Macro, Money and (International) Finance – Problem Set 1

September 8, 2020

Problem set prepared by Sebastian Merkel (smerkel@princeton.edu). Please let me know, if any tasks are unclear or you find mistakes in the problem descriptions.

The submission deadline is Monday, September 14 (end of day Princeton time). Please submit your group's solution via email to Fernando Mendo (fmendolopez@gmail.com).

1 The Basak-Cuoco Model with Heterogeneous Discount Rates

Consider the model from lecture 2, but unlike there assume that households are more patient than experts, i.e. they have a discount rate $\underline{\rho} < \rho$. This is the simplest way to generate both a nondegenerate stationary distribution and some endogenous capital price dynamics.

1. Derive closed-form expressions for ι, q, σ^q, μ^η and σ^η as a function of η and model parameters.[1] You do not actually have to follow the order of steps in the lecture. In this simple model it pays off to start with goods market clearing.

2. Replicate the figures from slide 50 (δ is not stated there, so choose just some parameter that generates similar numbers for the risk-free rate), then add to each plot the corresponding line for the model with $\underline{\rho} < \rho$ using $\underline{\rho} = 2\%$ (and all other parameters as before).

3. Assume $\phi > 0$. Show that in this model asset price movements mitigate exogenous risk. Explain economically, why this happens. Why does the effect disappear for $\phi = 0$?

4. Argue that the model must have a nondegenerate stationary distribution (just give some intuition, not a fully spelled-out formal proof). Compute the stationary density of η by numerically solving the ODE stated on page 16 of Yuliy Sannikov's stochastic calculus notes using the same parameters as in part 2.[2] What is the stationary density of q?

[1] As in the lecture, assume the specific functional form $\Phi(\iota) = \frac{1}{\phi}\log(1 + \phi\iota)$ for Φ.

[2] Have a look at Problem 2 before you do this. You find the stochastic calculus notes here: https://www.dropbox.com/s/6eoe0aywyz71rh6/Yuliy%20Stochastic%20Calculus.pdf

2 Solving ODEs

1. Read the section on ODEs in the differential equation document distributed with this problem set.
2. Solve the simple ODEs

$$y' = y \tag{1}$$
$$y' = x\cos\left(x^2\right) y^2 \tag{2}$$
$$y'' = -y \tag{3}$$

on the interval $[0, 10]$ with the initial condition $y(0) = 1$ for all three equations and the additional initial condition $y'(0) = 0$ for equation (3) using the following three methods:

   (a) explicit Euler method;
   (b) implicit Euler method;
   (c) a build-in solver of your numerical software or your favorite numerical library for your programming language.

For the first two methods, make a suitable discretization/step width choice yourself. For each of the three equations plot the results from the three approximation methods together with the respective true solution. These are given by (please verify):

$$y(x) = \exp(x), \qquad y(x) = \frac{1}{1 - \sin x^2/2}, \qquad y(x) = \cos(x).$$

3 Stability of Euler Methods

In this problem we study stability properties of Euler methods, both theoretically and numerically. This is useful for two reasons. First, as you will see in the numerical part of this problem, unstable methods have very bad error-propagation properties and are thus sometimes unsuitable (or require very small step widths and thus a long time to solve). Second, correct long-run convergence behavior will turn out to be important for the "iterative method" that we will use later in this course to solve continuous-time macro-finance models.

1. Consider the linear test equation

$$y' = \lambda y, \qquad y(0) = 1 \tag{4}$$

with a (complex) parameter $\lambda \in \mathbb{C}$. Its solution is $y(x) = e^{\lambda x}$ and if the real part of λ is negative, then $|y(x)|$ is bounded, strictly decreasing and converges to 0 as $x \to 0$. A numerical solution method (for a fixed step width) is called A-stable, if the numerical approximation to y has the same properties for all λ with negative real part.

   (a) Consider the explicit Euler method with a fixed step width Δx (that is $x_i - x_{i-1} = \Delta x$ for all i) for the test problem (4). Find an explicit formula for y_i and determine when $\{|y_i|\}_{i=1}^{N}$ is strictly decreasing and when it remains bounded and/or converges to 0 as $N \to \infty$. Conclude that the region of parameters λ, for which all three are satisfied is decreasing in Δx, but even for small Δx never includes all λ with negative real part – in particular, the explicit Euler method is not A-stable.

(b) Do the same analysis for the implicit Euler method for a fixed step width Δx. For which $\Delta x > 0$ is the implicit Euler method A-stable?

2. Choose $\lambda = -10$ and solve the test problem (4) using the explicit Euler method for $\Delta x = 0.05, 0.1, 0.19, 0.2, 0.21, 0.25$ over the time interval $[0, 2]$, plot the results together with the true solution. Repeat the same exercise with the the implicit method. Explain why your results are expected based on the analysis in part 1.