Financial and Monetary Economics

Eco529 Fall 2020

Lecture 04: Jumps and Runs

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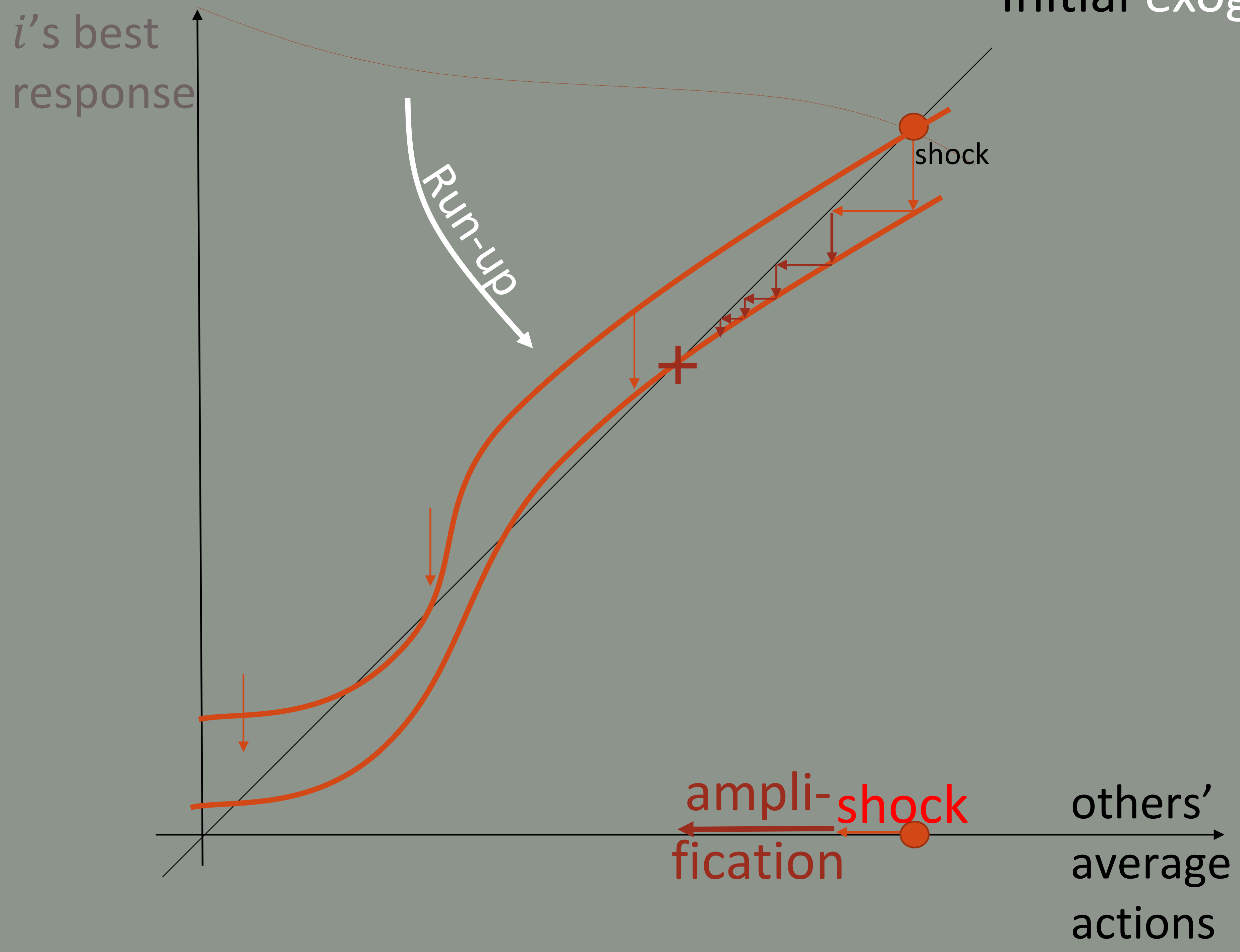
Jumps due to multiple equilibria

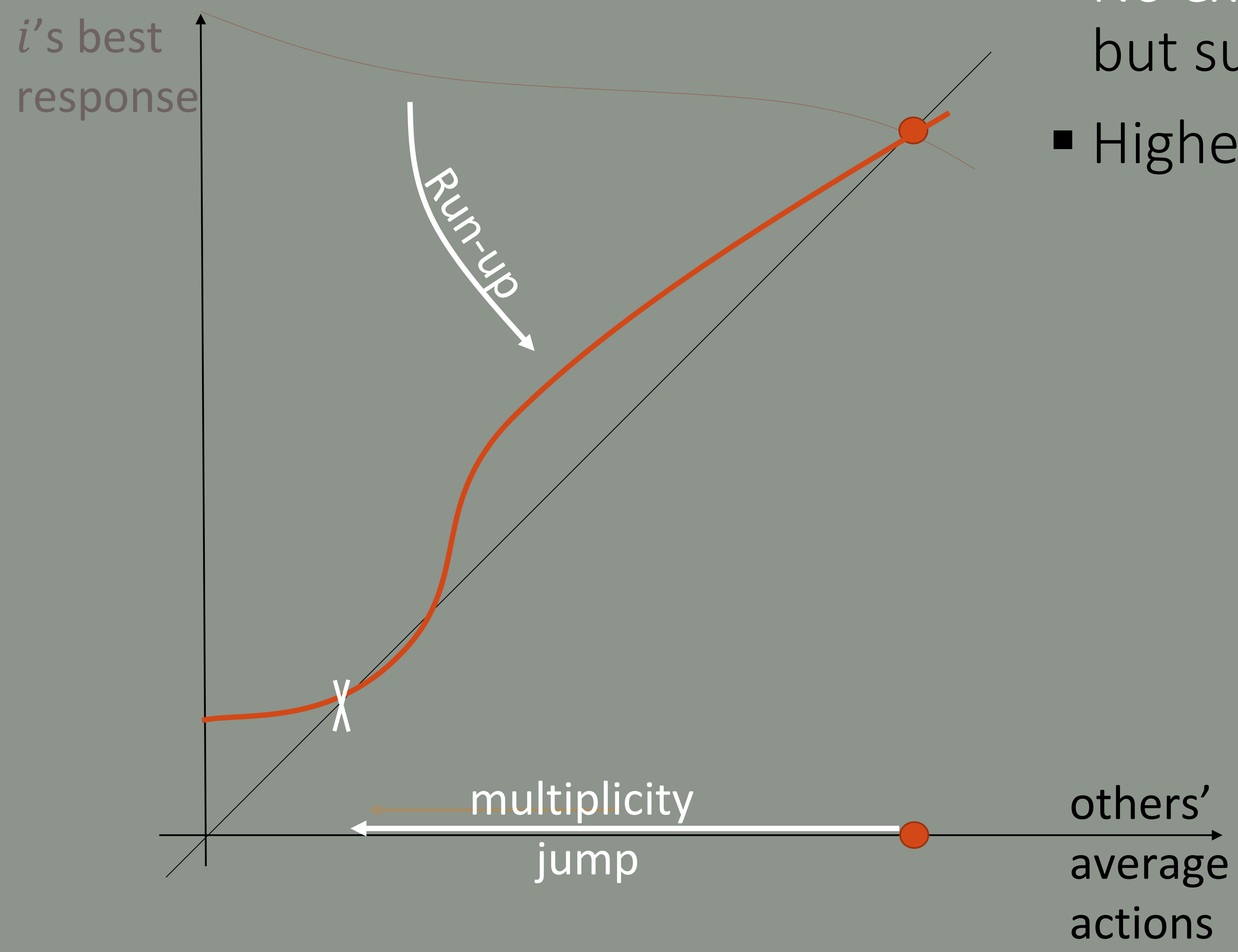
- Bank runs — Diamond Dybvig
- Liquidity spirals — Brunnermeier Pedersen
- Sudden stops — Calvo, Mendoza, …
- Currency attacks — Obstfeld (2nd generation models), Morris Shin
- Twin crisis models — Kaminsky Reinhart (3rd generation models)
- Loss of safe asset status — (after introducing safe asset in world with idiosyncratic risk)

Recall: Endogenous Risk due to Amplification

Initial exogenous shock σdZ_t /trigger



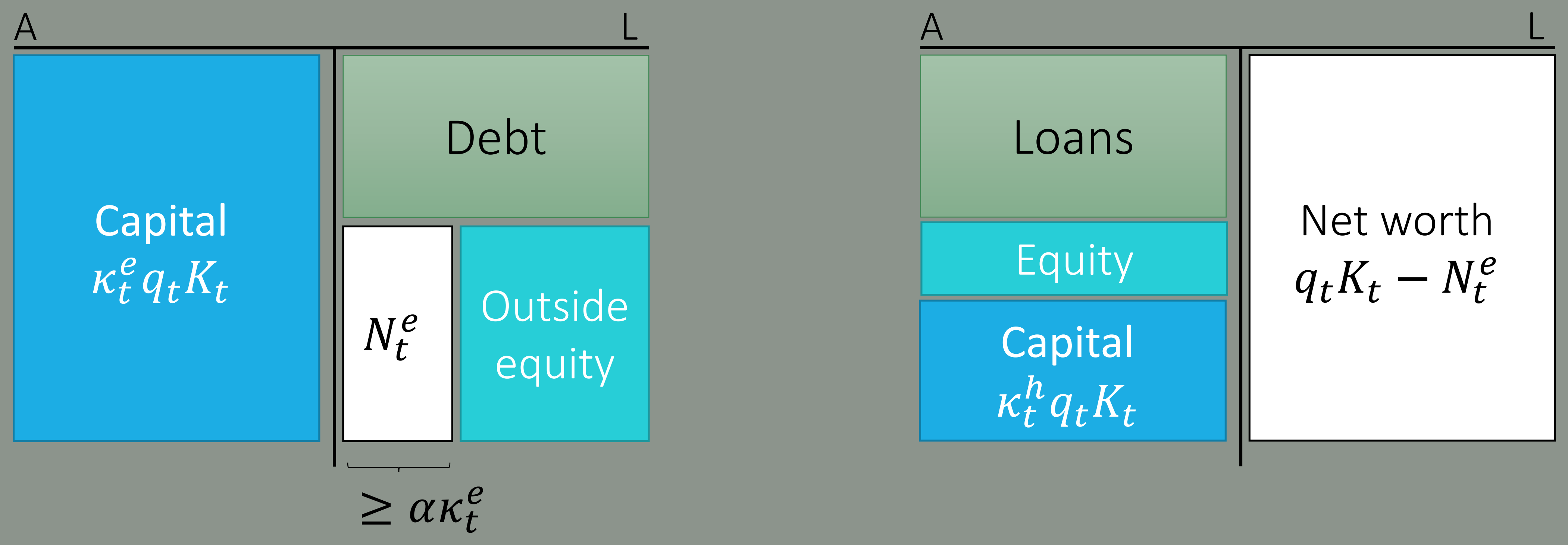
Recall: Endogenous Risk due to Multiple Equilibria Jumps



- No exogenous shock, but sunspot process
- Higher strategic complementarities

Two Type/Sector Model with Outside Equity

- Expert sector   Household sector



- Experts must hold fraction $\chi_t^e \geq \alpha \kappa_t^e$ (skin in the game constraint)
- Return on inside equity N_t can differ from outside equity
  - Issue outside equity at required return from HH
  - In related model, He and Krishnamurthy 2013 impose that inside and outside equity have same return

Two Type Model Setup

Expert sector

- Output: $y_t^e = a^e k_t^e$ $\quad a^e \geq a^h$
- Consumption rate: c_t^e
- Investment rate: ι_t^e

$$\frac{dk_t^{\tilde{\imath},e}}{k_t^{\tilde{\imath},e}} = \left(\Phi\left(\iota_t^{\tilde{\imath},e}\right) - \delta\right)dt + \sigma dZ_t + \tilde{\sigma} d\tilde{Z}_t^{\tilde{\imath}}$$

- $E_0[\int_0^\infty e^{-\rho^e t} \frac{(c_t^e)^{1-\gamma}}{1-\gamma} dt]$ $\quad \rho^e \geq \rho^h$

Household sector

- Output: $y_t^h = a^h k_t^h$
- Consumption rate: c_t^h
- Investment rate: ι_t^h

$$\frac{dk_t^{\tilde{\imath},h}}{k_t^{\tilde{\imath},h}} = \left(\Phi\left(\iota_t^{\tilde{\imath},h}\right) - \delta\right)dt + \sigma dZ_t + \tilde{\sigma} d\tilde{Z}_t^{\tilde{\imath}}$$

- $E_0[\int_0^\infty e^{-\rho^h t} \frac{(c_t^h)^{1-\gamma}}{1-\gamma} dt]$

Friction: Can only issue

- Risk-free debt
- Equity, but most hold $\chi_t^e \geq \alpha \kappa_t$

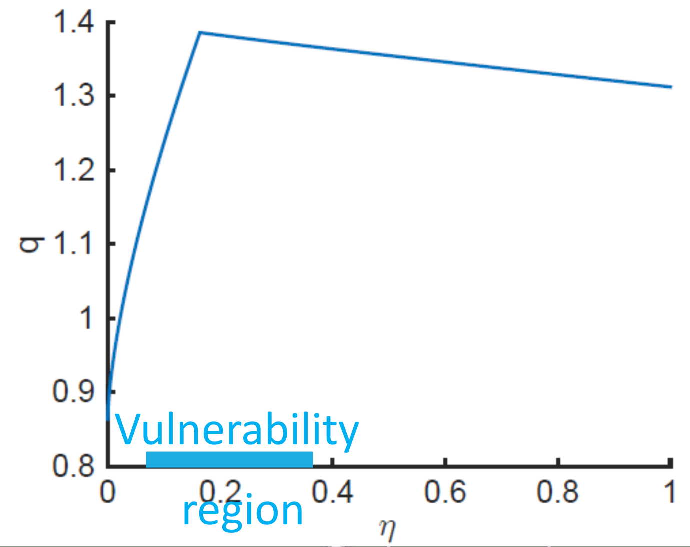
Unanticipated Run on Experts

- Can unanticipated withdrawal of all experts' funding be self-fulfilling?
- Unanticipated crash – jump to $\eta^e = 0$
  - Absent a run: solution as in Lecture 03, since unanticipated
  - When do jump capital losses wipe out experts net worth?

$$\left(q(\eta_t^e) - q(0)\right)\underbrace{\left(\theta_t^{e,K} + \theta_t^{e,OE}\right)\eta_t^e}_{\chi_t^e} K_t \geq \eta_t^e q(\eta_t^e) K_t$$

$$q(\eta_t^e)\left(1 - \frac{\eta_t^e}{\chi^e(\eta_t^e)}\right) \geq q(0) \text{ or } q(\eta_t^e)\left(1 - \frac{1}{\theta_t^{e,K} + \theta_t^{e,OE}}\right) \geq q(0)$$

  - Vulnerability region:
    - High price (not very low η^e)
    - "high risk-leverage" (not very high η^e)
  - After run: $\eta^0 = 0$ forever



2 Types of Runs and Modeling Challenges

- What type of run? What's the trigger?
  - Funding supply run: Depositor/households run
    - Household withdraw funding to experts
  - Funding demand run: Other experts run
    - Each expert tries to pay back debt and fire-sells assets
    - Drop in q is driver

- Model advantage: Always jump to the same point $q(\eta^e = 0)$!

- Modeling Challenges: (see Mendo (2020)

1. Experts are whipped out forever.
   - OLG structure:
     - Death: all agents die with Poisson rate λ^d,
     - Birth: fraction ψ of newborns are experts
2. With anticipated run, expert fear "infinite marginal utility state" $\eta^e = 0$.
   - Transfer of τK to bankrupt experts after run
   - Also fixes challenge 1.
   - To keep τ small also introduce relative performance penalty

From Ito to Levy and Cox Processes

- Ito process: $dX_t = \mu_t^X X_t dt + \sigma_t^X X_t dZ_t$ (geometric)
  the Brownian "shocks" dZ_t are i.i.d. and small s.t. continuous path
  - For non-normal shocks within dt one needs discontinuities
- Levy process: $dL_t = adt + bdZ_t + dJ_t$ – most general class with i.i.d. increments
  $$dX_t = \mu_t^X X_t dt + \sigma_t^X X_t dZ_t + j_t^X X_{t-} dJ_t$$
  - Restrict attention to Poisson processes:
    - Levy jump process can be written as integral w.r.t. Poisson random measures
    - Poisson process with arrival rate $\lambda > 0$:
    - J takes on values in $\mathbb{N}_0 = \{0,1,2,\ldots\}$
    - Increments $J_{t+\Delta t} - J_t$ are Poisson distributed with Parameter $\lambda\Delta t$
    - Stochastic integral w.r.t. Poisson process simply sums up the values of the integrand
    - $\int_0^T a_t dJ_t = \sum_{n=1}^{J_T} a_{\tau_n}$
  - Cox process: λ_t can be time-varying
  - Compensated Jump process: $J_t - \int_0^t \lambda_s ds$ is martingale
    - If $\int_0^t a_s dJ_s$ and a_t uses info only up to right before t then $J_t - \int_0^t a_s \lambda_s ds$ is martingale

Ito formulas

- $df(X_t) = f'(X_t)\left(\mu_t^X X_t dt + \sigma_t^X X_t dZ_t\right) + \frac{1}{2} f''(X_t)\left(\sigma_t^X X_t\right)^2 dt + (f(X_t) - f(X_{t-}))dJ_t$
  $= \left(f'(X_t)\mu_t^X X_t + \frac{1}{2} f''(X_t)\left(\sigma_t^X X_t\right)^2\right) dt + f'(X_t)\sigma_t^X X_t dZ_t + \left(f\left((1 + j_t^X)X_{t-}\right) - f(X_{t-})\right) dJ_t$
- Power rule:
  - $\frac{dX_t^\gamma}{X_{t-}^\gamma} = \left(\gamma\mu_t^X + \gamma(\gamma - 1)\left(\sigma_t^X\right)^2\right) dt + \gamma\sigma_t^X dZ_t + \left(\left(1 + j_t^X\right)^\gamma - 1\right) dJ_t$
- Product rule:
  - $\frac{d(X_t Y_t)}{X_{t-} Y_{t-}} = \left(\mu_t^X + \mu_t^Y + \sigma_t^X \sigma_t^Y\right)dt + \left(\sigma_t^X + \sigma_t^Y\right)dZ_t + \left(j_t^X + j_t^Y + j_t^X j_t^Y\right)dJ_t$
- Quotient rule:
  - $\frac{d(X_t/Y_t)}{X_{t-}/Y_{t-}} = \left(\mu_t^X - \mu_t^Y + \left(\sigma_t^Y\right)^2 - \sigma_t^X \sigma_t^Y\right) dt + \left(\sigma_t^X - \sigma_t^Y\right)dZ_t + \frac{j_t^X - j_t^Y}{1 + j_t^Y} dJ_t$
- Memorize simple rules:
  - $1 + j_t^X = \left(1 + j_t^X\right)^\gamma$
  - $1 + j_t^{XY} = \left(1 + j_t^X\right)\left(1 + j_t^Y\right)$
  - $1 + j_t^{X/Y} = \frac{1 + j_t^X}{1 + j_t^Y}$

Solving MacroModels Step-by-Step

0. Postulate aggregates, price processes & obtain return processes
1. For given C/N-ratio and SDF processes for each i *finance block*
   a. Real investment ι + Goods market clearing *(static)*
   - *Toolbox 1:* Martingale Approach, HJB vs. Stochastic Maximum Principle Approach

   b. Portfolio choice θ + Asset market clearing or
   Asset allocation κ & risk allocation χ
   - *Toolbox 2:* "price-taking social planner approach" – Fisher separation theorem
   - *Toolbox 3:* Change in numeraire to total wealth (including SDF)
2. Evolution of state variable η (and K) *forward equation*
3. Value functions *backward equation*
   a. Value fcn. as fcn. of individual investment opportunities ω
   - *Special cases: log-utility, constant investment opportunities*

   b. Separating value fcn. $V^i(n^{\tilde{i}};\eta,K)$ into $v^i(\eta)u(K)$
   c. Derive C/N-ratio and ς price of risk
4. Numerical model solution
   a. Transform BSDE for separated value fcn. $v^i(\eta)$ into PDE
   b. Solve PDE via value function iteration
5. KFE: Stationary distribution, Fan charts

0. Postulate Aggregates and Processes

- Individual capital evolution:

$$\frac{dk_t^{\tilde{\imath},i}}{k_t^{\tilde{\imath},i}} = \left(\Phi\left(\iota^{\tilde{\imath},i}\right) - \delta\right)dt + \sigma dZ_t + d\Delta_t^{k,\tilde{\imath},i}$$

  - Where $\Delta_t^{k,\tilde{\imath},i}$ is the individual cumulative capital purchase process
- Capital aggregation:
  - Within sector i: $K_t^i \equiv \int k_t^{\tilde{\imath},i} d\tilde{\imath}$
  - Across sectors: $K_t \equiv \sum_i K_t^i$
  - Capital share: $\kappa_t^i \equiv K_t^i / K_t$

$$\frac{dK_t}{K_t} = \left(\Phi\left(\iota_t^i\right) - \delta\right)dt + \sigma dZ_t$$

- Net worth aggregation:
  - Within sector i: $N_t^i \equiv \int n_t^{\tilde{\imath},i} d\tilde{\imath}$
  - Across sectors: $N_t \equiv \sum_i N_t^i$
  - Wealth share: $\eta_t^i \equiv N_t^i / N_t$
- Value of capital stock: $q_t K_t$

  Postulate $dq_t/q_t = \mu_t^q dt + \sigma_t^q dZ_t + j_t^q dJ_t$

Same Brownian

(c is numeraire)

0. Postulate Aggregates and Processes

- Individual capital evolution:

$$\frac{dk_t^{\tilde{\imath},i}}{k_t^{\tilde{\imath},i}} = \left(\Phi\left(\iota^{\tilde{\imath},i}\right) - \delta\right)dt + \sigma dZ_t + d\Delta_t^{k,\tilde{\imath},i}$$

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  Postulate $dq_t/q_t = \mu_t^q dt + \sigma_t^q dZ_t + j_t^q dJ_t$

Sunspot arrival rate

- Postulated SDF-process: $\frac{d\xi_t^i}{\xi_t^i} = \underbrace{\mu_t^{\xi^i}}_{\equiv -r_t^{F,i}} dt + \underbrace{\sigma_t^{\xi^i}}_{\equiv -\varsigma_t^i} dZ_t + \underbrace{j_t^{\xi^i}}_{\equiv -\nu_t^i} (dJ_t - \lambda_t dt)$ (c is numeraire)

Since only risky debt and not risk-free debt is traded

0. Postulate Aggregates and Processes

- … from price processes to return processes (using Ito)
  - Use Ito product rule to obtain capital gain rate (in absence of purchases/sales)
    - Define $\check{k}_t^{\tilde{\imath}}$: $\frac{d\check{k}_t^{\tilde{\imath},i}}{\check{k}_t^{\tilde{\imath},i}} = \left(\Phi\left(\iota_t^{\tilde{\imath},i}\right) - \delta\right)dt + \sigma dZ_t + d\Delta_t^{k,\tilde{\imath},i}$ without purchases/sales

Dividend yield

E[Capital gain rate]= $\frac{d(q_t\check{k}^i{}_t)}{(q_t\check{k}_t^i)}$

$$dr_t^k\left(\iota_t^{\tilde{\imath},i}\right) = \left(\frac{a^i - \iota_t^i}{q} + \Phi(\iota_t^i) - \delta + \mu_t^q + \sigma\sigma_t^q\right)dt + (\sigma + \sigma_t^q)dZ_t + j_t^q dJ_t$$

For aggregate capital return,
Replace a^i with $A(\kappa)$

- Return on defaultable debt

$$dr_t^D = r_t^i dt + j_t^{D,i} dJ_t$$

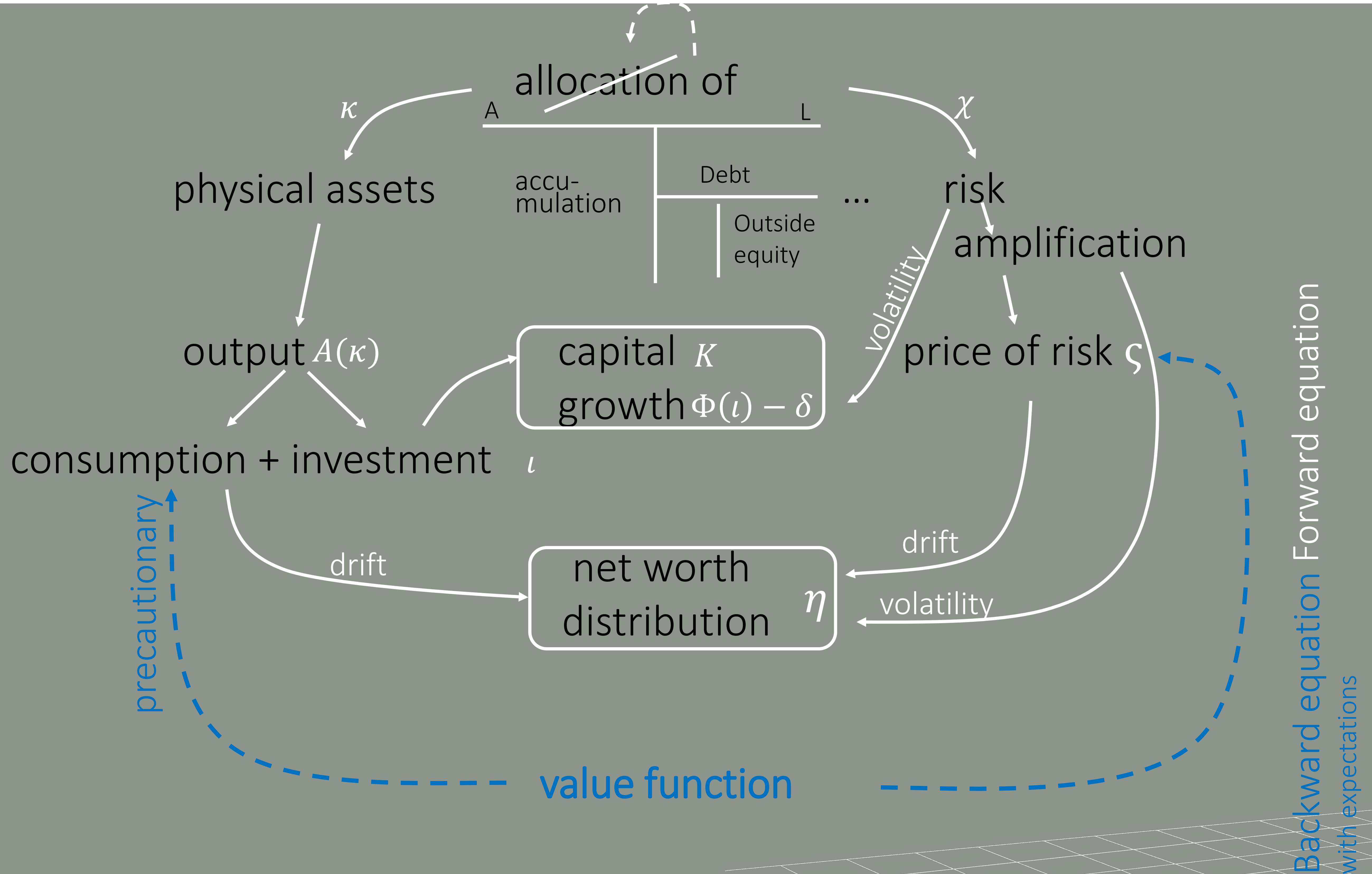
- Postulate SDF-process: (Example: $\xi_t^i = e^{-\rho t}V'(n_t^i)$.)

$$\frac{d\xi_t^i}{\xi_t^i} = -r_t^{F,i}dt - \varsigma_t^i dZ_t - \nu_t^i(dJ_t - \lambda_t dt)$$

Price of risk

Price of jump/run risk

The Big Picture



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1a. Individual Agent Choice of ι, θ, c/n

- Choice of ι is static problem (and separable) for each t
- $\max_{\iota_t^i} dr_t^k(\iota_t^i)$

$$= \max_{\iota_t^i} \left(\frac{a^i - \iota_t^i}{q_t} + \Phi(\iota_t^i) - \delta + \mu^q + \sigma\sigma^q \right) + (\sigma + \sigma_t^q) dZ_t + j_t^q dJ_t$$

For aggregate capital return,
Replace a^i with $A(\kappa)$

- FOC: $\frac{1}{q_t} = \Phi'(\iota_t^i)$ Tobin's q
  - All agents $\iota_t^i = \iota_t \Rightarrow \frac{dK_t}{K_t} = (\Phi(\iota_t) - \delta)\, dt + \sigma dZ_t$
  - Special functional form:
    - $\Phi(\iota) = \frac{1}{\phi}\log(\phi\iota + 1) \Rightarrow \phi\iota = q - 1$
- Goods market clearing: $(A(\kappa) - \iota_t)K_t = \sum_i C_i$.

$$\kappa_t a^e K_t + (1 - \kappa_t) a^h K_t - \iota(q_t) K_t = \eta_t^e \frac{C_t^e}{N_t^e} q_t K_t + (1 - \eta_t^e) \frac{C_t^h}{N_t^h} q_t K_t$$

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1a. Individual Agent Choice of ι, θ, c/n

$$\max_{\{\iota_t,\boldsymbol{\theta}_t,c_t\}_{t=0}^{\infty}} E\left[\int_0^{\infty} e^{-\rho t}u(c_t)dt\right]$$

s.t. $\frac{dn_t}{n_t} = -\frac{c_t}{n_t}dt + \sum_j \theta_t^j dr_t^j$ + labor income/endow/taxes

n_0 given

- Portfolio Choice: Martingale Approach
  - Let x_t^A be the value of a "self-financing trading strategy"(reinvest dividends)
- Theorem: $\xi_t x_t^A$ follows a Martingale, i.e. drift $= 0$.
  - Let $\frac{dx_t^A}{x_t^A} = \mu_t^A dt + \sigma_t^A dZ_t + j_t^A dJ_t,$
  - Recall SDF $\frac{d\xi_t^i}{\xi_t^i} = -r_t dt - \varsigma_t^i dZ_t - \nu_t^i(dJ_t - \lambda_t dt)$
  - By Ito product rule

$$\frac{d(\xi_t^i x_t^A)}{\xi_t^i x_t^A} = (-r_t + \mu_t^A - \varsigma_t^i\sigma_t^A - (1-\nu_t^i)\lambda_t)dt + (\sigma^A - \varsigma_t^i)dZ_t + \left(j_t^A - (1-\nu_t^i)(1+j_t^A)\right)dJ_t$$

$$\frac{d(\xi_t^i x_t^A)}{\xi_t^i x_t^A} = (-r_t + \mu_t^A - \varsigma_t^i\sigma_t^A - \nu_t^i\lambda_t j_t^A)dt + \underbrace{(\sigma^A - \varsigma_t^i)dZ_t + \left(j_t^A - (1-\nu_t^i)(1+j_t^A)\right)(dJ_t - \lambda_t dt)}_{martingale}$$

  - Expected return: $\mu_t^A + \lambda j_t^A = r_t + \varsigma_t^i\sigma_t^A + \nu_t^i\lambda j_t^A$

1a. Individual Agent Choice of ι, θ, c/n

- Expected return: $\mu_t^A + \lambda j_t^A = r_t^i + \varsigma_t^i \sigma_t^A + \nu_t^i \lambda j_t^A$
  - r_t^i is the shadow risk-free rate (need not to be same across groups)
  - ς_t^i is the price of Brownian risk of agents i,
    $\varsigma_t^i \sigma_t^A$ is the required Brownian risk premium of agents i
  - $\nu_t^i \lambda_t$ is the price of Poisson upside risk if $j^A > 0$
    For risk-neutral agents $\nu_t^i = 0$

- Remark:
  - $dr^{e,K}$ experts return on capital
  - $dr^{h,OE}$ households return on outside equity
  - $dr^{h,D}$ households' return on debt is risky (due to bankruptcy)

1a. Individual Agent Choice of ι, θ, c/n

- Expected return: $\mu_t^A + \lambda j_t^A = r_t^i + \varsigma_t^i \sigma_t^A + \nu_t^i \lambda j_t^A$
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- Remark:
  - For CRRA utility: SDF is $\xi_t = e^{-\rho} \omega_t^{1-\gamma} n_t^{-\gamma}$
    $$1 - \nu_t = (1 + j_t^{\omega})^{1-\gamma} (1 + j_t^n)^{-\gamma}$$
  - For log utility: $\nu_t = 1 - \frac{1}{1+j_t^n} = \frac{j_t^n}{1+j_t^n}$
  - For Epstein-Zin: part of ω_t-process

1a. Individual Agent Choice of ι, θ, c/n

- Of experts with outside equity issuance (after plugging in households' outside equity choice)

$$\frac{a^e-\iota_t}{q_t}+\Phi(\iota_t)-\delta+\mu_t^q+\sigma\sigma_t^q-\left[\frac{\chi_t^e}{\kappa_t^e}r_t^{F,e}+\left(1-\frac{\chi_t^e}{\kappa_t^e}\right)r_t^{F,h}\right]+\lambda_t\, j_t^q=$$
$$\left[\varsigma_t^e\frac{\chi_t^e}{\kappa_t^e}+\varsigma_t^h\left(1-\frac{\chi_t^e}{\kappa_t^e}\right)\right](\sigma+\sigma^q)+\left[\nu_t^e\frac{\chi_t^e}{\kappa_t^e}+\nu_t^h\left(1-\frac{\chi_t^e}{\kappa_t^e}\right)\right]\lambda_t\, j_t^q$$

- Of households' capital choice

$$\frac{a^h-\iota_t}{q_t}+\Phi(\iota_t)-\delta+\mu_t^q+\sigma\sigma_t^q-r_t^{F,h}+\lambda_t(j_t^q-j_t^D)$$
$$\leq\varsigma_t^h(\sigma+\sigma^q)+\nu_t^h\lambda_t(j_t^q-j_t^D)$$

with equality if $\kappa_t^e<1$

- Note: Later approach replaces this step with Fisher Separation Social Planners' choice (see below)

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1b. Asset/Risk Allocation across I Types

- Price-Taking Planner's Theorem:
  A social planner that takes prices as given chooses an physical asset allocation, κ_t, and Brownian risk allocation, χ_t, and a Jump risk allocation, ζ_t, that coincides with the choices implied by all individuals' portfolio choices.

Let $dN_t/N_t = \mu_t^N dt + \sigma_t^N dZ_t + j_t^N dJ_t$

$$\boldsymbol{\varsigma}_t = (\varsigma_t^1, \dots, \varsigma_t^I)$$
$$\boldsymbol{\chi}_t = (\chi_t^1, \dots, \chi_t^I)$$
$$\boldsymbol{\zeta}_t = (\zeta_t^1, \dots, \zeta_t^I)$$
$$\boldsymbol{\sigma}(\boldsymbol{\chi}_t) = (\chi_t^1 \sigma^N, \dots, \chi_t^I \sigma^N)$$
$$\boldsymbol{j}(\boldsymbol{\zeta}_t) = (\zeta_t^1 j_t^N, \dots, \zeta_t^I j_t^N)$$

Return on total wealth

- Planner's problem

$$\max_{\{\boldsymbol{\kappa}_t, \boldsymbol{\chi}_t, \boldsymbol{\zeta}_t\}} \frac{E_t[dr_t^N(\kappa_t)]}{dt} - \boldsymbol{\varsigma}_t \boldsymbol{\sigma}(\boldsymbol{\chi}_t) - \lambda \boldsymbol{\nu} \boldsymbol{j}(\boldsymbol{\zeta}_t)$$

$= dr^F/dt$ in equilibrium if risk free asset is tradable for everyone

subject to friction: $F(\boldsymbol{\kappa}_t, \boldsymbol{\chi}_t, \boldsymbol{\zeta}_t) \leq 0$

- Example:
  1. $\chi_t = \zeta_t = \kappa_t$ (can't issue outside equity to offload Brownian or risky debt to offload Jump risk)
  2. $\chi_t \geq \alpha \kappa_t$ (skin in the game constraint, outside equity up to a limit)

1b. Asset/Risk Allocation across I Types

- Sketch of Proof of Theorem

1. Fisher Separation Thm: (delegated portfolio choice by firm)
   - FOC yield the martingale approach solution
   - Each individual agent $(i, \tilde{\imath})$ portfolio maximization is equivalent to the maximization problem of a firm

$$\max_{\{\boldsymbol{\theta}^{j,i}\}} \frac{E_t\left[dr^{n^{(i,\tilde{\imath})}}\right]}{dt} - \varsigma\sigma^{r^n} - \lambda v^i j^{n^i}(\zeta_t)$$

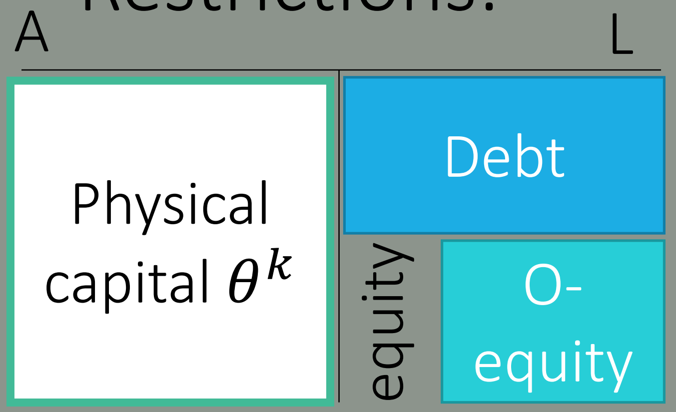
   - $dr^{n^{(i,\tilde{\imath})}} = \sum_j \theta^{j,i} dr^j = \sum_j \theta^{j,i} E[dr^j] + \sum_j \theta^{j,i}(\sigma^j dZ_t + v_t^i j_t^j (dJ_t - \lambda_t dt))$ is linear in θs
     - Either bang-bang solution for θs s.t. portfolio constraints bind
     - Or prices/returns/risk premia are s.t. that firm is indifferent

2. Aggregate
   - Taking η-weighted sum to obtain return on aggregate wealth

3. Use market clearing to relate θs to κs & χs & ζs (incl. θ-constraint)

1b. Allocation of Capital/Risk: 2 Types

- Expert: $\boldsymbol{\theta^e} = (\theta^{e,K}, \theta^{e,OE}, \theta^{e,D})$ for capital, outside equity, debt
  - Restrictions:



$\theta^{e,K} \geq 0,$

$\theta^{e,OE} \leq 0,$ only issue outside equity

$\theta^{e,OE} \geq -(1-\alpha)\theta^{e,K}$ skin in the game

maximize

$$\theta_t^{e,K} E[dr_t^{e,K}]dt + \theta_t^{e,OE}\left(E[dr_t^{OE}]dt\right) + \theta_t^{e,D} E[dr_t^{D,e}] - \varsigma_t^e(\theta_t^{e,K} + \theta_t^{e,OE})\sigma^{r^{e,K}}$$
$$- \lambda_t \nu_t^e\left((\theta_t^{e,K} + \theta_t^{e,OE}) j_t^{r^{eK}} + \theta_t^{e,D} j_t^{r^D}\right)$$

- Household: $\boldsymbol{\theta^h} = (\theta^{h,K}, \theta^{h,OE}, \theta^{h,D})$

$\theta^{h,K} \geq 0$

$\theta^{h,OE} \geq 0$

maximize

$$\theta^{h,K} E[dr_t^{h,K}]/dt + \theta^{h,OE} E[dr_t^{OE}]/dt + \theta^{h,D} E\left[dr_t^{D,h}\right] - \varsigma_t^h(\theta_t^{h,K} + \theta_t^{h,OE})\sigma^{r^{h,K}}$$
$$- \lambda_t \nu_t^h\left((\theta_t^{h,K} + \theta_t^{h,OE}) j_t^{r^{hK}} + \theta_t^{h,D} j_t^{r^D}\right)$$

1b. Allocation of Capital/Risk: 2 Types

- Example 2: 2 Type + with outside equity

$$\max_{\{\kappa_t^e,\chi_t^e\}} \left[\frac{\kappa_t^e a^e + (1-\kappa_t^e)a^h - \iota_t}{q_t} + \Phi(\iota_t) - \delta + \right] - \left(\chi_t^e \varsigma_t^e + (1-\chi_t^e)\varsigma_t^h\right)(\sigma + \sigma_t^q) + \cdots$$

- FOC_χ: Case 1: $\varsigma_t^e(\sigma+\sigma_t^q) + \cdots > \varsigma_t^h(\sigma+\sigma_t^q) + \cdots \Rightarrow \chi_t^e = \alpha\kappa_t^e$

  Case 2: $=$ $\chi_t^e > \alpha\kappa_t^e$

  - Case 1: plug $\chi_t^e = \alpha\kappa_t^e$ in objective

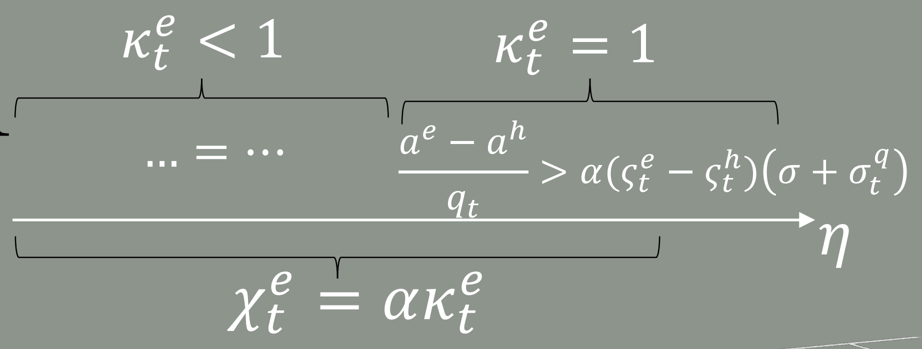
    a. $FOC_\kappa: \frac{a^e - a^h}{q_t} > \alpha(\varsigma_t^e - \varsigma_t^h)(\sigma + \sigma_t^q) + \cdots \Rightarrow \kappa_t^e = 1$

    b. $=$ $\Rightarrow \kappa_t^e < 1$

  - Case 2:

    a. $FOC_\kappa: \frac{a^e - a^h}{q_t} > 0$ $\Rightarrow \kappa_t^e = 1$

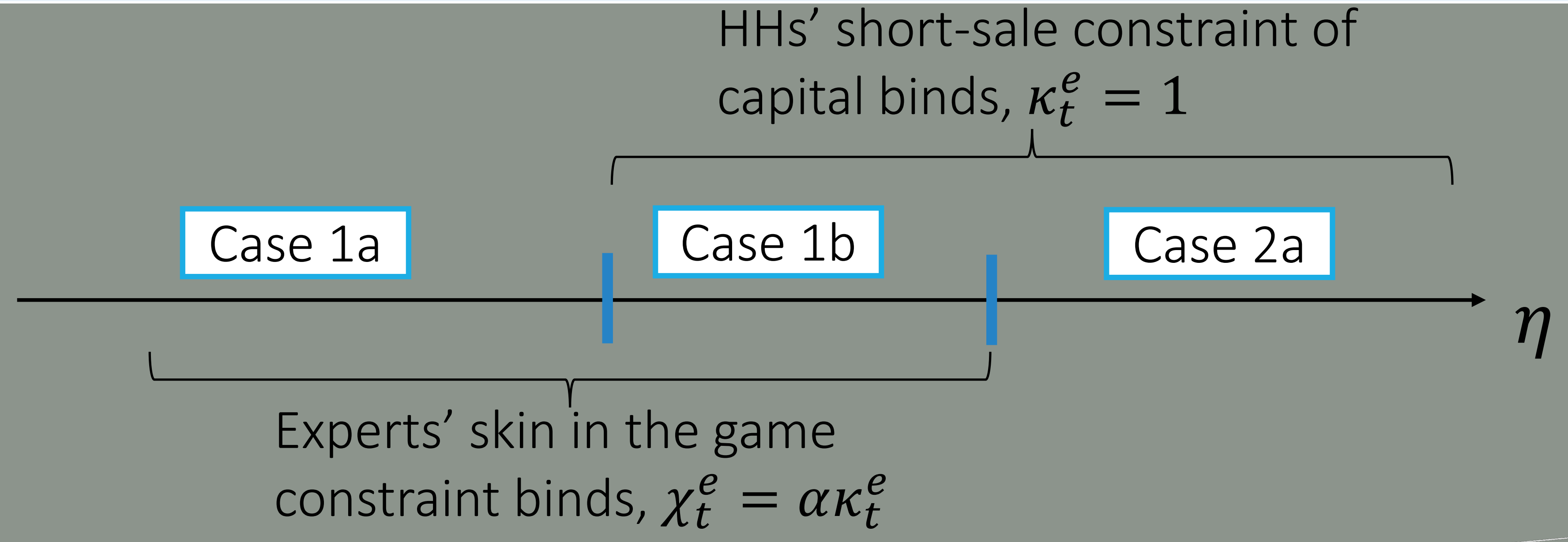
    b. $= 0 \Rightarrow \kappa_t^e < 1$ impossible



1b. Allocation of Capital, κ, and Risk, χ

- Summarizing previous slide (2 types with outside equity)

Cases	$\chi_t^e \geq \alpha\kappa_t^e$	$\kappa_t^e \leq 1$	$\frac{(a^e - a^h)}{q_t} \geq \alpha(\varsigma_t^e - \varsigma_t^h)(\sigma + \sigma_t^q) + \cdots$ Shift a capital unit to expert Benefit: LHS Cost: RHS	$(\varsigma_t^e - \varsigma_t^h)(\sigma + \sigma_t^q) + \cdots \geq 0$ Required risk premium of experts vs. HH
1a	=	<	=	>
1b	=	=	>	>
2a	>	=	>	=
impossible				



Invariance of Relative Capital Demand

- One of the insights of Mendo (2020) is that self-fulfilling jumps do not influence the relative demand for capital of experts relative to households.
  I.e. the excess market return that experts demand to hold capital is not affected.
- Subtract experts pricing condition from households
- $\mu_t^{r^{k,e}} - \mu_t^{r^{k,h}} \geq \frac{\chi_t^e}{\kappa_t^e}(\varsigma_t^e - \varsigma_t^h)(\sigma + \sigma_t^q) - \frac{\chi_t^e}{\kappa_t^e}\lambda_t(1 - \nu_t^h)\underbrace{\left(\frac{\partial j_t^D}{\partial \theta_t^{e,K}}(\theta_t^{e,K} - 1) + j_t^q - j_t^D\right)}_{=0}$
- Losses are split between experts and households (via defaultable debt)
- Since experts' losses are capped by their net worth due to limited liability, all additional losses from increasing capital holding, $\theta_t^{e,K}$, are born by households

Solving MacroModels Step-by-Step

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   - *Toolbox 1:* Martingale Approach, HJB vs. Stochastic Maximum Principle Approach

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   c. Derive C/N-ratio and ς price of risk
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5. KFE: Stationary distribution, Fan charts

Toolbox 3: Change of Numeraire

- x_t^A is a value of a self-financing strategy/asset in \$
- Y_t price of € in \$ (exchange rate)

$$\frac{dY_t}{Y_{t-}} = \mu_t^Y dt + \sigma_t^Y dZ_t + j_t^Y dJ_t$$

- x_t^A / Y_t value of the self-financing strategy/asset in €

$$\underbrace{e^{-\rho t} u'(c_t)}_{=\xi_t} Y_t \frac{x_t^A}{Y_t}$$ follows a martingale (+ SDF in new numeraire $\hat{\xi}_t = \xi_t Y_t$)

Frequency of sunspots, λ_t, are not dependent on numeraire

Recall $\mu_t^A - \mu_t^B + \lambda_t (j_t^A - j_t^B) = \underbrace{(-\sigma_t^\xi)}_{=\varsigma_t} \underbrace{(\sigma^A - \sigma_t^B)}_{risk} + \nu_t \lambda_t (j_t^A - j_t^B)$

$$\mu_t^{\frac{A}{Y}} - \mu_t^{\frac{B}{Y}} + \lambda_t \left(j_t^{\frac{A}{Y}} - j_t^{\frac{B}{Y}} \right) = \underbrace{\left(-\sigma_t^\xi - \sigma_t^Y\right)}_{price\ of\ risk} \underbrace{\left(\sigma^A - \sigma_t^B\right)}_{risk} + (\nu_t - j_t^Y - \nu_t j_t^Y)\, \lambda_t \frac{j_t^A - j_t^B}{1 + j_t^Y}$$

- Price of Brownian risk $\varsigma^€ = \varsigma^\$ - \sigma^Y$
- Price of Jump risk $\nu_t^€ = \nu_t^\$ - j_t^Y - \nu_t j_t^Y$

Change of Numeraire: SDF

- SDF in good numeraire is

$$d\xi_t^i/\xi_{t-}^i = -r_t^{F,i}dt - \varsigma_t^i dZ_t - \nu_t^i\,(dJ_t - \lambda_t dt)$$

- SDF in total net worth numeraire is

$$\begin{aligned} d\hat{\xi}_t^i/\hat{\xi}_{t-}^i &= \mu_t^{\hat{\xi}^i} dt - \left(\varsigma_t^i - \sigma_t^N\right)dZ_t - \left(\nu_t^i - j_t^N - \nu_t j_t^N\right)dJ_t \\ &= \hat{r}_t^{F,i} dt - \underbrace{\left(\varsigma_t^i - \sigma_t^N\right)}_{=\hat{\varsigma}_t^i} dZ_t - \left(\nu_t^i - j_t^N - \nu_t j_t^N\right)(dJ_t - \lambda_t dt) \end{aligned}$$

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2. GE: Markov States and Equilibria

- Equilibrium is a map

Histories of shocks - - - - → prices $q_t, \varsigma_t^i, \iota_t^i, \theta_t^e$

$\{\boldsymbol{Z}_s, s \in [0, t]\}$

net worth distribution

$$\eta_t^e = \frac{N_t^e}{q_t K_t} \in (0,1)$$

net worth share

- All agents maximize utility
  - Choose: portfolio, consumption, technology
- All markets clear
  - Consumption, capital, money, outside equity

2. Law of Motion of Wealth Share η_t

- Method 1: Using Ito's quotation rule $\eta_t^i = N_t^i/(q_t K_t)$
  - Recall

$$\frac{dN_t^i}{N_t^i} = -\frac{C_t^i}{N_t^i}dt + r_t^{bm}dt + \underbrace{\varsigma_t^i}_{price\ of\ risk}\underbrace{\left(\frac{\chi_t^i\kappa_t^i}{\eta_t^i}(\sigma+\sigma_t^q) - \sigma^{bm}\right)}_{excess\ risk}dt + +\nu\left(j_t^{N^i} - j_t^{bm}\right)dt$$

$$+\frac{\chi_t^i\kappa_t^i}{\eta_t^i}(\sigma+\sigma_t^q)dZ_t + \left(j_t^{N^i} - j_t^{bm}\right)dJ_t + \tau^i K_t/N_t^i dJ_t$$

bm = benchmark asset (tradable by everyone)

I ignored OLG terms for now

Transfers in case of Jump

  - $\frac{d\eta_t^i}{\eta_t^i} = \ldots$ (lots of algebra)

- Method 2: Change of numeraire + Martingale Approach
  - New numeraire: Total wealth in the economy, N_t
  - Apply Martingale Approach for value of i's portfolio
    - Simple algebra to obtain drift of η_t^i: $\mu_t^{\eta^i}$
      Note that change of numeraire does not affect ratio η^i!

2. μ^{η} Drift of Wealth Share: Many Types

- New Numeraire
  - **"Total net worth"** in the economy, N_t (without superscript)
  - Type i's portfolio net worth = net worth share

- Martingale Approach with new numeraire
  - Asset A $= i$'s portfolio return in terms of total wealth,

$$\left(\frac{C_t^i}{N_t^i} + \mu_t^{\eta^i} + \lambda_t j^{\eta^i}\right) dt + \sigma_t^{\eta^i} dZ_t$$

Dividend yield — E[capital gains] rate

  - Asset B (benchmark asset that everyone can hold, e.g. risk-free asset or money (in terms of total economy wide wealth as numeraire))

$$r_t^{bm} dt + \sigma_t^{bm} dZ_t$$

  - Apply our martingale asset pricing formula

$$\mu_t^A - \mu_t^B + \lambda_t (j_t^A - j_t^B) = \hat{\varsigma}_t^i (\sigma_t^A - \sigma_t^B) + \hat{\nu}_t (j_t^A - j_t^B)$$

Hat notation $\hat{}$ indicates total net worth numeraire

2. μ^{η} Drift of Wealth Share: Many Types

- Asset pricing formula (relative to benchmark asset)

$$\mu_t^{\eta^i} + \frac{C_t^i}{N_t^i} - r_t^{bm} + \lambda_t\left(j_t^{\eta^i} - j_t^{bm}\right) = \left(\varsigma_t^i - \sigma_t^N\right)\left(\sigma_t^{\eta^i} - \sigma_t^{bm}\right) + \hat{\nu}_t^i\left(j_t^{\eta^i} - j_t^{bm}\right)$$

due to change in numeraire

- Add up across types (weighted),
  (capital letters without superscripts are aggregates for total economy)

$$\underbrace{\sum_{i'}^{I} \eta_t^{i'} \mu_t^{\eta^{i'}}}_{=0} + \frac{C_t}{N_t} - r_t^{bm} + \underbrace{\sum_{i'}^{I} \eta_t^{i'} j_t^{\eta^{i'}}}_{=0} - \lambda_t d j_t^{bm} =$$

$$\sum_{i'} \eta_t^{i'} \varsigma_t^{i'}\left(\sigma_t^{\eta^{i'}} - \sigma_t^{bm}\right) + \sum_{i'} \eta_t^{i'} \hat{\nu}_t^{i'}\left(j_t^{\eta^{i'}} - j_t^{bm}\right)$$

- Subtract from first equation

$$\mu_t^{\eta^i} + \lambda_t j_t^{\eta^{i'}} = \frac{C_t}{N_t} - \frac{C_t^i}{N_t^i} - \varsigma_t^i\left(\sigma^{\eta^i} - \sigma^{bm}\right) - \sum_{i'} \eta_t^{i'} \varsigma_t^{i'}\left(\sigma_t^{\eta^{i'}} - \sigma_t^m\right)$$

$$+\hat{\nu}_t^i\left(j_t^{\eta^i} - j_t^{bm}\right) - \sum_{i'} \eta_t^{i'} \hat{\nu}_t^{i'}\left(j_t^{\eta^{i'}} - j_t^{bm}\right)$$

2. μ^η Drift of Wealth Share: Two Types $i \in \{e, h\}$

- Subtract from each other yield net worth share dynamics

$$\mu_t^{\eta^e} + \lambda_t j_t^{\eta^e} = \frac{C_t}{N_t} - \frac{C_t^e}{N_t^e} - (1-\eta_t^e)\hat{\varsigma}_t^e\left(\sigma_t^{\eta^e} - \sigma_t^{bm}\right) - (1-\eta_t^e)\hat{\varsigma}_t^h\left(\sigma_t^{\eta^h} - \sigma_t^{bm}\right)$$
$$+(1-\eta_t^e)\hat{\nu}_t^e\left(j_t^{\eta^e} - j_t^{bm}\right) - (1-\eta_t^e)\hat{\nu}_t^h\left(j_t^{\eta^h} - j_t^{bm}\right)$$

- In in our model, benchmark asset is risky debt,
  - $\sigma_t^{bm} = -\sigma_t^N$,
  - $j_t^{bm} = \frac{j^D - j^N}{1+j^N}$ (since j_t^D risky debt jump in c-numeraire, j_t^N wealth jump)
    - Apply quotient rule for jumps
- $\mu_t^{\eta^e} + \lambda_t j_t^{\eta^e} = \frac{C_t}{N_t} - \frac{C_t^e}{N_t^e} - (1-\eta_t^e)\hat{\varsigma}_t^e\left(\sigma_t^{\eta^e} + \sigma_t^{N}\right) - (1-\eta_t^e)\hat{\varsigma}_t^h\left(\sigma_t^{\eta^h} + \sigma_t^{bm}\right)$

$$+(1-\eta_t^e)\hat{\nu}_t^e\left(j_t^{\eta^e} - \frac{j^D - j^N}{1+j^N}\right) - (1-\eta_t^e)\hat{\nu}_t^h\left(j_t^{\eta^h} - \frac{j^D - j^N}{1+j^N}\right)$$

2. σ^η Volatility of Wealth Share

- Since $\eta_t^i = N_t^i / N_t$,

$$\sigma_t^{\eta^i} = \sigma_t^{N^i} - \sigma_t^N = \sigma_t^{N^i} - \sum_{i'} \eta_t^{i'} \sigma_t^{N^{i'}}$$

$$= \left(1 - \eta_t^i\right) \sigma_t^{N^i} - \sum_{i^- \neq i} \eta_t^{i^-} \sigma_t^{N^{i^-}}$$

$$j_t^{\eta^i} = \frac{j_t^{N^i} - j_t^N}{1 + j_t^N} = \frac{j_t^{N^i} - \sum_{i'} \eta_t^{i'} j_t^{N^{i'}}}{1 + \sum_{i'} \eta_t^{i'} j_t^{N^{i'}}} = \frac{\left(1 - \eta_t^i\right) j_t^{N^i} - \sum_{i^- \neq i} \eta_t^{i^-} j_t^{N^{i^-}}}{1 + \sum_{i'} \eta_t^{i'} j_t^{N^{i'}}}$$

- Note for 2 types example

$$j_t^{\eta^e} = \frac{(1 - \eta_t^e)(j_t^{N^e} - j_t^{N^h})}{1 + \eta_t^e j_t^{N^e} + (1 - \eta_t^e) j_t^{N^h}}$$

- …

- Note:
  - OLG structure and
  - transfers τK_t

  also affects net worth evolution and still has to be incorporated!

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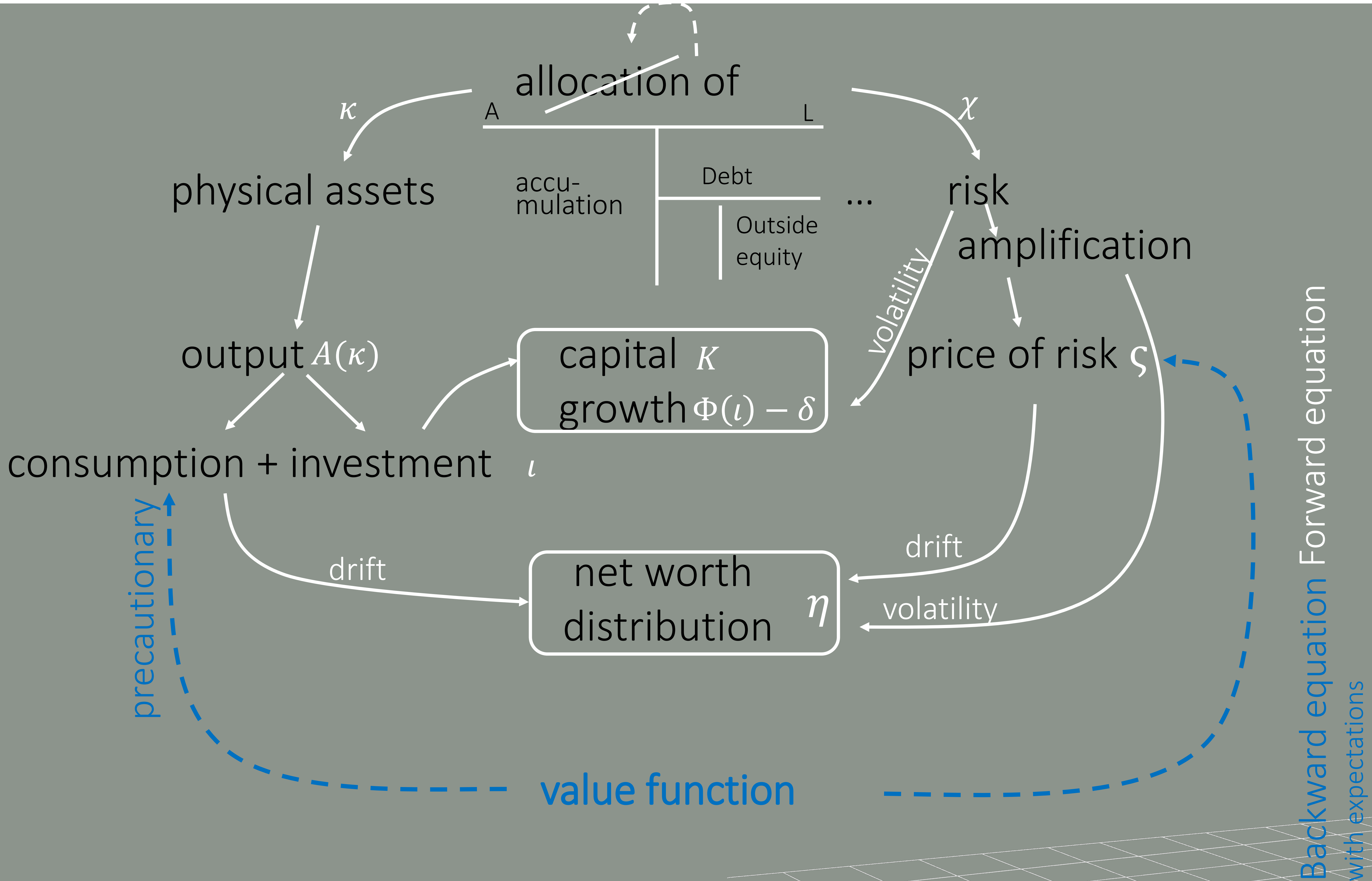
The Big Picture



3a. CRRA Value Function Applies separately for each type of agent

- Martingale Approach: works best in endowment economy
- Here: mix Martingale approach with value function (envelop condition)

recursive utility

$$U_t = E_t\left[\int_t^\infty f(c_s, U_s)\,ds\right]$$

$$f(c,U) = (1-\gamma)\rho U\left(\log(c) - \frac{1}{1-\gamma}\log((1-\gamma)U)\right)$$

- $V^i(n_t^i; \boldsymbol{\eta}_t, K_t)$ for individuals i
- For CRRA/power utility $u(c_t^i) = \frac{(c_t^i)^{1-\gamma}-1}{1-\gamma}$

$\Rightarrow$ increase net worth by factor, optimal c^i for all future states increases by this factor $\Rightarrow \left(\frac{c_t^i}{n_t^i}\right)$-ratio is invariant in n_t^i

- $\Rightarrow$ value function can be written as $V^i(n_t^i; \boldsymbol{\eta}_t, K_t) = \frac{u(\omega^i(\boldsymbol{\eta}_t, K_t) n_t^i)}{\rho^i}$
- ω_t^i Investment opportunity/ "net worth multiplier"
  - $\omega^i(\boldsymbol{\eta}_t, K_t)$-function turns out to be independent of K_t
  - Change notation from $\omega^i(\boldsymbol{\eta}_t, K_t)$-function to ω_t^i-process

3a. Special case: log utility

- Result: $q(\eta^e)$-function is invariant to run risk, i.e. same as in Lecture 03.
  - … but expected returns are different.
- Proof (sketch)
  - Log utility implies, prices of risk:
    - $\varsigma_t^i = \sigma_t^{n^i}$
    - $\lambda_t \nu_t^i = \lambda_t / (1 + j_t^{n^i})$
  - Goods market clearing
  - Brownian amplification equation

$$\sigma + \sigma_t^q = \frac{\sigma}{1 - \frac{q'}{q}(\kappa - \eta)}$$

  - Relative asset pricing equation

$$\frac{a^e - a^h}{q_t} \geq \left(\frac{\kappa_t}{\eta_t} - \frac{1-\kappa_t}{1-\eta_t}\right)\left(\sigma + \sigma_t^q\right)^2$$

3a. Value function in OLG setting

- Note: with OLG structure we have to take care that individual value function differs from sector wide.
- $V_t^i = \frac{1}{\rho^i}\frac{\left(\omega_t^i n_t^i\right)^{1-\gamma}}{1-\gamma} = \frac{1}{\rho^i}\frac{\left(\omega_t^i \eta_t^{i,\tilde{\imath}} N^i\right)^{1-\gamma}}{1-\gamma}$
- Where $\eta_t^{i,\tilde{\imath}}$ is the net worth share of individual $(\tilde{\imath}, i)$ within sector i
- It is time-varying deterministically, and hence does not affect asset pricing.

3a. CRRA Value Function: relate to ω

- $\Rightarrow$ value function can be written as $\frac{u\left(\omega_t^i n_t^i\right)}{\rho}$, that is

$$= \frac{1}{\rho^i} \frac{\left(\omega_t^i n_t^i\right)^{1-\gamma} - 1}{1-\gamma} = \frac{1}{\rho^i} \frac{\left(\omega_t^i\right)^{1-\gamma} \left(n_t^i\right)^{1-\gamma} - 1}{1-\gamma}$$

- $\frac{\partial V}{\partial n^i} = u'(c^i)$ by optimal consumption (if no corner solution)

$$\frac{\left(\omega_t^i\right)^{1-\gamma} \left(n_t^i\right)^{-\gamma}}{\rho^i} = (c_t^i)^{-\gamma} \Leftrightarrow \frac{c_t^i}{n_t^i} = (\rho^i)^{1/\gamma} (\omega_t^i)^{1-1/\gamma}$$

Optimal consumption is different:

$$\omega^{1-\gamma} n^{-\gamma} = \frac{\partial V}{\partial n} = \frac{\partial f}{\partial c} = \rho(\omega n)^{1-\gamma} \frac{1}{c}$$

$$\Longrightarrow \frac{c}{n} = \rho$$

3a. CRRA Value Function: relate to ω

- $\Rightarrow$ value function can be written as $\frac{u\left(\omega_t^i n_t^i\right)}{\rho}$, that is

$$= \frac{1}{\rho^i} \frac{\left(\omega_t^i n_t^i\right)^{1-\gamma} - 1}{1-\gamma} = \frac{1}{\rho^i} \frac{\left(\omega_t^i\right)^{1-\gamma}\left(n_t^i\right)^{1-\gamma} - 1}{1-\gamma}$$

- SDF now

$$\xi_t = e^{\int_0^t \frac{\partial f}{\partial V}(c_s, V_s)ds} \frac{\partial V}{\partial n} = e^{\int_0^t \frac{\partial f}{\partial V}(c_s, V_s)ds} \omega_t^{1-\gamma} n_t^{-\gamma}$$

- Get new discounting term

$$e^{-\int_0^t \frac{\partial f}{\partial V}(c_s, V_s)ds} \xi_t n_t = (1-\gamma) V_t$$

$$\Longrightarrow E_t[dV_t]/V_t = (-\partial f/\partial V_t - c_t/n_t)dt$$

3a. CRRA Value Function: Special Cases

$$\frac{c_t^i}{n_t^i} = (\rho^i)^{1/\gamma}(\omega_t^i)^{1-1/\gamma}$$

- For log utility $\gamma = 1$:
  $\xi_t^i = e^{-\rho^i t}/c_t^i = e^{-\rho^i t}/(\rho n_t^i)$ for any $\omega_t^i \Rightarrow \sigma_t^{n^i} = \sigma_t^{c^i} = \varsigma_t^i$
  - Expected excess return: $\mu_t^A - r_t^F = \sigma_t^{n^i}\sigma_t^A$
  - Recall $\frac{dn_t^i}{n_t^i} = -\frac{c_t^i}{n_t^i}dt + \left(1-\theta^i\right)dr_t^K + \theta^i dr_t$

3a. CRRA Value Function: Special Cases

$$\frac{c_t^i}{n_t^i} = (\rho^i)^{1/\gamma}(\omega_t^i)^{1-1/\gamma}$$

- For log utility $\gamma = 1$:
  $\xi_t^i = e^{-\rho^i t}/c_t^i = e^{-\rho^i t}/(\rho n_t^i)$ for any $\omega_t^i \Rightarrow \sigma_t^{n^i} = \sigma_t^{c^i} = \varsigma_t^i$
  - Expected excess return: $\mu_t^A - r_t^F = \sigma_t^{n^i}\sigma_t^A$
  - Recall $\frac{dn_t^i}{n_t^i} = -\frac{c_t^i}{n_t^i}dt + \left(1-\theta^i\right)dr_t^K + \theta^i dr_t$
- For constant investment opportunities $\omega_t^i = \omega^i$,
  $\Rightarrow c^i/n^i$ is constant and hence $\sigma_t^{c^i} = \sigma^{n^i}$
  - Expected excess return: $\mu_t^A - r_t^F = \gamma\sigma_t^{n^i}\sigma_t^A$

Poll 49: Which term refers to (dynamic/Mertonian) hedging demand?

a) γ
b) σ_t^n
c) hidden in risk-free rate
d) none of the above

Solving MacroModels Step-by-Step

0. Postulate aggregates, price processes & obtain return processes
1. For given C/N-ratio and SDF processes for each i *finance block*
   a. Real investment ι + Goods market clearing *(static)*
   - *Toolbox 1:* Martingale Approach, HJB vs. Stochastic Maximum Principle Approach

   b. Portfolio choice θ + Asset market clearing or
   Asset allocation κ & risk allocation χ
   - *Toolbox 2:* "price-taking social planner approach" – Fisher separation theorem
   - *Toolbox 3:* Change in numeraire to total wealth (including SDF)
2. Evolution of state variable η (and K) *forward equation*
3. Value functions *backward equation*
   a. Value fcn. as fcn. of individual investment opportunities ω
   - *Special cases: log-utility, constant investment opportunities*

   b. Separating value fcn. $V^i(n^{\tilde{i}};\eta,K)$ into $v^i(\eta)u(K)$
   c. Derive C/N-ratio and ς price of risk
4. Numerical model solution
   a. Transform BSDE for separated value fcn. $v^i(\eta)$ into PDE
   b. Solve PDE via value function iteration
5. KFE: Stationary distribution, Fan charts