Stationary distribution, Fan charts

Postulate Aggregates and Processes

- $q_t^K K_t$ value of physical capital
- $q_t^B K_t$ value of nominal capital/outside money/gov. debt
  - $\mathcal{B}_t/\wp_t = q_t^B K_t$ price level (inverse of “value of money”)
- $\vartheta_t := \frac{q_t^B}{q_t^K + q_t^B}$ fraction of nominal wealth

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- $\vartheta_t := \frac{q_t^B}{q_t^K + q_t^B}$ fraction of nominal wealth

0. Postulate
   - q^K-price process $dq_t^K/q_t^K = \mu_t^{q^K} dt + \sigma_t^{q^K} dZ_t,$
   - q^B-price process $dq_t^B/q_t^B = \mu_t^{q^B} dt + \sigma_t^{q^B} dZ_t,$
   - SDF for each $\tilde{\imath}$ agent $d\xi_t^{\tilde{\imath}}/\xi_t^{\tilde{\imath}} = -r_t^f dt - \varsigma_t dZ_t - \tilde{\varsigma}_t^{\tilde{\imath}} d\tilde{Z}_t^{\tilde{\imath}}$

Poll 13: Why is risk-free rate and aggregate price of risk the same for all $\tilde{\imath}$?

a) Because risk free debt can be traded.

b) Because they identical up to size (scaled versions of each other).

0. Return on Gov. Bond/Money

- Number of Bonds/coins follows:

$$\frac{dB_t}{B_t} = \breve{\mu}_t^B dt + idt + \sigma_t^B dZ_t$$

  - Where i is interest paid on government bonds/outside money (reserves)
- Return on Gov. Bond/Money: in output numeraire

$$dr_t^B = idt + \underbrace{\frac{d\left(q_t^B K_t/B_t\right)}{q_t^B K_t/B_t}}_{-inflation} = \frac{d\left(q_t^B K_t\right)}{q_t^B K_t} - \breve{\mu}_t^B dt - \sigma_t^B dZ_t + \sigma_t^B\left(\sigma_t^B - \sigma\right)dt$$

- Seigniorage (excluding interest paid to money holders)

0. Return on Capital (with seigniorage rebate terms)

- $dr_t^{K,\tilde{\imath}} = \frac{a-\iota_t^{\tilde{\imath}}}{q_t^K}dt + \frac{d\left(q_t^K k_t^{\tilde{\imath}}\right)}{q_t^K k_t^{\tilde{\imath}}} - \frac{k^{\tilde{\imath}} d\tau}{q_t^K k_t^{\tilde{\imath}}}$

$$= \left(\frac{a-\iota_t^{\tilde{\imath}}}{q_t^K} + \Phi\left(\iota_t^{\tilde{\imath}}\right) - \delta + \mu_t^{q^K}\right)dt + \left(\sigma + \sigma_t^{q^K}\right)dZ_t + \tilde{\sigma}_t d\tilde{Z}_t^{\tilde{\imath}} - \frac{k^{\tilde{\imath}} d\tau}{q_t^K k_t^{\tilde{\imath}}}$$

- Use government budget constraint to substitute out $d\tau_t$ (and $\mathcal{B}_t/\wp_t = q_t^B K_t$)

$$K_t d\tau_t + \overbrace{\Big(\underbrace{\frac{1}{\wp_t}}_{\substack{value\ of \\ a\ bond/coin}} + \underbrace{d\frac{1}{\wp_t}}_{=d\frac{B_t}{q_t^B K_t}}\Big)(dB_t - i_t B_t dt)}^{seigniorage} = K_t d\mathscr{g}_t$$

$$K_t d\tau_t = K_t \mu_t^{\mathscr{g}} dt + K_t \sigma_t^{\mathscr{g}} dZ_t - q_t^B K_t \left\{\left[\mu_t^B - i_t + \left(\sigma + \sigma_t^{q^B} - \sigma_t^B\right)\sigma_t^B\right]dt + \sigma_t^B dZ_t\right\}$$

$$dr_t^{K,\tilde{\imath}} = \left(\frac{\overbrace{a-\mu_t^{\mathscr{g}}}^{\breve{a}:=}-\iota_t^{\tilde{\imath}}}{q_t^K} + \Phi\left(\iota_t^{\tilde{\imath}}\right) - \delta + \mu_t^{q^K} + \frac{q_t^B}{q_t^K}\left(\check{\mu}_t^B + \left(\sigma + \sigma_t^{q^B} - \sigma_t^B\right)\sigma_t^B\right)\right)dt$$
$$+ \left(\sigma + \sigma_t^{q^K} + \frac{q_t^B \sigma_t^B - \sigma_t^{\mathscr{g}}}{q_t^K}\right)dZ_t + \tilde{\sigma}_t d\tilde{Z}_t^{\tilde{\imath}}$$

Household Problem

- Wealth evolution (budget constraint)

$$\frac{dn_t^{\tilde{\iota}}}{n_t^{\tilde{\iota}}} = -\frac{c_t^{\tilde{\iota}}}{n_t^{\tilde{\iota}}} dt + dr_t^{\mathcal{B}} + \left(1 - \theta_t^{\tilde{\iota}}\right)\left(dr_t^{K,\tilde{\iota}}\left(\iota_t^{\tilde{\iota}}\right) - dr_t^{\mathcal{B}}\right)$$

- HJB equation of household

$$\rho V_t(n) = \max_{c,\theta,\tilde{\iota}} \left\{ \begin{array}{c} \frac{\left(c_t^{\tilde{\iota}}\right)^{1-\gamma}}{1-\gamma} \\ +V_t'(n)\left[-c_t^{\tilde{\iota}} + n\left(\underbrace{\Phi(\iota_t) - \delta + \mu_t^p - \breve{\mu}^{\mathcal{B}}}_{= E\left[dr_t^{\mathcal{B}}\right]/dt} + (1-\theta)\underbrace{\left(\frac{\breve{a} - \tilde{\iota}}{q_t} + \Phi(\tilde{\iota}) - \Phi(\iota_t) - \frac{\mu_t^{\vartheta} - \breve{\mu}^{\mathcal{B}}}{1 - \vartheta_t}\right)}_{= E\left[dr_t^{K,\tilde{\iota}} - dr_t^{\mathcal{B}}\right]/dt}\right)\right] \\ +\frac{1}{2}V_t''(n) n^2 (1-\theta)^2 \tilde{\sigma}^2 \end{array} \right\}$$

Optimal Choices

- Guess (and verify) value function $V_t(n) = \alpha_t + \frac{1}{\rho}\log n_t$

- Optimal investment rate
  - $\phi\iota_t = q_t - 1$

$\frac{1}{q_t} = \Phi'(\iota_t^{\tilde{i}})$ Tobin's q
All agents $\iota_t^{\tilde{i}} = \iota_t$
Special functional form:
$\Phi(\iota_t) = \frac{1}{\phi}\log(\phi\iota_t + 1) \Rightarrow \phi\iota_t = q_t - 1$

Optimal Choices

- Guess (and verify) value function $V_t(n) = \alpha_t + \frac{1}{\rho} \log n_t$
- Optimal investment rate
  - $\phi\iota_t = q_t - 1$
- Consumption
  - $\frac{c_t}{n_t} =: \check{\rho}_t \quad \Rightarrow C_t = \check{\rho}_t \left(q_t^B + q_t^K\right) K_t$

- Looking ahead to Step 3:
  When is $\frac{c}{n}$ constant? Recall $\frac{c}{n} = \rho^{1/\gamma} \omega^{1-1/\gamma}$
  - Log utility, $\gamma = 1$: $\check{\rho} = \rho$
  - In steady state:
    ω investment opportunity/net worth multiplier is constant

Optimal Choices & Market Clearing

- Optimal investment rate
  - $\phi\iota_t = q_t - 1$
- Consumption
  - $\frac{c_t}{n_t} =: \check{\rho}_t \quad \Rightarrow C_t = \check{\rho}_t\left(q_t^B + q_t^K\right)K_t$

Goods market

$= (\check{a} - \iota_t)K_t$

- Portfolio
  - Solve for θ_t later

Capital market

$1 - \theta_t = 1 - \vartheta_t$

Debt market

clears by Walras law

Equilibrium (before solving for portfolio choice)

Equilibrium
$q_t^B = \vartheta_t \dfrac{1+\phi a}{(1-\vartheta_t)+\phi\check{\rho}_t}$
$q_t^K = (1-\vartheta_t)\dfrac{1+\phi a}{(1-\vartheta_t)+\phi\check{\rho}_t}$
$\iota_t = \dfrac{(1-\vartheta_t)a-\check{\rho}_t}{(1-\vartheta_t)+\phi\check{\rho}_t}$

- Moneyless equilibrium with $q_t^B = 0 \Rightarrow \vartheta_t = 0$
- Next, determine portfolio choice.

Portfolio choice θ (consumption numeraire)

Price capital relative to money

- Asset pricing equation (martingale method) for $\sigma_t^B = \sigma^\tau = 0$

$$\frac{E\left[dr_t^{K^{\tilde{i}}}\right]}{dt} = \frac{a-\iota_t}{q_t^K} + \Phi(\iota_t) - \delta + \mu_t^{q^K} + \sigma\sigma_t^{q^K} + \frac{q_t^B}{q_t^K}\check{\mu}_t^B \qquad = r_t^f + \varsigma_t\left(\sigma + \sigma_t^{q^K}\right) + \tilde{\varsigma}_t\tilde{\sigma}$$

$$\frac{E\left[dr_t^B\right]}{dt} = \qquad \Phi(\iota_t) - \delta + \mu_t^{q^B} + \sigma\sigma_t^{q^B} \quad - \check{\mu}_t^B \qquad = r_t^f + \varsigma_t\left(\sigma + \sigma_t^{q^B}\right)$$

$$\frac{a-\iota_t}{q_t^K} + \mu_t^{q^K} - \mu_t^{q^B} + \sigma\left(\sigma_t^{q^K} - \sigma_t^{q^B}\right) + \frac{1}{1-\vartheta_t}\check{\mu}_t^B = \qquad \varsigma_t\left(\sigma_t^{q^K} - \sigma_t^{q^B}\right) + \tilde{\varsigma}_t\tilde{\sigma}$$

Price of Risk:	$\varsigma_t = -\sigma_t^v + \sigma_t^{q^B+q^K} + \gamma\sigma,$	$\tilde{\varsigma}_t = \gamma\tilde{\sigma}_t^n = \gamma(1-\theta_t)\tilde{\sigma}$
- for log-utility	$\varsigma_t = \sigma^n$	$\tilde{\varsigma}_t = \tilde{\sigma}_t^n = (1-\theta_t)\tilde{\sigma}$

Risk-free rate: $r_t = \Phi(\iota_t) - \delta - \check{\mu}_t^B - \gamma\sigma^2$ since $\varsigma_t = \gamma\sigma$

- For stationary equilibria

$$\frac{a-\iota_t}{q_t^K} + \mu_t^{q^K} - \mu_t^{q^B} + \sigma\left(\sigma_t^{q^K} - \sigma_t^{q^B}\right) + \frac{1}{1-\vartheta_t}\check{\mu}_t^B = \varsigma_t\left(\sigma_t^{q^K} - \sigma_t^{q^B}\right) + (1-\theta_t)\gamma\tilde{\sigma}$$

- $\frac{(a-\iota)/q^K + \check{\mu}^B}{\gamma\tilde{\sigma}^2} = 1 - \theta = 1 - \vartheta$ capital market clearing
- $\check{\rho}\underbrace{\left(q^B + q^K\right)/q^K}_{\frac{1}{1-\vartheta}} = (a-\iota)/q^K$ goods market clearing

$$(1-\vartheta) = \sqrt{\check{\rho} + \check{\mu}^B}/(\gamma\tilde{\sigma})$$

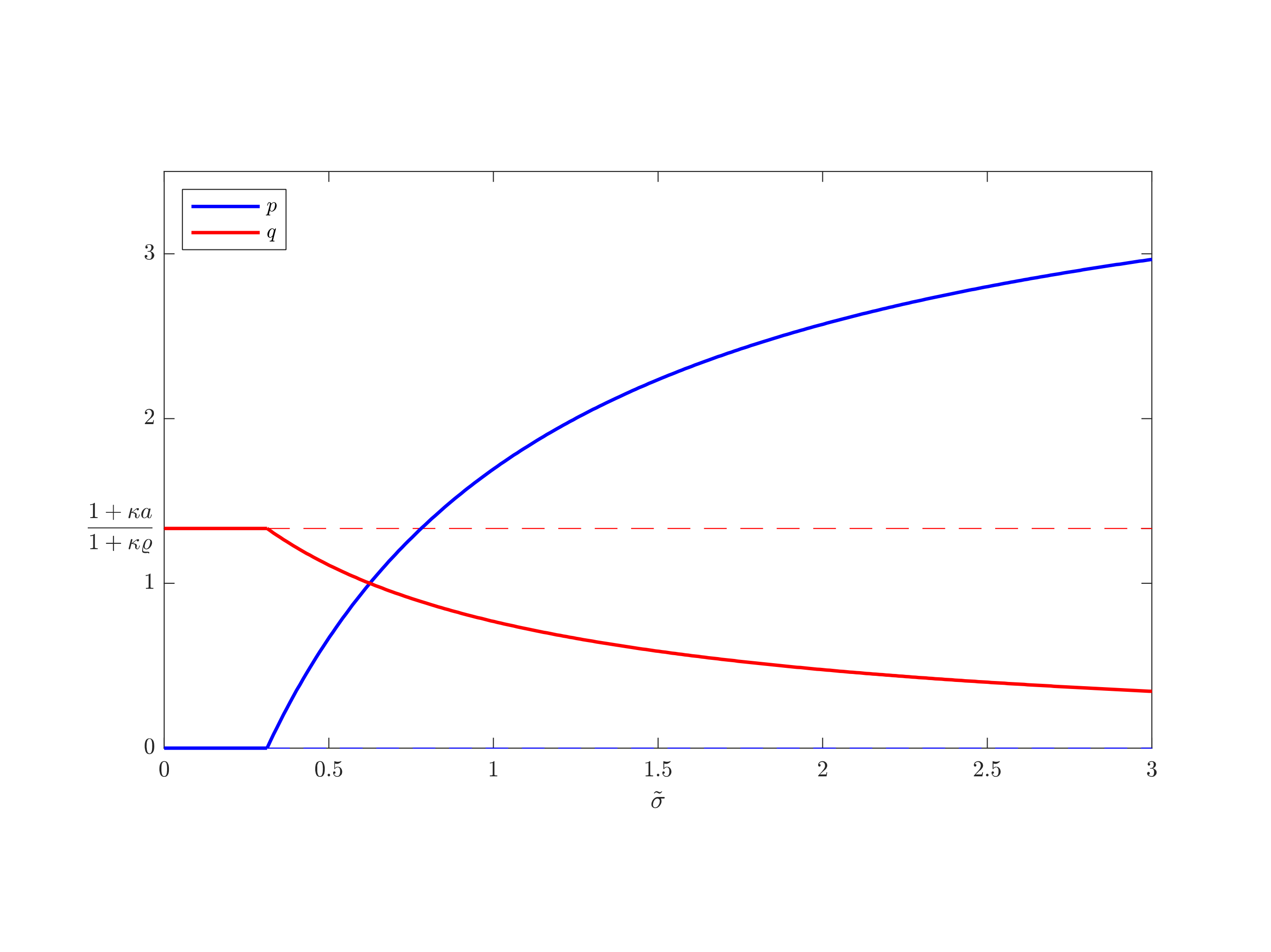
Two Stationary Equilibria

Non-Monetary	Monetary
$q_0^B = 0$	$q^B = \frac{\left(\sqrt{\gamma}\tilde{\sigma} - \sqrt{\check{\rho} + \check{\mu}^B}\right)(1+\phi a)}{\sqrt{\check{\rho} + \check{\mu}^B} + \kappa\sqrt{\gamma}\tilde{\sigma}\check{\rho}}$
$q_0^K = \frac{1 + \phi a}{1 + \phi\check{\rho}_0}$	$q^K = \frac{\sqrt{\check{\rho} + \check{\mu}^B}\ (1+\phi a)}{\sqrt{\check{\rho} + \check{\mu}^B} + \phi\sqrt{\gamma}\tilde{\sigma}\check{\rho}}$
$\iota = \frac{a - \check{\rho}_0}{1 + \phi\check{\rho}_0}$	$\iota = \frac{a\sqrt{\check{\rho} + \check{\mu}^B} - \sqrt{\gamma}\tilde{\sigma}\check{\rho}}{\sqrt{\check{\rho} + \check{\mu}^B} + \phi\sqrt{\gamma}\tilde{\sigma}\check{\rho}}$

- For log utility
  - $\check{\rho} = \check{\rho}_0 = \rho$
  - $\gamma = 1$

Remark

- Money is a bubble
  - But provides store of value/insurance role
- Comparative static: As $\tilde{\sigma}$ increases
  - Flight to safety to bubbly money
    - q^B rises (disinflation)
    - q^K falls and so does
      - ι and
      - growth rate of economy



- Can be extended to a model with stochastic idiosyncratic volatility ($\tilde{\sigma}_t$ becomes state variable)
- how is investment rate affected?

Remark: Pecuniary externalities

1. Agents' portfolio choice takes r^B as given, but
   … it is affected by agents' portfolio choice θ,
   which affects q^K, which in turn affects ι (esp. for low κ),
   which affects the real return on money
2. Agents' portfolio choice takes q^K as given, but
   … tilting the portfolio towards money,
   lower q^K (esp. for high ϕ), which in turn
   reduces risk *per unit of output/capital*

- Government acts like a "diversifier"
  - Individual tax liability is idiosyncratically risky, but "dividend" is not
  - Tax = co-ownership with dividends paid in form of r^B

Solving MacroModels Step-by-Step

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1. For given C/N-ratio and SDF processes for each i *finance block*
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   c. "Money evaluation equation" ϑ
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   a. Value fcn. as fcn. of individual investment opportunities ω
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   b. Separating value fcn. $V^i(n^{\tilde{i}};\eta,K)$ into $v^i(\eta)\left(\tilde{\eta}^{\tilde{i}}\right)^{1-\gamma}u(K)$
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   b. Solve PDE via value function iteration
5. KFE: Stationary distribution, Fan charts

Remark: Portfolio choice Money vs. N_t

- Portfolio choice problem is simplified if money vs. total net worth (instead of money vs. capital)
  - Seigniorage is part of N_t and doesn't have to be broken out

$$\frac{E\left[dr_t^{K^{\tilde{i}}}\right]}{dt} = \frac{a-\iota_t}{q_t} + \Phi(\iota_t) - \delta + \mu_t^{q^K} + \sigma\sigma_t^{q^K} + \frac{q_t^B}{q_t^K}\check{\mu}_t^B = r_t^f + \varsigma_t\left(\sigma + \sigma_t^{q^K}\right) + \tilde{\varsigma}_t\tilde{\sigma}$$

$$\frac{E\left[dr_t^{N^{\tilde{i}}}\right]}{dt} = \check{\rho}_t + \Phi(\iota_t) - \delta + \mu_t^{q^K+q^B} + \sigma\sigma_t^{q^K+q^B} + 0 = r_t^f + \varsigma_t\left(\sigma + \sigma_t^{q^K+q^B}\right) + \tilde{\varsigma}_t(1-\theta_t)\tilde{\sigma}$$

$$\frac{E\left[dr_t^B\right]}{dt} = \Phi(\iota_t) - \delta + \mu_t^p + \sigma\sigma_t^p - \mu_t^M = r_t^f + \varsigma_t\left(\sigma + \sigma_t^p\right)$$

Change to total net worth numeraire N_t

- SDF in consumption numeraire

$$\frac{d\xi_t^{\tilde{\imath}}}{\xi_t^{\tilde{\imath}}} = -r_t^f dt - \varsigma_t dZ_t - \tilde{\varsigma}_t^{\tilde{\imath}} d\tilde{Z}_t^{\tilde{\imath}}$$

- SDF in N_t-numeraire

$$\frac{d\hat{\xi}_t^{\tilde{\imath}}}{\hat{\xi}_t^{\tilde{\imath}}} = \frac{d(\xi_t^{\tilde{\imath}} N_t)}{(\xi_t^{\tilde{\imath}} N_t)} = -(r_t^f - \mu_t^N + \varsigma_t \sigma_t^N)dt - (\varsigma_t - \sigma_t^N)dZ_t - \tilde{\varsigma}_t^{\tilde{\imath}} d\tilde{Z}_t^{\tilde{\imath}}$$

- Return in consumption numeraire:

$$dr_t^j = \mu_t^{r^j} dt + \sigma_t^{r^j} dZ_t - \tilde{\sigma}_t^{r^j} d\tilde{Z}_t^{\tilde{\imath}}$$

- Return in N_t-numeraire

$$dr_{t,N}^j = \left(\mu_t^{r^j} - \mu_t^N - \sigma_t^N \left(\sigma_t^{r^j} - \sigma_t^N\right)\right) dt + \left(\sigma_t^{r^j} - \sigma_t^N\right) dZ_t - \tilde{\sigma}_t^{r^j} d\tilde{Z}_t^{\tilde{\imath}}$$

  - Value of self-financing strategy investing in asset in the consumption numeraire, e.g. x^j satisfies $dx_t^j / x_t^j = dr_t^j$. The same holds in the N_t-numeraire, but now the value is x_t^j / N_t.

Portfolio choice θ (consumption numeraire)

Total net worth N_t return relative to bond/money return

- Asset pricing equation (martingale method)

$$\frac{E\left[dr_t^{N^{\tilde{i}}}\right]}{dt} = \check{\rho}_t + \Phi(\iota_t) - \delta + \mu_t^{q+p} + \sigma\sigma_t^{q+p} + 0 = r_t^f + \varsigma_t(\sigma + \sigma_t^{q+p}) + \tilde{\varsigma}_t(1 - \theta_t)\tilde{\sigma}$$

$$\frac{E[dr_t^B]}{dt} = \Phi(\iota_t) - \delta + \mu_t^p + \sigma\sigma_t^p - \check{\mu}_t^B = r_t^f + \varsigma_t(\sigma + \sigma_t^p)$$

Portfolio choice θ (N_t-numeraire)

Total net worth N_t relative to single bond/coin

- Asset pricing equation (martingale method)

$$\frac{E\left[dr_t^{\tilde{\eta}^{\tilde{\imath}}}\right]}{dt} = \check{\rho}_t + 0 = \left(r_t^f - (\Phi(\iota_t) - \delta) - \mu_t^{q^K+q^B} - \sigma\sigma_t^{q^K+q^B} + \varsigma_t(\sigma + \sigma_t^{q^K+q^B})\right) + (\varsigma_t - \sigma_t^N)0 + \tilde{\varsigma}_t(1-\theta_t)\tilde{\sigma}$$

$$\frac{E\left[dr_t^{\vartheta/B}\right]}{dt} = \mu_t^{\vartheta/B} = \underbrace{\left(r_t^f - (\Phi(\iota_t) - \delta) - \mu_t^{q^K+q^B} - \sigma\sigma_t^{q^K+q^B} + \varsigma_t(\sigma + \sigma_t^{q^K+q^B})\right)}_{risk-free\ rate\ in\ N_t-numeraire} + \underbrace{(\varsigma_t - \sigma_t^N)\sigma_t^{\vartheta/B}}_{\substack{price\ of\ risk \\ in\ N_t-numeraire}}$$

Portfolio choice θ (N_t-numeraire)

Total net worth N_t relative to a single bond/coin of money

- Asset pricing equation (martingale method)

$$\frac{E\left[dr_t^{\tilde{\eta}^{\tilde{\imath}}}\right]}{dt} = \check{\rho}_t + 0 = \left(r_t^f - (\Phi(\iota_t) - \delta) - \mu_t^{q^K+q^B} - \sigma\sigma_t^{q^K+q^B} + \varsigma_t(\sigma + \sigma_t^{q^K+q^B})\right) + (\varsigma_t - \sigma_t^N)0 + \tilde{\varsigma}_t(1-\theta_t)\tilde{\sigma}$$

$$\frac{E\left[dr_t^{\vartheta/B}\right]}{dt} = \mu_t^{\vartheta/B} = \underbrace{\left(r_t^f - (\Phi(\iota_t) - \delta) - \mu_t^{q^K+q^B} - \sigma\sigma_t^{q^K+q^B} + \varsigma_t(\sigma + \sigma_t^{q^K+q^B})\right)}_{risk-free\ rate\ in\ N_t-numeraire} + \underbrace{(\varsigma_t - \sigma_t^N)\sigma_t^{\vartheta/B}}_{\substack{price\ of\ risk \\ in\ N_t-numeraire}}$$

$$\check{\rho}_t - \mu_t^{\vartheta/B} = -(\varsigma_t - \sigma_t^N)\sigma_t^{\vartheta/B} + \tilde{\varsigma}_t(1-\theta_t)\tilde{\sigma}$$

- Remark:
  - Value of a single bond/coin in N_t-numeraire

$$\frac{d(\vartheta_t/B_t)}{\vartheta_t/B_t} = \mu_t^{\vartheta} + \sigma_t^{\vartheta} dZ_t - \mu_t^B dt - \sigma_t^B dZ_t + \sigma_t^B(\sigma_t^B - \sigma_t^{\vartheta})dt$$

$$= \mu_t^{\vartheta/B} dt + \sigma_t^{\vartheta/B} dZ_t$$ (defining return-drift and volatility)

  - Terms are shifted into risk-free rate in N_t-numeraire, which drop out when differencing

Portfolio choice θ (N_t-numeraire)

Total net worth N_t relative to single bond/coin of money

- Asset pricing equation (martingale method)

$$\frac{E\left[dr_t^{\tilde{\eta}^{\tilde{\iota}}}\right]}{dt} = \check{\rho}_t + 0 = \left(r_t^f - (\Phi(\iota_t) - \delta) - \mu_t^{q^K+q^B} - \sigma\sigma_t^{q^K+q^B} + \varsigma_t(\sigma + \sigma_t^{q^K+q^B})\right) + (\varsigma_t - \sigma_t^N)0 + \tilde{\varsigma}_t(1-\theta_t)\tilde{\sigma}$$

$$\frac{E\left[dr_t^{\vartheta/B}\right]}{dt} = \mu_t^{\vartheta/B} = \underbrace{\left(r_t^f - (\Phi(\iota_t) - \delta) - \mu_t^{q^K+q^B} - \sigma\sigma_t^{q^K+q^B} + \varsigma_t(\sigma + \sigma_t^{q^K+q^B})\right)}_{risk-free\ rate\ in\ N_t-numeraire} + \underbrace{(\varsigma_t - \sigma_t^N)\sigma_t^{\vartheta/B}}_{\substack{price\ of\ risk \\ in\ N_t-numeraire}}$$

$$\check{\rho}_t - \mu_t^{\vartheta/B} = -(\varsigma_t - \sigma_t^N)\sigma_t^{\vartheta/B} + \tilde{\varsigma}_t(1-\theta_t)\tilde{\sigma}$$

- Price of Risk: $\varsigma_t = -\sigma_t^v + \sigma_t^{p+q} + \gamma\sigma,\ \tilde{\varsigma}_t = \gamma\tilde{\sigma}_t^n = \gamma(1-\theta_t)\tilde{\sigma}$

$$\check{\rho}_t - \mu_t^{\vartheta/B} = (\sigma_t^v - (\gamma-1)\sigma)\sigma_t^{\vartheta/B} + \gamma(1-\theta_t)^2\tilde{\sigma}^2$$

- Capital market clearing: $1-\theta = 1-\vartheta$
- For stationary equilibria and $\sigma^M = 0$:

$$(1-\vartheta) = \sqrt{\check{\rho} + \check{\mu}^B}/(\gamma\tilde{\sigma})$$

Recall
$\mu_t^{\vartheta/B} = \mu_t^{\vartheta} - \mu_t^B + \sigma_t^B(\sigma_t^B - \sigma_t^{\vartheta})$
$\sigma_t^{\vartheta/B} = \sigma_t^{\vartheta} - \sigma_t^B$

Solving MacroModels Step-by-Step

0. Postulate aggregates, price processes & obtain return processes
1. For given C/N-ratio and SDF processes for each i *finance block*
   a. Real investment ι + Goods market clearing *(static)*
   - *Toolbox 1:* Martingale Approach, HJB vs. Stochastic Maximum Principle Approach

   b. Portfolio choice θ + Asset market clearing or
   Asset allocation κ & risk allocation χ
   - *Toolbox 2:* "price-taking social planner approach" – Fisher separation theorem

   c. "Money evaluation equation" ϑ
   - *Toolbox 3:* Change in numeraire to total wealth (including SDF)
2. Evolution of state variable ~~η (and~~ K) *forward equation*
3. Value functions *backward equation*
   a. Value fcn. as fcn. of individual investment opportunities ω
   - *Special cases: log-utility, constant investment opportunities*

   b. Separating value fcn. $V^i(n^{\tilde{i}};\eta,K)$ into $v^i(\eta)\left(\tilde{\eta}^{\tilde{i}}\right)^{1-\gamma}u(K)\left(n^{\tilde{i}}/n^i\right)^{1-\gamma}$

   c. Derive $\breve{\rho} = C/N$-ratio and ς price of risk
4. Numerical model solution
   a. Transform BSDE for separated value fcn. $v^i(\eta)$ into PDE

   b. Solve PDE via value function iteration
5. KFE: Stationary distribution, Fan charts

3a.+b. + Isolating Idio. Risk

For CRRA utility fcn

- Rephrase the conjecture value function as

$$V_t^{\tilde{i}} = \frac{\left(\omega_t^i n_t^{\tilde{i}}\right)^{1-\gamma}}{1-\gamma} = \underbrace{\left(\omega_t^i \frac{N_t^i}{K_t}\right)^{1-\gamma}}_{=:v_t^i} \underbrace{\left(\frac{n_t^{\tilde{i}}}{N_t^i}\right)^{1-\gamma}}_{=:\left(\tilde{\eta}_t^{\tilde{i}}\right)^{1-\gamma}} \frac{K_t^{1-\gamma}}{(1-\gamma)}$$

- v_t^i depend only on aggregate state η_t
- Ito's quotation rule

$$\frac{d\tilde{\eta}_t^{\tilde{i}}}{\tilde{\eta}_t^{\tilde{i}}} = \frac{d\left(n_t^{\tilde{i}}/N_t\right)}{n_t^{\tilde{i}}/N_t} = \left(\mu_t^{n^{\tilde{i}}} - \mu_t^{N^i} + \left(\sigma_t^{N^i}\right)^2 - \sigma^{N^i}\sigma^{n^{\tilde{i}}}\right)dt + \left(\sigma_t^{n^{\tilde{i}}} - \sigma_t^{N^i}\right)dZ_t + \tilde{\sigma}^{n^i} d\tilde{Z}_t^{\tilde{i}}$$
$$= \tilde{\sigma}^{n^i} d\tilde{Z}_t^{\tilde{i}}$$

- Ito's Lemma

$$\frac{d\left(\tilde{\eta}_t^{\tilde{i}}\right)^{1-\gamma}}{\left(\tilde{\eta}_t^{\tilde{i}}\right)^{1-\gamma}} = -\frac{1}{2}\gamma(1-\gamma)\left(\tilde{\sigma}^{n^{\tilde{i}}}\right)^2 dt + (1-\gamma)\tilde{\sigma}^{n^{\tilde{i}}} d\tilde{Z}_t^{\tilde{i}}$$

3b. BSDE for v_t^i

$$\frac{dV_t^{\tilde{i}}}{V_t^i} = \frac{d\left(v_t^i(\tilde{\eta}_t^{\tilde{i}})^{1-\gamma}(K_t)^{1-\gamma}\right)}{v_t^i(\tilde{\eta}_t^{\tilde{i}})^{1-\gamma}(K_t)^{1-\gamma}}$$

- By Ito's product rule

$$= \left(\mu_t^{v^i} + (1-\gamma)(\Phi(\iota_t) - \delta) - \frac{1}{2}\gamma(1-\gamma)\left(\sigma^2 + \left(\tilde{\sigma}^{n^i}\right)^2\right) + (1-\gamma)\sigma\sigma_t^{v^i}\right)dt$$

$$+ \, volatility \; terms$$

- Recall by consumption optimality

$\frac{dV_t^{\tilde{i}}}{V_t^{\tilde{i}}} - \rho dt + \frac{c_t^{\tilde{i}}}{n_t^{\tilde{i}}} dt$ follows a martingale

  - Hence, drift above $= \rho - \frac{c_t^{\tilde{i}}}{n_t^{\tilde{i}}}$

- BSDE:

$$\mu_t^{v^i} + (1-\gamma)(\Phi(\iota_t) - \delta) - \frac{1}{2}\gamma(1-\gamma)\left(\sigma^2 + \left(\tilde{\sigma}_t^{n^i}\right)^2\right) + (1-\gamma)\sigma\sigma_t^{v^i} = \rho - \frac{c_t^{\tilde{i}}}{n_t^{\tilde{i}}}$$

3. Deriving C/N-ratio $\check{\rho}$ in stationary setting

- In stationary equilibrium

$$\underbrace{\mu_t^{v^i}}_{=0} + (1-\gamma)(\Phi(\iota_t) - \delta) - \frac{1}{2}\gamma(1-\gamma)\left(\sigma^2 + \left(\tilde{\sigma}^{n^i}\right)^2\right) + \underbrace{(1-\gamma)\sigma\sigma_t^{v^i}}_{=0} = \rho - \underbrace{\frac{c_t^{\tilde{i}}}{n_t^{\tilde{i}}}}_{=\check{\rho}}$$

- Recall and plug in
  - $\tilde{\sigma}^{n^i} = (1-\vartheta)\tilde{\sigma} = \sqrt{\check{\rho} + \mu^M}/\gamma$ using $(1-\vartheta) = \sqrt{\check{\rho} + \mu^M}/(\gamma\tilde{\sigma})$
  - $\iota = \frac{a\sqrt{\check{\rho}+\mu^M} - \sqrt{\gamma}\tilde{\sigma}\check{\rho}}{\sqrt{\check{\rho}+\mu^M} + \kappa\sqrt{\gamma}\tilde{\sigma}\check{\rho}}$

  yields an equation for $\check{\rho}$

$$(1-\gamma)\left(\frac{1}{\kappa}\log\frac{\sqrt{\check{\rho} + \mu^M}\;(1+\phi a)}{\sqrt{\check{\rho} + \mu^M} + \phi\sqrt{\gamma}\tilde{\sigma}\check{\rho}} - \delta\right) - \frac{1}{2}\gamma(1-\gamma)\left(\sigma^2 + \frac{\check{\rho} + \mu^M}{\gamma^2}\right) = \rho - \check{\rho}$$

  - For $\gamma = 1$: $\check{\rho} = \rho$

Roadmap

- Solve one-sector money model
  - Different ways to derive money evaluation equation
  - Value function with idiosyncratic risk
  - Bubble/Ponzi scheme, r vs. g vs. ς and transfersality condition
  - Mining the bubble and MMT

- Fiscal Theory of the Price Level (with a bubble)
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Remark: Bubble/Ponzi Scheme and Transversality

- Gov. Debt/Money is a Ponzi scheme/bubble
  - But provides store of value/insurance role

- Why does the transversality condition not rule out the bubble?
  - High individual discount rate (low SDF) since net worth (and also optimal money holdings) is idiosyncratically risky

$$\lim_{T\to\infty} E\left[\xi_T n_T^{\tilde{i}}\right] = 0$$

  - Low "social" discount rate (high SDF)

$$\lim_{T\to\infty} E\left[e^{-r^F T} n_T^{\tilde{i}}\right] > 0$$

- Similarity to OLG/perpetual youth models

Remark: r_t^f

- From $r^f = E[dr^{\mathcal{B}}]/dt$

$$r^f = \Phi(\iota) - \delta - \breve{\mu}^{\mathcal{B}}$$

$$r^f = \frac{1}{\kappa}\log\frac{\sqrt{\breve{\rho}+\breve{\mu}^{\mathcal{B}}}\,(1+\phi a)}{\sqrt{\breve{\rho}+\breve{\mu}^{\mathcal{B}}}+\phi\tilde{\sigma}\breve{\rho}} - \delta - \breve{\mu}^{\mathcal{B}}$$

- Remark: bond supply growth
  - Increases ι as portfolio choice is tilted towards capital
  - Depresses real r^f one-to-one because …

$$r^f = \rho + \mu^c - \left((\sigma^c)^2 + (\tilde{\sigma}^c)^2\right)$$

$$r^f = \rho + \left(\Phi\left(\iota(\breve{\mu}^{\mathcal{B}})\right) - \delta\right) - \left(\sigma^2 + \left(1 - \vartheta(\breve{\mu}^{\mathcal{B}})\right)^2 \tilde{\sigma}^2\right)$$

  - … agents hold more idiosyncratic risk

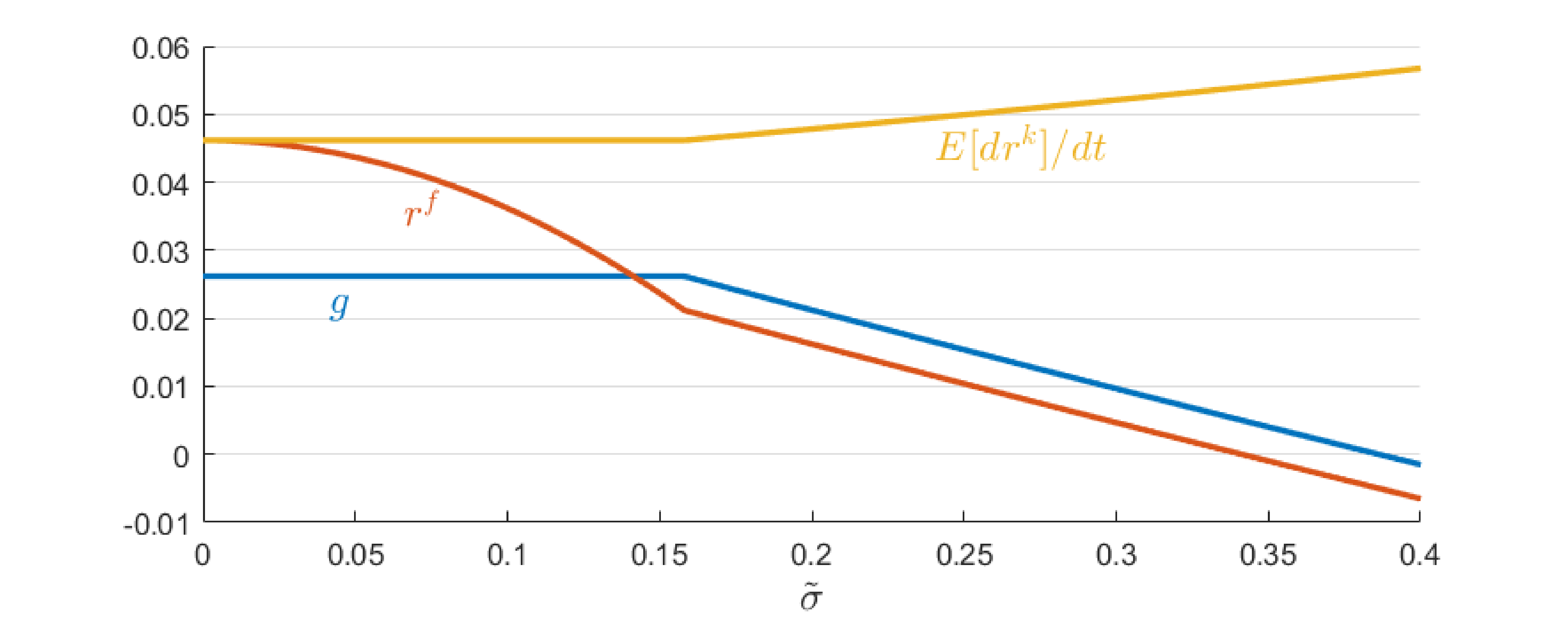
Remark: “sticky price of risk $\tilde{\varsigma}$”

- The price of risk $\tilde{\varsigma}$ depends on $\tilde{\sigma}$ only via ρ

$$\tilde{\varsigma} = \tilde{\sigma}^{n^i} = (1-\vartheta)\tilde{\sigma} = \sqrt{\rho + \breve{\mu}^{\mathcal{B}}}$$ using $(1-\vartheta) = \sqrt{\breve{\rho} + \breve{\mu}^{\mathcal{B}}}/\tilde{\sigma}$

  - $\tilde{\varsigma}$ is independent of $\tilde{\sigma}$
  - Intuition:
    - Increasing $\tilde{\sigma}$ the value of money adjusts in such a way that $\tilde{\sigma}^c$ is not affected
      - For large κ, higher risk translate into smaller q that reduces idiosyncratic risk driven by capital shocks.
      - For small κ, ι is lower and capital in the long-run is reduced

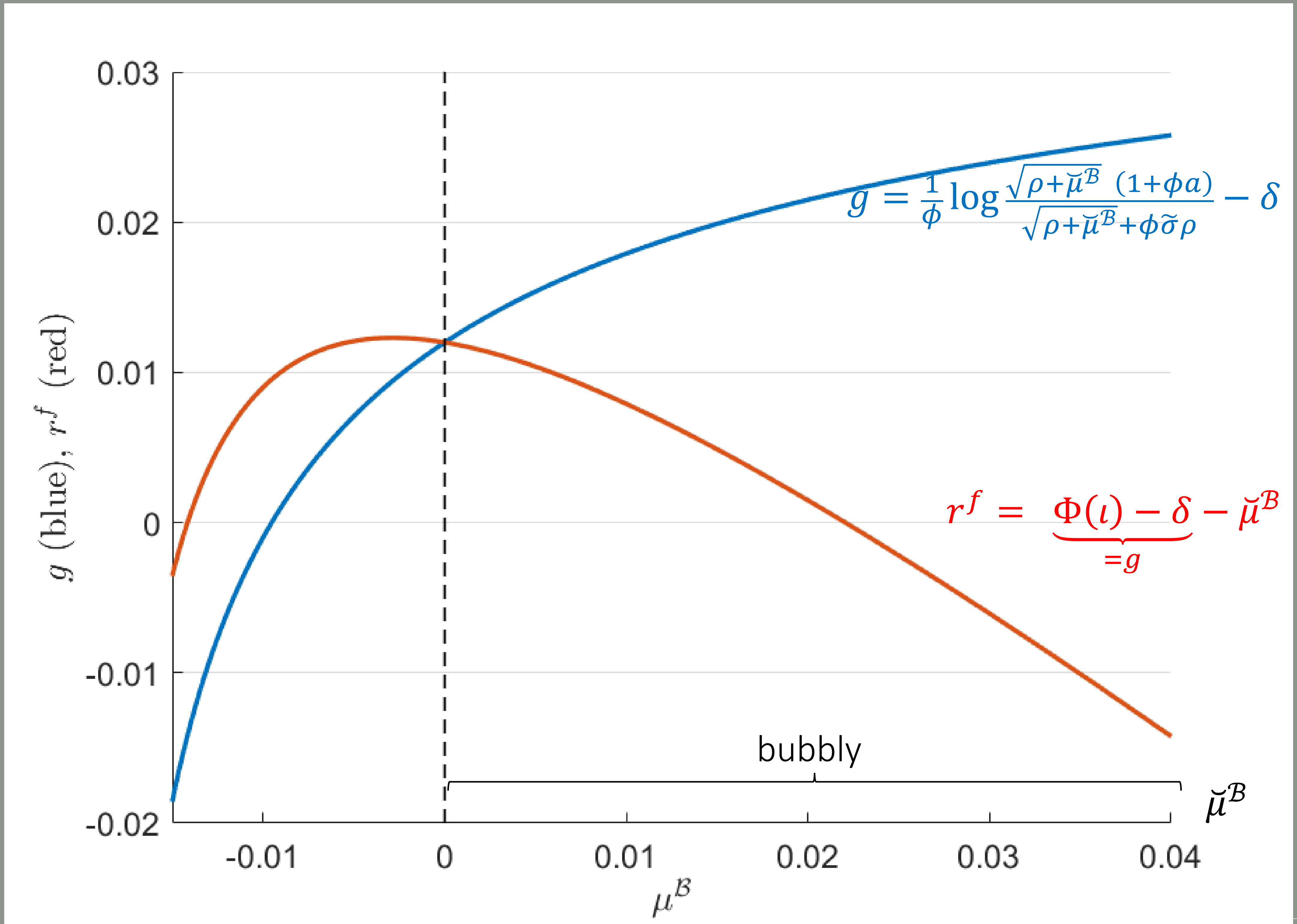
r^f vs. g vs. $E[dr^K]/dt$ for different $\tilde{\sigma}$



$$a = .27, \not{g} = \frac{a}{3}, \delta = .1, \rho = .02, \kappa = 3\,, \breve{\mu}^{\mathcal{B}} = .005$$

- $E[dr^K]/dt > r^f > g$ for small $\tilde{\sigma}/\sqrt{\rho}$
- $E[dr^K]/dt > g > r^f$ for large $\tilde{\sigma}/\sqrt{\rho}$
- $E[dr^K]/dt < g$ can never happen

r^f versus g for different $\breve{\mu}^{\mathcal{B}}$



$a = .27, g = \frac{a}{3}, \delta = .1,$
$\rho = .02, \tilde{\sigma} = .25, \phi = 3\,,$

Bubble and Transversality

- Government debt is a bubble: provides risk-free store of value
- Bonds allow for self-insurance through trading
    - $d\tilde{Z}_t^i < 0 \Rightarrow$ buy capital, sell bonds
    - $d\tilde{Z}_t^i > 0 \Rightarrow$ sell capital, buy bonds

  $\Rightarrow$ lowers volatility of total wealth n_t^i, but increases volatility of bond wealth $n_t^{b,i} := \theta_t^i n_t^i$
- Why does the transversality condition (TVC) not rule out the bubble?
    - TVC for bond wealth: $\lim_{T\to\infty} \mathbb{E}[\xi_T^i n_T^{b,i}] = 0$
    - effective discount rate in TVC = discount rate for stochastic bond portfolio $n^{b,i}$

      $= \text{risk-free rate } r^f + (\text{risk premium for idiosyncratic } n^{b,i}\text{-fluctuations})$
    - discount rate for individual bond = discount rate for aggregate bond stock $\int n^{b,i} di$

      $= \text{risk-free rate } r^f$
    - risk premium: (self-insurance) service flow from retrading bonds (like a convenience yield)
- More general point: beneficial equilibrium trades are essential feature of (rational) bubbles

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"Mining the Fiscal Bubble"

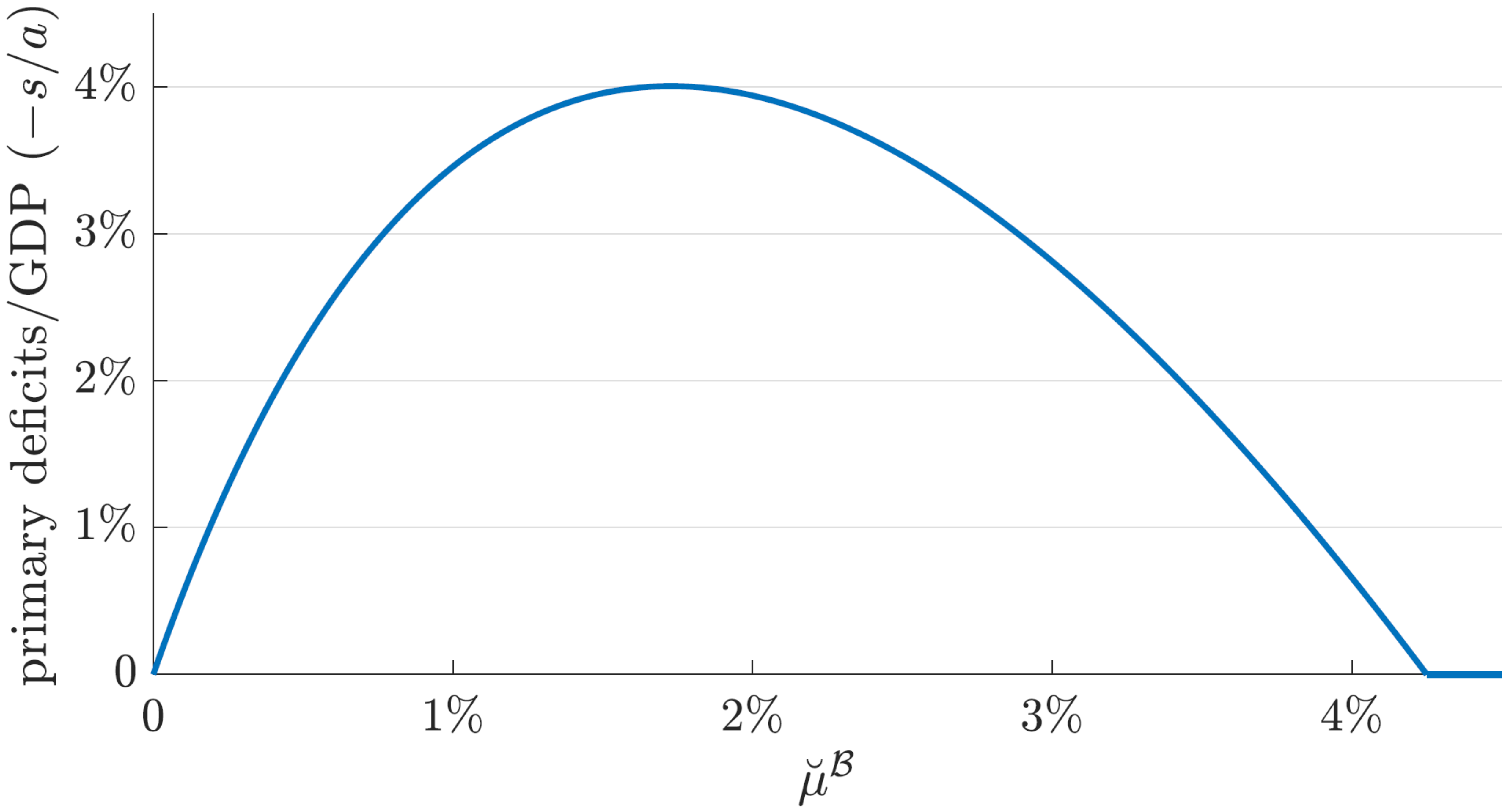
$s > 0$	$s = 0$	$s < 0$	
$r^f > g$	$r^f = g$	$r^f < g$	$\breve{\mu}^{\mathcal{B}}$
$PVS > 0$	$PVS = 0$	$PVS < 0$	
no bubble	bubble > 0	bubble > 0	

In all three cases, the bubble – or its mere possibility – grants government some leeway:

- $s < 0$: perpetual deficits are funded out of the bubble, never have to raise taxes ("bubble mining")
- $s = 0$: government debt enjoys positive value despite zero surpluses (debt "backed" by the bubble)
- $s > 0$: no equilibrium bubble, yet possibility of bubble makes debt more sustainable

  unexpected (persistent) drop in surpluses below zero
  $\Rightarrow$ bubble emerges instead of collapse of the value of debt

Bubble Mining Laffer Curve



see Brunnermeier, Merkel, Sannikov (2020): "The Limits of Modern Monetary Theory"

MMT connection?

- MMT: "as long as inflation is not rising, budget constraints don't matter"

- Can we lower primary surplus (more negative), s/a, without causing inflation?
  - $\pi = \mu^{\mathcal{B}} - \underbrace{[\Phi\left(\iota(\mu^{\mathcal{B}} - i)\right) - \delta\,]}_{g}$
  - Increase $\mu^{\mathcal{B}}$
    - Direct π-effect: higher
    - Indirect π-effect: lower since growth rate g increases (q,ι rises)
  - Lower i to increase g further.
  - Steady state inflation will be the same, but <u>jump in price level</u>
- No clear MMT connection (full employment/utilization in our setting)

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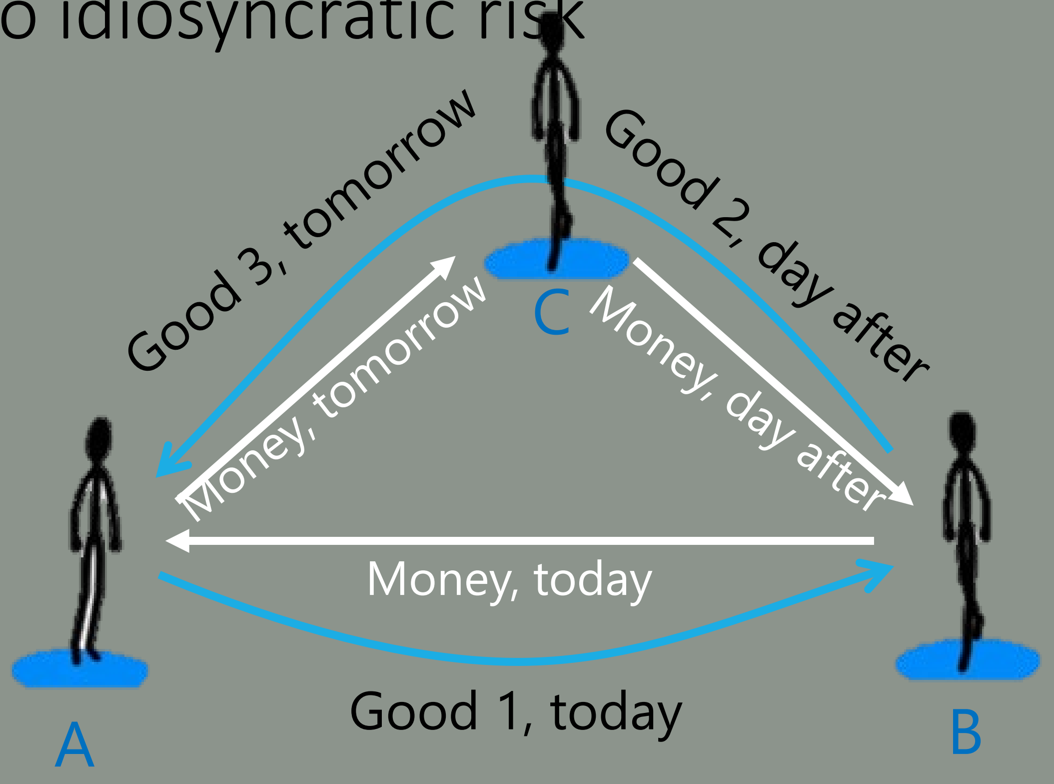
The 4 Roles of Money

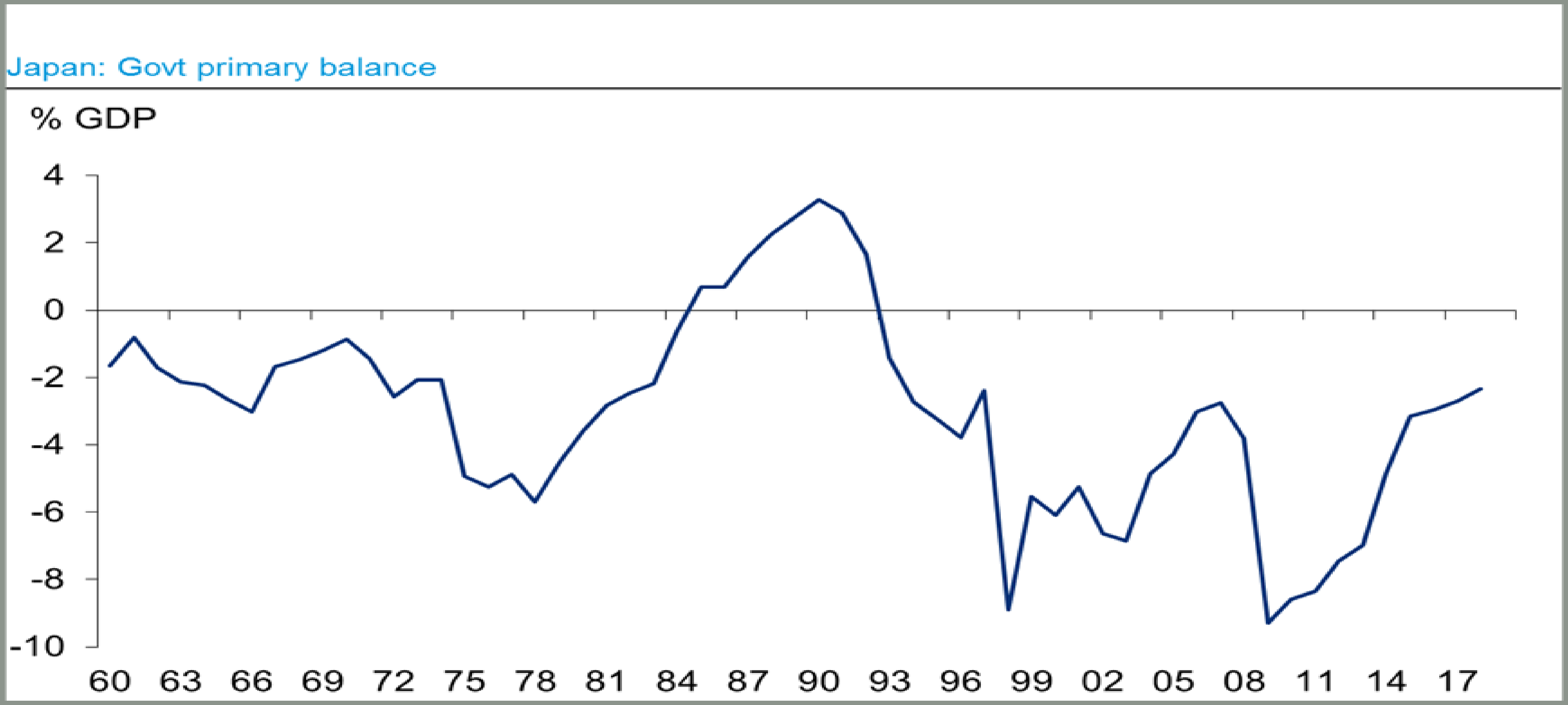
- Unit of account
  - Intratemporal: Numeraire — bounded rationality/price stickiness
  - Intertemporal: Debt contract — incomplete markets
- Store of value
  - "I Theory of Money without I"
    Less risky than other "capital" – no idiosyncratic risk
  - Fiscal theory of the price level
- Medium of exchange
  - Overcome double-coincidence of wants problem



- Record keeping device – money is memory
  - Virtual ledger

Motivation

- Can a country permanently run primary fiscal deficits without destabilizing its currency? (MMT?)
- The Japan FTPL critique: E.g. Noah Smith (2017)
  - $\frac{\mathcal{B}_t+\mathcal{M}_t}{\wp_t} = E_t \int_t^\infty \frac{\xi_s}{\xi_t}(T_s - G_s)ds + E_t \int_t^\infty \frac{\xi_s}{\xi_t}\Delta i_s \frac{\mathcal{M}_s}{\wp_s} ds$, ($\xi_t$= SDF)



  - How to rescue the FTPL?

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  - $\frac{\mathcal{B}_t+\mathcal{M}_t}{\wp_t} = E_t \int_t^\infty \frac{\xi_s}{\xi_t}(T_s - G_s)ds + E_t \int_t^\infty \frac{\xi_s}{\xi_t}\Delta i_s \frac{\mathcal{M}_s}{\wp_s} ds$, ($\xi_t$= SDF)

- $\Delta i_s = i_s - i_s^m$ goes towards zero
  - Low interest rate environment
  - Digital money + Narrow banking/FinTech

- How to rescue the FTPL?

- How to rescue the FTPL?

Deriving FTPL equation (in cts time)

- Nominal government budget constraint

$$\left(\mu_t^{\mathcal{B}}\mathcal{B}_t + \mu_t^{\mathcal{M}}\mathcal{M}_t + \wp_t T_t\right)dt = (i_t\mathcal{B}_t + i_t^m\mathcal{M}_t + \wp_t G_t)dt$$

- Multiply by nominal SDF $\xi_t/\wp_t$, rearrange

$$[\left(\mu_t^{\mathcal{B}} - i_t\right)\frac{\xi_t}{\wp_t}\mathcal{B}_t + \left(\mu_t^{\mathcal{M}} - i_t\right)\frac{\xi_t}{\wp_t}\mathcal{M}_t]dt = -\xi_t(T_t - G_t - \underbrace{(i_t - i_t^m)}_{\Delta i_t :=}\frac{\mathcal{M}_t}{\wp_t})dt$$

- Suppose $\xi_t/\wp_t$ prices the nominal bond
  - Then $E_t\left[\frac{d(\xi_t/\wp_t)}{(\xi_t/\wp_t)}\right] = i_t dt$
  - Substitute into above, use product rule, take expectations

$$E_t\left[d\left(\frac{\xi_t}{\wp_t}(\mathcal{B}_t + \mathcal{M}_t)\right)\right] = -E_t[\xi_t\left(T_t - G_t - \Delta i_t\frac{\mathcal{M}_t}{\wp_t}\right)dt]$$

- In integral form

$$\frac{\mathcal{B}_t + \mathcal{M}_t}{\wp_t} = E_t\int_t^T \frac{\xi_s}{\xi_t}(T_s - G_s)ds + E_t\int_t^T \frac{\xi_s}{\xi_t}\Delta i_s\frac{\mathcal{M}_s}{\wp_s}ds + \frac{\xi_T}{\xi_t}\frac{\mathcal{B}_T + \mathcal{M}_T}{\wp_T}$$

Deriving FTPL equation (in cts time)

- Take limit $T \to \infty$

$$\frac{\mathcal{B}_t + \mathcal{M}_t}{\wp_t} = E_t \int_t^\infty \frac{\xi_s}{\xi_t}(T_s - G_s)ds + E_t \int_t^\infty \frac{\xi_s}{\xi_t} \Delta i_s \frac{\mathcal{M}_s}{\wp_s} ds + \lim_{T\to\infty} E_t \frac{\xi_T}{\xi_t} \frac{\mathcal{B}_T + \mathcal{M}_T}{\wp_T}$$

- Remark 1:
  - Literature focuses on settings, in which private-sector transversality eliminates the bubble term
  - Here: fiscal theory in setting, in which where transversality does not rule out bubbles
- Remark 2:
  - The sum of the three limits on the right may not be well-defined mathematically, because they can be infinite with opposite signs
  - The limit of the sum may nevertheless exist and be finite
    - This is what matters economically (cannot separately trade the bubble and fundamental components)

3 Forms of Seigniorage

$$\frac{\mathcal{B}_t + \mathcal{M}_t}{\wp_t} = E_t \int_t^\infty \frac{\xi_s}{\xi_t}(T_s - G_s)ds + E_t \int_t^\infty \frac{\xi_s}{\xi_t} \Delta i_s \frac{\mathcal{M}_s}{\wp_s} ds + \lim_{T\to\infty} E_t \frac{\xi_T}{\xi_t} \frac{\mathcal{B}_T + \mathcal{M}_T}{\wp_T}$$

1. Surprise devaluation
   - Irrational expectations
   - Small (Hilscher, Raviv, Reis 2014)
     - Inflation options imply likelihood of exceeding 5% of GDP is less than 1%

2. Exploiting liquidity benefits of “narrow” cash
   - Only for “narrow” cash that provides medium-of-exchange services
   - $\Delta i = i - i^M$
   - 0.36 % of GDP, NPV = 20% (at most 30%) of GDP, (Reis 2019)

3. “Money bubble mining”

FTPL Equation with a Bubble in BruSan notation (& $\Delta i = 0$)

- Primary surplus $sK_t = \tau a K_t - \mathscr{g} K_t = -\breve{\mu}^B p K_t$

Government spending $\mathscr{g} \neq g$

- FTPL equation:

$$\frac{\mathcal{B}_0}{\wp_0} = q^B K_0 = \lim_{T\to\infty} \int_0^T e^{-(r^f-g)t} sK_0 dt + \underbrace{\lim_{T\to\infty} e^{-(r^f-g)T} pK_0}_{\text{Bubble}}$$

- $r^f = \underbrace{(\Phi(\iota) - \delta)}_{=g} - \breve{\mu}^{\mathcal{B}}$

$s > 0$ | $s = 0$ | $s < 0$

$\breve{\mu}^{\mathcal{B}}$

$r^f > g$ | $r^f = g$ | $r^f < g$

PV(s) > 0 | PV(s) $= 0$ | PV(s) < 0

No Bubble | Bubble > 0 | Bubble > 0

Mining the "FTPL-Bubble"

- $\breve{\mu}^{\mathcal{B}} > 0$: perpetual deficits are funded out of the bubble
- PV of surpluses is $-\infty$, bubble is ∞, so consider finite-horizon version of FTPL equation

$$q^B = \underbrace{\int_0^T e^{-(r^f-g)t} dt \cdot s}_{=\left(1-e^{\breve{\mu}^B T}\right)q^B} + \underbrace{e^{-(r^f-g)T} q^B}_{=e^{\breve{\mu}^B T} q^B}$$

- For$\breve{\mu}^B > 0$: as $\breve{\mu}^B$ increases
  - PV of surpluses over $[0, T]$ decreases
  - For T large, the continuation value $e^{\breve{\mu}^B T} p$ increases
- In this sense, mining the bubble increases its value

Determination of Price Level

1. In a particular equilibrium:
   FTPL equation with bubble alone doesn't determine price level (because size of bubble is not determined),
   Goods market clearing determines price level
   - … and FTPL equation determines size of the bubble, because
   - Bubble generates a consumption demand from wealth effects

2. Multiple equilibria:
   - Off-equilibrium fiscal backing is sufficient

FTPL: Resolving Equilibrium Multiplicity

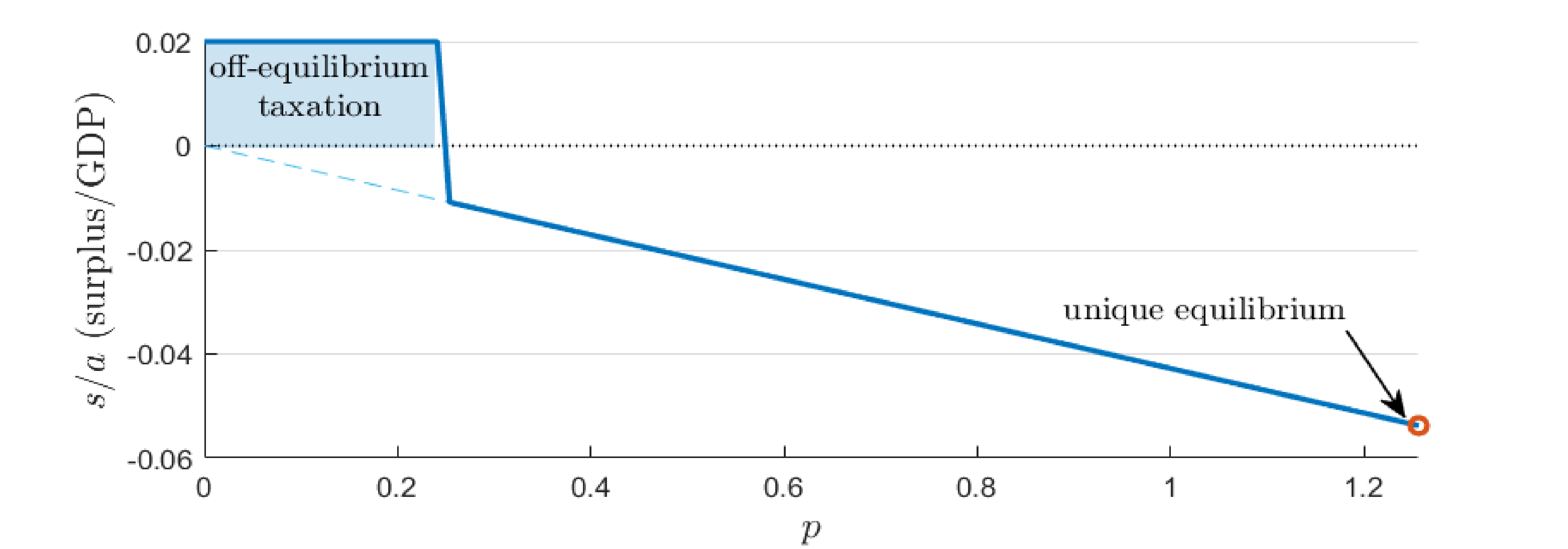
- In any equilibrium (not necessarily with constant $\breve{\mu}_t^{\mathcal{B}}$), the path of ϑ must satisfy
$$\dot{\vartheta}_t = \left(\rho - (1-\vartheta_t)^2\tilde{\sigma}^2 + \breve{\mu}_t^{\mathcal{B}}\right)\vartheta_t$$
- With constant $\breve{\mu}_t^{\mathcal{B}}$, there is continuum of solution paths in $[0,1]$ (even for positive surpluses, $\breve{\mu}^{\mathcal{B}} < 0$)
  - Fiscal policy adjusts surpluses $s_t = -(q_t^B + q_t^K)\breve{\mu}_t^{\mathcal{B}}\vartheta_t$ proportionally with ϑ_t
- All but one solution asymptotically approach $\vartheta_t = 0$
- If surpluses $s_t = -\left(q_t^B + q_t^K\right)\breve{\mu}_t^{\mathcal{B}}\vartheta_t > \varepsilon > 0$ whenever ϑ_t is close to 0, these solutions are all eliminated

FTPL: Resolving Equilibrium Multiplicity

- Thus: uniqueness requires fiscal commitment to tax, if inflation breaks out
- If credible, off-equilibrium commitment is sufficient

$$\dot{\vartheta}_t = \begin{cases} \left(\rho - (1-\vartheta_t)^2\tilde{\sigma}^2 + \breve{\mu}^{\mathcal{B}}\right)\vartheta_t, & \vartheta_t > \underline{\vartheta} \\ \left(\rho - (1-\vartheta_t)^2\tilde{\sigma}^2\right)\vartheta_t - \dfrac{\tau a}{p_t + q_t}, & \vartheta_t \leq \underline{\vartheta} \end{cases}$$

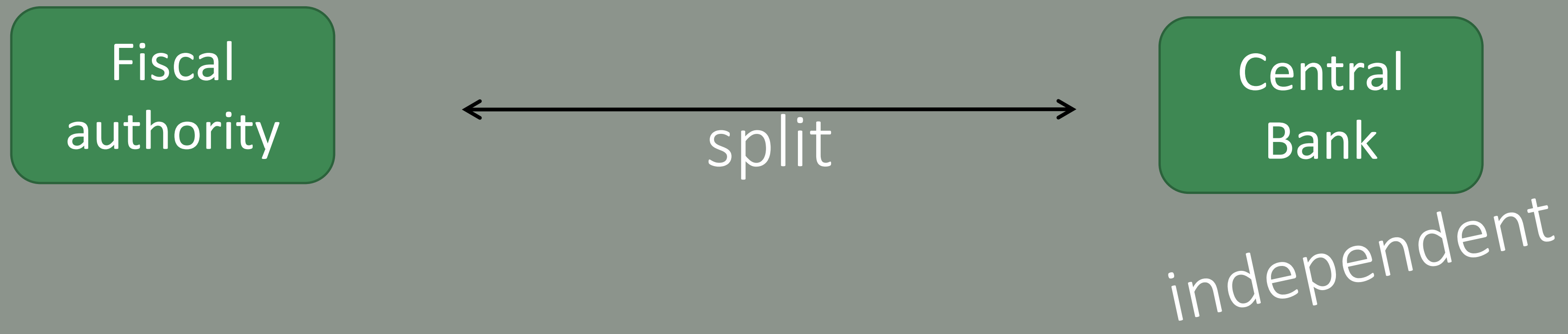
With $\tau, \underline{\vartheta} > 0$



FTPL: Resolving Equilibrium Multiplicity

- Equilibria
  - Moneyless steady state with $p^0 = 0$
  - Price p_t converges over time to zero (hyperinflation)
- With $\varepsilon > 0$ fiscal backing $p_t > \varepsilon$,
  these equilibria are eliminated
  ⇒ only steady state money equilibrium remains
- Off equilibrium fiscal backing suffices
  to rule out moneyless and hyperinflation equilibria
  - If after a hypothetical jump into the moneyless equilibrium,
    one can pay (a small amount) of taxes with money.
    Hence, money is not worthless and
    the moneyless equilibrium does not exist.

FTPL: Who controls inflation?



- Monetary dominance
  - Fiscal authority is forced to adjust budget deficits
- Fiscal dominance
  - Inability or unwillingness of fiscal authorities to control long-run expenditure/GDP ratio
  - Limits monetary authority to raise interest rates
- 0/1 Dominance vs. battle: “dynamic game of chicken”

See YouTube video 4, minute 4:15

Roadmap

- Solve one-sector money model
  - Different ways to derive money evaluation equation
  - Value function with idiosyncratic risk
  - Bubble/Ponzi scheme, r vs. g vs. ς and transfersality condition
  - Mining the bubble and MMT

- Fiscal Theory of the Price Level (with a bubble)
  - FTPL equation
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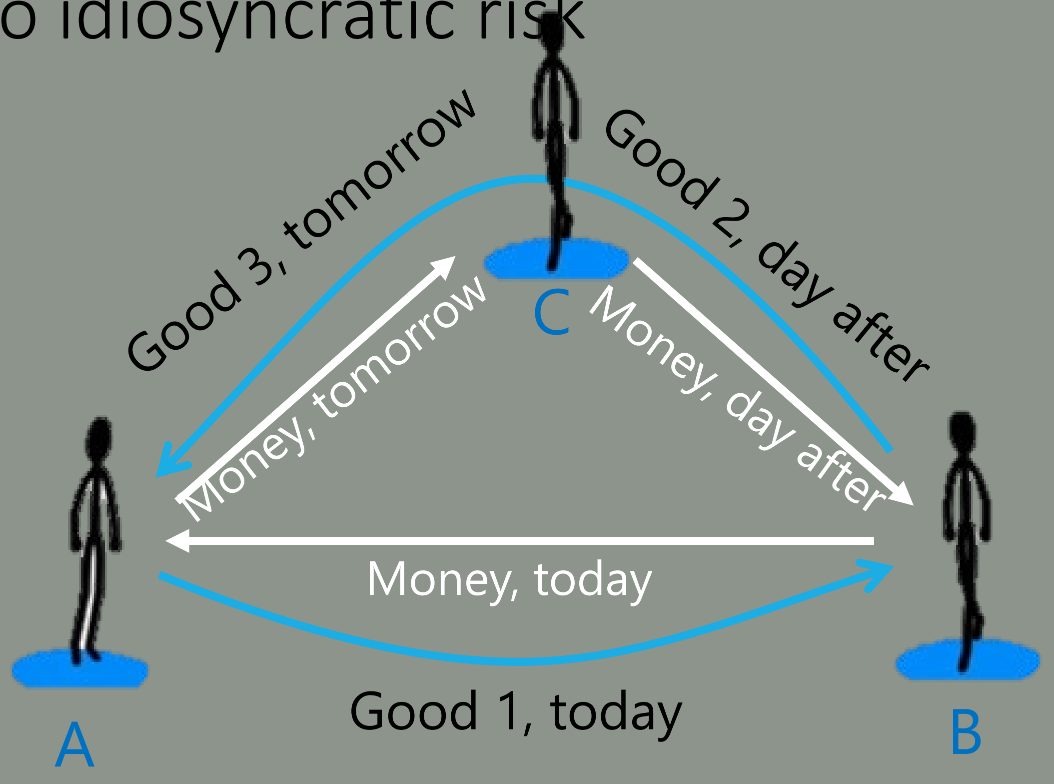
- Medium of Exchange Role of Money

The 4 Roles of Money

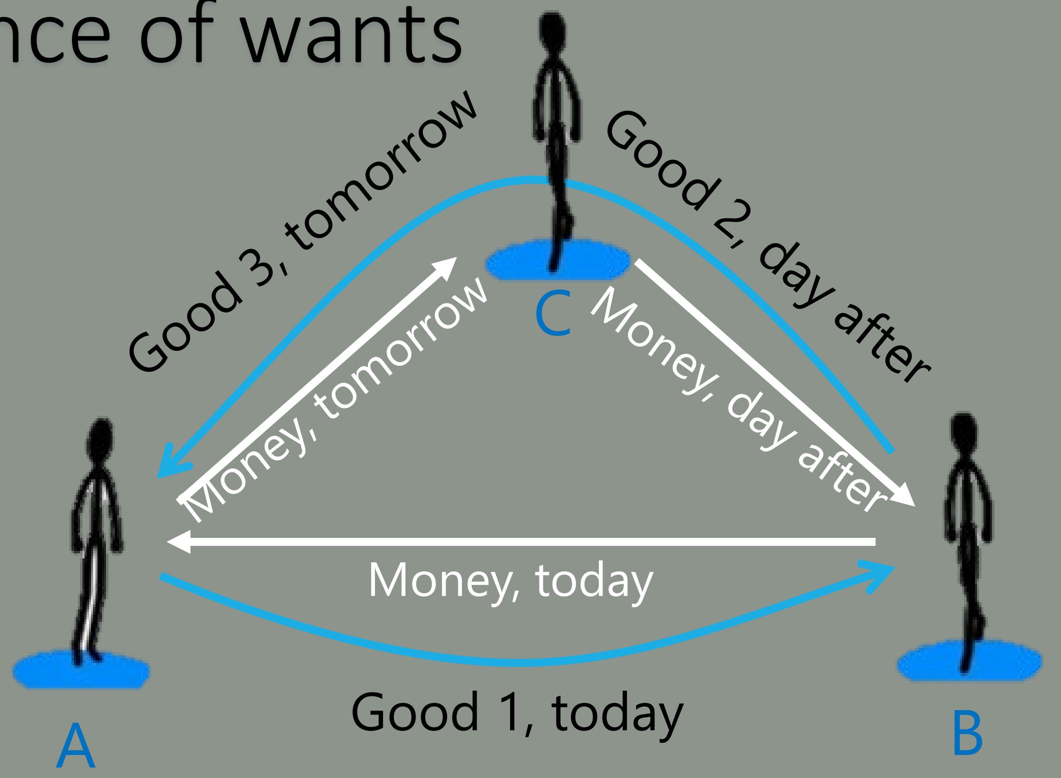
- Unit of account
  - Intratemporal: Numeraire — bounded rationality/price stickiness
  - Intertemporal: Debt contract — incomplete markets
- Store of value
  - "I Theory of Money without I"
    Less risky than other "capital" – no idiosyncratic risk
  - Fiscal theory of the price level
- Medium of exchange
  - Overcome double-coincidence of wants problem



- Record keeping device – money is memory
  - Virtual ledger

Medium of Exchange – Transaction Role

- Overcome double-coincidence of wants



- Quantity equation: $\wp_t T_t = \nu M_t$
  - ν (nu) is velocity (Monetarism: ν exogenous, constant)
  - T transactions
    - Consumption C } Y
    - New investment production ιK } Y — produce own machines
    - Transaction of physical capital $d\Delta^k$ — infinite velocity
    - Transaction of financial claims $\boldsymbol{d\theta^{j \notin M}}$ — infinite velocity

Models of Medium of Exchange

- Reduced form models
  - Cash in advance $T_t = \nu \frac{M_t}{\wp_t}$

    $c_t \leq \sum_{j \in M} \nu^j \theta^j n_t$

    Only assets $j \in M$ with money-like features
  - Shopping time models $c = (c^c, l)$
  - Money in the utility function consume money CES
    - New Keynesian Models
    - No satiation point
  - New Monetary Economics

For generic setting encompassing all models:
see Brunnermeier-Niepelt 2018

Cash in Advance

- Liquidity/cash in advance constraint
  - $c_t \leq \sum_{j\in M} \nu^j \theta^j n_t$ Lagrange multiplier $\hat{\lambda}_t$
  - Asset $j \in M$ which relaxes liquidity/CIA constraint

- Price of liquid/money asset

Relaxes constraint

Value of money yields extra "liquidity service"

$$q_t^{j\in M} = E_t\left[\frac{\xi_{t+\Delta}}{\xi_t}\left(x_{t+\Delta} + q_{t+\Delta}^{j}\right)\right] - \hat{\lambda}_t \nu^j p_t^{j\in M}$$

$$q_t^{j\in M} = E_t\left[\frac{\xi_{t+\Delta}}{\xi_t}\frac{1}{\underbrace{1+\hat{\lambda}_t \nu^j}_{\Lambda_{t+\Delta}^{j}/\Lambda_t^{j} :=}}\left(x_{t+\Delta} + q_{t+\Delta}^{j}\right)\right]$$

$$q_t^{j\in M} = \lim_{T\to\infty} E_t\left[\sum_{\tau=1}^{(T-t)/\Delta} \frac{\xi_{t+\tau\Delta}}{\xi_t}\frac{\Lambda^j{}_{t+\tau\Delta}}{\Lambda^j{}_t} x_{t+\tau\Delta}\right] + \lim_{T\to\infty} E_t\left[\frac{\xi_T}{\xi_t}\frac{\Lambda^j{}_T}{\Lambda^j{}_t} q_T\right]$$

As if SDF is multiplied by "liquidity multiplier" (Brunnermeier-Niepelt)

Cash in Advance

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$$q_t^{j \in M} = \lim_{T \to \infty} E_t \left[\int_t^T \frac{\xi_\tau}{\xi_t} \frac{\Lambda^j{}_\tau}{\Lambda^j{}_t} x_\tau d\tau \right] + \underbrace{\lim_{T \to \infty} E_t \left[\frac{\xi_T}{\xi_t} \frac{\Lambda^j{}_T}{\Lambda^j{}_t} q_T \right]}_{\text{Bubble}}$$

  - "Money bubble" easier to obtain due to liquidity service
    - Condition absent aggregate risk: $r^M < g$ easier to obtain since $r^M < r^f$
- HJB approach (Problem Set #3)

$$\mu_t^{r,j} = r_t^f + \varsigma_t \sigma_t^{r,j} + \tilde{\varsigma}_t \tilde{\sigma}^{r,j} - \lambda_t \nu^j$$

where $\lambda_t = \hat{\lambda}_t / V'(n_t)$

↑ (Shadow) risk-free rate of illiquid asset

Add Cash in Advance to BruSan Model

- Return on money
  - Store of value – as before
  - Liquidity service

$$\frac{E\left[dr_t^M\right]}{dt} = \Phi(\iota_t) - \delta + \mu_t^p + \sigma\sigma_t^p - \mu^M = r_t^f + \varsigma_t\left(\sigma + \sigma_t^p\right) - \lambda_t \nu^M$$

- In steady state

$$\Phi(\iota) - \delta - \underbrace{\left(\mu^M - \lambda\nu^M\right)}_{\ddot{\mu}^M :=} = r^f + \varsigma\sigma$$

- Solving the model as before …
  - By simply replace μ^M with $\mu^M - \lambda_t \nu_t^M$
  - Special case: $\ddot{\mu}^M = 0$, i.e. $\mu^M = \lambda\nu^M$, $\gamma = 1$ $\Rightarrow$ explicit solution as fcn of $\check{\rho}$
    - Same q and p as a function of ζ,
    - But $\check{\rho} \neq \rho$ if CIA constraint binds in steady state, otherwise $\check{\rho} = \rho$
      1. Assume it binds, i.e. $\zeta = \nu\vartheta$
      2. Recall from slide 21 for $\hat{\mu}^M = 0$ and $\gamma = 1$, $\vartheta = \frac{\tilde{\sigma} - \sqrt{\zeta}}{\tilde{\sigma}}$
      3. Equate 1. and 2. to obtain quadratic solution for $\check{\rho}$
         1. If $< \rho$, then solution equals $\check{\rho}$
         2. If $> \rho$, then $\check{\rho} = \rho$ and hence CIA doesn't bind, $\lambda = 0$, above solution
- "Occasionally" binding CIA constraint (outside of steady state)
- for sufficiently high $\tilde{\sigma}$, store of value (insurance motive) $\Rightarrow \lambda_t = 0$

Add Money in Utility to BruSan Model

- Money in utility function $u(c, M/\wp) = u(c, \theta n)$

- Can be expressed as equality constraint
  - Difference to CIA inequality: No satiation point

- DiTella add MiU to BruSan 2016 AER PP
  - https://web.stanford.edu/~sditella/Papers/Di_Tella_Liquidity_Traps.pdf

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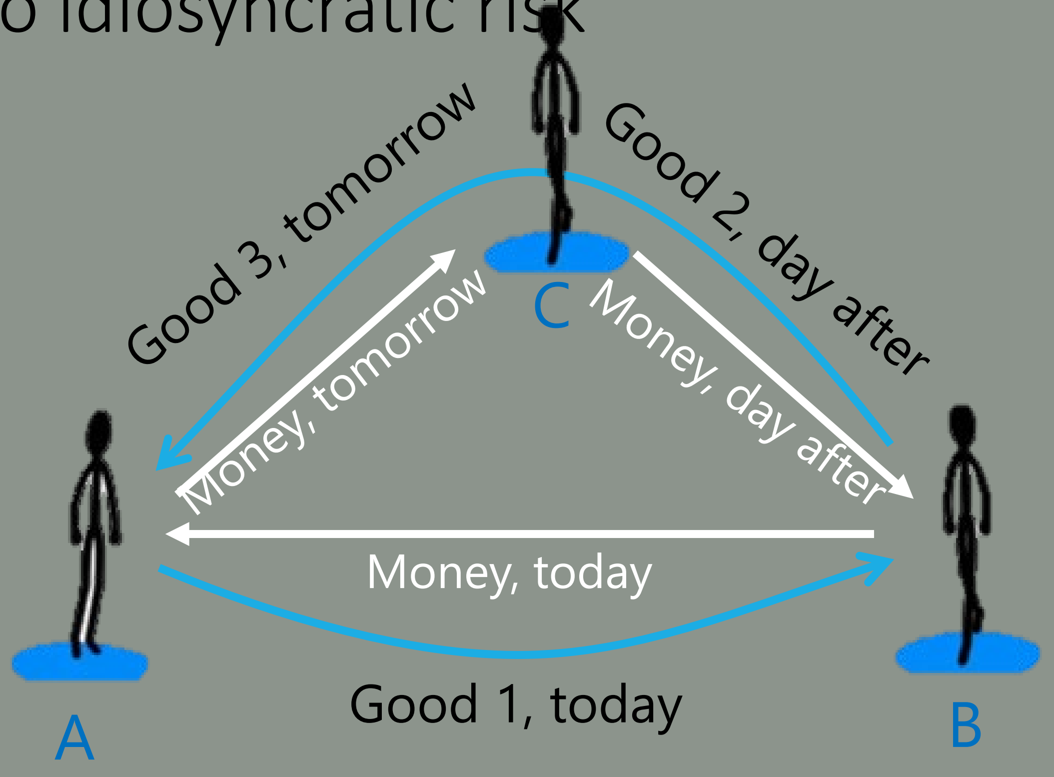

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THE END