Financial and Monetary Economics

Eco529 Fall 2020

Lecture 03: Endogenous Risk Dynamics

Markus K. Brunnermeier
Princeton University

Princeton, September 2020

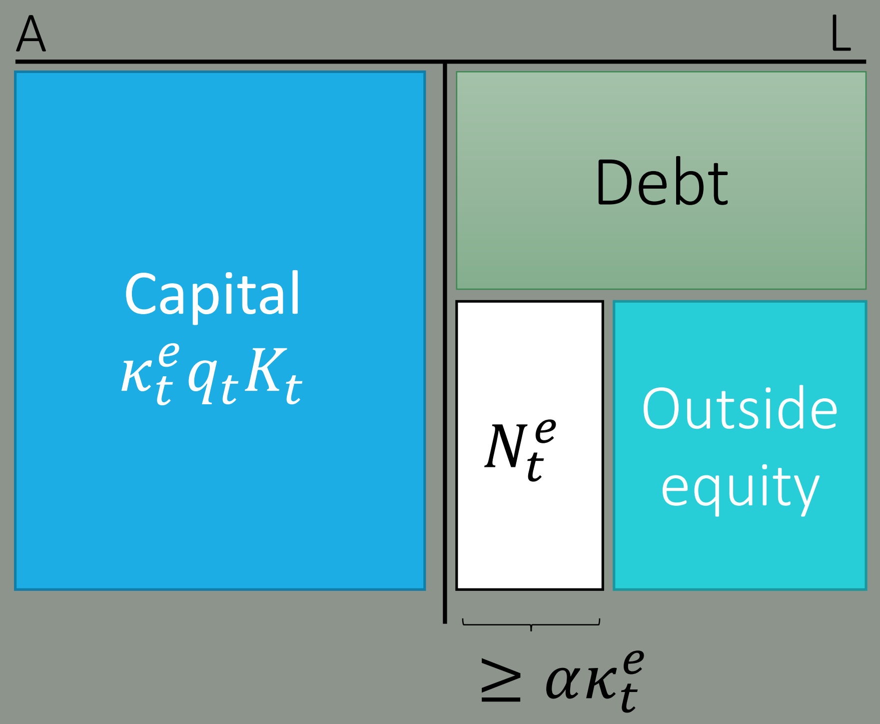
Desired Model Properties

- Normal regime: stable around steady state
  - Experts are adequately capitalized
  - Experts can absorb macro shock
- Endogenous risk and price of risk
  - Fire-sales, liquidity spirals, fat tails
  - Spillovers across assets and agents
  - Market and funding liquidity connection
  - SDF vs. cash-flow news
- Volatility paradox
- Financial innovation less stable economy
- ("Net worth trap" double-humped stationary distribution)

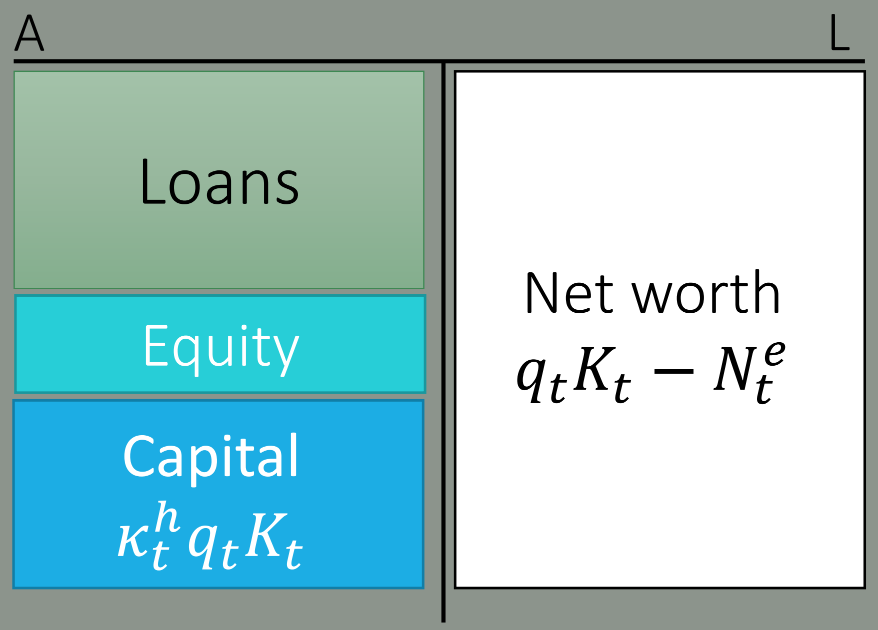
Two Type/Sector Model with Outside Equity

BruSan 2017:
Handbook of Macroeconomics,
Lecture Notes, Chatper 3

- Expert sector



Household sector



- Experts must hold fraction $\chi_t^e \geq \alpha\kappa_t^e$ (skin in the game constraint)
- Return on inside equity N_t can differ from outside equity
  - Issue outside equity at required return from HH
  - In related model, He and Krishnamurthy 2013 impose that inside and outside equity have same return

Two Type Model Setup

Expert sector

- Output: $y_t^e = a^e k_t^e$ $\quad a^e \geq a^h$

Household sector

- Output: $y_t^h = a^h k_t^h$

$$A(\boldsymbol{\kappa}) = \kappa^e a^e + \kappa^h a^h$$

Capital share
of experts

Poll 4: Why is it important that households can hold capital?

a) to capture fire-sales
b) for households to speculate
c) to obtain stationary distribution

Two Type Model Setup

Expert sector

- Output: $y_t^e = a^e k_t^e$ $a^e \geq a^h$

Household sector

- Output: $y_t^h = a^h k_t^h$

$$\underset{\substack{\uparrow \\ \text{Capital share of experts}}}{A(\boldsymbol{\kappa})} = \kappa^e a^e + \kappa^h a^h$$

Poll 5: What are the modeling tricks to obtain stationary distribution?

a) switching types

b) agents die, OLG/perpetual youth models (without bequest motive)

c) different preference discount rates

Two Type Model Setup

Expert sector

- Output: $y_t^e = a^e k_t^e$ $a^e \geq a^h$
- Consumption rate: c_t^e
- Investment rate: ι_t^e

$$\frac{dk_t^{\tilde{\imath},e}}{k_t^{\tilde{\imath},e}} = \left(\Phi\left(\iota_t^{\tilde{\imath},e}\right) - \delta\right)dt + \sigma dZ_t + \tilde{\sigma} d\tilde{Z}_t^{\tilde{\imath}}$$

Household sector

- Output: $y_t^h = a^h k_t^h$
- Consumption rate: c_t^h
- Investment rate: ι_t^h

$$\frac{dk_t^{\tilde{\imath},h}}{k_t^{\tilde{\imath},h}} = \left(\Phi\left(\iota_t^{\tilde{\imath},h}\right) - \delta\right)dt + \sigma dZ_t + \tilde{\sigma} d\tilde{Z}_t^{\tilde{\imath}}$$

Physical capital evolution absent market transactions/fire-sales

Two Type Model Setup

Expert sector

- Output: $y_t^e = a^e k_t^e$ $a^e \geq a^h$
- Consumption rate: c_t^e
- Investment rate: ι_t^e

$$\frac{dk_t^{\tilde{\iota},e}}{k_t^{\tilde{\iota},e}} = \left(\Phi\left(\iota_t^{\tilde{\iota},e}\right) - \delta\right) dt + \sigma dZ_t + \tilde{\sigma} d\tilde{Z}_t^{\tilde{\iota}}$$

- $E_0[\int_0^\infty e^{-\rho^e t} \frac{(c_t^e)^{1-\gamma}}{1-\gamma} dt]$ $\rho^e \geq \rho^h$

Household sector

- Output: $y_t^h = a^h k_t^h$
- Consumption rate: c_t^h
- Investment rate: ι_t^h

$$\frac{dk_t^{\tilde{\iota},h}}{k_t^{\tilde{\iota},h}} = \left(\Phi\left(\iota_t^{\tilde{\iota},h}\right) - \delta\right) dt + \sigma dZ_t + \tilde{\sigma} d\tilde{Z}_t^{\tilde{\iota}}$$

- $E_0[\int_0^\infty e^{-\rho^h t} \frac{(c_t^h)^{1-\gamma}}{1-\gamma} dt]$

Two Type Model Setup

Expert sector

- Output: $y_t^e = a^e k_t^e$ $a^e \geq a^h$
- Consumption rate: c_t^e
- Investment rate: ι_t^e

$$\frac{dk_t^{\tilde{\imath},e}}{k_t^{\tilde{\imath},e}} = \left(\Phi\left(\iota_t^{\tilde{\imath},e}\right) - \delta\right)dt + \sigma dZ_t + \tilde{\sigma} d\tilde{Z}_t^{\tilde{\imath}}$$

- $E_0[\int_0^\infty e^{-\rho^e t}\frac{(c_t^e)^{1-\gamma}}{1-\gamma}dt]$ $\rho^e \geq \rho^h$

Household sector

- Output: $y_t^h = a^h k_t^h$
- Consumption rate: c_t^h
- Investment rate: ι_t^h

$$\frac{dk_t^{\tilde{\imath},h}}{k_t^{\tilde{\imath},h}} = \left(\Phi\left(\iota_t^{\tilde{\imath},h}\right) - \delta\right)dt + \sigma dZ_t + \tilde{\sigma} d\tilde{Z}_t^{\tilde{\imath}}$$

- $E_0[\int_0^\infty e^{-\rho^h t}\frac{(c_t^h)^{1-\gamma}}{1-\gamma}dt]$

Friction: Can only issue

- Risk-free debt
- Equity, but most hold $\chi_t^e \geq \alpha\kappa_t$

Solving MacroModels Step-by-Step

0. Postulate aggregates, price processes & obtain return processes
1. For given C/N-ratio and SDF processes for each i *finance block*
    a. Real investment ι + Goods market clearing *(static)*
    - *Toolbox 1:* Martingale Approach, HJB vs. Stochastic Maximum Principle Approach

    b. Portfolio choice θ + Asset market clearing or
    Asset allocation κ & risk allocation χ
    - *Toolbox 2:* "price-taking social planner approach" – Fisher separation theorem

    - *Toolbox 3:* Change in numeraire to total wealth (including SDF)
2. Evolution of state variable η (and K) *forward equation*
3. Value functions *backward equation*
    a. Value fcn. as fcn. of individual investment opportunities ω
    - *Special cases: log-utility, constant investment opportunities*

    b. Separating value fcn. $V^i(n^{\tilde{i}};\eta,K)$ into $v^i(\eta)u(K)$
    c. Derive C/N-ratio and ς price of risk
4. Numerical model solution
    a. Transform BSDE for separated value fcn. $v^i(\eta)$ into PDE
    b. Solve PDE via value function iteration
5. KFE: Stationary distribution, Fan charts

0. Postulate Aggregates and Processes

- Individual capital evolution:

$$\frac{dk_t^{\tilde{\imath},i}}{k_t^{\tilde{\imath},i}} = \left(\Phi\left(\iota^{\tilde{\imath},i}\right) - \delta\right)dt + \sigma dZ_t + d\Delta_t^{k,\tilde{\imath},i}$$

  - Where $\Delta_t^{k,\tilde{\imath},i}$ is the individual cumulative capital purchase process

(c is numeraire)

0. Postulate Aggregates and Processes

- Individual capital evolution:

$$\frac{dk_t^{\tilde{\imath},i}}{k_t^{\tilde{\imath},i}} = \left(\Phi\left(\iota^{\tilde{\imath},i}\right) - \delta\right)dt + \sigma dZ_t + d\Delta_t^{k,\tilde{\imath},i}$$

  - Where $\Delta_t^{k,\tilde{\imath},i}$ is the individual cumulative capital purchase process
- Capital aggregation:
  - Within sector i: $K_t^i \equiv \int k_t^{\tilde{\imath},i} d\tilde{\imath}$
  - Across sectors: $K_t \equiv \sum_i K_t^i$
  - Capital share: $\kappa_t^i \equiv K_t^i / K_t$

$$\frac{dK_t}{K_t} = \left(\Phi\left(\iota_t^i\right) - \delta\right)dt + \sigma dZ_t$$

(c is numeraire)

0. Postulate Aggregates and Processes

- Individual capital evolution:

$$\frac{dk_t^{\tilde{\imath},i}}{k_t^{\tilde{\imath},i}} = \left(\Phi\left(\iota^{\tilde{\imath},i}\right) - \delta\right)dt + \sigma dZ_t + d\Delta_t^{k,\tilde{\imath},i}$$

  - Where $\Delta_t^{k,\tilde{\imath},i}$ is the individual cumulative capital purchase process
- Capital aggregation:
  - Within sector i: $K_t^i \equiv \int k_t^{\tilde{\imath},i} d\tilde{\imath}$
  - Across sectors: $K_t \equiv \sum_i K_t^i$
  - Capital share: $\kappa_t^i \equiv K_t^i / K_t$

$$\frac{dK_t}{K_t} = \left(\Phi\left(\iota_t^i\right) - \delta\right)dt + \sigma dZ_t$$

- Net worth aggregation:
  - Within sector i: $N_t^i \equiv \int n_t^{\tilde{\imath},i} d\tilde{\imath}$
  - Across sectors: $N_t \equiv \sum_i N_t^i$
  - Wealth share: $\eta_t^i \equiv N_t^i / N_t$

(c is numeraire)

0. Postulate Aggregates and Processes

- Individual capital evolution:

$$\frac{dk_t^{\tilde{\iota},i}}{k_t^{\tilde{\iota},i}} = \left(\Phi\left(\iota^{\tilde{\iota},i}\right) - \delta\right)dt + \sigma dZ_t + d\Delta_t^{k,\tilde{\iota},i}$$

  - Where $\Delta_t^{k,\tilde{\iota},i}$ is the individual cumulative capital purchase process
- Capital aggregation:
  - Within sector i: $K_t^i \equiv \int k_t^{\tilde{\iota},i} d\tilde{\iota}$
  - Across sectors: $K_t \equiv \sum_i K_t^i$
  - Capital share: $\kappa_t^i \equiv K_t^i / K_t$

$$\frac{dK_t}{K_t} = \left(\Phi\left(\iota_t^i\right) - \delta\right)dt + \sigma dZ_t$$

- Net worth aggregation:
  - Within sector i: $N_t^i \equiv \int n_t^{\tilde{\iota},i} d\tilde{\iota}$
  - Across sectors: $N_t \equiv \sum_i N_t^i$
  - Wealth share: $\eta_t^i \equiv N_t^i / N_t$
- Value of capital stock: $q_t K_t$

  Postulate $dq_t/q_t = \mu_t^q dt + \sigma_t^q dZ_t$

Poll 13: How many Brownian motions span prob. space?

a) one

b) two

c) one + number of sectors

d) two + number of sectors

(c is numeraire)

0. Postulate Aggregates and Processes

- Individual capital evolution:

$$\frac{dk_t^{\tilde{\imath},i}}{k_t^{\tilde{\imath},i}} = \left(\Phi\left(\iota^{\tilde{\imath},i}\right) - \delta\right)dt + \sigma dZ_t + d\Delta_t^{k,\tilde{\imath},i}$$

  - Where $\Delta_t^{k,\tilde{\imath},i}$ is the individual cumulative capital purchase process
- Capital aggregation:
  - Within sector i: $K_t^i \equiv \int k_t^{\tilde{\imath},i} d\tilde{\imath}$
  - Across sectors: $K_t \equiv \sum_i K_t^i$
  - Capital share: $\kappa_t^i \equiv K_t^i / K_t$

$$\frac{dK_t}{K_t} = \left(\Phi\left(\iota_t^i\right) - \delta\right)dt + \sigma dZ_t$$

- Net worth aggregation:
  - Within sector i: $N_t^i \equiv \int n_t^{\tilde{\imath},i} d\tilde{\imath}$
  - Across sectors: $N_t \equiv \sum_i N_t^i$
  - Wealth share: $\eta_t^i \equiv N_t^i / N_t$
- Value of capital stock: $q_t K_t$

  Postulate $dq_t / q_t = \mu_t^q dt + \sigma_t^q dZ_t$

Same Brownian

(c is numeraire)

0. Postulate Aggregates and Processes

- Individual capital evolution:

$$\frac{dk_t^{\tilde{\imath},i}}{k_t^{\tilde{\imath},i}} = \left(\Phi\left(\iota^{\tilde{\imath},i}\right) - \delta\right)dt + \sigma dZ_t + d\Delta_t^{k,\tilde{\imath},i}$$

  - Where $\Delta_t^{k,\tilde{\imath},i}$ is the individual cumulative capital purchase process
- Capital aggregation:
  - Within sector i: $K_t^i \equiv \int k_t^{\tilde{\imath},i} d\tilde{\imath}$
  - Across sectors: $K_t \equiv \sum_i K_t^i$
  - Capital share: $\kappa_t^i \equiv K_t^i / K_t$

$$\frac{dK_t}{K_t} = \left(\Phi\left(\iota_t^i\right) - \delta\right)dt + \sigma dZ_t$$

- Net worth aggregation:
  - Within sector i: $N_t^i \equiv \int n_t^{\tilde{\imath},i} d\tilde{\imath}$
  - Across sectors: $N_t \equiv \sum_i N_t^i$
  - Wealth share: $\eta_t^i \equiv N_t^i / N_t$
- Value of capital stock: $q_t K_t$

  Postulate $dq_t/q_t = \mu_t^q dt + \sigma_t^q dZ_t$
- Postulated SDF-process: $\frac{d\xi_t^i}{\xi_t^i} = \underbrace{\mu_t^{\xi}}_{\equiv -r_t} dt + \underbrace{\sigma_t^{\xi^i}}_{\equiv -\varsigma_t^i} dZ_t$ (c is numeraire)

0. Postulate Aggregates and Processes

- … from price processes to return processes (using Ito)
  - Use Ito product rule to obtain capital gain rate (in absence of purchases/sales)
    - Define $\check{k}_t^{\tilde{\imath}}$: $\frac{d\check{k}_t^{\tilde{\imath},i}}{\check{k}_t^{\tilde{\imath},i}} = \left(\Phi\left(\iota_t^{\tilde{\imath},i}\right) - \delta\right) dt + \sigma dZ_t \cancel{+ d\Delta_t^{k,\tilde{\imath},i}}$ without purchases/sales

$$dr_t^k\left(\iota_t^{\tilde{\imath},i}\right) = \left(\overbrace{\frac{a^i - \iota_t^i}{q}}^{\text{Dividend yield}} + \overbrace{\Phi(\iota_t^i) - \delta + \mu_t^q + \sigma\sigma_t^q}^{\text{E[Capital gain rate]}= \frac{d(q_t\check{k}^i{}_t)}{(q_t\check{k}_t^i)}}\right) dt + \left(\sigma + \sigma_t^q\right) dZ_t$$

For aggregate capital return, Replace a^i with $A(\kappa)$

- Postulate SDF-process: (Example: $\xi_t^i = e^{-\rho t} V'(n_t^i)$.)

$$\frac{d\xi_t^i}{\xi_t^i} = -r_t dt - \underset{\substack{\uparrow \\ \text{Price of risk}}}{\varsigma_t^i} dZ_t$$

Recall discrete time $e^{-r^F} = E[SDF]$

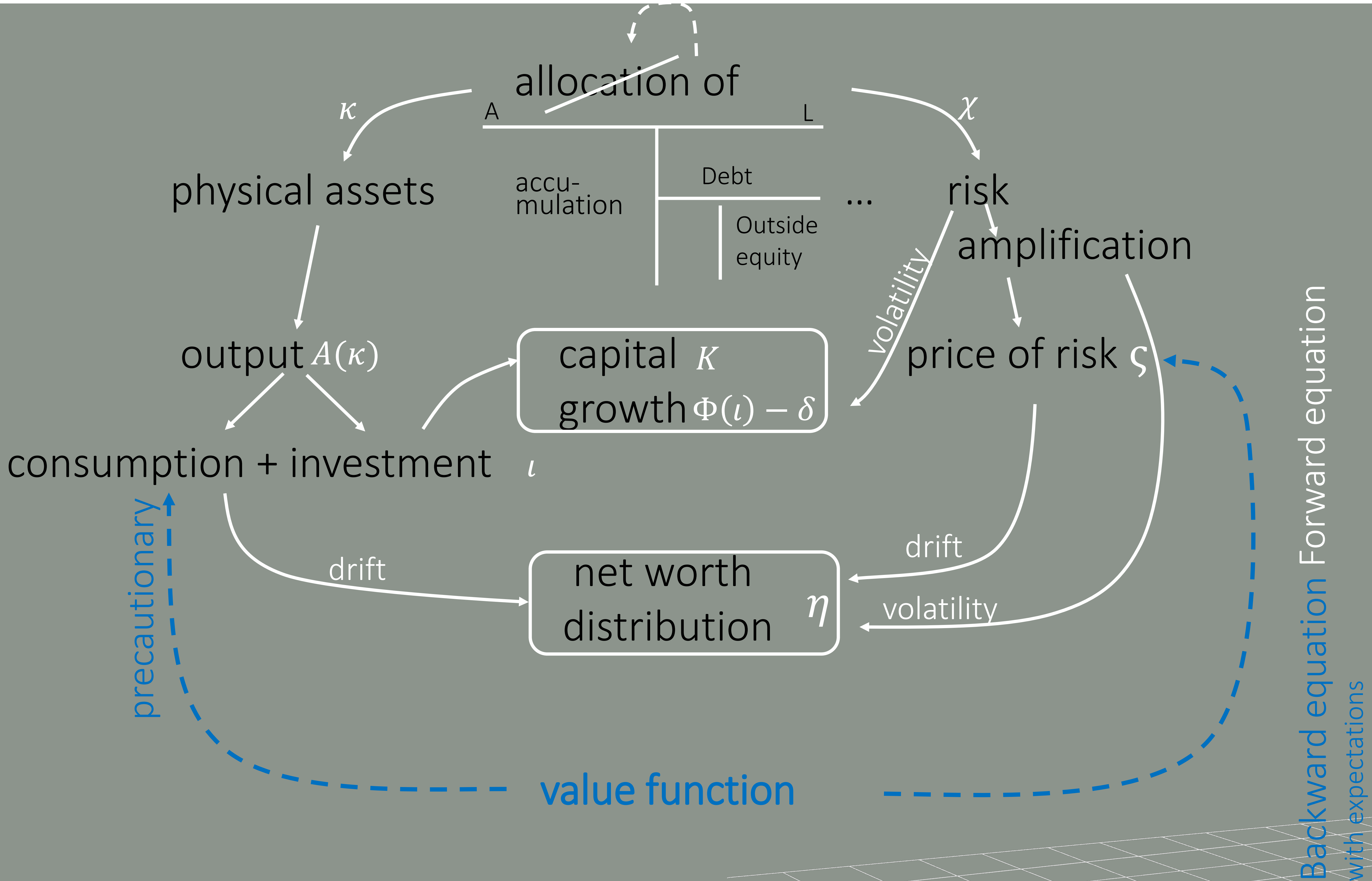
Poll 16: Why does drift of SDF equal risk-free rate

a) no idio risk

b) $e^{-r^F} = E[SDF]$*1

c) no jump in consumption

The Big Picture



allocation of

A

L

κ

χ

physical assets

accu-mulation

Debt

Outside equity

…

risk

amplification

volatility

output $A(\kappa)$

capital K

growth $\Phi(\iota) - \delta$

price of risk ς

consumption + investment ι

drift

net worth distribution η

drift

volatility

precautionary

value function

Backward equation with expectations

Forward equation

Solving MacroModels Step-by-Step

0. Postulate aggregates, price processes & obtain return processes
1. For given C/N-ratio and SDF processes for each i *finance block*
    a. Real investment ι + Goods market clearing *(static)*
    - *Toolbox 1:* Martingale Approach, HJB vs. Stochastic Maximum Principle Approach

    b. Portfolio choice θ + Asset market clearing or
    Asset allocation κ & risk allocation χ
    - *Toolbox 2:* "price-taking social planner approach" – Fisher separation theorem

    - *Toolbox 3:* Change in numeraire to total wealth (including SDF)
2. Evolution of state variable η (and K) *forward equation*
3. Value functions *backward equation*
    a. Value fcn. as fcn. of individual investment opportunities ω
    - *Special cases: log-utility, constant investment opportunities*

    b. Separating value fcn. $V^i\left(n^{\tilde{i}}; \eta, K\right)$ into $v^i(\eta)u(K)$
    c. Derive C/N-ratio and ς price of risk
4. Numerical model solution
    a. Transform BSDE for separated value fcn. $v^i(\eta)$ into PDE
    b. Solve PDE via value function iteration
5. KFE: Stationary distribution, Fan charts

1a. Individual Agent Choice of ι, θ, c

- Choice of ι is static problem (and separable) for each t
- $\max_{\iota_t^i} dr_t^k(\iota_t^i)$

$$= \max_{\iota_t^i} \left(\frac{a^i - \iota_t^i}{q_t} + \Phi(\iota_t^i) - \delta + \mu^q + \sigma\sigma^q \right)$$

For aggregate capital return,
Replace a^i with $A(\kappa)$

- FOC: $\frac{1}{q_t} = \Phi'(\iota_t^i)$ Tobin's q
  - All agents $\iota_t^i = \iota_t \Rightarrow \frac{dK_t}{K_t} = (\Phi(\iota_t) - \delta)\, dt + \sigma dZ_t$
  - Special functional form:
    - $\Phi(\iota) = \frac{1}{\phi} \log(\phi\iota + 1) \Rightarrow \phi\iota = q - 1$
- Goods market clearing: $(A(\kappa) - \iota_t)K_t = \sum_i C_i$.

Solving MacroModels Step-by-Step

0. Postulate aggregates, price processes & obtain return processes
1. For given C/N-ratio and SDF processes for each i *finance block*
   a. Real investment ι + Goods market clearing *(static)*
   - *Toolbox 1:* Martingale Approach, HJB vs. Stochastic Maximum Principle Approach

   b. Portfolio choice θ + Asset market clearing or
   Asset allocation κ & risk allocation χ
   - *Toolbox 2:* "price-taking social planner approach" – Fisher separation theorem

   - *Toolbox 3:* Change in numeraire to total wealth (including SDF)
2. Evolution of state variable η (and K) *forward equation*
3. Value functions *backward equation*
   a. Value fcn. as fcn. of individual investment opportunities ω
   - *Special cases: log-utility, constant investment opportunities*

   b. Separating value fcn. $V^i\left(n^{\tilde{i}};\eta,K\right)$ into $v^i(\eta)u(K)$
   c. Derive C/N-ratio and ς price of risk
4. Numerical model solution
   a. Transform BSDE for separated value fcn. $v^i(\eta)$ into PDE
   b. Solve PDE via value function iteration
5. KFE: Stationary distribution, Fan charts

1a. Individual Agent Choice of ι, θ, c

- Of experts with outside equity issuance (after plugging in households' outside equity choice)

$$\frac{a^e - \iota_t}{q_t} + \Phi(\iota_t) - \delta + \mu_t^q + \sigma\sigma_t^q - r_t =$$

$$[\varsigma_t^e \chi_t^e / \kappa_t^e + \varsigma_t^h (1 - \chi_t^e / \kappa_t^e)](\sigma + \sigma^q)$$

New compared to Basac-Cuoco

- Of households' capital choice

$$\frac{a^h - \iota_t}{q_t} + \Phi(\iota_t) - \delta + \mu_t^q + \sigma\sigma_t^q - r_t \leq \varsigma_t^h (\sigma + \sigma^q)$$

with equality if $\kappa_t^e < 1$

- Note: Later approach replaces this step with Fisher Separation Social Planners' choice (see below)

1a. Individual Agent Choice of ι, θ, c

- Consumption Choice: Martingale Approach
  - Consider a self-financing trading strategy consisting of agent’s net worth *with consumption reinvested*.
  - $\frac{d\left(\xi_t^i n_t^i\right)}{\xi_t^i n_t^i} + \frac{c_t^i}{n_t^i} dt = \Bigg(\underbrace{-r_t + \mu_t^{n^i} - \varsigma_t^i \sigma_t^{n^i} + \frac{c_t^i}{n_t^i}}_{=0} \Bigg) dt + \sigma \ldots$
  - $\frac{c_t^i}{n_t^i} = r_t - \mu_t^{n^i} + \varsigma_t^i \sigma_t^{n^i}$
  - (only) useful for steady state characterization

Solving MacroModels Step-by-Step

0. Postulate aggregates, price processes & obtain return processes
1. For given C/N-ratio and SDF processes for each i *finance block*
    a. Real investment ι + Goods market clearing *(static)*
    - *Toolbox 1:* Martingale Approach, HJB vs. Stochastic Maximum Principle Approach

    b. Portfolio choice θ + Asset market clearing or
    Asset allocation κ & risk allocation χ
    - *Toolbox 2:* "price-taking social planner approach" – Fisher separation theorem

    - *Toolbox 3:* Change in numeraire to total wealth (including SDF)
2. Evolution of state variable η (and K) *forward equation*
3. Value functions *backward equation*
    a. Value fcn. as fcn. of individual investment opportunities ω
    - *Special cases: log-utility, constant investment opportunities*

    b. Separating value fcn. $V^i\left(n^{\tilde{i}};\eta,K\right)$ into $v^i(\eta)u(K)$
    c. Derive C/N-ratio and ς price of risk
4. Numerical model solution
    a. Transform BSDE for separated value fcn. $v^i(\eta)$ into PDE
    b. Solve PDE via value function iteration
5. KFE: Stationary distribution, Fan charts

1b. Asset/Risk Allocation across I Types

- Price-Taking Planner's Theorem:
  A social planner that takes prices as given chooses an physical asset allocation, κ_t and risk allocation χ_t that coincides with the choices implied by all individuals' portfolio choices.

$$\boldsymbol{\varsigma}_t = (\varsigma_t^1, \dots, \varsigma_t^I)$$
$$\boldsymbol{\chi}_t = (\chi_t^1, \dots, \chi_t^I)$$
$$\boldsymbol{\sigma}(\boldsymbol{\chi}_t) = (\chi_t^1 \sigma^N, \dots, \chi_t^I \sigma^N)$$

- Planner's problem

Return on total wealth

$$\max_{\{\boldsymbol{\kappa}_t, \boldsymbol{\chi}_t\}} E_t\left[dr_t^N(\kappa_t)\right]/dt - \boldsymbol{\varsigma}_t \sigma(\boldsymbol{\chi}_t)$$

$= dr^F/dt$ in equilibrium

subject to friction: $F(\boldsymbol{\kappa}_t, \boldsymbol{\chi}_t) \leq 0$

  - Example:
    1. $\chi_t = \kappa_t$ (if one holds capital, one has to hold risk)
    2. $\chi_t \geq \alpha\kappa_t$ (skin in the game constraint, outside equity up to a limit)

1b. Asset/Risk Allocation across I Types

- Sketch of Proof of Theorem

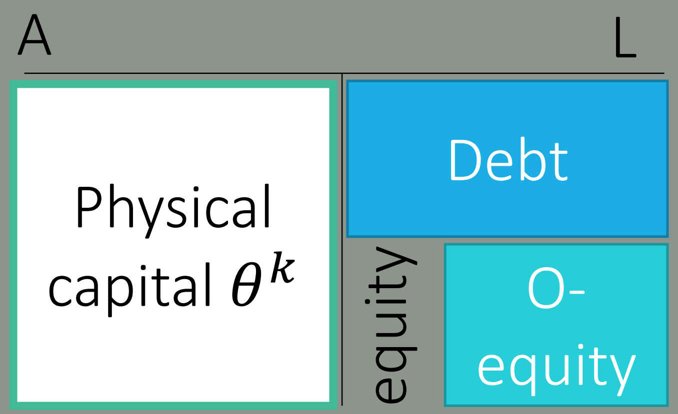
1. Fisher Separation Thm: (delegated portfolio choice by firm)
    - FOC yield the martingale approach solution
    - Each individual agent $(i,\tilde{\imath})$ portfolio maximization is equivalent to the maximization problem of a firm

$$\max_{\{\boldsymbol{\theta}^{j,i}\}} E_t\left[dr^{n^{(i,\tilde{\imath})}}\right]/dt - \varsigma\sigma^{r^n}$$

    - $dr^{n^{(i,\tilde{\imath})}} = \sum_j \theta^{j,i} dr^j = \sum_j \theta^{j,i} E[dr^j] + \sum_j \theta^{j,i} \sigma^j dZ_t$ is linear in θs
        - Either bang-bang solution for θs s.t. portfolio constraints bind
        - Or prices/returns/risk premia are s.t. that firm is indifferent
2. Aggregate
    - Taking η-weighted sum to obtain return on aggregate wealth
3. Use market clearing to relate θs to κs & χs (incl. θ-constraint)

1b. Allocation of Capital/Risk: 2 Types

- Expert: $\boldsymbol{\theta^e} = (\theta^{e,K}, \theta^{e,OE}, \theta^{e,D})$ for capital, outside equity, debt
  - Restrictions: $\theta^{e,K} \geq 0,$
    $\theta^{e,OE} \leq 0,$ only issue outside equity
    $\theta^{e,OE} \geq -(1-\alpha)\theta^{e,K}$ skin in the game

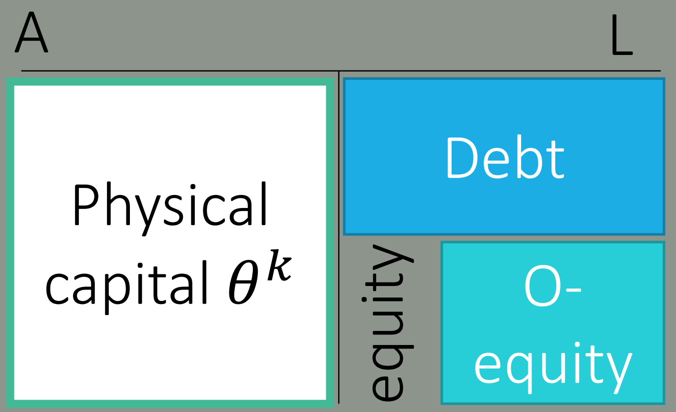


maximize

$$\theta_t^{e,K} E\left[dr_t^{e,K}\right]/dt + \theta_t^{e,OE} E\left[dr_t^{OE}\right]/dt + \theta_t^{e,D} r_t - \varsigma_t^e\left(\theta_t^{e,K} + \theta_t^{e,OE}\right)\sigma^{r^{e,K}}$$

1b. Allocation of Capital/Risk: 2 Types

- Expert: $\boldsymbol{\theta^e} = (\theta^{e,K}, \theta^{e,OE}, \theta^{e,D})$ for capital, outside equity, debt
  - Restrictions: $\theta^{e,K} \geq 0,$



A

L

Physical capital θ^k

Debt

equity

O-equity

$\theta^{e,OE} \leq 0,$ only issue outside equity

$\theta^{e,OE} \geq -(1-\alpha)\theta^{e,K}$ skin in the game

maximize

$$\theta_t^{e,K} E[dr_t^{e,K}]/dt + \theta_t^{e,OE} E[dr_t^{OE}]/dt + \theta_t^{e,D} r_t - \varsigma_t^e (\theta_t^{e,K} + \theta_t^{e,OE}) \sigma^{r^{e,K}}$$

$\theta^{h,K} \geq 0$

$\theta^{h,OE} \geq 0$

- Household: $\boldsymbol{\theta^h} = (\theta^{h,K}, \theta^{h,OE}, \theta^{h,D})$

maximize

$$\theta^{h,K} E[dr_t^{h,K}]/dt + \theta^{h,OE} E[dr_t^{OE}]/dt + \theta^{h,D} r_t - \varsigma_t^e (\theta_t^{h,K} + \theta_t^{h,OE}) \sigma^{r^{h,K}}$$

1b. Allocation of Capital/Risk: 2 Types

- Aggregate η-weighted sum of expert + HH max problem

$$\eta^e\{\dots\}+\eta^h\{\dots\}$$

- $\underbrace{\eta_t^e\theta_t^{e,K}}_{\kappa_t^e:=} E\left[dr_t^{e,K}\right]/dt + \underbrace{\eta_t^h\theta_t^{hK}}_{\kappa_t^h:=} E\left[dr_t^{h,K}\right]/dt +$

$$\underbrace{\left(\eta_t^e\theta_t^{e,OE}+\eta_t^h\theta_t^{h,OE}\right)}_{=0} E\left[dr_t^{OE}\right]/dt + \underbrace{\left(\eta_t^e\theta_t^{e,D}+\eta_t^h\theta_t^{e,D}\right)}_{=0} r_t$$

$$-\varsigma_t^e \underbrace{\eta_t^e\left(\theta_t^{e,K}+\theta_t^{e,OE}\right)}_{=:\chi_t^e}\sigma_t^{r^K} - \varsigma_t^h \underbrace{\eta_t^h\left(\theta_t^{h,K}+\theta_t^{h,OE}\right)}_{=:\chi_t^h}\sigma_t^{r^k}$$

1b. Allocation of Capital/Risk: 2 Types

- Aggregate η-weighted sum of expert + HH max problem

$$\eta^e\{\dots\}+\eta^h\{\dots\}$$

- $\underbrace{\eta_t^e\theta_t^{e,K}}_{\kappa_t^e:=} E\left[dr_t^{e,K}\right]/dt+\underbrace{\eta_t^h\theta_t^{hK}}_{\kappa_t^h:=} E\left[dr_t^{h,K}\right]/dt+$

$$\underbrace{\left(\eta_t^e\theta_t^{e,OE}+\eta_t^h\theta_t^{h,OE}\right)}_{=0}E\left[dr_t^{OE}\right]/dt+\underbrace{\left(\eta_t^e\theta_t^{e,D}+\eta_t^h\theta_t^{e,D}\right)}_{=0}r_t$$

$$-\varsigma_t^e\underbrace{\eta_t^e\left(\theta_t^{e,K}+\theta_t^{e,OE}\right)}_{=:\chi_t^e}\sigma_t^{r^K}-\varsigma_t^h\underbrace{\eta_t^h\left(\theta_t^{h,K}+\theta_t^{h,OE}\right)}_{=:\chi_t^h}\sigma_t^{r^k}$$

Poll 29: Why $=0$?

a) because marginal benefits
= marginal costs at optimum

b) due to martingale behavior

c) because outside equity and
debt are in zero net supply

1b. Allocation of Capital/Risk: 2 Types

- Translate constraints:
  - $\chi_t^e \leq \kappa_t^e$ experts cannot buy outside equity of others

    only important for the case with idio risk
  - $\chi_t^e = \underbrace{\eta_t^e \theta_t^{e,K}}_{\kappa_t^e} + \underbrace{\eta_t^e \theta_t^{e,OE}}_{\geq -\kappa_t^e(1-\alpha)} \geq \alpha \kappa_t^e$
- Price-taking social planers problem

$$\max_{\{\kappa_t^e, \kappa_t^h = 1-\kappa_t^e, \chi_t^e \in [\alpha\kappa_t^e, \kappa_t^h], \chi_t^h = 1-\chi_t^e\}} \left[\frac{\kappa_t^e a^e + \kappa_t^h a^h - \iota_t}{q_t} + \Phi(\iota_t) - \delta\right] - (\varsigma_t^e \chi_t^e + \varsigma_t^h \chi_t^h)\sigma_t^{r^K}$$

End of Proof. Q.E.D.

  - Linear objective (if frictions take form of constraints)
    - Price of risk adjust such that objective becomes flat *or*
    - Bang-bang solution hitting constraints

1b. Allocation of Capital/Risk: 2 Types

- Example 1: 2 Types + no outside equity $(\alpha = 1)$

$$\max_{\{\kappa_t^e, \chi_t^e\}} \left[\frac{\kappa_t^e a^e + (1-\kappa_t^e) a^h - \iota_t}{q_t} + \Phi(\iota_t) - \delta \right] - \left(\chi_t^e \varsigma_t^e + (1-\chi_t^e) \varsigma_t^h \right) \left(\sigma + \sigma_t^q \right)$$

s.t. friction $\chi_t^e = \kappa_t^e$ if no outside equity can be issued

- FOC_χ: $\frac{a^e - a^h}{q_t} = (\varsigma_t^e - \varsigma_t^h)\left(\sigma + \sigma_t^q\right)$

- May hold only with inequality (≥), if at constraint $\kappa_t^e = 1$

1b. Allocation of Capital/Risk: 2 Types

- Example 2: 2 Type + with outside equity

$$\max_{\{\kappa_t^e,\chi_t^e\}}\left[\frac{\kappa_t^e a^e+(1-\kappa_t^e)a^h-\iota_t}{q_t}+\Phi(\iota_t)-\delta\right]-\left(\chi_t^e\varsigma_t^e+(1-\chi_t^e)\varsigma_t^h\right)(\sigma+\sigma_t^q)$$

- FOC_χ: Case 1: $\varsigma_t^e(\sigma+\sigma_t^q)>\varsigma_t^h(\sigma+\sigma_t^q)\Rightarrow\chi_t^e=\alpha\kappa_t^e$

  Case 2: $=$ $\chi_t^e>\alpha\kappa_t^e$

  - Case 1: plug $\chi_t^e=\alpha\kappa_t^e$ in objective
    - *a.* FOC_κ: $\frac{a^e-a^h}{q_t}>\alpha(\varsigma_t^e-\varsigma_t^h)(\sigma+\sigma_t^q)\Rightarrow\kappa_t^e=1$
    - *b.* $=$ $\Rightarrow\kappa_t^e<1$
  - Case 2:
    - *a.* FOC_κ: $\frac{a^e-a^h}{q_t}>0$ $\Rightarrow\kappa_t^e=1$
    - *b.* $=\ 0\Rightarrow\kappa_t^e<1$ impossible

1b. Allocation of Capital/Risk: 2 Types

- Example 2: 2 Type + with outside equity

$$\max_{\{\kappa_t^e,\chi_t^e\}}\left[\frac{\kappa_t^e a^e+(1-\kappa_t^e)a^h-\iota_t}{q_t}+\Phi(\iota_t)-\delta\right]-\left(\chi_t^e\varsigma_t^e+(1-\chi_t^e)\varsigma_t^h\right)(\sigma+\sigma_t^q)$$

- FOC_χ: Case 1: $\varsigma_t^e(\sigma+\sigma_t^q) > \varsigma_t^h(\sigma+\sigma_t^q) \Rightarrow \chi_t^e = \alpha\kappa_t^e$

  Case 2: $=$ $\chi_t^e > \alpha\kappa_t^e$

  - Case 1: plug $\chi_t^e = \alpha\kappa_t^e$ in objective

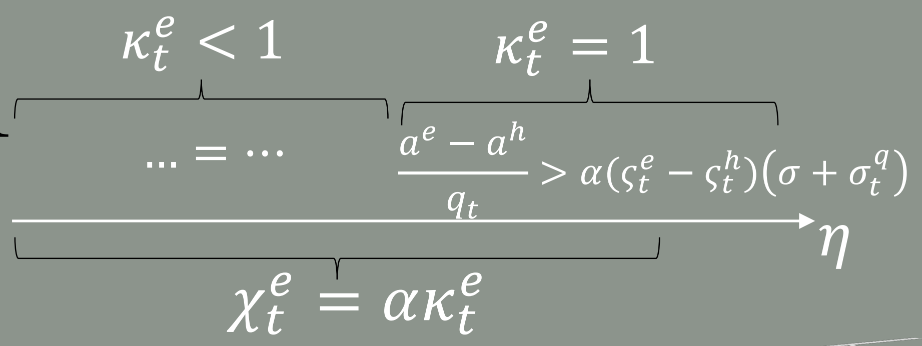
    a. $FOC_\kappa: \frac{a^e-a^h}{q_t} > \alpha(\varsigma_t^e-\varsigma_t^h)(\sigma+\sigma_t^q) \Rightarrow \kappa_t^e = 1$

    b. $= \Rightarrow \kappa_t^e < 1$

  - Case 2:

    a. $FOC_\kappa: \frac{a^e-a^h}{q_t} > 0 \Rightarrow \kappa_t^e = 1$

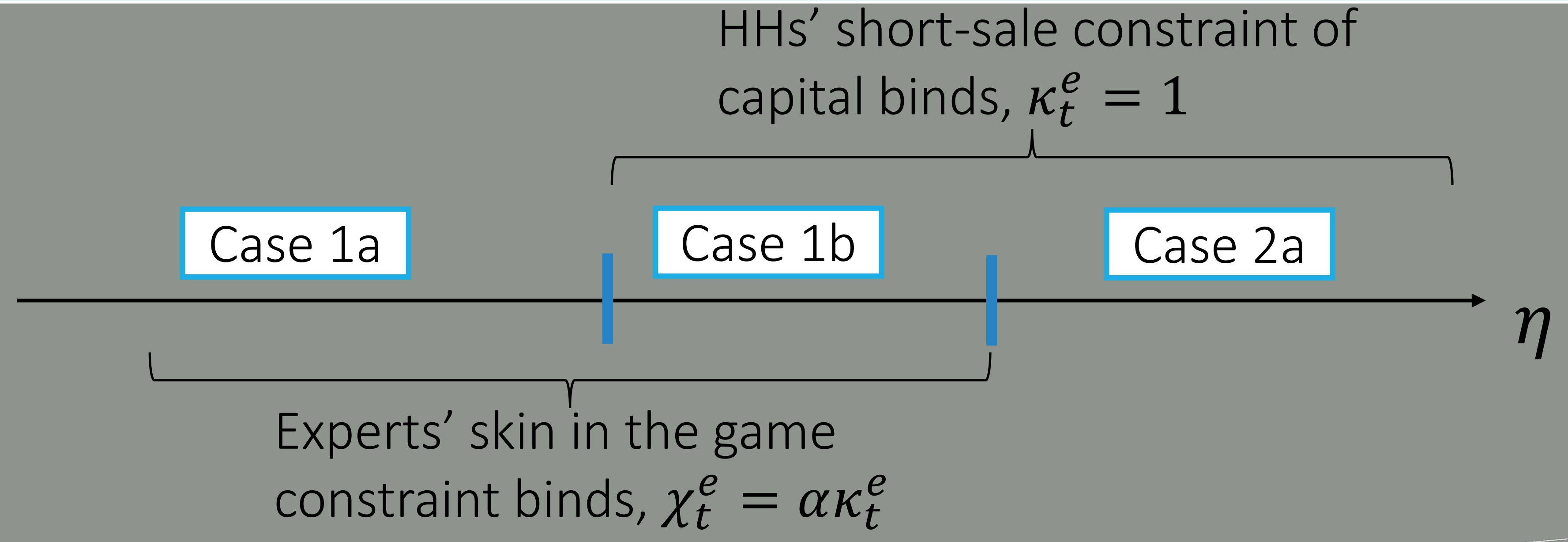
    b. $= 0 \Rightarrow \kappa_t^e < 1$ impossible



1b. Allocation of Capital, κ, and Risk, χ

- Summarizing previous slide (2 types with outside equity)

Cases	$\chi_t^e \geq \alpha\kappa_t^e$	$\kappa_t^e \leq 1$	$\frac{(a^e - a^h)}{q_t} \geq \alpha(\varsigma_t^e - \varsigma_t^h)(\sigma + \sigma_t^q)$ Shift a capital unit to expert Benefit: LHS Cost: RHS	$(\varsigma_t^e - \varsigma_t^h)(\sigma + \sigma_t^q) \geq 0$ Required risk premium of experts vs. HH
1a	=	<	=	>
1b	=	=	>	>
2a	>	=	>	=
impossible				



HHs' short-sale constraint of capital binds, $\kappa_t^e = 1$

Case 1a

Case 1b

Case 2a

η

Experts' skin in the game constraint binds, $\chi_t^e = \alpha\kappa_t^e$

Solving MacroModels Step-by-Step

0. Postulate aggregates, price processes & obtain return processes
1. For given C/N-ratio and SDF processes for each i *finance block*
   a. Real investment ι + Goods market clearing *(static)*
   - *Toolbox 1:* Martingale Approach, HJB vs. Stochastic Maximum Principle Approach

   b. Portfolio choice θ + Asset market clearing or
   Asset allocation κ & risk allocation χ
   - *Toolbox 2:* "price-taking social planner approach" – Fisher separation theorem

   - *Toolbox 3:* Change in numeraire to total wealth (including SDF)
2. Evolution of state variable η (and K) *forward equation*
3. Value functions *backward equation*
   a. Value fcn. as fcn. of individual investment opportunities ω
   - *Special cases: log-utility, constant investment opportunities*

   b. Separating value fcn. $V^i\left(n^{\tilde{i}};\eta,K\right)$ into $v^i(\eta)u(K)$
   c. Derive C/N-ratio and ς price of risk
4. Numerical model solution
   a. Transform BSDE for separated value fcn. $v^i(\eta)$ into PDE
   b. Solve PDE via value function iteration
5. KFE: Stationary distribution, Fan charts

Toolbox 3: Change of Numeraire

- x_t^A is a value of a self-financing strategy/asset in \$
- Y_t price of € in \$ (exchange rate)

$$\frac{dY_t}{Y_t} = \mu_t^Y dt + \sigma_t^Y dZ_t$$

- x_t^A / Y_t value of the self-financing strategy/asset in €

$\underbrace{e^{-\rho t} u'(c_t)}_{=\xi_t} Y_t \frac{x_t^A}{Y_t}$ follows a martingale

Recall $\mu_t^A - \mu_t^B = \underbrace{(-\sigma_t^\xi)}_{=\varsigma_t} \underbrace{(\sigma^A - \sigma_t^B)}_{risk}$

$$\mu_t^{A/Y} - \mu_t^{B/Y} = \underbrace{(-\sigma_t^\xi - \sigma_t^Y)}_{price\ of\ risk} \underbrace{(\sigma^A - \cancel{\sigma_t^Y} - \sigma_t^B + \cancel{\sigma_t^Y})}_{risk}$$

- Price of risk $\varsigma^€ = \varsigma^\$ - \sigma^Y$

Toolbox 3: Change of Numeraire

- x_t^A is a value of a self-financing strategy/asset in \$
- Y_t price of € in \$ (exchange rate)

$$\frac{dY_t}{Y_t} = \mu_t^Y dt + \sigma_t^Y dZ_t$$

- x_t^A / Y_t value of the self-financing strategy/asset in €

$\underbrace{e^{-\rho t} u'(c_t)}_{=\xi_t} Y_t \frac{x_t^A}{Y_t}$ follows a martingale

Recall $\mu_t^A - \mu_t^B = \underbrace{(-\sigma_t^\xi)}_{=\varsigma_t} \underbrace{(\sigma^A - \sigma_t^B)}_{risk}$

$$\mu_t^{A/Y} - \mu_t^{B/Y} = \underbrace{(-\sigma_t^\xi - \sigma_t^Y)}_{price\ of\ risk} \underbrace{(\sigma^A - \cancel{\sigma_t^Y} - \sigma_t^B + \cancel{\sigma_t^Y})}_{risk}$$

- Price of risk $\varsigma^{\text{€}} = \varsigma^{\$} - \sigma^Y$

Poll 37: Why does the price of risk change, though real risk remains the same

a) because risk-free rate might not stay risk-free

b) because covariance structure changes

Solving MacroModels Step-by-Step

0. Postulate aggregates, price processes & obtain return processes
1. For given C/N-ratio and SDF processes for each i *finance block*
    a. Real investment ι + Goods market clearing *(static)*
    - *Toolbox 1:* Martingale Approach, HJB vs. Stochastic Maximum Principle Approach

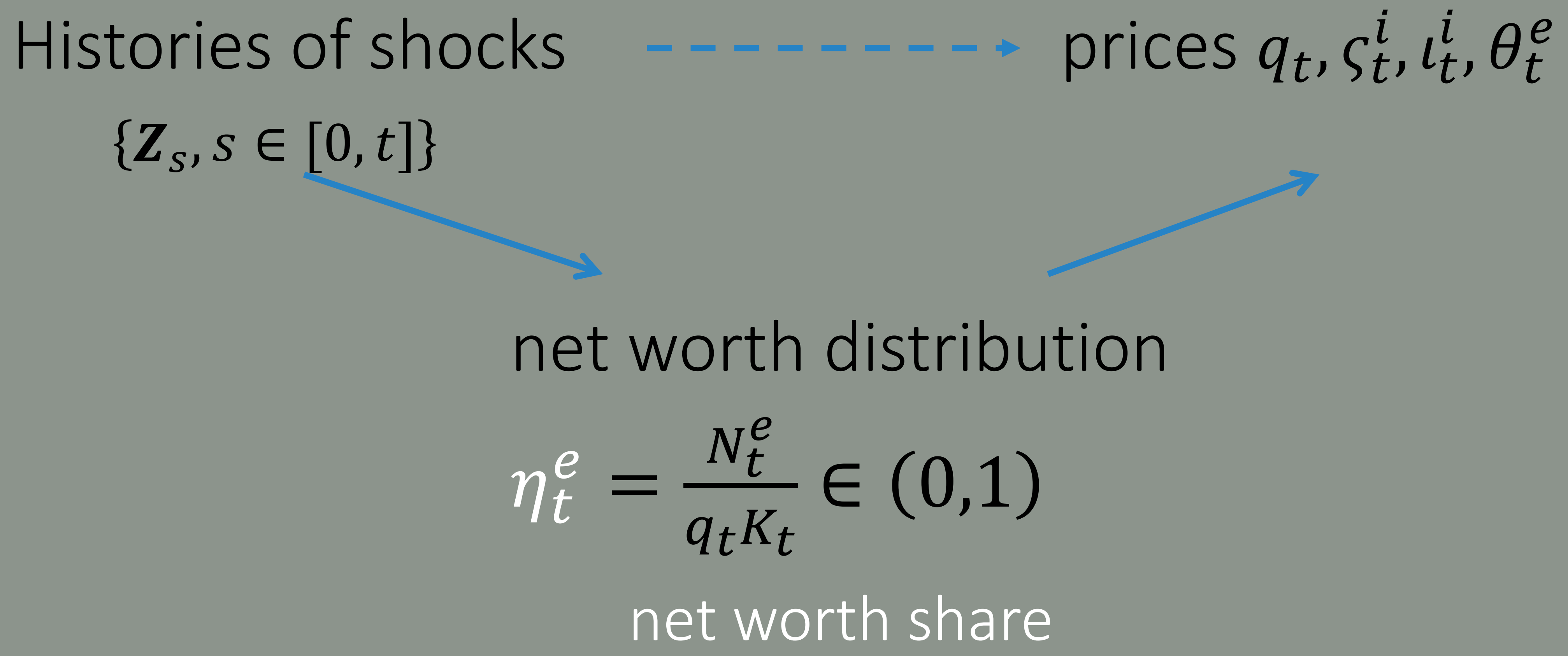
    b. Portfolio choice θ + Asset market clearing or
    Asset allocation κ & risk allocation χ
    - *Toolbox 2:* "price-taking social planner approach" – Fisher separation theorem

    - *Toolbox 3:* Change in numeraire to total wealth (including SDF)
2. Evolution of state variable η (and K) *forward equation*
3. Value functions *backward equation*
    a. Value fcn. as fcn. of individual investment opportunities ω
    - *Special cases: log-utility, constant investment opportunities*

    b. Separating value fcn. $V^i\left(n^{\tilde{i}};\eta,K\right)$ into $v^i(\eta)u(K)$
    c. Derive C/N-ratio and ς price of risk
4. Numerical model solution
    a. Transform BSDE for separated value fcn. $v^i(\eta)$ into PDE
    b. Solve PDE via value function iteration
5. KFE: Stationary distribution, Fan charts

2. GE: Markov States and Equilibria

- Equilibrium is a map



Histories of shocks $\{\boldsymbol{Z}_s, s \in [0,t]\}$

prices $q_t, \varsigma_t^i, \iota_t^i, \theta_t^e$

net worth distribution

$$\eta_t^e = \frac{N_t^e}{q_t K_t} \in (0,1)$$

net worth share

- All agents maximize utility
  - Choose: portfolio, consumption, technology
- All markets clear
  - Consumption, capital, money, outside equity

2. Law of Motion of Wealth Share η_t

- **Method 1:** Using Ito's quotation rule $\eta_t^i = N_t^i/(q_t K_t)$
  - Recall
    $$\frac{dN_t^i}{N_t^i} = r_t dt + \underbrace{\frac{\chi_t^i \kappa_t^i}{\eta_t^i}(\sigma+\sigma_t^q)}_{risk} \underbrace{\varsigma_t^i}_{price\ of\ risk} dt + \frac{\chi_t^i \kappa_t^i}{\eta_t^i}(\sigma+\sigma_t^q) dZ_t - \frac{C_t^i}{N_t^i} dt$$
  - $\frac{d\eta_t^i}{\eta_t^i} = \ldots$ (lots of algebra)

- **Method 2:** Change of numeraire + Martingale Approach
  - New numeraire: Total wealth in the economy, N_t
  - Apply Martingale Approach for value of i's portfolio
    - Simple algebra to obtain drift of η_t^i: $\mu_t^{\eta^i}$
      Note that change of numeraire does not affect ratio η^i!

2. μ^{η} Drift of Wealth Share: Many Types

- New Numeraire
  - **"Total net worth"** in the economy, N_t (without superscript)
  - Type i's portfolio net worth = net worth share

- Martingale Approach with new numeraire
  - Asset A $= i$'s portfolio return in terms of total wealth,

$$\left(\underbrace{\frac{C_t^i}{N_t^i}}_{\text{Dividend yield}} + \underbrace{\mu_t^{\eta^i}}_{\text{E[capital gains] rate}}\right) dt + \sigma_t^{\eta^i} dZ_t$$

  - Asset B (benchmark asset that everyone can hold, e.g. risk-free asset or money (in terms of total economy wide wealth as numeraire))

$$r_t^m dt + \sigma_t^m dZ_t$$

  - Apply our martingale asset pricing formula

$$\mu_t^A - \mu_t^B = \varsigma_t^i (\sigma_t^A - \sigma_t^B)$$

Poll 41: Is risk-free asset, risk free in the new numeraire?

a) Yes

b) No

2. μ^η Drift of Wealth Share: Many Types

- Asset pricing formula (relative to benchmark asset)

$$\mu_t^{\eta^i} + \frac{C_t^i}{N_t^i} - r_t^m = \left(\varsigma_t^i - \sigma_t^N\right)\left(\sigma_t^{\eta^i} - \sigma_t^m\right)$$

due to change in numeraire

- Add up across types (weighted),

(capital letters without superscripts are aggregates for total economy)

$$\underbrace{\sum_{i'}^{I} \eta_t^{i'} \mu_t^{\eta^{i'}}}_{=0} + \frac{C_t}{N_t} - r_t^m = \sum_{i'} \eta_t^{i'} \left(\varsigma_t^{i'} - \sigma_t^N\right)\left(\sigma_t^{\eta^{i'}} - \sigma_t^m\right)$$

Poll 42: Why $= 0$*?*

a) Because we have stationary distribution

b) Because η*s sum up to* 1

c) Because η*s follow martingale*

Benchmark asset everyone can trade

$$\sigma_t^m = -\sigma_t^N$$

2. μ^η Drift of Wealth Share: Two Types

- Asset pricing formula (relative to benchmark asset)

$$\mu_t^{\eta^i} + \frac{C_t^i}{N_t^i} - r_t^m = \left(\varsigma_t^i - \sigma_t^N\right)\left(\sigma_t^{\eta^i} - \sigma_t^m\right)$$

For benchmark asset: risk-free debt

$$\sigma_t^m = -\sigma_t^N$$

- Add up across types (weighted),

(capital letters without superscripts are aggregates for total economy)

$$\underbrace{\left(\eta_t^e \mu_t^{\eta^e} + \eta_t^h \mu_t^{\eta^h}\right)}_{=0} + \frac{C_t}{N_t} - r_t^m$$

$$= \eta_t^e \left(\varsigma_t^e - \sigma_t^N\right)\left(\sigma_t^{\eta^e} - \sigma_t^m\right) + \eta_t^h \left(\varsigma_t^h - \sigma_t^N\right)\left(\sigma_t^{\eta^h} - \sigma_t^m\right)$$

- Subtract from each other yield net worth share dynamics

$$\mu_t^{\eta^e} = (1 - \eta_t^e)(\varsigma_t^e - \sigma_t^N)\left(\sigma_t^{\eta^e} - \sigma_t^m\right) - (1 - \eta_t^e)\left(\varsigma_t^h - \sigma_t^{N^h}\right)\left(\sigma_t^{\eta^h} - \sigma_t^m\right)$$

$$-\left(\frac{C_t^e}{N_t^e} - \frac{C_t}{q_t K_t}\right)$$

2. σ^{η} Volatility of Wealth Share

- Since $\eta_t^i = N_t^i / N_t$,

$$\sigma_t^{\eta^i} = \sigma_t^{N^i} - \sigma_t^{N} = \sigma_t^{N^i} - \sum_{i'} \eta_t^{i'} \sigma_t^{N^{i'}} = \left(1 - \eta_t^i\right)\sigma_t^{N^i} - \sum_{i^- \neq i} \eta_t^{i^-} \sigma_t^{N^{i^-}}$$

- Note for 2 types example

$$\sigma_t^{\eta^e} = (1 - \eta_t^e)(\sigma_t^{n^e} - \sigma_t^{n^h})$$

Change in notation in 2 type setting
Type-net worth is $n^i = N^i$

$$\sigma_t^{n^e} = \underbrace{\chi_t^e / \eta_t^e}_{=\theta^{e,K} + \theta^{e,OE}} \quad (\sigma + \sigma_t^q) \qquad\qquad \sigma_t^{n^h} = \frac{\chi_t^h}{\eta_t^h}\left(\sigma + \sigma_t^q\right) = \frac{1-\chi_t^e}{1-\eta_t^e}\left(\sigma + \sigma_t^q\right)$$

Hence,

$$\sigma_t^{\eta^e} = \frac{\chi_t^e - \eta_t^e}{\eta_t^e} \, (\sigma + \sigma_t^q)$$

- Note also, $\eta_t^e \sigma_t^{\eta^e} + \eta_t^h \sigma_t^{\eta^h} = 0 \Rightarrow \sigma_t^{\eta^h} = -\frac{\eta_t^e}{\eta_t^h} \sigma_t^{\eta^e} = -\frac{\eta_t^e}{1-\eta_t^e} \sigma_t^{\eta^e}$

2. Amplification Formula: Loss Spiral

- Recall $$\sigma_t^{\eta^e} = \underbrace{\frac{\chi_t^e - \eta_t^e}{\eta_t^e}}_{\text{leverage}} (\sigma + \sigma_t^q)$$

- By Ito's Lemma on $q(\eta^e)$ $$\sigma_t^q = \frac{q'(\eta_t^e)}{q(\eta_t^e)} \eta_t^e \sigma_t^{\eta^e}$$

$$\sigma_t^q = \frac{q'(\eta_t^e)}{\underbrace{q/\eta_t^e}_{elasticity}} \frac{\chi_t^e - \eta_t^e}{\eta_t^e} (\sigma + \sigma_t^q)$$

- Total volatility

$$\sigma + \sigma_t^q = \frac{\sigma}{1 - \frac{q'(\eta_t^e)}{q/\eta_t^e} \frac{\chi_t^e - \eta_t^e}{\eta_t^e}}$$

- Loss spiral
  - Market illiquidity (price impact elasticity)

2. Amplification Formula: Loss Spiral

- Recall

$$\left.\begin{array}{l} \sigma_t^{\eta^e} = \underbrace{\frac{\chi_t^e - \eta_t^e}{\eta_t^e}}_{\text{leverage}} (\sigma + \sigma_t^q) \\ \\ \sigma_t^q = \frac{q'(\eta_t^e)}{q(\eta_t^e)} \eta_t^e \sigma_t^{\eta^e} \end{array}\right\}$$

- By Ito's Lemma on $q(\eta^e)$

$$\sigma_t^q = \underbrace{\frac{q'(\eta_t^e)}{q/\eta_t^e}}_{elasticity} \frac{\chi_t^e - \eta_t^e}{\eta_t^e} (\sigma + \sigma_t^q)$$

- Total volatility

$$\sigma + \sigma_t^q = \frac{\sigma}{1 - \frac{q'(\eta_t^e)}{q/\eta_t^e} \frac{\chi_t^e - \eta_t^e}{\eta_t^e}}$$

Poll 46: Where is the spiral?

a) Sum of infinite geometric series (denominator)

b) in q', since with constant price, no spiral

- Loss spiral
  - Market illiquidity (price impact elasticity)

Solving MacroModels Step-by-Step

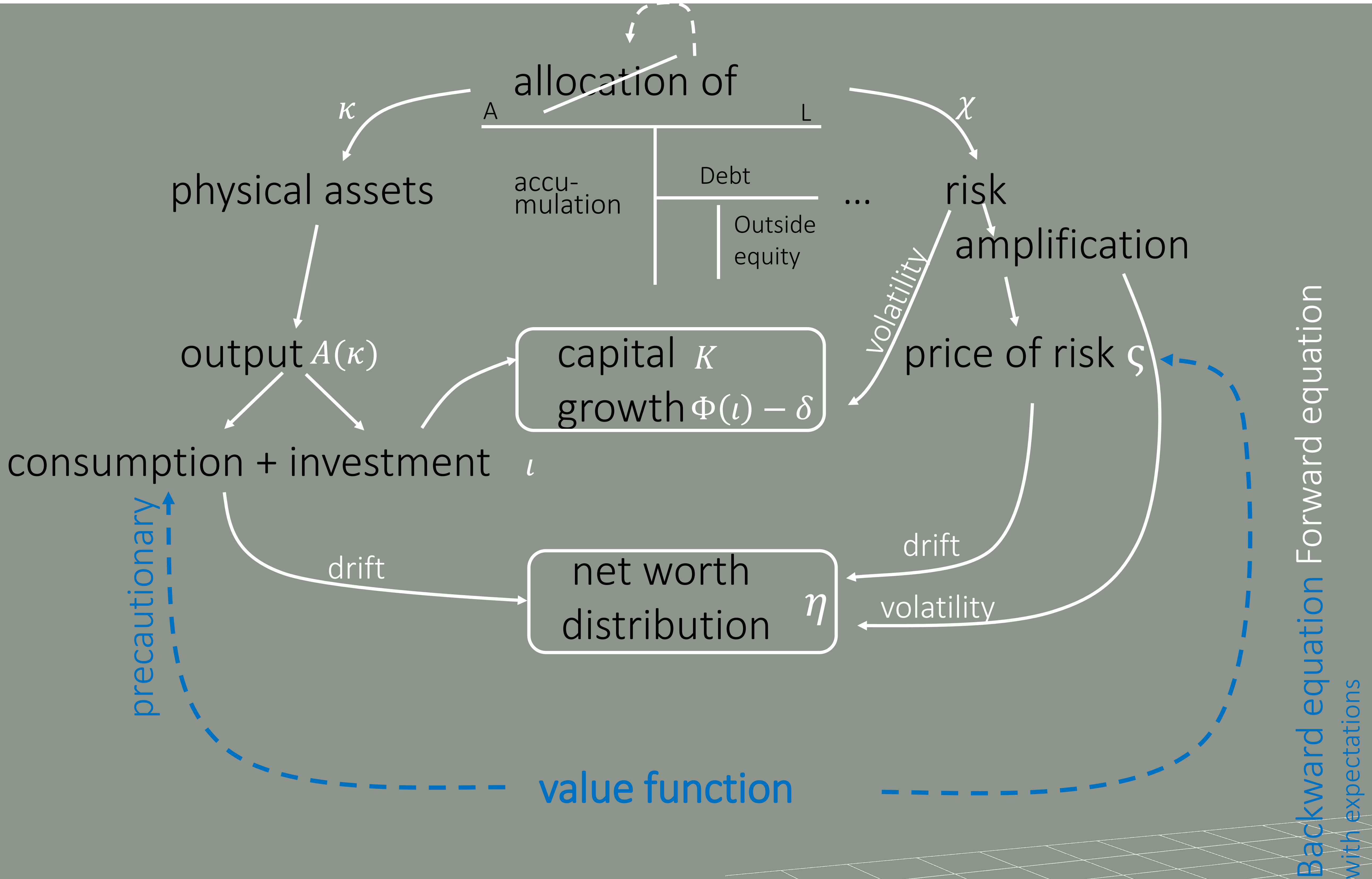
0. Postulate aggregates, price processes & obtain return processes
1. For given C/N-ratio and SDF processes for each i *finance block*
    a. Real investment ι + Goods market clearing *(static)*
    - *Toolbox 1:* Martingale Approach, HJB vs. Stochastic Maximum Principle Approach

    b. Portfolio choice θ + Asset market clearing or
    Asset allocation κ & risk allocation χ
    - *Toolbox 2:* "price-taking social planner approach" – Fisher separation theorem

    - *Toolbox 3:* Change in numeraire to total wealth (including SDF)
2. Evolution of state variable η (and K) *forward equation*
3. Value functions *backward equation*
    a. Value fcn. as fcn. of individual investment opportunities ω
    - *Special cases: log-utility, constant investment opportunities*

    b. Separating value fcn. $V^i(n^{\tilde{i}};\eta,K)$ into $v^i(\eta)u(K)\left(n^{\tilde{i}}/n^i\right)^{1-\gamma}$
    c. Derive C/N-ratio and ς price of risk
4. Numerical model solution
    a. Transform BSDE for separated value fcn. $v^i(\eta)$ into PDE
    b. Solve PDE via value function iteration
5. KFE: Stationary distribution, Fan charts

The Big Picture



allocation of

A

L

κ

χ

physical assets

accu-mulation

Debt

Outside equity

…

risk

amplification

volatility

output $A(\kappa)$

capital K

growth $\Phi(\iota) - \delta$

price of risk ς

consumption + investment ι

drift

net worth distribution η

drift

volatility

precautionary

value function

Backward equation Forward equation

with expectations

3a. CRRA Value Function Applies separately for each type of agent

- Martingale Approach: works best in endowment economy
- Here: mix Martingale approach with value function (envelop condition)

- $V^i(n_t^i; \boldsymbol{\eta_t}, K_t)$ for individuals $\boldsymbol{i}$
- **For** CRRA/power utility $u\left(c_t^i\right) = \frac{(c_t^i)^{1-\gamma}-1}{1-\gamma}$

⇒ increase net worth by factor, optimal c^i for all future states increases by this factor ⇒ $\left(\frac{c_t^i}{n_t^i}\right)$-ratio is invariant in $\boldsymbol{n_t^i}$

- ⇒ value function can be written as $V^i\left(n_t^i; \boldsymbol{\eta_t}, K_t\right) = \frac{u\left(\omega^i(\boldsymbol{\eta_t}, K_t) n_t^i\right)}{\rho^i}$
- ω_t^i Investment opportunity/ “net worth multiplier”
  - $\omega^i(\boldsymbol{\eta_t}, K_t)$-function turns out to be independent of K_t
  - Change notation from $\omega^i(\boldsymbol{\eta_t}, K_t)$-function to ω_t^i-process

3a. CRRA Value Function: relate to ω

- ⇒ value function can be written as $\frac{u\left(\omega_t^i n_t^i\right)}{\rho}$, that is

$$= \frac{1}{\rho^i}\frac{\left(\omega_t^i n_t^i\right)^{1-\gamma} - 1}{1-\gamma} = \frac{1}{\rho^i}\frac{\left(\omega_t^i\right)^{1-\gamma}\left(n_t^i\right)^{1-\gamma} - 1}{1-\gamma}$$

- $\frac{\partial V}{\partial n^i} = u'(c^i)$ by optimal consumption (if no corner solution)

$$\frac{\left(\omega_t^i\right)^{1-\gamma}\left(n_t^i\right)^{-\gamma}}{\rho^i} = (c_t^i)^{-\gamma} \Leftrightarrow \boxed{\frac{c_t^i}{n_t^i} = (\rho^i)^{1/\gamma}(\omega_t^i)^{1-1/\gamma}}$$

Next step:

a) Special simple cases

b) replace ω_t with something scale invariant

3a. CRRA Value Function: Special Cases

$$\boxed{\frac{c_t^i}{n_t^i} = (\rho^i)^{1/\gamma}(\omega_t^i)^{1-1/\gamma}}$$

- For log utility $\gamma = 1$:
  $\xi_t^i = e^{-\rho^i t}/c_t^i = e^{-\rho^i t}/(\rho n_t^i)$ for any $\omega_t^i \Rightarrow \sigma_t^{n^i} = \sigma_t^{c^i} = \varsigma_t^i$
  - Expected excess return: $\mu_t^A - r_t^F = \sigma_t^{n^i}\sigma_t^A$
  - Recall $\frac{dn_t^i}{n_t^i} = -\frac{c_t^i}{n_t^i}dt + \left(1 - \theta^i\right)dr_t^K + \theta^i dr_t$

3a. CRRA Value Function: Special Cases

$$\boxed{\frac{c_t^i}{n_t^i} = (\rho^i)^{1/\gamma}(\omega_t^i)^{1-1/\gamma}}$$

- For log utility $\gamma = 1$:
  $\xi_t^i = e^{-\rho^i t}/c_t^i = e^{-\rho^i t}/(\rho n_t^i)$ for any $\omega_t^i \Rightarrow \sigma_t^{n^i} = \sigma_t^{c^i} = \varsigma_t^i$
  - Expected excess return: $\mu_t^A - r_t^F = \sigma_t^{n^i}\sigma_t^A$
  - Recall $\frac{dn_t^i}{n_t^i} = -\frac{c_t^i}{n_t^i}dt + \left(1 - \theta^i\right)dr_t^K + \theta^i dr_t$
- For constant investment opportunities $\omega_t^i = \omega^i$,
  $\Rightarrow c^i/n^i$ is constant and hence $\sigma_t^{c^i} = \sigma^{n^i}$
  - Expected excess return: $\mu_t^A - r_t^F = \gamma\sigma_t^{n^i}\sigma_t^A$

Poll 52: Which term refers to (dynamic/Mertonian) hedging demand?

a) γ
b) σ_t^n
c) hidden in risk-free rate
d) none of the above

3a. CRRA Value Function: Special Cases

$$\boxed{\frac{c_t^i}{n_t^i} = (\rho^i)^{1/\gamma}(\omega_t^i)^{1-1/\gamma}}$$

- For log utility $\gamma = 1$:
  $\xi_t^i = e^{-\rho^i t}/c_t^i = e^{-\rho^i t}/(\rho n_t^i)$ for any $\omega_t^i \Rightarrow \sigma_t^{n^i} = \sigma_t^{c^i} = \varsigma_t^i$
  - Expected excess return: $\mu_t^A - r_t^F = \sigma_t^{n^i}\sigma_t^A$
  - Recall $\frac{dn_t^i}{n_t^i} = -\frac{c_t^i}{n_t^i}dt + \left(1-\theta^i\right)dr_t^K + \theta^i dr_t$
- For constant investment opportunities $\omega_t^i = \omega^i$,
  $\Rightarrow c^i/n^i$ is constant and hence $\sigma_t^{c^i} = \sigma^{n^i}$
  - Expected excess return: $\mu_t^A - r_t^F = \gamma\sigma_t^{n^i}\sigma_t^A$
  - Now $\frac{dn_t^i}{n_t^i} = r^F dt + \frac{(\varsigma^i)^2}{\gamma}dt + \frac{\varsigma^i}{\gamma}dZ_t - \frac{c_t^i}{n_t^i}dt$
  - $r^F = \rho^i + \gamma\left(r^F + \frac{(\varsigma^i)^2}{\gamma} - \frac{c_t^i}{n_t^i}\right) - \frac{\gamma+1}{2}\frac{(\varsigma^i)^2}{\gamma}$

$$= \rho^i + \gamma\left(r^F - \frac{c_t^i}{n_t^i}\right) + \frac{\gamma-1}{\gamma}\frac{(\varsigma^i)^2}{2}$$

$$\Rightarrow \frac{c_t^i}{n_t^i} = \rho^i + \frac{\gamma-1}{\gamma}\left(r^F - \rho^i + \frac{(\varsigma^i)^2}{2\gamma}\right)$$

Also holds if ω_t^i evolves deterministically

3a. CRRA Value Function: Special Cases

$$\boxed{\frac{c_t^i}{n_t^i} = (\rho^i)^{1/\gamma}(\omega_t^i)^{1-1/\gamma}}$$

- For log utility $\gamma = 1$:
  $\xi_t^i = e^{-\rho^i t}/c_t^i = e^{-\rho^i t}/(\rho n_t^i)$ for any $\omega_t^i \Rightarrow \sigma_t^{n^i} = \sigma_t^{c^i} = \varsigma_t^i$
  - Expected excess return: $\mu_t^A - r_t^F = \sigma_t^{n^i}\sigma_t^A$
  - Recall $\frac{dn_t^i}{n_t^i} = -\frac{c_t^i}{n_t^i}dt + \left(1-\theta^i\right)dr_t^K + \theta^i dr_t$
- For constant investment opportunities $\omega_t^i = \omega^i$,
  $\Rightarrow c^i/n^i$ is constant and hence $\sigma_t^{c^i} = \sigma^{n^i}$
  - Expected excess return: $\mu_t^A - r_t^F = \gamma\sigma_t^{n^i}\sigma_t^A$
  - Now $\frac{dn_t^i}{n_t^i} = r^F dt + \frac{(\varsigma^i)^2}{\gamma}dt + \frac{\varsigma^i}{\gamma}dZ_t - \frac{c_t^i}{n_t^i}dt$
  - $r^F = \rho^i + \gamma\left(r^F + \frac{(\varsigma^i)^2}{\gamma} - \frac{c_t^i}{n_t^i}\right) - \frac{\gamma+1}{2}\frac{\left(\varsigma^i\right)^2}{\gamma}$

$$\Rightarrow \frac{c_t^i}{n_t^i} = \rho^i + \frac{\gamma-1}{\gamma}\left(r^F - \rho^i + \frac{(\varsigma^i)^2}{2\gamma}\right)$$

Also holds if ω_t^i evolves deterministically

$$= \rho^i + \gamma\left(r^F - \frac{c_t^i}{n_t^i}\right) + \frac{\gamma-1}{\gamma}\frac{(\varsigma^i)^2}{2}$$

Way to compute c_t^i/n_t^i if one can obtain from some other source r^F (omega can we avoided)

3b. CRRA Value Fcn. & State Variable η

- Recall Martingale approach: if x_t is the value of a portfolio with return $\frac{dn_t^i}{n_t^i} + \frac{c_t^i}{n_t^i}dt$, then $\xi_t^i x_t^i$must be a martingale

$$\frac{d(\xi_t^i n_t^i)}{\xi_t^i n_t^i} = -\frac{c_t^i}{n_t^i}dt + martingale$$

- Optimal consumption implies with CRRA $V_t^i = \frac{1}{\rho^i}\frac{\left(\omega_t^i n_t^i\right)^{1-\gamma}}{1-\gamma}$:

$$\frac{\partial u^i}{\partial c^i} = \frac{\partial V^i}{\partial n^i} \Leftrightarrow (c_t^i)^{-\gamma} = \frac{1}{\rho^i}\left(\omega^i\right)^{1-\gamma}(n_t^i)^{-\gamma} \Leftrightarrow e^{\rho^i t}\underbrace{e^{-\rho^i t}(c_t^i)^{-\gamma}}_{=\xi_t^i} n_t^i = \underbrace{\frac{1}{\rho^i}\left(\omega^i\right)^{1-\gamma}\left(n_t^i\right)^{1-\gamma}}_{(1-\gamma)V_t^i}$$

- Hence,

$$\frac{dV_t^i}{V_t^i} = \frac{d(e^{\rho^i t}\xi_t^i n_t^i)}{e^{\rho^i t}\xi_t^i n_t^i} = \left(\rho^i - \frac{c_t^i}{n_t^i}\right)dt + martingale$$

- Next, let's compute the drift of $\frac{dV_t^i}{V_t^i}$

3b. CRRA Value Fcn: De-scale by K_t

- Drift of $\frac{dV_t^i}{V_t^i}$, we could use Ito on $V_t^i = \frac{1}{\rho^i} \frac{\left(\omega_t^i n_t^i\right)^{1-\gamma}}{1-\gamma}$, but
  - *Poll 56: What could be the problem?*
    a. *Net worth n_t is unbounded*
    b. *Net worth $n_t(\eta_t)$ and N-multiplier $\omega_t(\eta_t)$ are not differentiable (if $q(\eta_t)$, $q^B(\eta_t)$ have a kink).*
    c. *N-multiplier is not scale invariant*

3b. CRRA Value Fcn: De-scale by K_t

- Drift of $\frac{dV_t^i}{V_t^i}$, we could use Ito on $V_t^i = \frac{1}{\rho^i}\frac{\left(\omega_t^i n_t^i\right)^{1-\gamma}}{1-\gamma}$, but
  - *Poll 57: What could be the problem?*
    - *a. Net worth n_t is unbounded*
    - *b. Net worth $n_t(\eta_t)$ and N-multiplier $\omega_t(\eta_t)$ are not differentiable (if $q(\eta_t)$, $q^B(\eta_t)$ have a kink).*
    - *c. N-multiplier is not scale invariant*
  - Answer: b.
- In equilibrium $n^i = N^i$ (all experts/HH are the same)
- Let’s de-scale the problem w.r.t. K_t

$$V_t^i = \frac{1}{\rho^i}\frac{\left(\omega_t^i n_t^i\right)^{1-\gamma}}{1-\gamma} = \underbrace{\frac{\left(\omega_t^i N_t^i / K_t\right)^{1-\gamma}}{\rho^i}}_{v_t^i :=} \underbrace{\frac{K_t^{1-\gamma}}{1-\gamma}}_{u(K) :=}$$

and define v_t^i (which is twice differentiable in η_t)

- state variable K_t is easy to handle due to scale invariance

3b. CRRA Value Function

$$\frac{dV_t^i}{V_t^i} = \frac{d\left(v_t^i K_t^{1-\gamma}\right)}{v_t^i K_t^{1-\gamma}}$$

- By Ito's product rule

$$= \left(\mu_t^{v^i} + (1-\gamma)(\Phi(\iota_t) - \delta) - \frac{1}{2}\gamma(1-\gamma)\left(\sigma^2\right) + (1-\gamma)\sigma\sigma_t^{v^i}\right)dt$$
$$+ \, volatility \; terms$$

- Recall by consumption optimality
$\frac{dV_t^i}{V_t^i} - \rho^i dt + \frac{c_t^i}{n_t^i} dt$ follows a martingale

Poll 58: Why martingale?
- *a) Because we can "price" net worth with SDF*
- *b) because ρ^i and c_t^i/n_t^i cancel out*

- Hence, drift above $= \rho^i - \frac{c_t^i}{n_t^i}$ Still have to solve for $\mu_t^{v^i}$, $\sigma_t^{v^i}$

3b. CRRA Value Fcn BSDE

- Only conceptual interim solution
  - We will transform it into a PDE in Step 4 below
- From last slide

$$\underbrace{\mu_t^{v^i} + (1-\gamma)(\Phi(\iota_t) - \delta) - \frac{1}{2}\gamma(1-\gamma)\sigma^2 + (1-\gamma)\sigma\sigma_t^{v^i}}_{=:\mu_t^{V^i}} = \rho - \frac{c_t^i}{n_t^i}$$

- Can solve for $\mu_t^{v^i}$, then v_t^i must follow

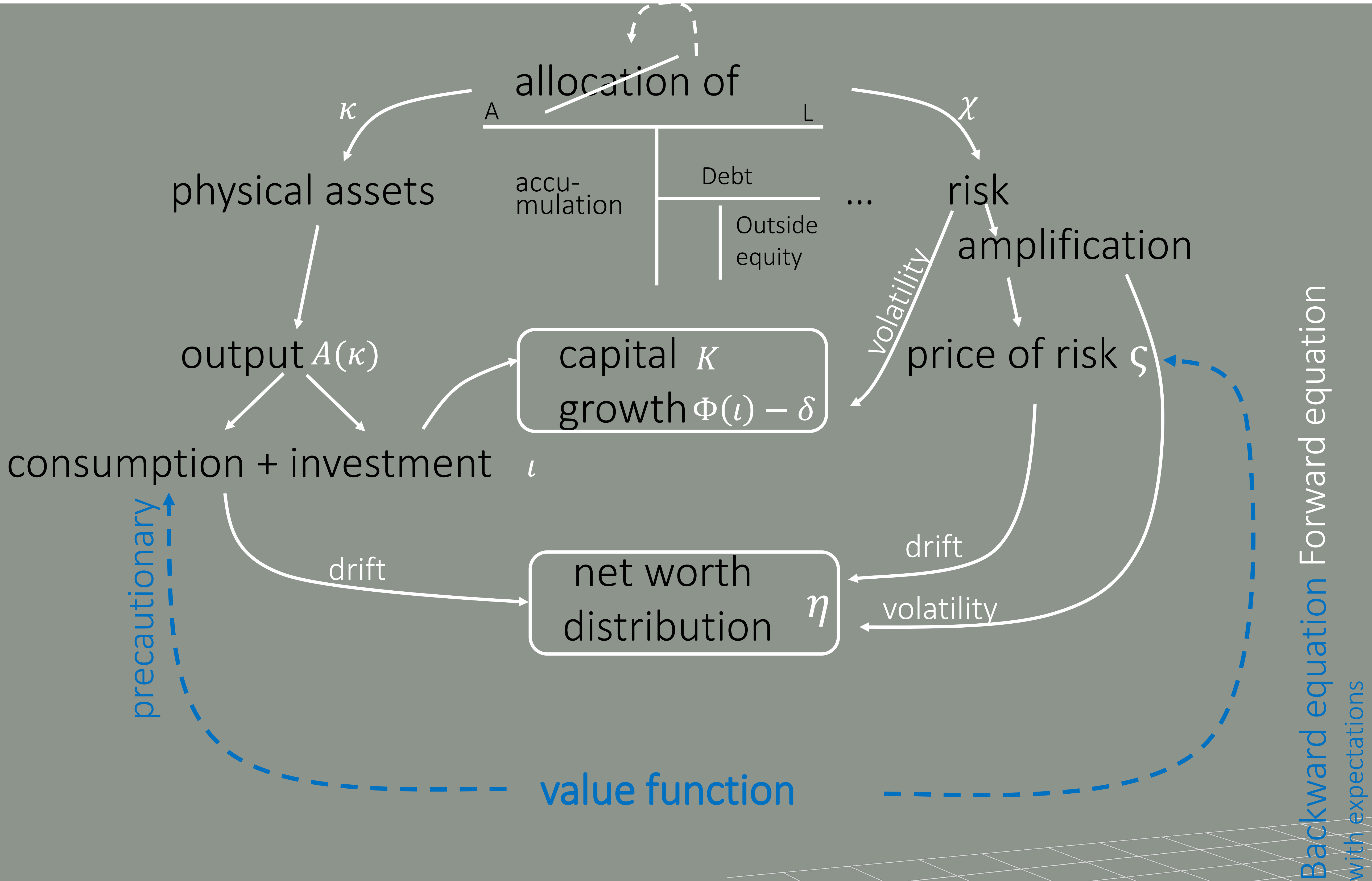
$$\frac{dv_t^i}{v_t^i} = f\left(\eta_t^i, v_t^i, \sigma_t^{v^i}\right)dt + \sigma_t^{v^i} dZ_t$$

with

$$f\left(\eta_t^i, v_t^i, \sigma_t^{v^i}\right) = \rho^i - \frac{c_t^i}{n_t^i} - (1-\gamma)(\Phi(\iota_t) - \delta) + \frac{1}{2}\gamma(1-\gamma)\left(\sigma^2\right) - (1-\gamma)\sigma\sigma_t^{v^i}$$

- Together with terminal condition v_T^i (possibly a constant for 1000 periods ahead), this is a backward stochastic differential equation (BSDE)
- A solution consists of processes v^i and σ^{v^i}
- Can use numerical BSDE solution methods (as random objects, so only get simulated paths)
- To solve this via a PDE we also need to get state evolution

The Big Picture



allocation of

A

L

κ

χ

physical assets

accu-mulation

Debt

Outside equity

…

risk

amplification

volatility

output $A(\kappa)$

capital K

growth $\Phi(\iota) - \delta$

price of risk ς

consumption + investment ι

drift

net worth distribution η

drift

volatility

precautionary

value function

Backward equation Forward equation

with expectations

3c. Get ςs from Value Function Envelop

- Recall $V^i\left(n_t^i; \boldsymbol{\eta_t}, K_t\right) = \frac{u\left(\omega^i(\boldsymbol{\eta_t}, K_t) n_t^i\right)}{\rho^i}$
- For envelop condition $\frac{\partial V_t}{\partial n_t} = \frac{\partial u(c_t)}{\partial c_t}$
  - To obtain $\frac{\partial V^i\left(n_t^i; \boldsymbol{\eta_t}, K_t\right)}{\partial n_t^i} = \frac{(\omega^i(\boldsymbol{\eta_t}, K_t))^{1-\gamma}}{\rho^i}\left(n_t^i\right)^{-\gamma}$
  - $= \underbrace{\frac{\left(\omega_t^i n_t^i / K_t\right)^{1-\gamma}}{\rho^i}}_{v_t^i :=} \left(\frac{K_t}{n_t^i}\right)^{1-\gamma} \left(n_t^i\right)^{-\gamma},$
  - $\Rightarrow \frac{\partial V_t}{\partial n_t^i} = v_t^i \left(\frac{K_t}{n_t^i}\right)^{1-\gamma} \left(n_t^i\right)^{-\gamma} = (c_t^i)^{-\gamma} = \frac{\partial u\left(c_t^i\right)}{\partial c_t^i}$
- In equilibrium $N_t^i = n_t^i$ and $C_t^i = c_t^i$ & using $N_t^i = \eta_t^i q_t K_t$

$$\frac{v_t^i}{\eta_t^i q_t} K_t^{-\gamma} = (C_t^i)^{-\gamma}$$

- Ito's quotient rule $\sigma_t^{v^i} - \sigma_t^{\eta^i} - \sigma_t^q - \gamma\sigma = -\gamma\sigma_t^{c^i} = -\varsigma_t^i$

3c. Get $\frac{C_t^i}{N_t^i}$ from Value Function Envelop

- Recall Envelop condition $v_t^i \left(\frac{K_t}{N_t^i}\right)^{1-\gamma} \left(n_t^i\right)^{-\gamma} = (c_t^i)^{-\gamma}$
- using $K_t/N_t^i = 1/\eta_t^i q_t$

$$\frac{C_t^i}{N_t^i} = \frac{c_t^i}{n_t^i} = \frac{(\eta_t^i q_t)^{1/\gamma-1}}{(v_t^i)^{1/\gamma}}$$

- Aggregate level (two agents case)

$$\frac{C_t}{N_t} = \frac{C_t^e + C_t^h}{N_t^e + N_t^h} = \eta_t^e \frac{C_t^e}{N_t^e} + \eta_t^h \frac{C_t^h}{N_t^h} = \frac{1}{q_t}\left[\left(\frac{\eta_t^e q_t}{v_t^e}\right)^{1/\gamma} + \left(\frac{\eta_t^h q_t}{v_t^h}\right)^{1/\gamma}\right]$$

Solving MacroModels Step-by-Step

0. Postulate aggregates, price processes & obtain return processes
1. For given C/N-ratio and SDF processes for each i *finance block*
   - a. Real investment ι + Goods market clearing *(static)*
   - *Toolbox 1:* Martingale Approach, HJB vs. Stochastic Maximum Principle Approach
   - b. Portfolio choice θ + Asset market clearing or
     Asset allocation κ & risk allocation χ
   - *Toolbox 2:* "price-taking social planner approach" – Fisher separation theorem

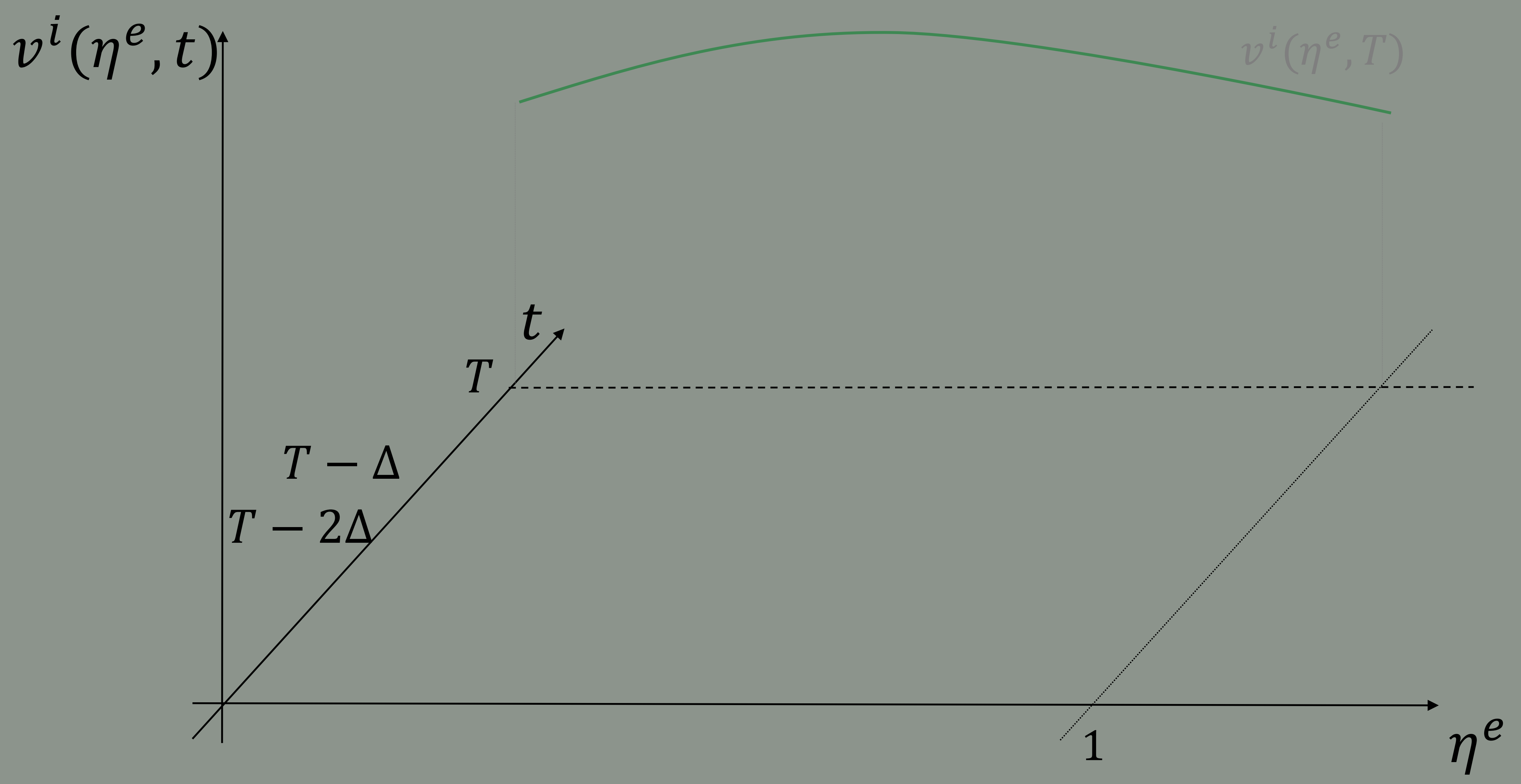
   - *Toolbox 3:* Change in numeraire to total wealth (including SDF)
2. Evolution of state variable η (and K) *forward equation*
3. Value functions *backward equation*
   - a. Value fcn. as fcn. of individual investment opportunities ω
   - *Special cases: log-utility, constant investment opportunities*
   - b. Separating value fcn. $V^i(n^{\tilde{i}};\eta,K)$ into $v^i(\eta)u(K)$
   - c. Derive C/N-ratio and ς price of risk
4. Numerical model solution
   - a. Transform BSDE for separated value fcn. $v^i(\eta)$ into PDE
   - b. Solve PDE via value function iteration
5. KFE: Stationary distribution, Fan charts

4. Value function Iteration - Big picture

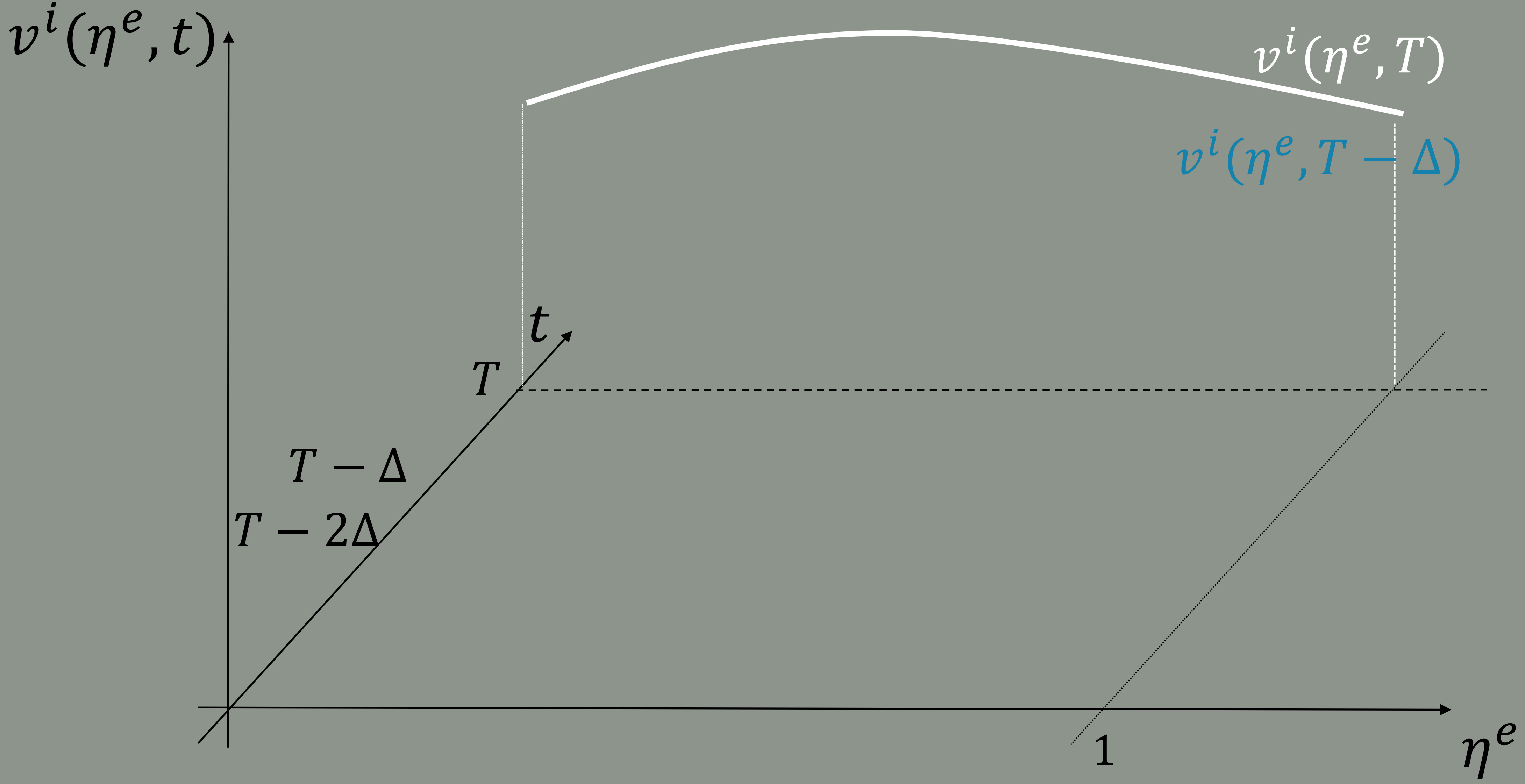
- Add time, t, as an additional state variable $v^e(\eta^e, t)$, $v^h(\eta^e, t)$
- Convert BSDE into PDE using Ito’s Lemma

Note in 2 type model we only use η^e as state variable

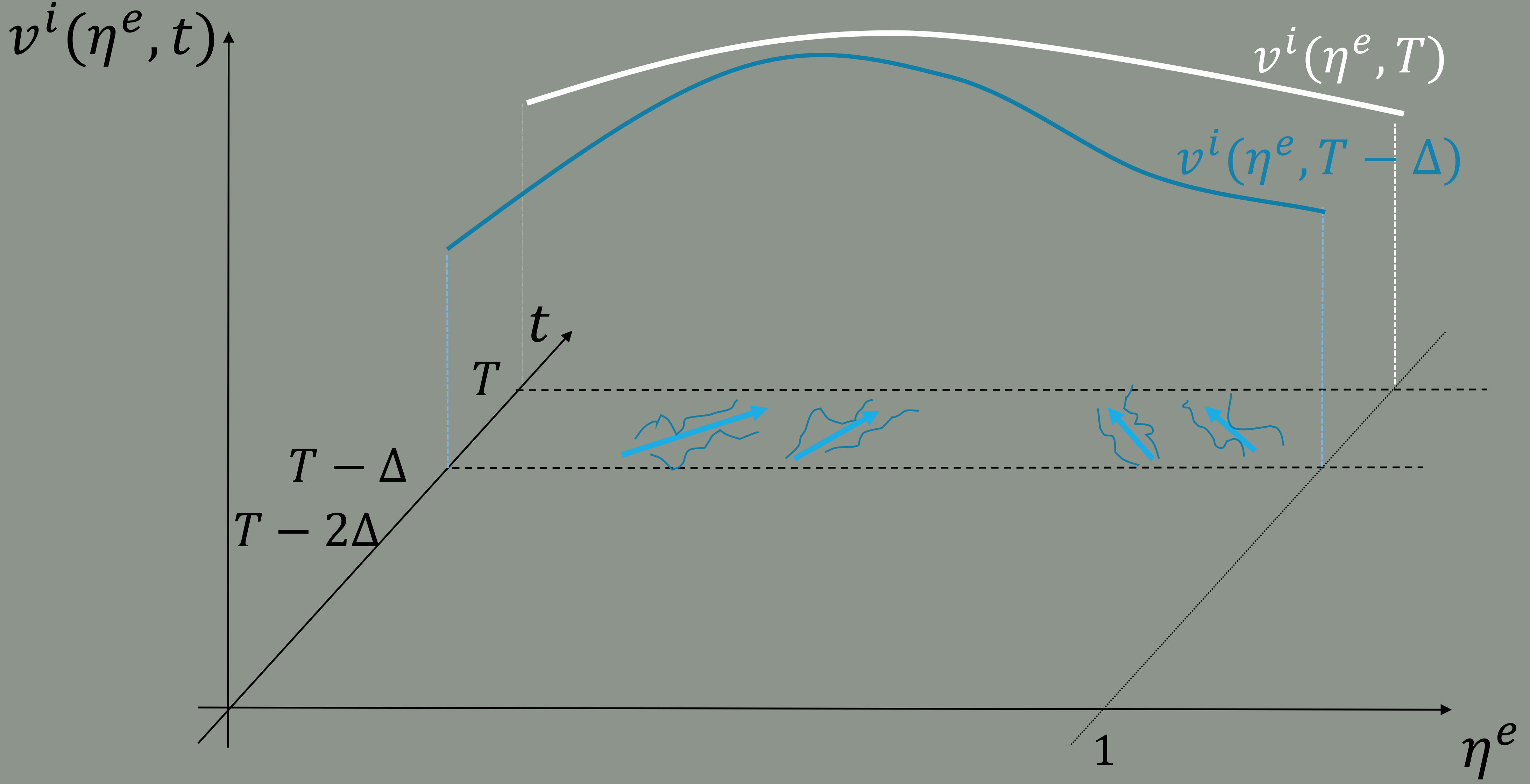
4. Value Function Iteration – Big Picture



4. Value Function Iteration – Big Picture



4. Value Function Iteration – Big Picture



4. Value function Iteration - Big picture

- Add time, t, as an additional state variable $v^e(\eta^e, t)$, $v^h(\eta^e, t)$
- Convert BSDE into PDE using Ito's Lemma
  - $\mu_t^{v^e} v_t^e = \partial_t v_t^e + \eta_t^e \mu_t^{\eta^e} \partial_\eta v_t^e + \frac{1}{2}\left(\eta_t^e \sigma_t^{\eta^e} \right)^2 \partial_{\eta\eta} v_t^e$
  - $\mu_t^{v^h} v_t^h = \partial_t v_t^h + \eta_t^e \mu_t^{\eta^e} \partial_\eta v_t^h + \frac{1}{2}\left(\eta_t^e \sigma_t^{\eta^e} \right)^2 \partial_{\eta\eta} v_t^h$

Short-hand notation:
$\partial_x f$ for $\partial f / \partial x$

- Guess terminal value functions $v^e(\eta^e, T)$ and $v^h(\eta^e, T)$ (far in the future $t = T$)
- … and iterate back to $t = 0$
  - In each step use
    - From Step 3: $\mu_t^{v^e} v_t^e$, $\mu_t^{v^h} v_t^h$
    - From Step 2: $\eta_t^e \mu_t^{\eta^e}$ and $\eta_t^e \sigma_t^{\eta^e}$ (η-evolution)
    - Portfolio choice, planners' problem, (static conditions)
    - Market clearing
  - To calculate all terms in these $\mu_{t-\Delta}^{v^i} v_{t-\Delta}^i$, $\eta_{t-\Delta}^e \mu_{t-\Delta}^{\eta^e}$ and $\eta_{t-\Delta}^e \sigma_{t-\Delta}^{\eta^e}$

4a. PDE Value Function Iteration

Short-hand notation:
$\partial_x f$ for $\partial f / \partial x$

- Postulate $v_t^i = v^i(\eta_t^e, t)$
- By Ito's Lemma
- $$\frac{dv_t^i}{v_t^i} = \underbrace{\frac{\partial_t v_t^i + (\eta^e \mu_t^{\eta^e})\partial_\eta v_t^i + \frac{1}{2}\left(\eta_t^e \sigma_t^{\eta^e}\right)^2 \partial_{\eta\eta} v_t^i}{v_t^i}}_{\mu_t^{v^i}} dt + \underbrace{\frac{(\eta^e \sigma_t^{\eta^e})\partial_\eta v_t^i}{v_t^i}}_{\sigma_t^{v^i}} dZ_t$$
- That is,
- $\mu_t^{v^i} v_t^i = \partial_t v_t^i + (\eta^e \mu_t^{\eta^e})\partial_\eta v_t^i + \frac{1}{2}\left(\eta_t^e \sigma_t^{\eta^e}\right)^2 \partial_{\eta\eta} v_t^i$
- $\sigma_t^{v^i} v_t^i = (\eta^e \sigma_t^{\eta^e})\, \partial_\eta v_t^i$
- Equating with Step 3 (plug in $\mu_t^{v^i}$)$\Rightarrow$ "growth equation"

$$\boxed{\begin{aligned}&\partial_t v_t^i + \left(\eta^e \mu_t^{\eta^e} + (1-\gamma)\sigma \eta_t^e \sigma_t^{\eta^e}\right)\partial_\eta v_t^i + \frac{1}{2}\left(\eta_t^e \sigma_t^{\eta^e}\right)^2 \partial_{\eta\eta} v_t^i \\ &= \left(\rho^i - (1-\gamma)(\Phi(\iota_t) - \delta) + \frac{1}{2}\gamma(1-\gamma)\sigma^2\right) v_t^i - \frac{c_t^i}{n_t^i} v_t^i\end{aligned}}$$

4a. PDE Value Fcn: Replacing Terms

$$\partial_t v_t^i + \left(\eta^e \mu_t^{\eta^e} + (1-\gamma)\sigma\eta_t^e \sigma_t^{\eta^e}\right)\partial_\eta v_t^i + \frac{1}{2}\left(\eta_t^e \sigma_t^{\eta^e}\right)^2 \partial_{\eta\eta} v_t^i = \left(\rho^i - (1-\gamma)(\Phi(\iota_t) - \delta) + \frac{1}{2}\gamma(1-\gamma)\sigma^2\right) v_t^i - \frac{c_t^i}{n_t^i} v_t^i$$

1. Replace "blue terms" using results from Step 2.

$$\mu_t^{\eta^e} = (1-\eta_t^e)\left(\varsigma_t^e - \sigma_t^q - \sigma\right)\left(\sigma_t^{\eta^e} - \underbrace{\sigma_t^M}_{=0}\right) - (1-\eta_t^e)\left(\varsigma_t^h - \sigma_t^q - \sigma\right)\left(\sigma_t^{\eta^h} - \underbrace{\sigma_t^M}_{=0}\right) - \left(\frac{C_t^e}{N_t^e} - \frac{C_t}{N_t}\right)$$

$$\sigma_t^{\eta^e} = \frac{\chi_t^e - \eta_t^e}{\eta_t^e}\left(\sigma + \sigma_t^q\right) \qquad \sigma_t^{\eta^h} = -\frac{\eta_t^e}{1-\eta_t^e}\sigma_t^{\eta^e}$$

2. Replace "white terms" using results from Step 3c.

$$\varsigma_t^e = -\sigma_t^{v^e} + \sigma_t^{\eta^e} + \sigma_t^q + \gamma\sigma, \qquad \varsigma_t^h = -\sigma_t^{v^h} + \sigma_t^{\eta^h} + \sigma_t^q + \gamma\sigma$$

$$\frac{C_t^i}{N_t^i} = \frac{(\eta_t^i q_t)^{1/\gamma - 1}}{(v_t^i)^{1/\gamma}}$$

Recall from Ito's Lemma

$$\sigma_t^{v^i} v_t^i = (\eta^e \sigma_t^{\eta^e})\partial_\eta v_t^i$$

$$\frac{C_t}{N_t} = \frac{1}{q_t}\left[\left(\frac{\eta_t^e q_t}{v_t^e}\right)^{1/\gamma} + \left(\frac{(1-\eta_t^e) q_t}{v_t^h}\right)^{1/\gamma}\right]$$

3. Replace "red terms" $\iota_t, \sigma_t^q, \chi_t^e$ (see below)

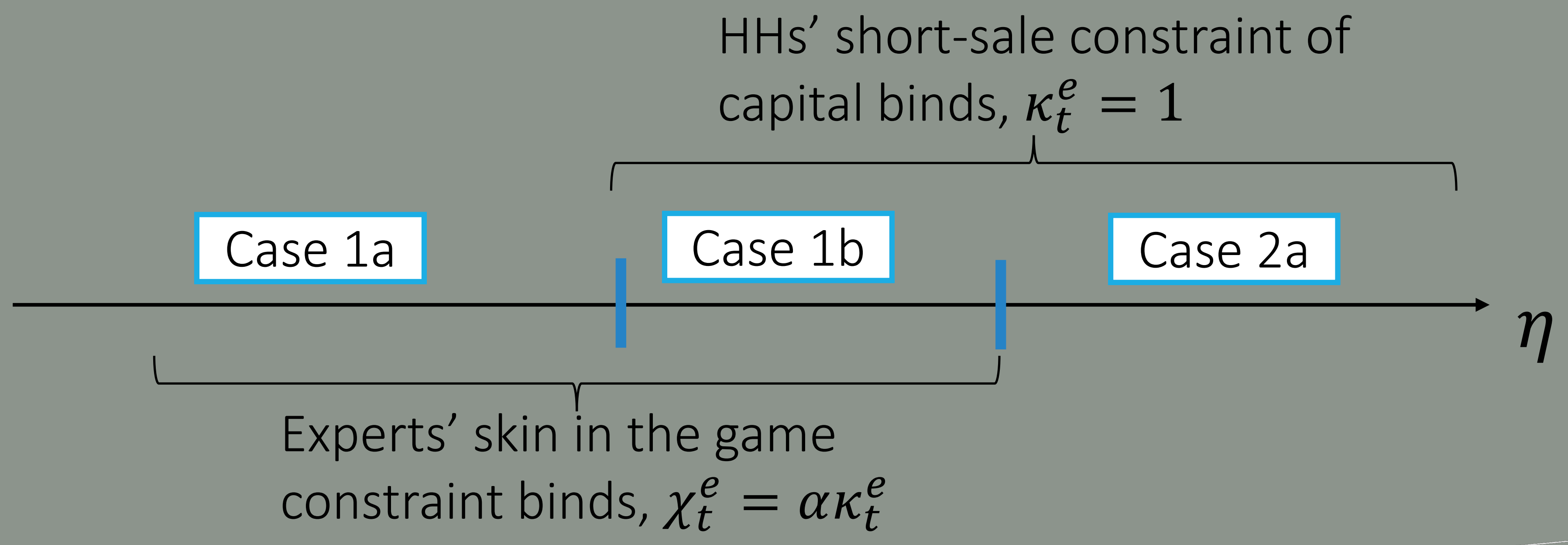
4a. Replacing ι_t

- Recall from optimal re-investment $\Phi'(\iota_t) = 1/q_t$
  - For $\Phi(\iota) = \frac{1}{\phi}\log(\phi\iota + 1) \Rightarrow \boxed{\phi\iota = q - 1}$

4a. Replacing χ, obtain $\boldsymbol{\kappa}$ for good mkt clearing

- Recall from planner's problem (Step 1b)

Cases	$\chi_t^e \geq \alpha \boldsymbol{\kappa}_t^e$	$\kappa_t^e \leq 1$	$\frac{(a^e - a^h)}{q_t} \geq \alpha(\varsigma_t^e - \varsigma_t^h)(\sigma + \sigma_t^q)$	$(\varsigma_t^e - \boldsymbol{\varsigma}_t^h)(\sigma + \sigma_t^q) \geq \mathbf{0}$
1a	=	<	=	>
1b	=	=	>	>
2a	>	=	>	=
impossible				



HHs' short-sale constraint of capital binds, $\kappa_t^e = 1$

Experts' skin in the game constraint binds, $\chi_t^e = \alpha\kappa_t^e$

4a. Replacing χ, obtain κ for good mkt clearing

- Need to determine diff in risk premia $(\varsigma_t^e - \varsigma_t^h)(\sigma + \sigma_t^q)$:
- Recall
  - diff in price of risk: $\varsigma_t^e - \varsigma_t^h = -\sigma_t^{v^e} + \sigma_t^{v^h} + \frac{\sigma_t^{\eta^e}}{1-\eta_t^e}$
  - By Ito's lemma $\sigma_t^{v^e} = \frac{\partial_\eta v_t^e}{v_t^e}\eta_t^e \sigma_t^{\eta^e}$ and $\sigma_t^{v^h} = \frac{\partial_\eta v_t^h}{v_t^h}\eta_t^e \sigma_t^{\eta^e}$

$$\Rightarrow (\varsigma_t^e - \varsigma_t^h)(\sigma + \sigma_t^q) = \left(-\frac{\partial_\eta v_t^e}{v_t^e} + \frac{\partial_\eta v_t^h}{v_t^h} + \frac{1}{(1-\eta_t^e)\eta_t^e}\right)\eta_t^e \sigma_t^{\eta^e}(\sigma + \sigma_t^q)$$

$$= \left(-\frac{\partial_\eta v_t^e}{v_t^e} + \frac{\partial_\eta v_t^h}{v_t^h} + \frac{1}{(1-\eta_t^e)\eta_t^e}\right)(\chi_t^e - \eta_t^e)(\sigma + \sigma_t^q)^2$$

- Note, since $-\frac{\partial_\eta v_t^e}{v_t^e} + \frac{\partial_\eta v_t^h}{v_t^h} + \frac{1}{(1-\eta_t^e)\eta_t^e} > 0$,

$$(\varsigma_t^e - \varsigma_t^h)(\sigma + \sigma_t^q) > 0 \Leftrightarrow \chi_t^e > \eta_t^e \Leftrightarrow \alpha\psi_t^e > \eta_t^e$$

4a. Replacing χ, obtain κ for good mkt clearing

- Determination of κ_t

$$(a^e - a^h)/q_t \geq \alpha\left(-\frac{\partial_\eta v_t^e}{v_t^e} + \frac{\partial_\eta v_t^h}{v_t^h} + \frac{1}{(1-\eta_t^e)\eta_t^e}\right)(\chi_t^e - \eta_t^e)(\sigma + \sigma_t^q)^2$$

with equality if $\kappa_t^e < 1$

- Determination of χ_t^e

$$\chi_t^e = \max\{\alpha\kappa_t^e, \eta_t^e\}$$

4a. Market Clearing

- Output good market

$$\left(\kappa_t^e a^e + (1-\kappa_t^e)a^h - \iota_t\right)K_t = C_t$$

- … jointly restricts κ_t and q_t

$$\kappa_t a^e + (1-\kappa_t)a^h - \iota(q_t) = \underbrace{\left(\frac{\eta_t^e q_t}{v_t^e}\right)^{1/\gamma}}_{C_t^e/K_t} + \underbrace{\left(\frac{(1-\eta_t^e)q_t}{v_t^h}\right)^{1/\gamma}}_{C_t^h/K_t}$$

4a. Market Clearing

- Output good market

$$\left(\kappa_t^e a^e + (1 - \kappa_t^e) a^h - \iota_t\right) K_t = C_t$$

- Capital market is taken care off by price taking social planner approach

$$1 - \theta_t^e = \frac{\kappa_t^e q_t K_t}{\eta_t^e q_t K_t}$$

- Risk-free debt also taken care off by price taking social planner approach
  (would be cleared by Walras Law anyways)

4a. $\sigma^q(q, q')$

- Recall from “amplification slide” – Step 2

$$\sigma + \sigma_t^q = \frac{\sigma}{1 - \frac{q'\left(\eta_t^e\right)}{q/\eta_t^e}\frac{\chi_t^e - \eta_t^e}{\eta_t^e}}$$

$$\boxed{\sigma^q = \frac{q'(\eta_t^e)}{q\left(\eta_t^e\right)}(\chi_t^e - \eta_t^e)(\sigma + \sigma_t^q)}$$

- Now all red terms are replaced and we can solve …

4b. Algorithm – Static Step

- Suppose we know functions $v^e(\eta^e), v^h(\eta)$, have five static conditions:

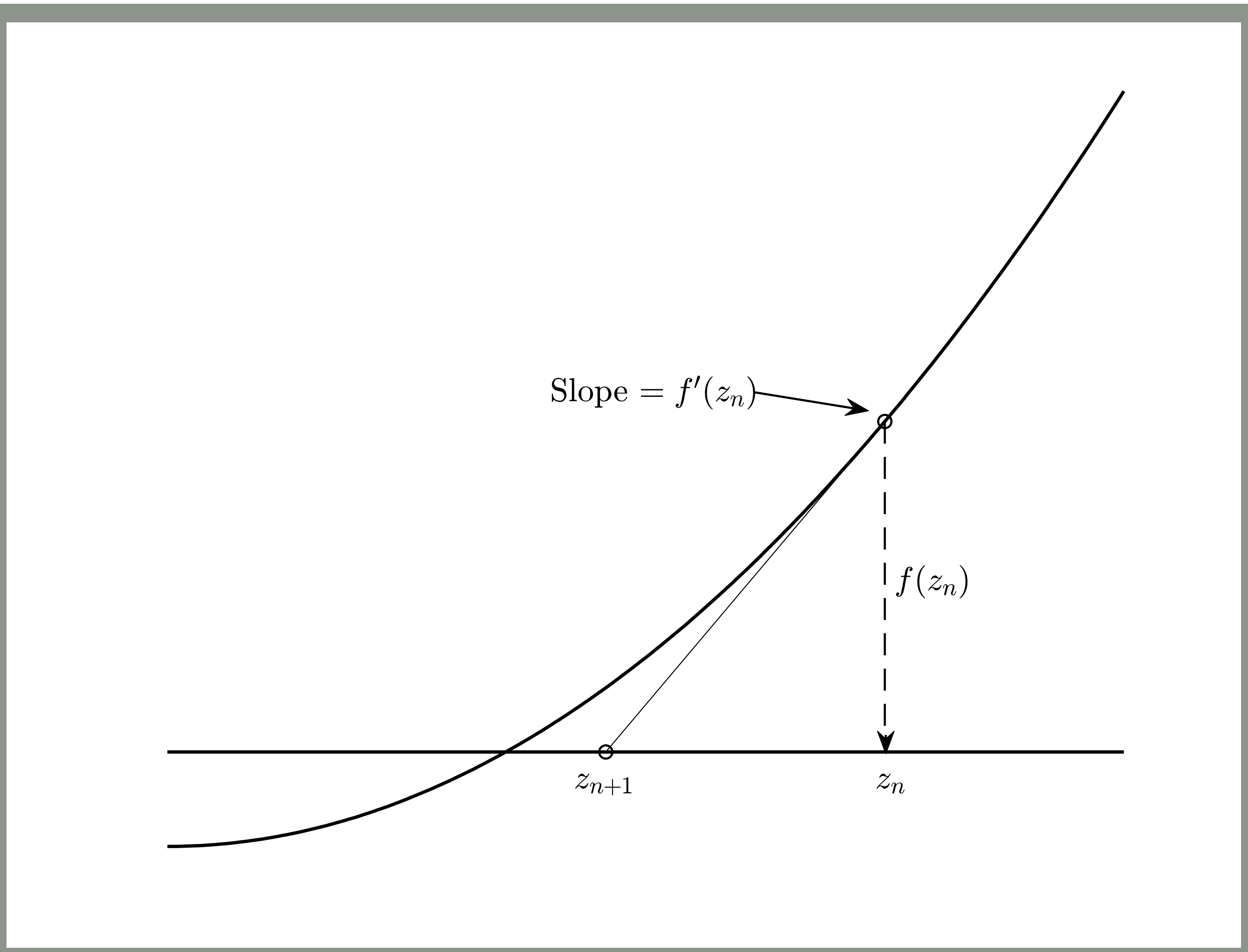
1. $\phi\iota_t = q_t - 1$
2. Planner condition for κ_t^e
3. Planner condition for χ_t^e
4. $\kappa_t^e a^e + (1-\kappa_t^e)a^h - \iota(q_t) = \underbrace{\left(\frac{\eta_t^e q_t}{v_t^e}\right)^{\frac{1}{\gamma}}}_{C_t^e/K_t} + \underbrace{\left(\frac{(1-\eta_t^e)q_t}{v_t^h}\right)^{\frac{1}{\gamma}}}_{C_t^h/K_t}$
5. $\sigma^q = \frac{q'(\eta_t^e)}{q(\eta_t^e)}(\chi_t^e - \eta_t^e)(\sigma + \sigma_t^q)$

$\Rightarrow$ Get $q(\eta^e)$, $\kappa^e(\eta^e)$, $\sigma^{\eta^e}(\eta^e)$

- Start at $q(0)$, solve to the right, use different procedure for two η regions depending on κ:

1. While $\kappa^e < 1$, solve ODE for $q(\eta^e)$:
   - For given $q(\eta)$, plug optimal investment (1) into (4)
   - Plug planner condition (3) into (2) and (5)
   - Solve ODE using three equilibrium condition (2),(4) and (5) via Newton's method (see next slide)
2. When $\kappa = 1$, (2) is no longer informative, solve (1) and (4) for $q(\eta)$

4b. *Aside:* Newton's Method



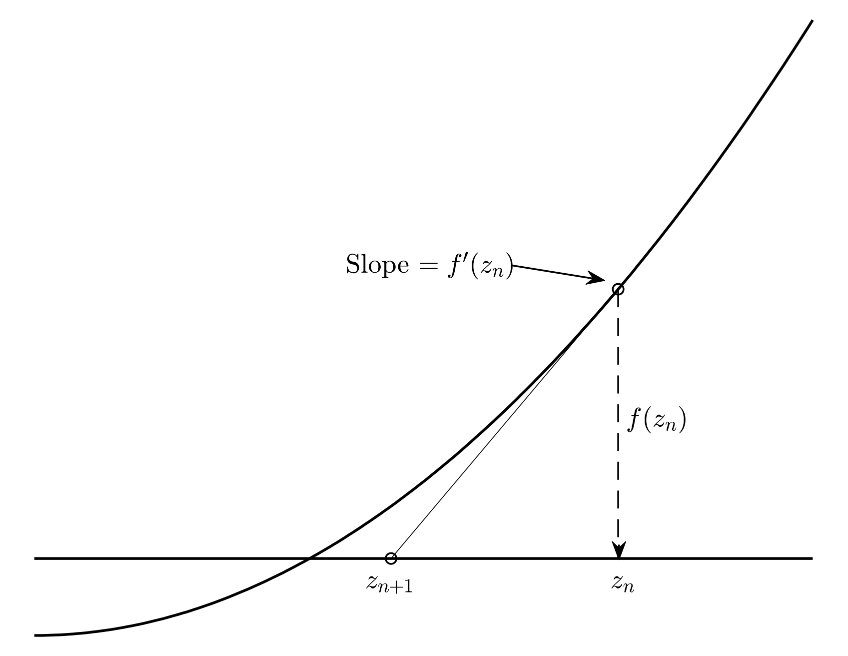
- Find the root of equation system $F(\boldsymbol{z}_n) = 0$ via iterative method

$$\mathrm{z}_{n+1} = \mathrm{z}_\mathrm{n} - J_n^{-1} F(\boldsymbol{z}_n)$$

Where J_n is the Jacobian matrix, i.e., $J_{ij} = \partial f_i(\boldsymbol{z}) / \partial z_j$.

- Newton's method does not guarantee global convergence.
- commonly take several-step iteration.

4b. *Aside:* Newton's Method



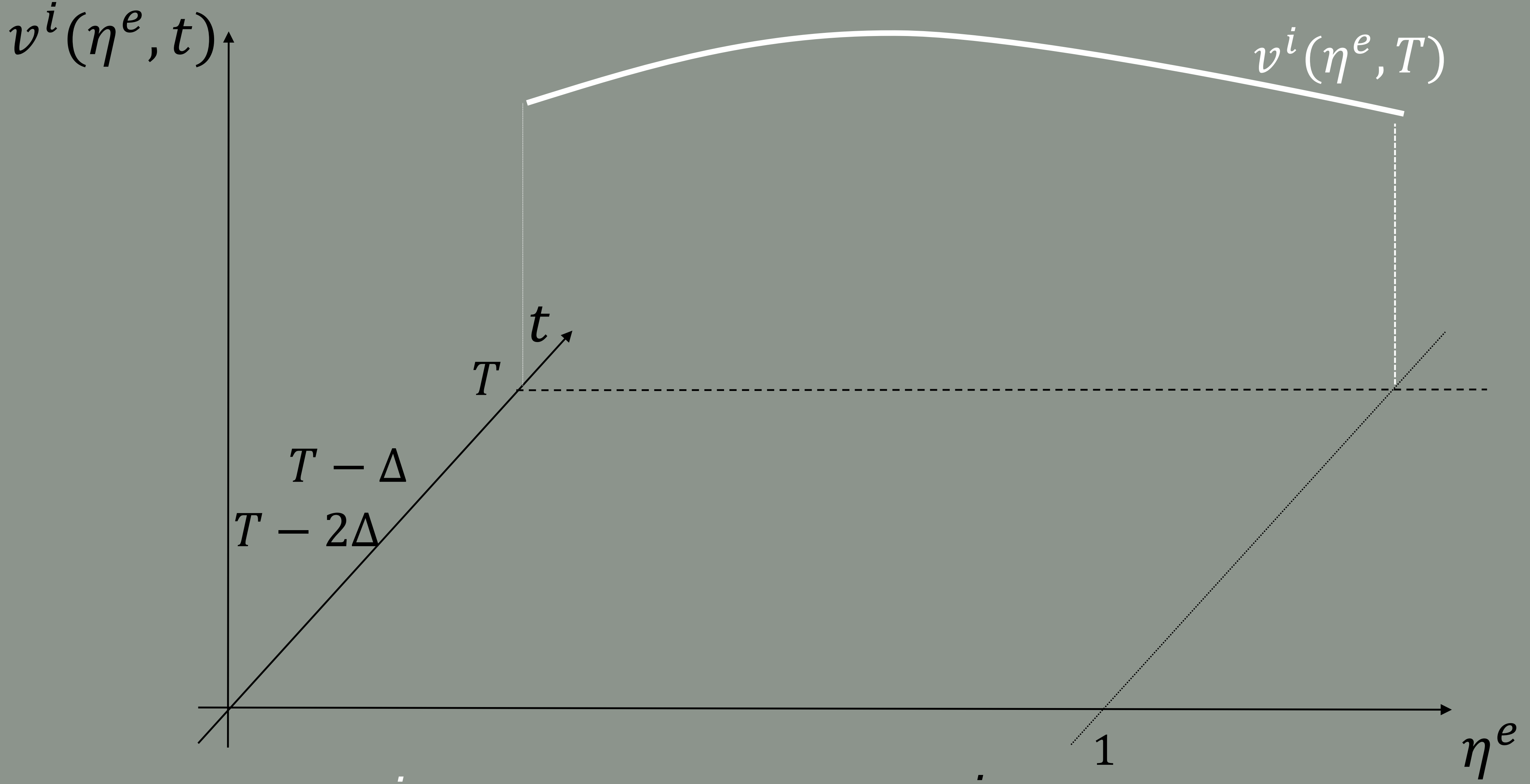
$$\begin{bmatrix} \text{market clearing condtion} \\ \text{amplification condition} \\ \text{Planner condition for } \kappa_t^e \end{bmatrix}$$

$$\boldsymbol{z}_n = \begin{bmatrix} q_t \\ \kappa_t^e \\ \sigma + \sigma_t^q \end{bmatrix},$$

$$F(\boldsymbol{z}_n) = \begin{bmatrix} \kappa_t^e a^e + (1-\kappa_t^e)a^h - \iota(q_t) - \left(\frac{\eta_t^e q_t}{v_t^e}\right)^{\frac{1}{\gamma}} + \left(\frac{(1-\eta_t^e)q_t}{v_t^h}\right)^{\frac{1}{\gamma}} \\ q'(\eta_t^e)(\chi_t^e - \eta_t^e)(\sigma + \sigma_t^q) - \sigma^q q(\eta_t^e) \\ (a^e - a^h) - \alpha q_t \left(-\frac{\partial_\eta v_t^e}{v_t^e} + \frac{\partial_\eta v_t^h}{v_t^h} + \frac{1}{(1-\eta_t^e)\eta_t^e}\right)(\chi_t^e - \eta_t^e)(\sigma + \sigma_t^q)^2 \end{bmatrix}$$

Plug in blue terms from optimal investment and Planner condition for χ_t^e

4. Value Function Iteration – Big Picture

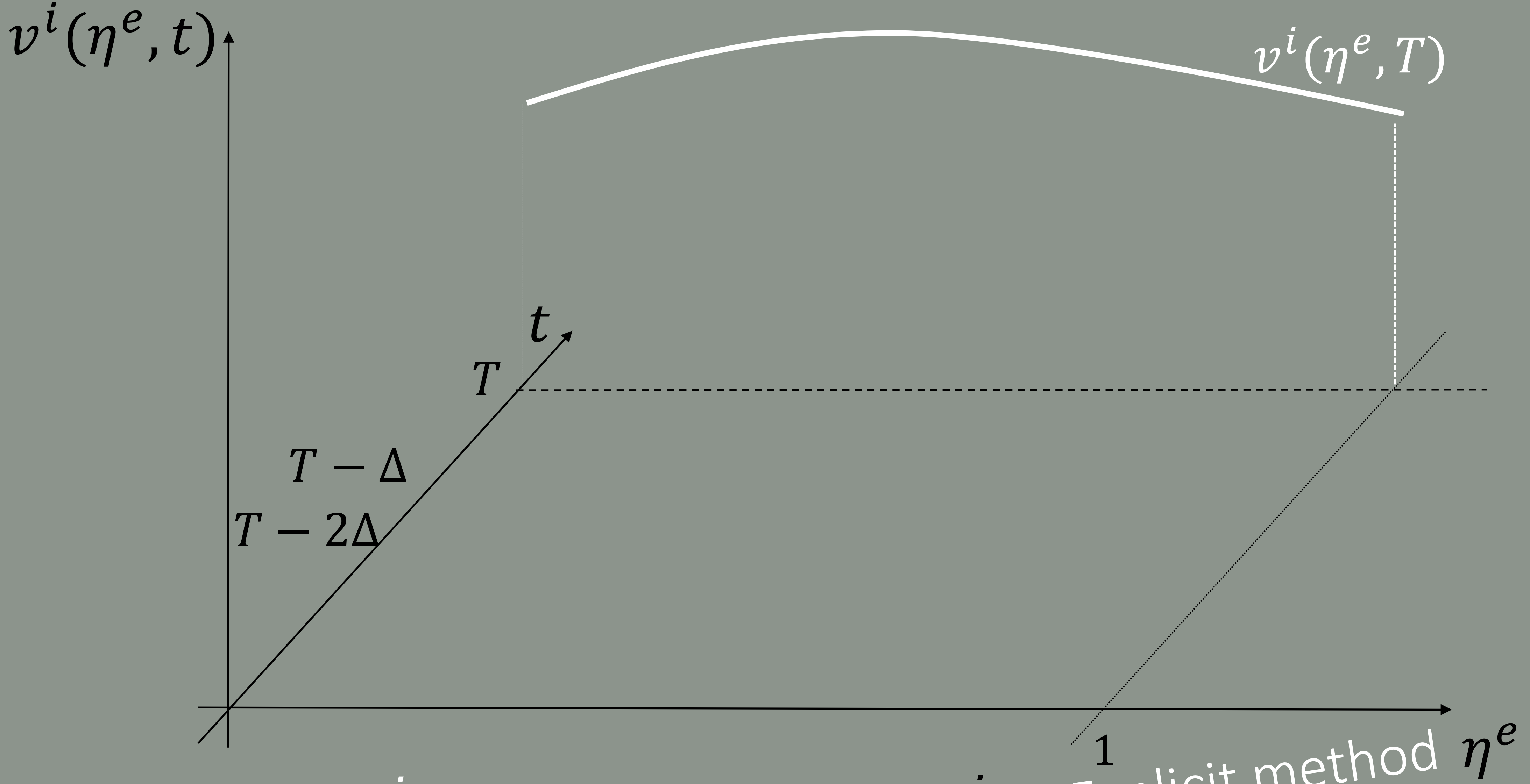


- For given $v^i(\eta^e,T)$, derive SDF ξ^i_T
- Optimal investment, portfolio, consumption, at T as fcn. of η^e

4. Market clearing at T obtain PDE coefficient at T

(pretend they are constant between T & $T-\Delta$)

4. Value Function Iteration – Big Picture



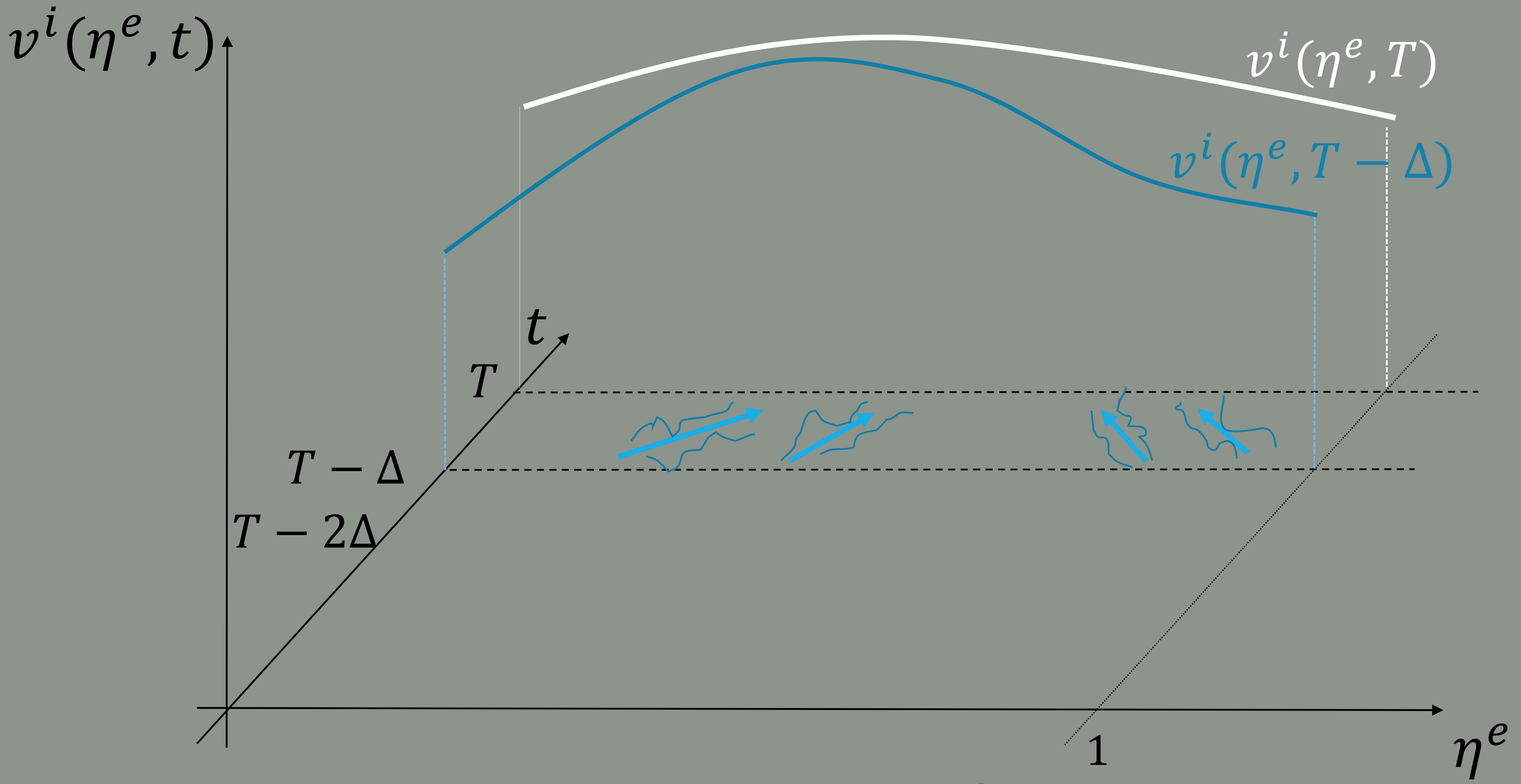
Explicit method

Implicit method uses $T-\Delta$

- For given $v^i(\eta^e,T)$, derive SDF ξ_T^i
- Optimal investment, portfolio, consumption, at T as fcn. of η^e

4. Market clearing at T obtain PDE coefficient at T
(pretend they are constant between T & $T-\Delta$)

4. Value Function Iteration – Big Picture



- Obtain descaled value function $v^i(\eta^e, T-\Delta)$
- Repeat previous steps

4b. Pseudocode

```
1. Initialize two terminal functions $v^e(\boldsymbol{\eta^e}, T), v^h(\boldsymbol{\eta^e}, T)$ over $\boldsymbol{\eta^e}$-grid $(\eta_1^e, \eta_2^e, \cdots \eta_n^e)$
2. For $t \in \{T, T - \Delta t, T - 2\Delta t, \cdots 0\}$
   a. Compute $\partial_\eta v_t^i$ by first-order difference
   b. Start at $\eta_1^e = 0$ (autarky economy), find $q(0,t), \kappa^e(0,t), \sigma^q(0,t)$.
   c. For $\eta_i^e \in \{\eta_2^e, \eta_3^e, \cdots, \eta_n^e\}$
      i.  If $\kappa^e(\eta_i^e, t) < 1$, solve ODE for $q(\eta_i^e, t), \kappa^e(\eta_i^e, t), \sigma^q(\eta_i^e, t)$ using Newton's method.
      ii. If $\kappa^e(\eta_i^e, t) = 1$, solve ODE for $q(\eta_i^e, t)$ from market clearing equation via Newton's
          method. Then find $\sigma^q(\eta_i^e, t)$ using amplification function
   d. Find $\mu^{\eta^e}(\boldsymbol{\eta^e}, t), \sigma^{\eta^e}(\boldsymbol{\eta^e}, t), \mu^{v^i}(\boldsymbol{\eta^e}, t)$.
   e. Update: obtain $v^e(\boldsymbol{\eta^e}, t - \Delta t)$ from $v^e(\boldsymbol{\eta^e}, t)$ via finite difference method (do $\mu_t^v v_t =$
      $\partial_t v_t^i + \mu_t^{\eta^e} \eta_t^e (\partial_\eta v_t^i) + \frac{1}{2}\left(\sigma_t^{\eta^e} \eta_t^e\right)^2 (\partial_{\eta\eta} v_t^i)$ for one time-step)
```

Upwind scheme: $\partial_\eta f(n,t) = \begin{cases} \dfrac{f(\eta+1,t) - f(\eta,t)}{\Delta\eta} & \text{for} \quad \mu^\eta \eta > 0 \\ \dfrac{f(\eta,t) - f(\eta-1,t)}{\Delta\eta} & \text{for} \quad \mu^\eta \eta < 0 \end{cases}$

Implicit scheme: $\partial_t f(\eta, t) = \frac{f(\eta,t+1) - f(\eta,t)}{\Delta t}$

2-order difference: $\partial_n f(\eta, t) = \frac{f(\eta+1) - 2f(\eta) + f(\eta-1)}{(\Delta\eta)^2}$

Financial and Monetary Economics

Eco529 Fall 2020

Lecture 03: Endogenous Risk Dynamics

Solutions

Markus K. Brunnermeier

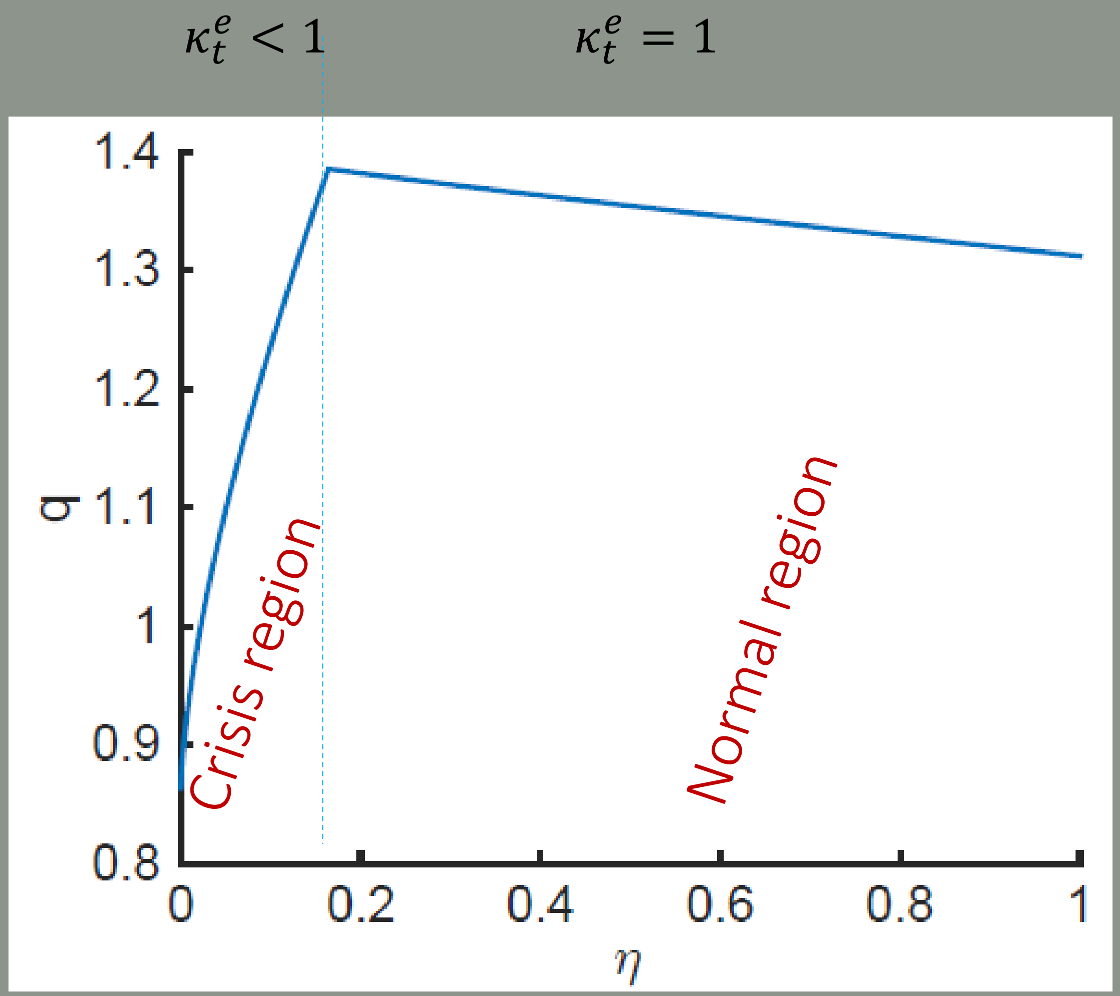
Princeton University

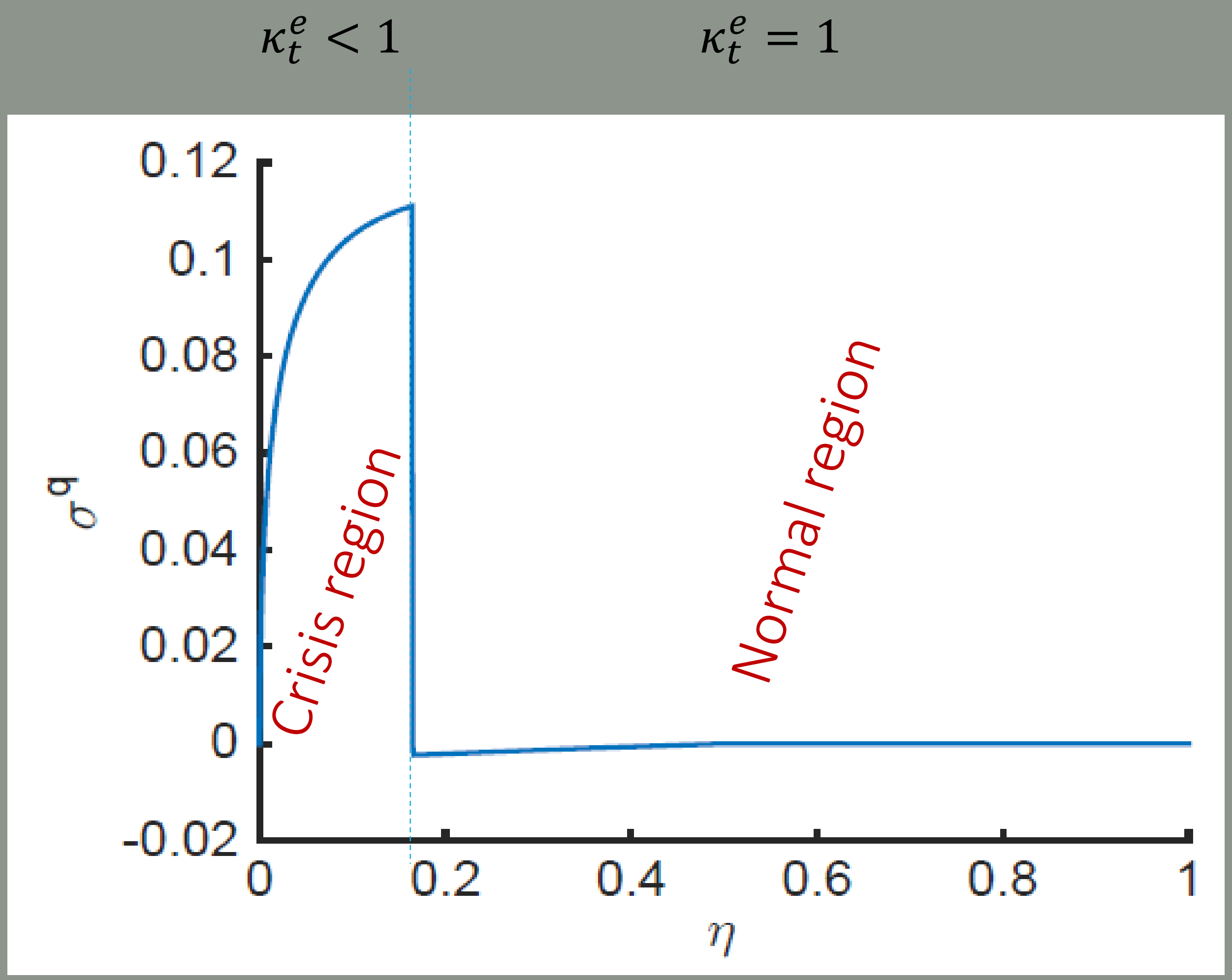
Princeton, September 2020

Solution

- Price of capital

Amplification

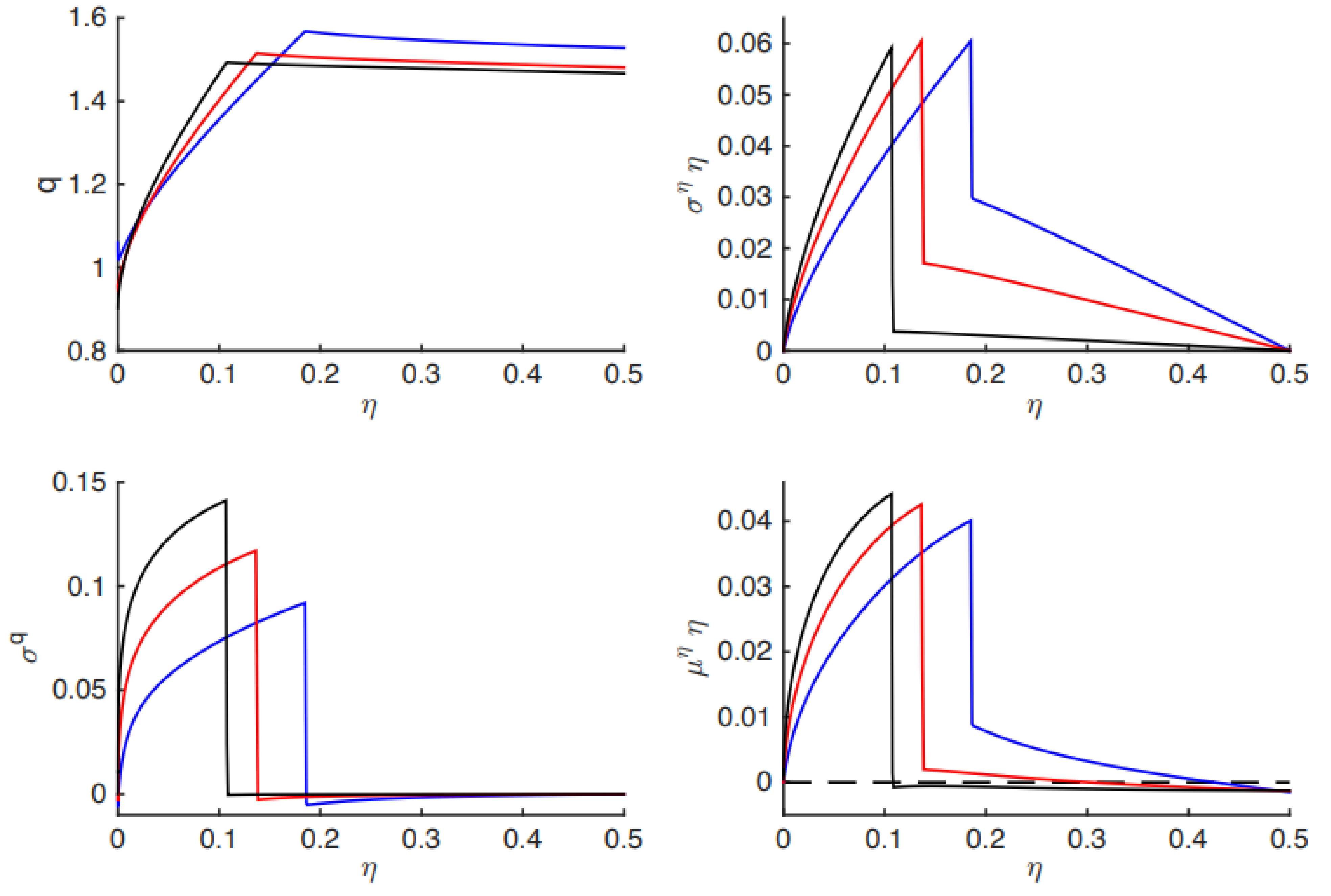




Parameters: $\rho^e = .06, \rho^h = .05, a^e = .11, a^h = .03,$
$\delta = .05, \sigma = .01, \alpha = .50, \gamma = 2, \phi = 10$

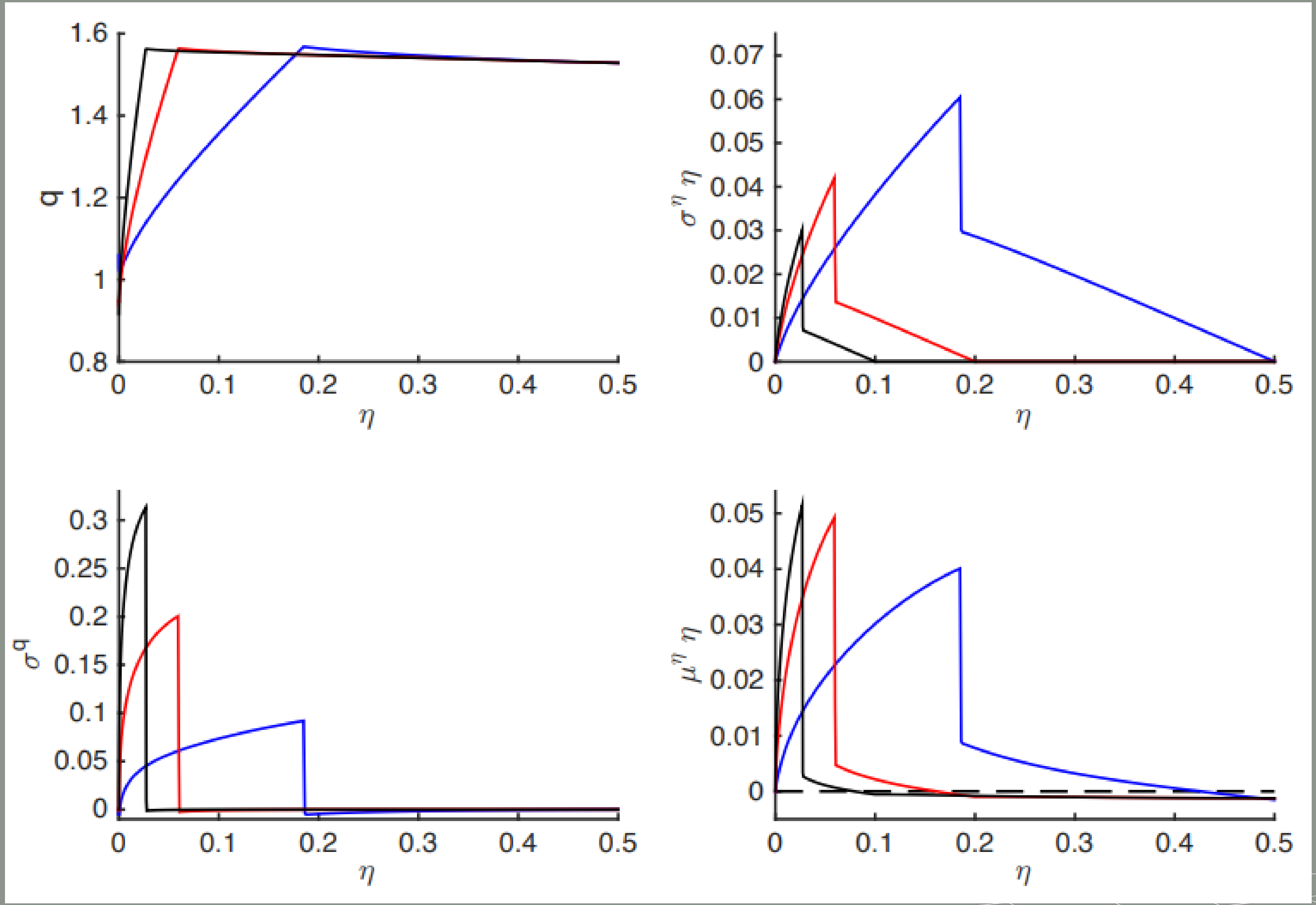
Volatility Paradox

- Comparative Static w.r.t. $\sigma = .01, .05, .1$



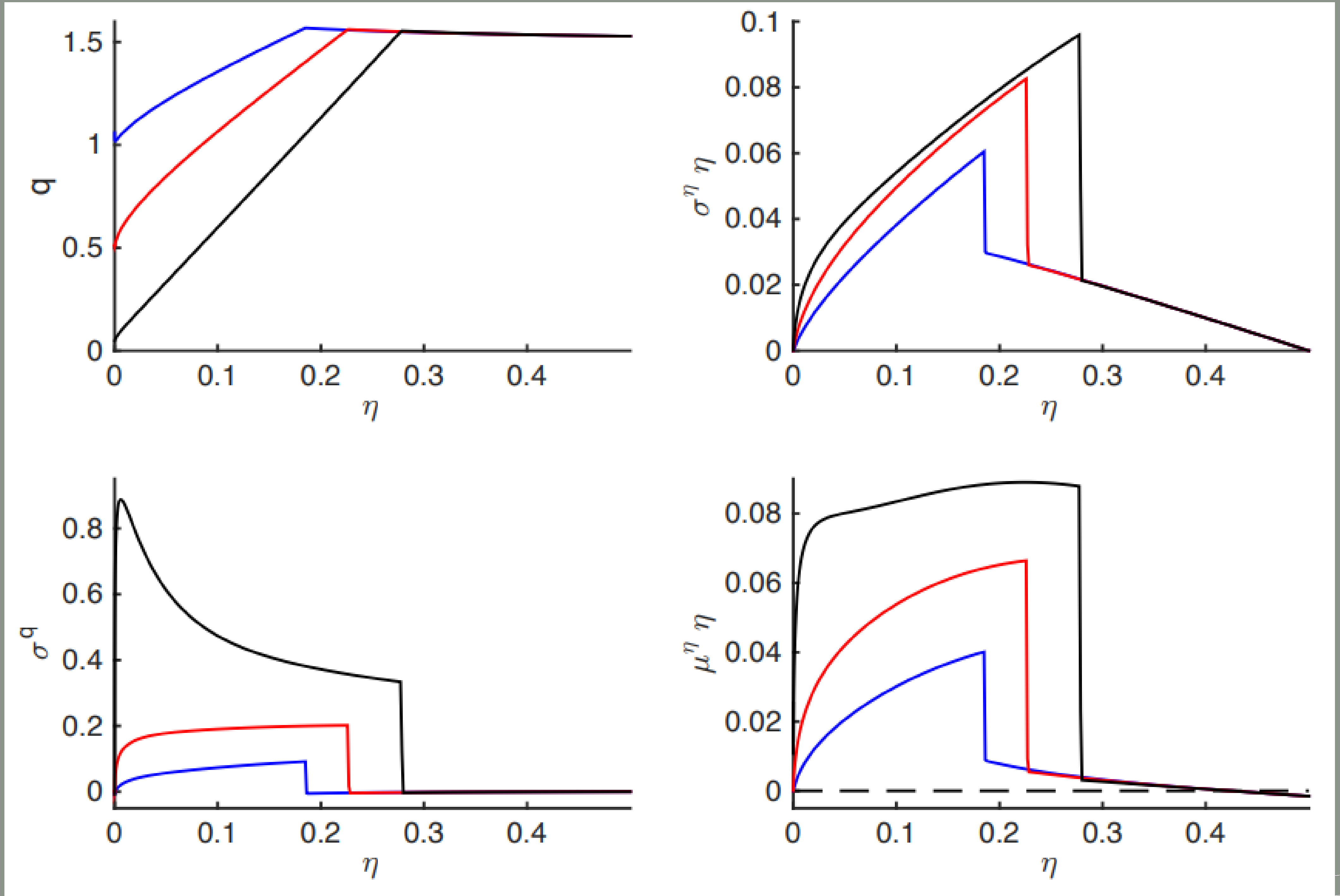
Risk Sharing via Outside Equity

- Comparative Static w.r.t. Risk sharing $\alpha = .1, .2, .5$ (skin the game constraint)



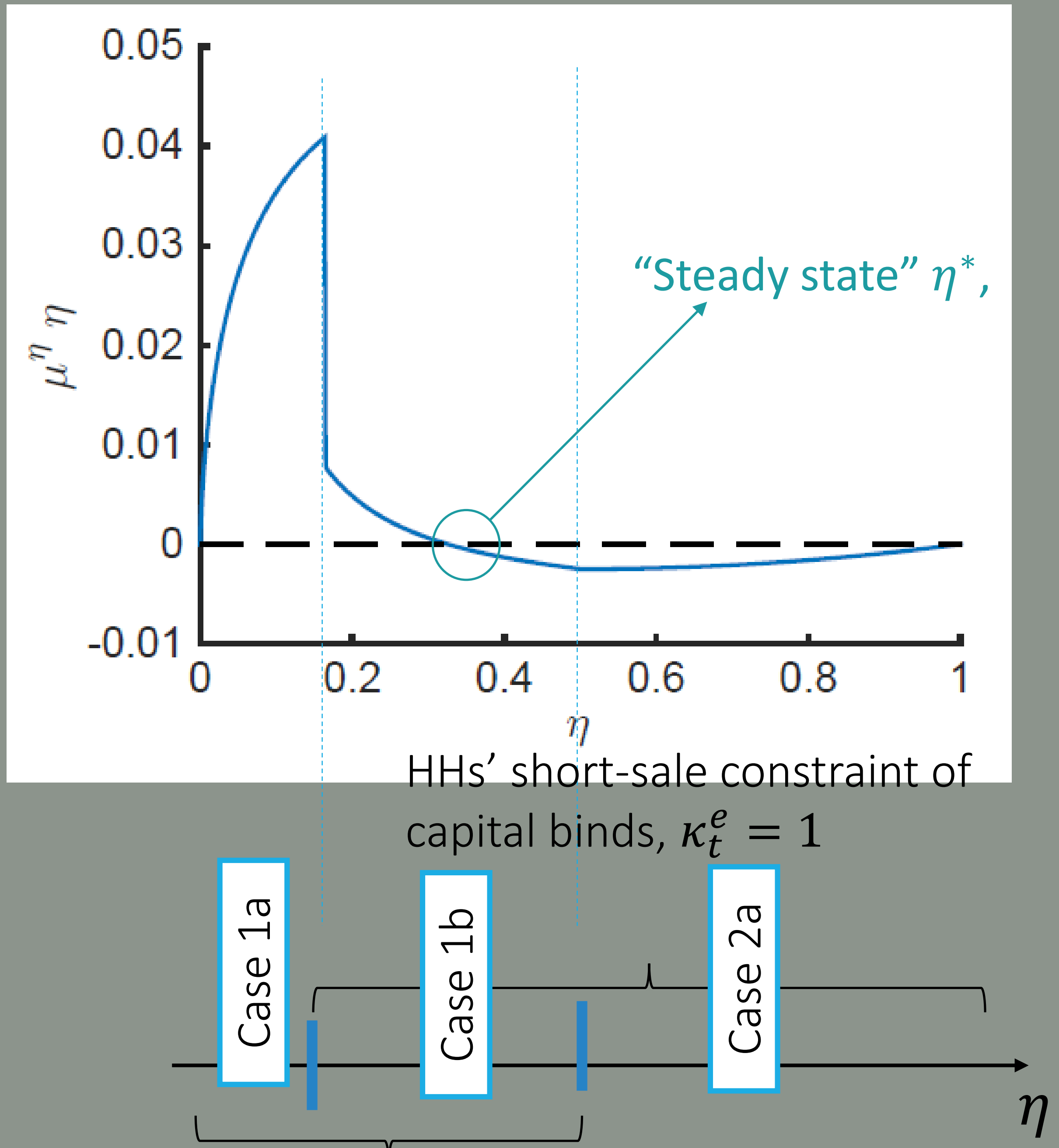
Market Liquidity

- Comparative static w.r.t. $a^h = .03, -.03, -.09$



From $\mu^{\eta^e}(\eta^e)$ & $\sigma^{\eta^e}(\eta^e)$ to Stationary Distribution

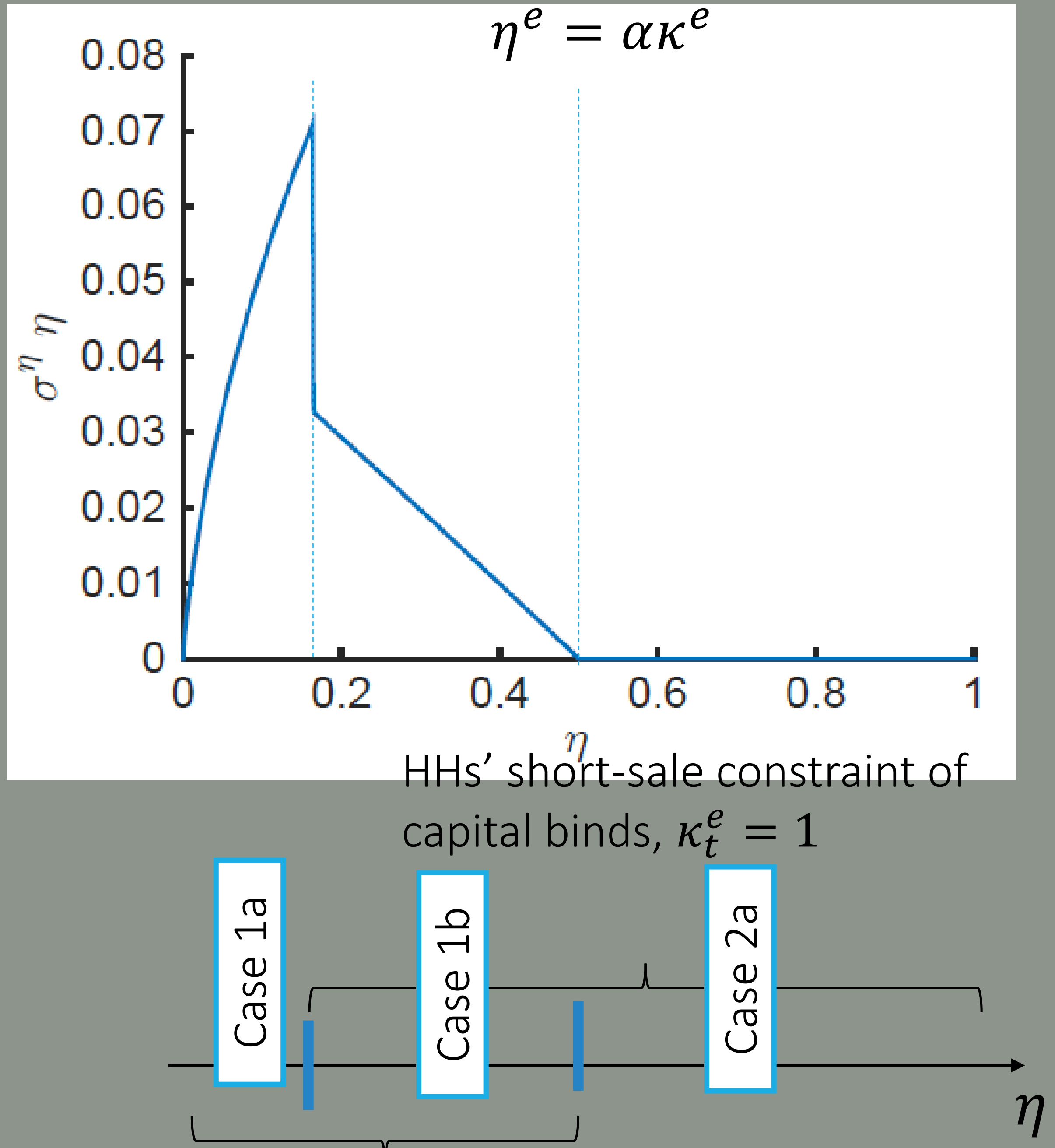
- Drift and Volatility of η^e



“Steady state” η^*,

HHs’ short-sale constraint of capital binds, $\kappa_t^e = 1$

Experts’ skin in the game constraint binds, $\chi_t^e = \alpha\kappa_t^e$



HHs’ short-sale constraint of capital binds, $\kappa_t^e = 1$

Experts’ skin in the game constraint binds, $\chi_t^e = \alpha\kappa_t^e$

Solving MacroModels Step-by-Step

0. Postulate aggregates, price processes & obtain return processes
1. For given C/N-ratio and SDF processes for each i *finance block*
    a. Real investment ι + Goods market clearing *(static)*
    - *Toolbox 1:* Martingale Approach, HJB vs. Stochastic Maximum Principle Approach

    b. Portfolio choice θ + Asset market clearing or
    Asset allocation κ & risk allocation χ
    - *Toolbox 2:* "price-taking social planner approach" – Fisher separation theorem

    - *Toolbox 3:* Change in numeraire to total wealth (including SDF)
2. Evolution of state variable η (and K) *forward equation*
3. Value functions *backward equation*
    a. Value fcn. as fcn. of individual investment opportunities ω
    - *Special cases: log-utility, constant investment opportunities*

    b. Separating value fcn. $V^i(n^{\tilde{i}}; \eta, K)$ into $v^i(\eta)u(K)$
    c. Derive C/N-ratio and ς price of risk
4. Numerical model solution
    a. Transform BSDE for separated value fcn. $v^i(\eta)$ into PDE
    b. Solve PDE via value function iteration
5. KFE: Stationary distribution, Fan charts

5. Kolmogorov Forward Equation

- Given an initial distribution $f(\eta,0)=f_0(\eta)$, the density diffusion follows PDE

$$\frac{\partial f(\eta,t)}{\partial t}=\frac{\partial[f(\eta,t)\mu(\eta)]}{\partial\eta}+\frac{1}{2}\frac{\partial^2[f(\eta,t)\sigma^2(\eta)]}{\partial\eta^2}$$

- “Kolmogorov Forward Equation” is in physics referred to as “Fokker-Planck Equation”

- Corollary: if stationary distribution $f(\eta)$ exists, it satisfies the ODE

$$0=\frac{\partial[f(\eta,t)\mu(\eta)]}{\partial\eta}+\frac{1}{2}\frac{\partial^2[f(\eta,t)\sigma^2(\eta)]}{\partial\eta^2}$$

5. Kolmogorov Forward Equation

- Kolmogorov forward differential operator T (hard to discretize)

$$Tf := \frac{\partial}{\partial \eta}[\mu f] + \frac{1}{2}\frac{\partial^2}{\partial \eta^2}[\sigma^2 f]$$

- Shortcut: Kolmogorov backward differential operator S

$$Sg := \mu \frac{\partial}{\partial \eta} g + \sigma^2 \frac{1}{2}\frac{\partial^2}{\partial \eta^2} g$$

- KFE is the adjoint equation to the KBE, T is the adjoint of S.
- Approximate operator S with discretization matrix A using finite difference method.
- A can be interpreted as the transition matrix of a continuous-time Markov chain.
- adjoints in finite-dimensional space are matrix transposes, A^T can approximate operator T.

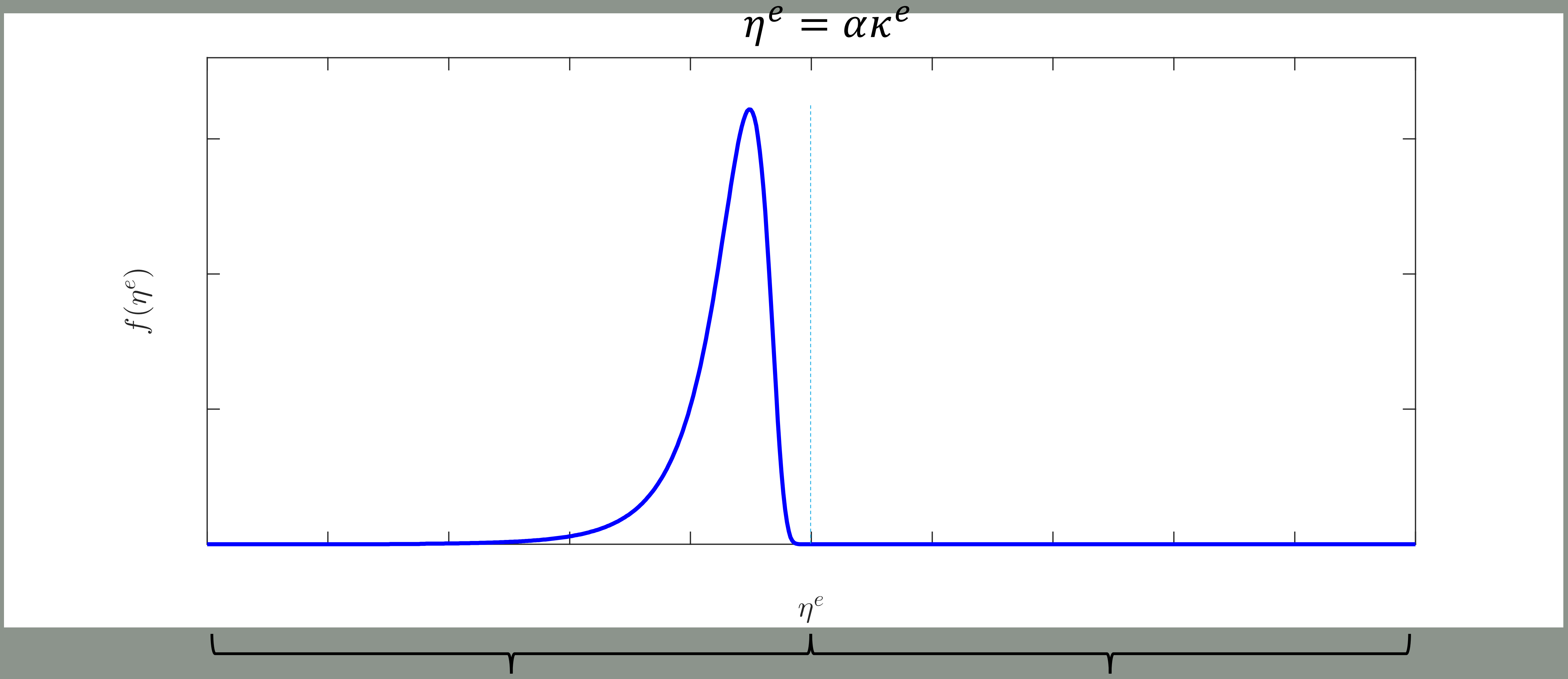
5. Kolmogorov Forward Equation

- Solving method:
  1. Approximate operator S with A using finite difference method.
  2. For stationary distribution, $A^T f = 0$,
     - find kernel space of A^T
     - normalized the space vector to density function.
  3. For time-dependent KFE, $A^T f = f_t$,
     - Solve the PDE as we did with value function, but move forward.
- Alternative method: Monte Carlo Simulation of SDE

  (high computing complexity for high dimensions…)

5. Stationary Distribution

- Stationary distribution of η^e

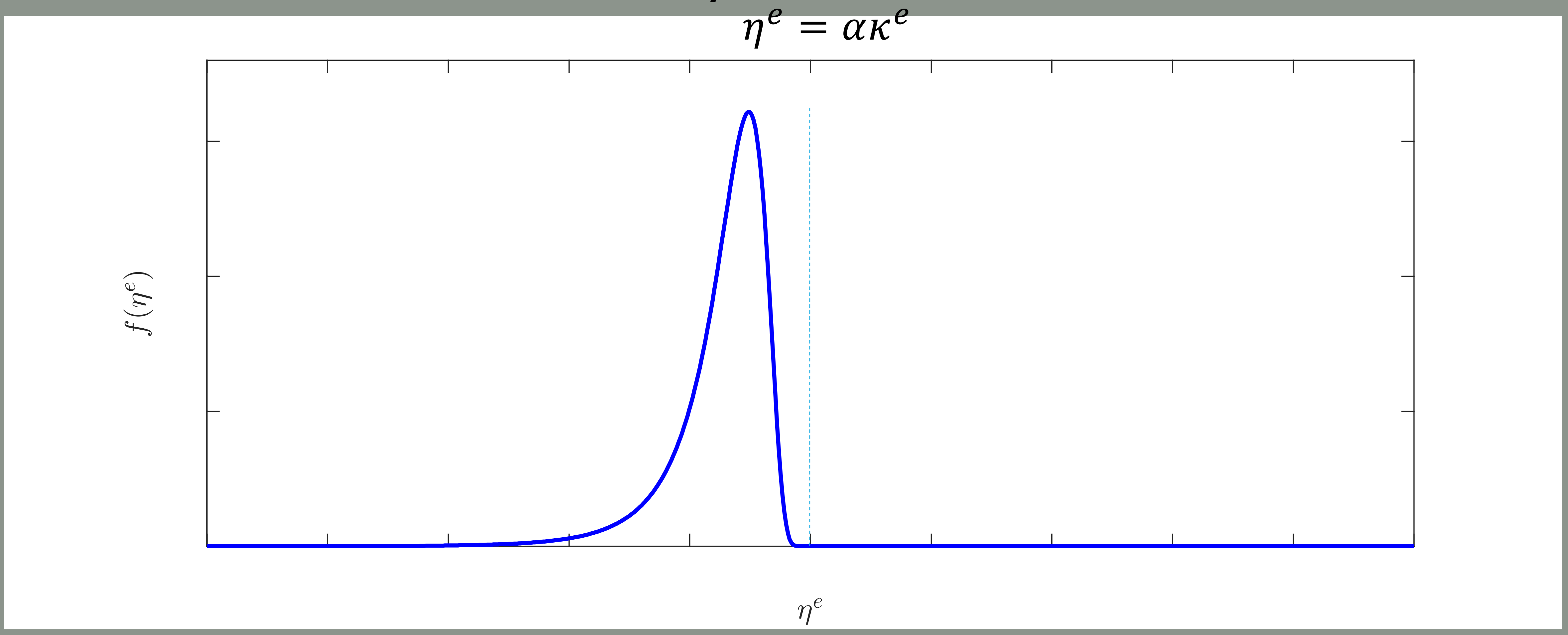


Experts' skin in the game constraint binds $\chi_t^e = \alpha\kappa_t^e$

Perfect risk-sharing region (infeasible)

5. Stationary Distribution

- Stationary distribution of η^e



Experts' skin in the game constraint binds $\chi_t^e = \alpha\kappa_t^e$

Perfect risk-sharing region (infeasible)

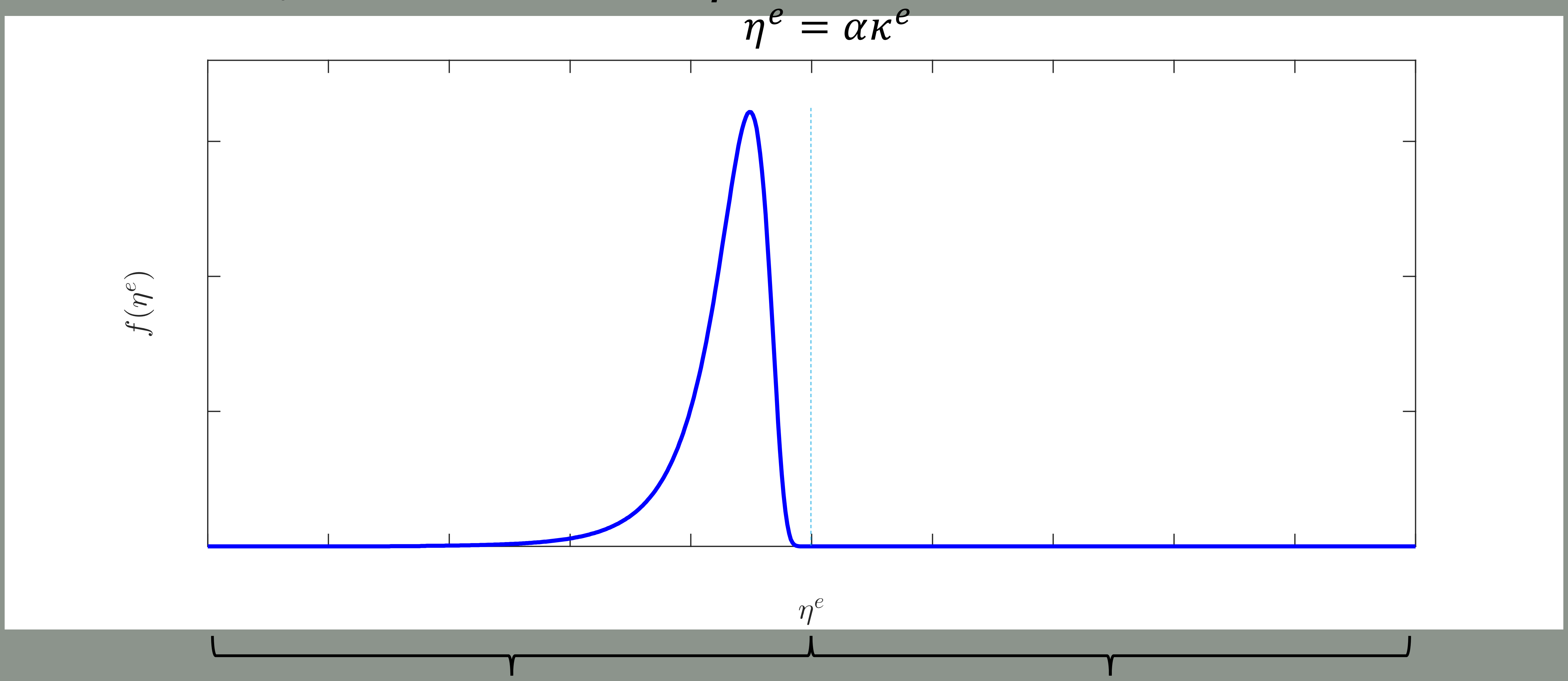
Poll 97: Is the constraint always (not a occasionally binding)

a) yes

b) no, only for some parameters $\rho^e > \rho^h$

5. Stationary Distribution

- Stationary distribution of η^e



Experts' skin in the game constraint binds $\chi_t^e = \alpha\kappa_t^e$

Perfect risk-sharing region (infeasible)

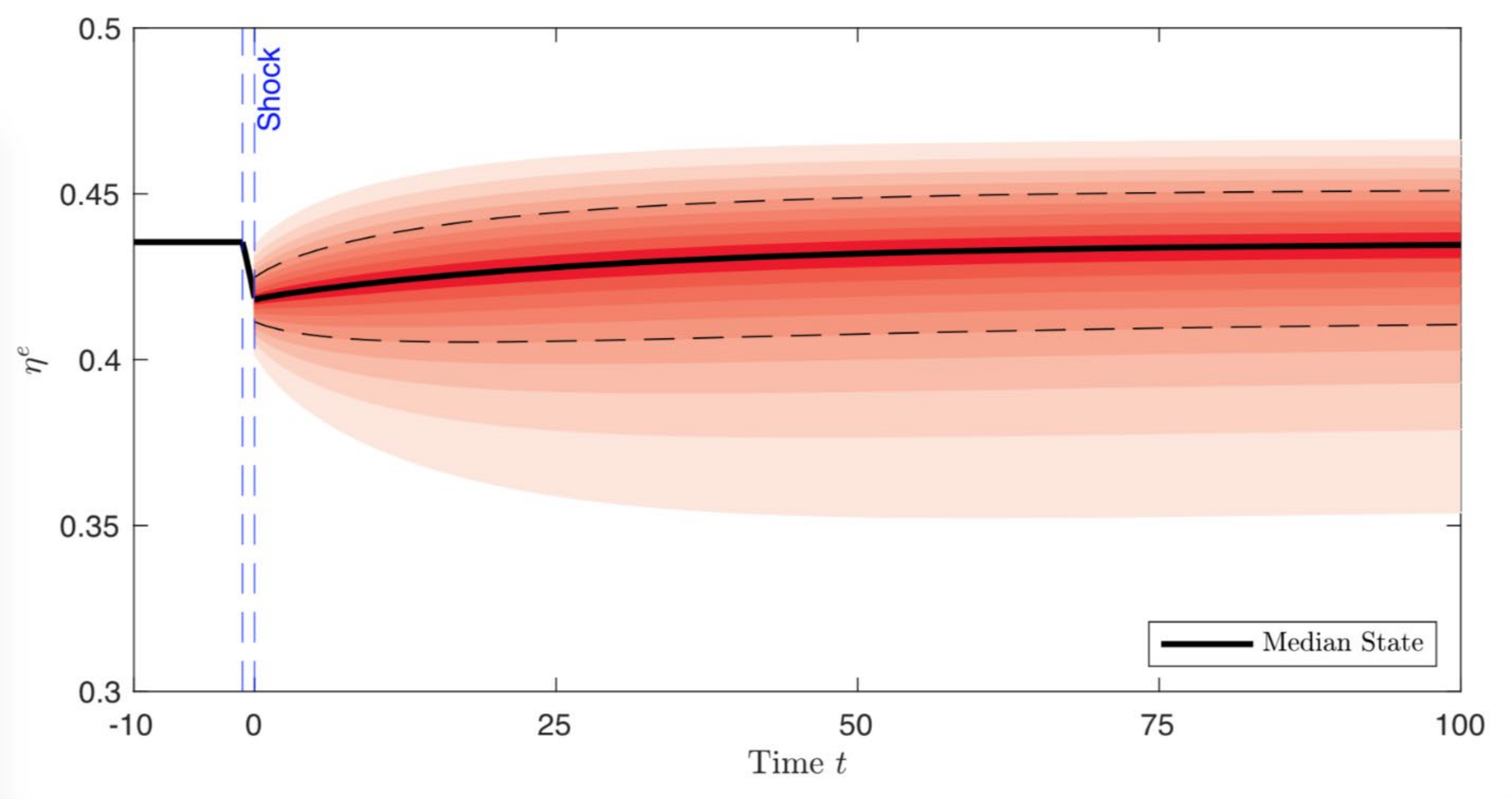
Poll 98: What happens for $\rho^e = \rho^h$

a) experts take over the economy, $\eta \to 1$

b) there is a steady state at $\eta = \alpha$

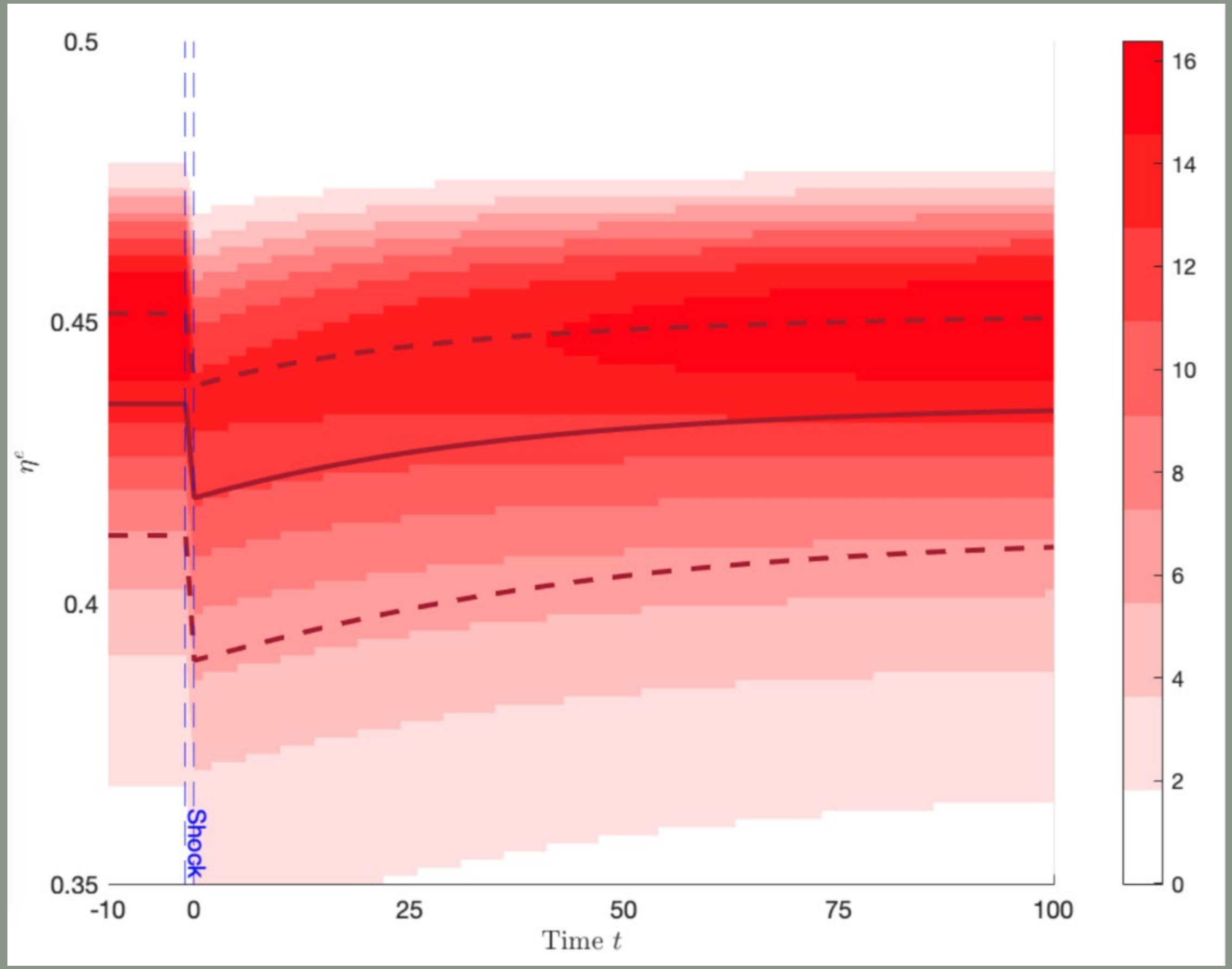
5. Fan chart and distributional impulse response

- … the theory to Bank of England's empirical fan charts
- Starts at η_0, the median of stationary distribution
- Simulate a shock at 1% quantile of original Brownian shock ($dZ_t = -2.32\, dt$) for a period of $\Delta t = 1$.
- Converges back to stationary distribution



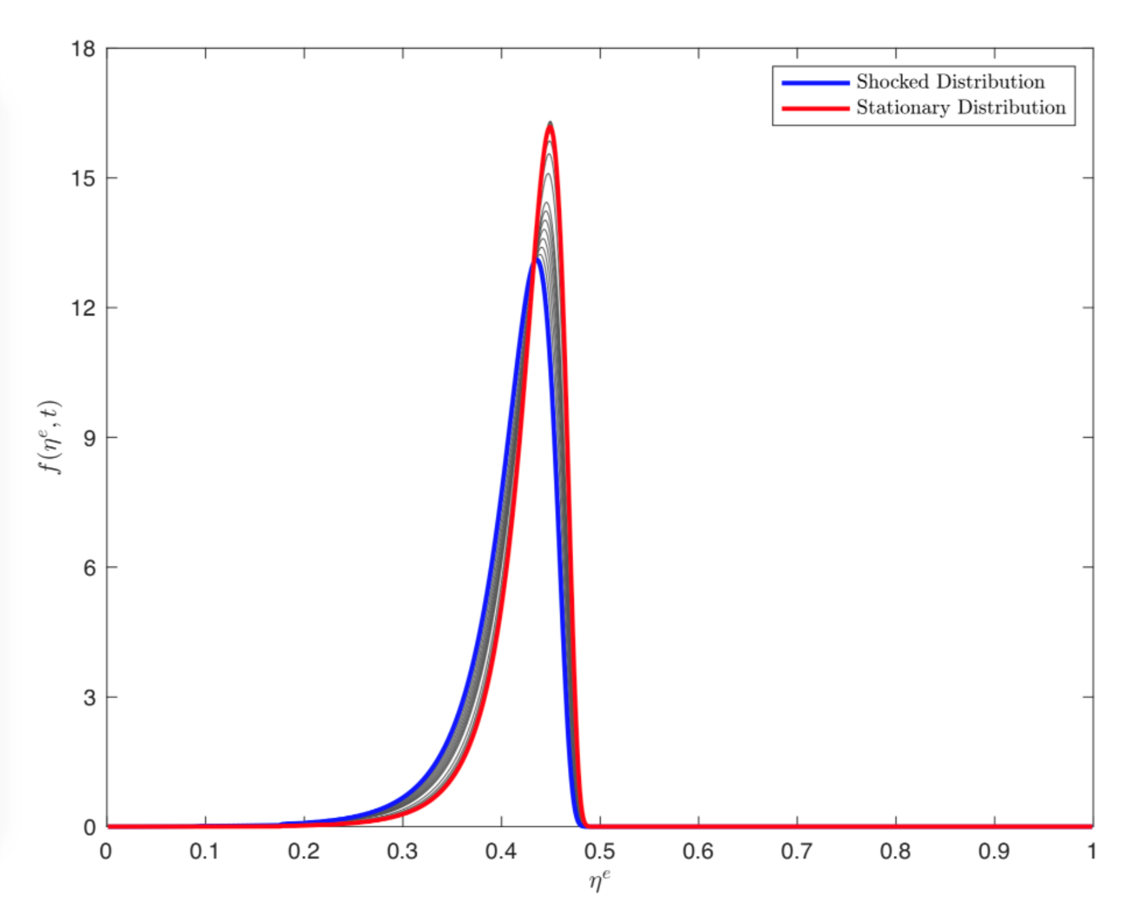
5. Fan chart and distributional impulse response

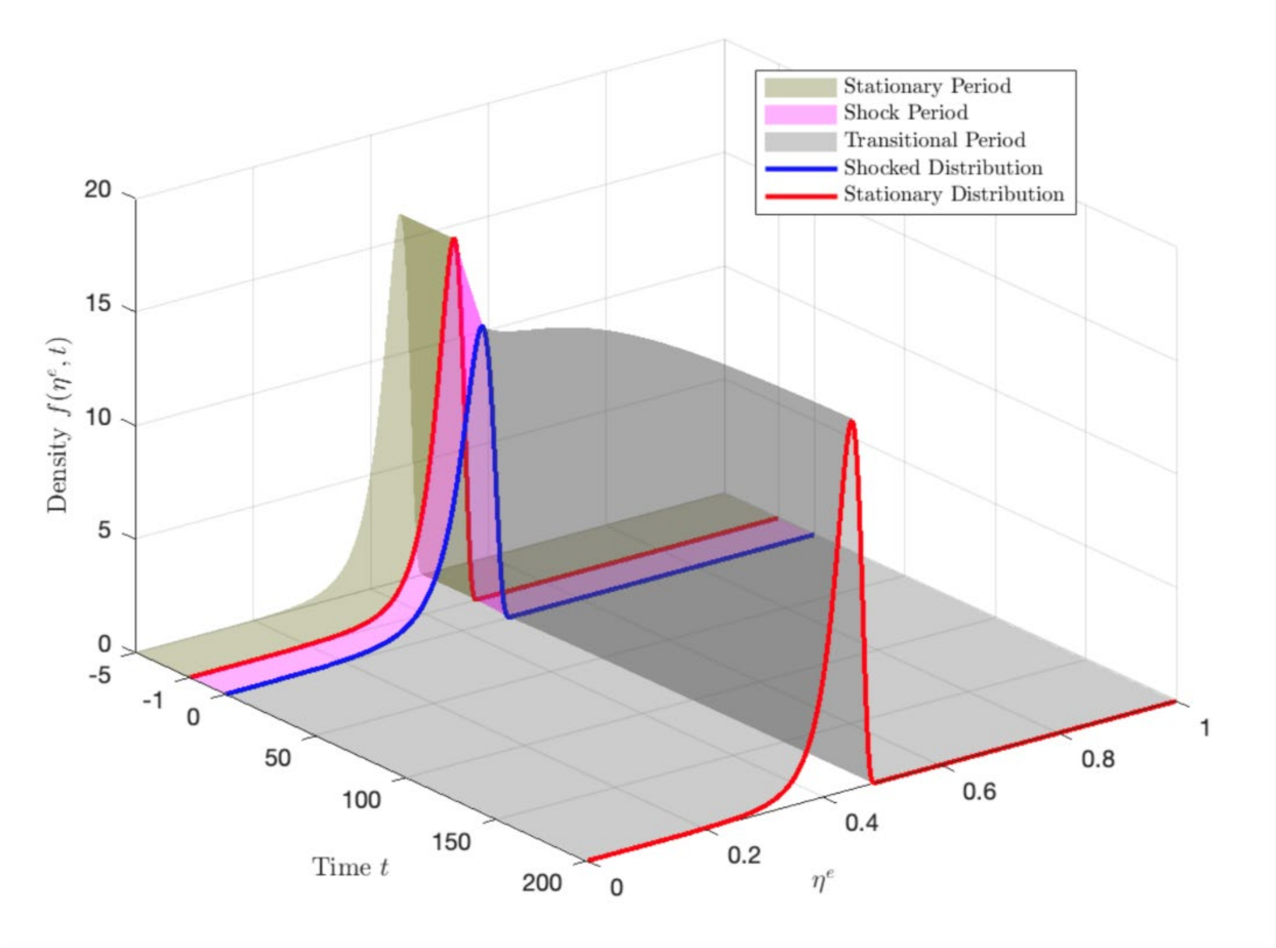
- Starts at stationary distribution
- Simulate a shock at 1% quantile of original Brownian shock ($dZ_t = -2.32\ dt$) for a period of $\Delta t = 1$.
- Converges back to stationary distribution



5. Density Diffusion

- Starts at stationary distribution
- Simulate a shock at 1% quantile of original Brownian shock ($dZ_t = -2.32\ dt$) for a period of $\Delta t = 1$.
- Converges back to stationary distribution

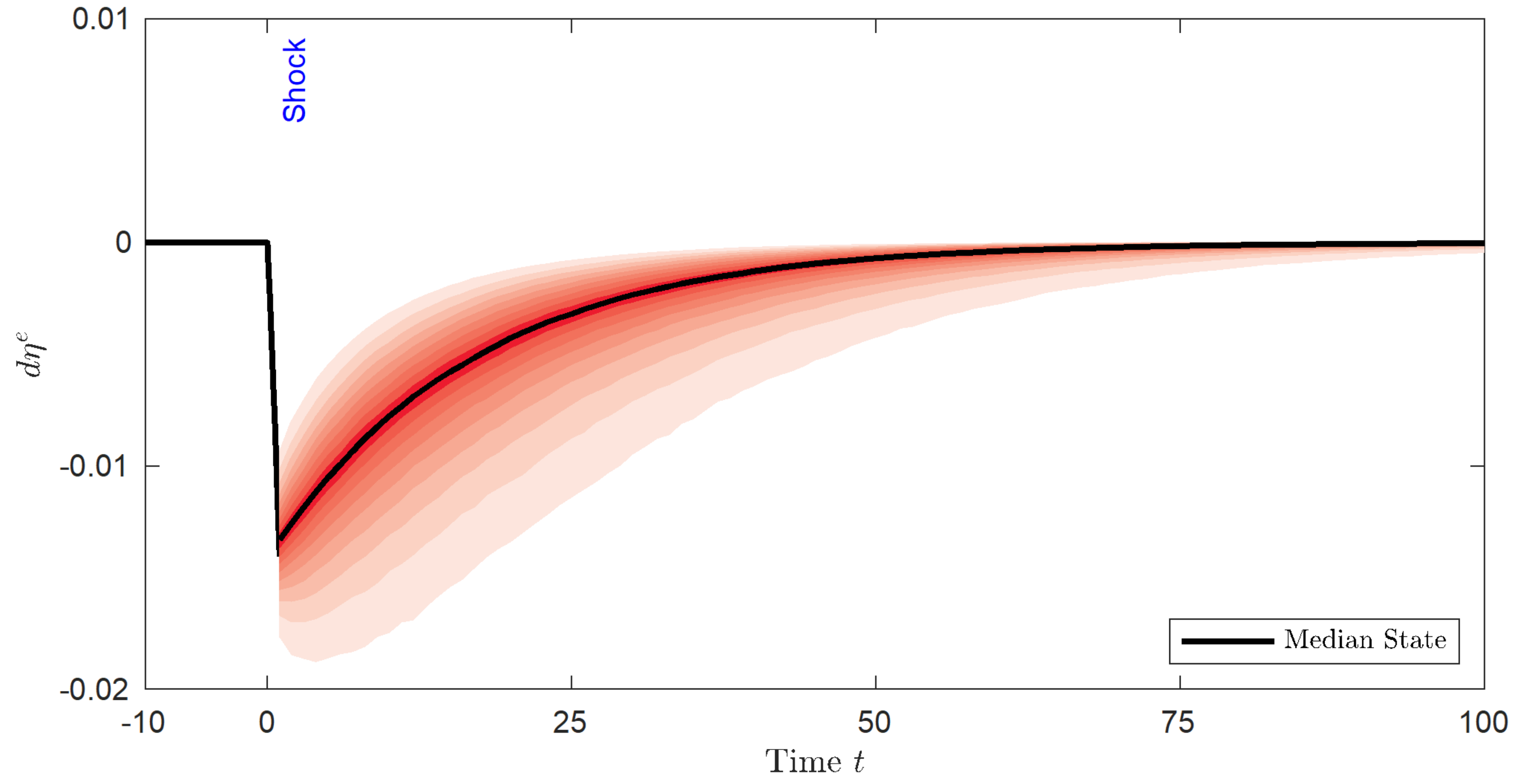




5.Density Diffusion Movies

5. Distributional Impulse Response

- Difference between path with and without shock
- Difference converges to zero in the long-run



$\sigma = 0.15$

Solving MacroModels Step-by-Step

0. Postulate aggregates, price processes & obtain return processes
1. For given C/N-ratio and SDF processes for each i *finance block*
   a. Real investment ι + Goods market clearing *(static)*
   - *Toolbox 1:* Martingale Approach, HJB vs. Stochastic Maximum Principle Approach

   b. Portfolio choice θ + Asset market clearing or
   Asset allocation κ & risk allocation χ
   - *Toolbox 2:* “price-taking social planner approach” – Fisher separation theorem

   - *Toolbox 3:* Change in numeraire to total wealth (including SDF)
2. Evolution of state variable η (and K) *forward equation*
3. Value functions *backward equation*
   a. Value fcn. as fcn. of individual investment opportunities ω
   - *Special cases: log-utility, constant investment opportunities*

   b. Separating value fcn. $V^i\left(n^{\tilde{i}};\eta,K\right)$ into $v^i(\eta)u(K)$
   c. Derive C/N-ratio and ς price of risk
4. Numerical model solution
   a. Transform BSDE for separated value fcn. $v^i(\eta)$ into PDE
   b. Solve PDE via value function iteration
5. KFE: Stationary distribution, Fan charts

Recent Macro-finance Literature (in cts. time)

- Core
  - BrunSan (2014), Basak & Cuoco (1998) He & Krishnamurthy (2012,13), DiTella (2013), Isohätälä et al. (2014)
- Intermediation/shadow banking
  - Phelan (2014), Adrian & Boyarchenko (2012,13), Huang (2014), Moreira & Savov (2014), Klimenko & Rochet (2015)
- Quantification
  - He & Krishnamurthy (2014), Mittnik & Semmler (2013)
- International
  - BruSan (2015), Maggiori (2013)
- Monetary
  - "The I Theory of Money" (2012), Drechsler et al. (2014)
- ….