Macro, Money and Finance

Problem Set 1 – Solutions (selective)

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Problem Set 1 – Problem 2 (Basak Cuoco with $\underline{\rho} \neq \rho$)

1. Solve model, closed-form expressions for
$$\iota, q, \sigma^q, \mu^\eta, \sigma^\eta$$

2. Plot $q, r^f, \sigma^\eta \eta, \mu^\eta \eta$

3. Asset prices attenuate risk – why?

4. Stationary distribution

Problem Set 1 – Problem 2 – Model Solution

- Start with goods market clearing

$$\left(\rho\eta + \underline{\rho}\left(1-\eta\right)\right) q = a - \iota$$

- Use optimal investment ($q = 1 + \kappa\iota$), solve for ι

$$\iota\left(\eta\right) = \frac{a - \rho\eta - \underline{\rho}\left(1-\eta\right)}{1 + \kappa\rho\eta + \kappa\underline{\rho}\left(1-\eta\right)}$$

- Recover q

$$q\left(\eta\right) = \frac{1 + \kappa a}{1 + \kappa\rho\eta + \kappa\underline{\rho}\left(1-\eta\right)}$$

Problem Set 1 – Problem 2 – Model Solution

- Return on capital, expert portfolio choice, laws of motion of N and qK all do not depend on $\underline{\rho}$

  → same as in lecture

$$\frac{d\eta}{\eta} = \left(\frac{a-\iota}{q} - \rho + \theta^2(\sigma+\sigma^q)^2\right)dt - \theta(\sigma+\sigma^q)dZ$$

- Capital market clearing

$$1-\theta = \underbrace{\frac{qK}{N}}_{\frac{1}{\eta}} \Rightarrow \theta = -\frac{1-\eta}{\eta}$$

- Thus

$$\frac{d\eta}{\eta} = \left(\frac{a-\iota}{q} - \rho + \left(\frac{1-\eta}{\eta}\right)^2 (\sigma+\sigma^q)^2\right) dt + \frac{1-\eta}{\eta}(\sigma+\sigma^q)\, dZ$$

Problem Set 1 – Problem 2 – Model Solution

- Left to find σ^q
- Apply Ito's formula to $q(\eta)$

$$\frac{dq(\eta)}{q(\eta)} = \frac{q'(\eta)\mu^\eta\eta + \frac{1}{2}q''(\eta)(\sigma^\eta\eta)^2}{q(\eta)}dt + \frac{q'(\eta)}{q(\eta)}\sigma^\eta\eta dZ$$

- Recall,
  - $$q(\eta) = \frac{1+\kappa a}{1+\kappa\rho\eta+\kappa\underline{\rho}(1-\eta)}$$
  - $$\sigma^\eta = \frac{1-\eta}{\eta}(\sigma+\sigma^q)$$
- So,

$$\sigma^q = \frac{q'(\eta)}{q(\eta)}(1-\eta)(\sigma+\sigma^q) \Rightarrow \sigma^q = \frac{\frac{q'(\eta)}{q(\eta)}(1-\eta)}{1-(1-\eta)\frac{q'(\eta)}{q(\eta)}}\sigma$$

$$\sigma^q(\eta) = -\frac{(1-\eta)\kappa\left(\rho-\underline{\rho}\right)}{1+\kappa\rho}\sigma$$

Problem Set 1 – Problem 2 – Model Solution

- Conclusion:

$$\iota(\eta) = \frac{a - \rho\eta - \underline{\rho}(1-\eta)}{1 + \kappa\rho\eta + \kappa\underline{\rho}(1-\eta)}$$

$$q(\eta) = \frac{1 + \kappa a}{1 + \kappa\rho\eta + \kappa\underline{\rho}(1-\eta)}$$

$$\sigma^q(\eta) = -\frac{(1-\eta)\kappa\left(\rho - \underline{\rho}\right)}{1 + \kappa\rho}\sigma$$

$$\mu^\eta(\eta) = \left(\frac{1-\eta}{\eta}\frac{1 + \kappa\rho\eta + \kappa\underline{\rho}(1-\eta)}{1 + \kappa\rho}\sigma\right)^2 - \left(\rho - \underline{\rho}\right)(1-\eta)$$

$$\sigma^\eta(\eta) = \frac{1-\eta}{\eta}\frac{1 + \kappa\rho\eta + \kappa\underline{\rho}(1-\eta)}{1 + \kappa\rho}\sigma.$$

Problem Set 1 – Problem 2 – Solution Plots

- Risk-free rate (from experts' portfolio choice)

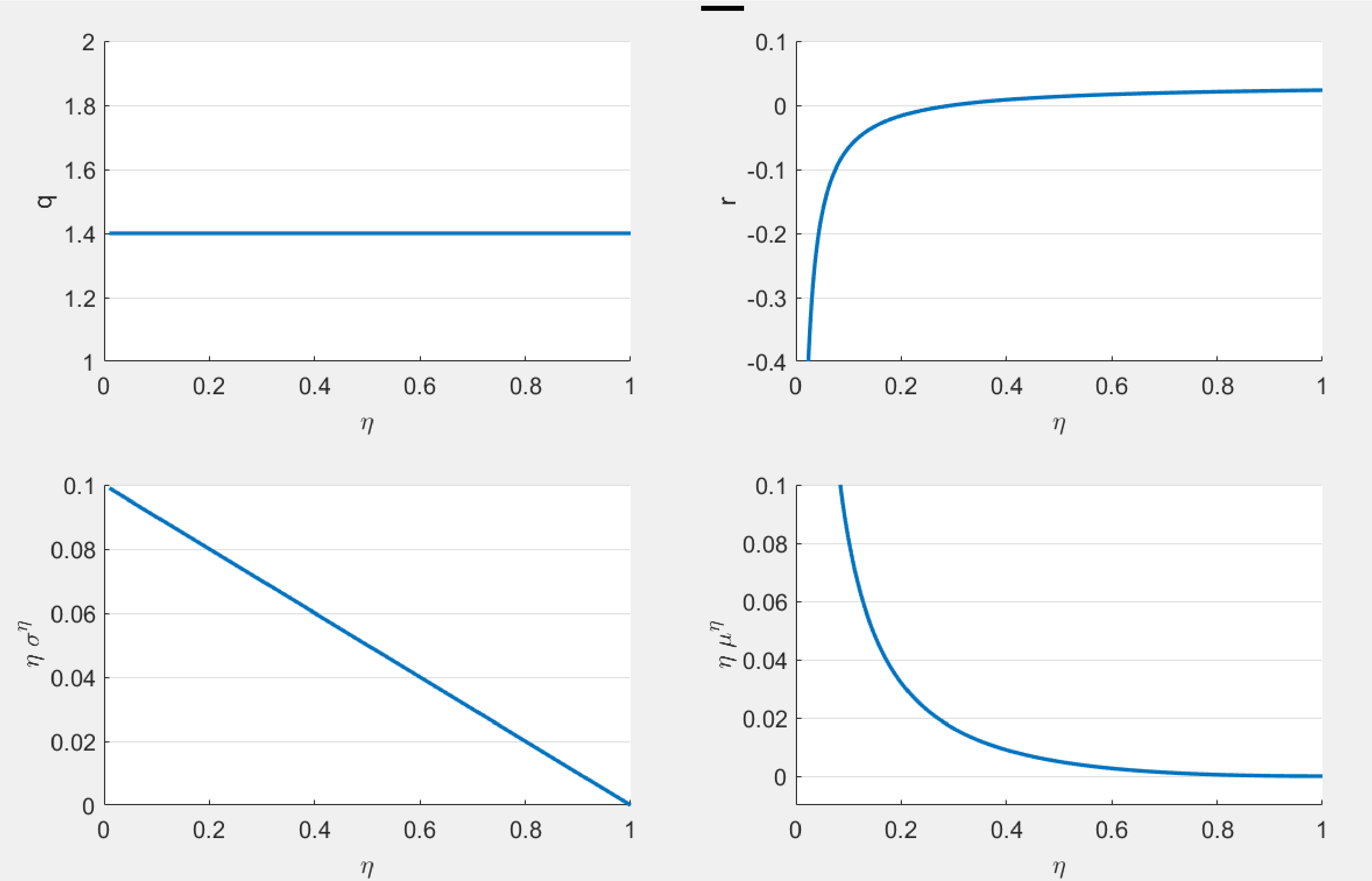
$$r^f = \frac{a-\iota}{q} + \Phi(\iota) - \delta + \mu^q + \sigma\sigma^q - \varsigma\left(\sigma + \sigma^q\right)$$

- ς? μ^q?

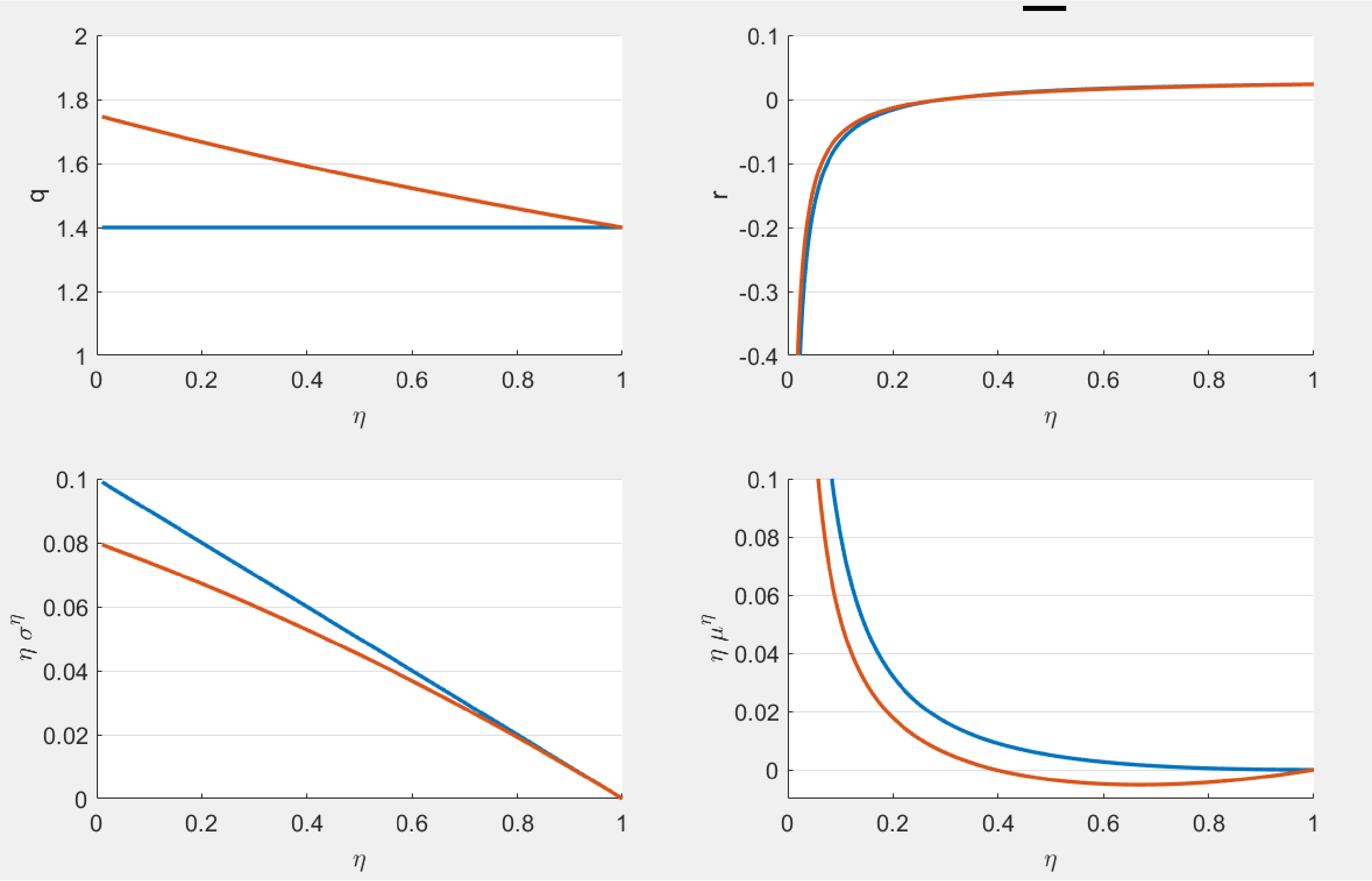
$$\varsigma\left(\eta\right) = \frac{1}{\eta}\left(\sigma + \sigma^q\left(\eta\right)\right)$$

$$\mu^q\left(\eta\right) = \frac{q'\left(\eta\right)\mu^\eta\left(\eta\right)\eta + \frac{1}{2}q''\left(\eta\right)\left(\sigma^\eta\left(\eta\right)\eta\right)^2}{q\left(\eta\right)}$$

Problem Set 1 – Problem 2 – Solution Plots Replication of Lecture ($\underline{\rho} = \rho$)



Problem Set 1 – Problem 2 – Solution Plots Heterogeneous Time Preference ($\underline{\rho} < \rho$)



Problem Set 1 – Problem 2 – Amplification?

- Is there endogenous amplification in this model?
  - No,

$$\sigma + \sigma^q = \left(1 - \frac{(1-\eta)\,\kappa\,(\rho - \underline{\rho})}{1 + \kappa\rho}\right)\sigma < \sigma$$

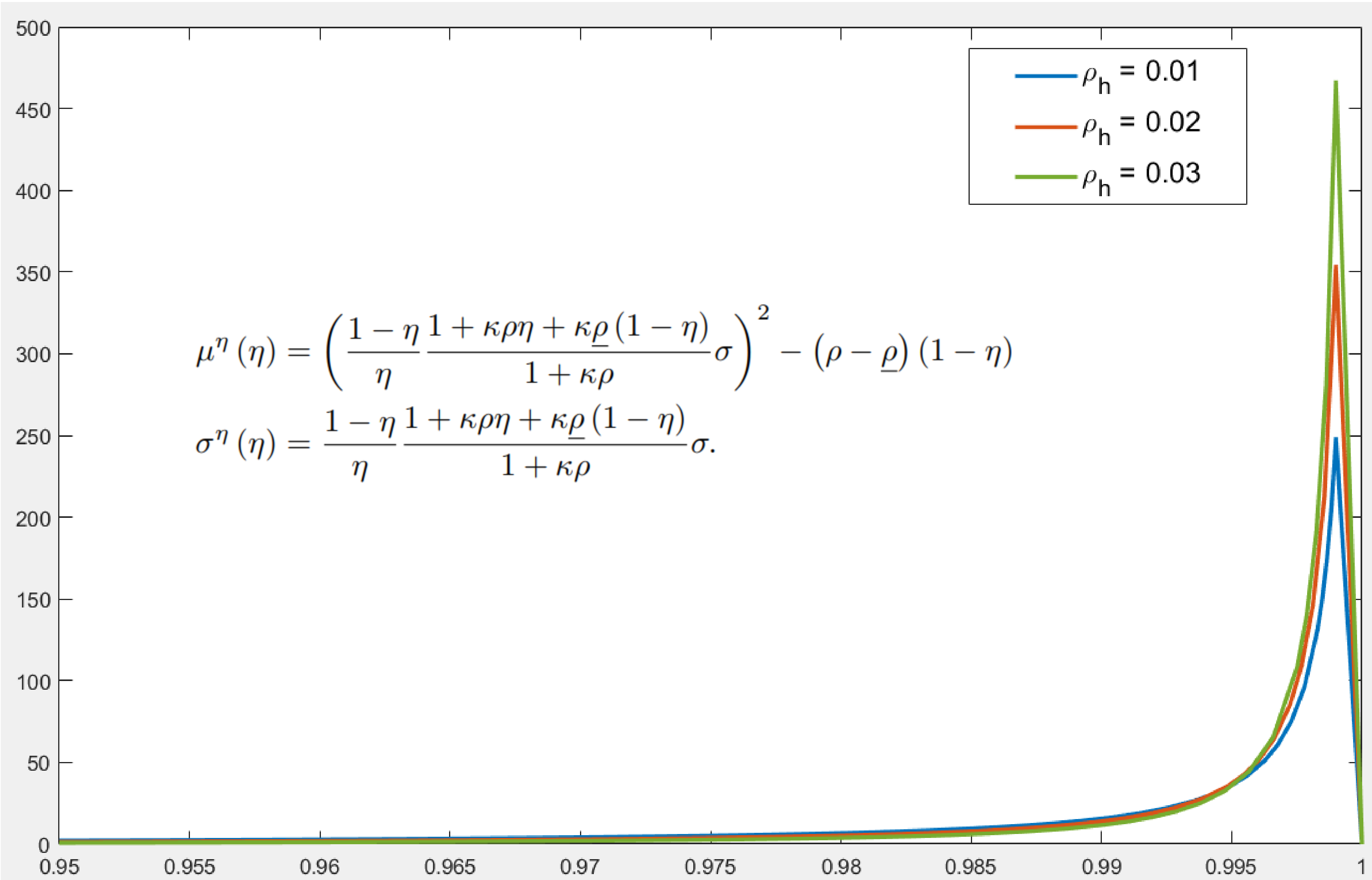
- Reason:

1. Technical:
   - q is decreasing in η
   - generates negative amplification term

$$\sigma + \sigma^q = \frac{\sigma}{1 - (1-\eta)\,\frac{q'(\eta)}{q(\eta)}}$$

2. Economic Answer:
   - After negative shock, expert wealth is reduced by more than output (due to leverage)
   - If households have lower MPC than experts (due to higher patience), aggregate consumption demand falls
   - Relative price of capital (relative to output good) and physical investment must rise

Problem Set 1 – Problem 2
Stationary Distribution



$$\mu^{\eta}(\eta) = \left(\frac{1-\eta}{\eta}\frac{1+\kappa\rho\eta+\kappa\underline{\rho}(1-\eta)}{1+\kappa\rho}\sigma\right)^2 - (\rho-\underline{\rho})(1-\eta)$$

$$\sigma^{\eta}(\eta) = \frac{1-\eta}{\eta}\frac{1+\kappa\rho\eta+\kappa\underline{\rho}(1-\eta)}{1+\kappa\rho}\sigma.$$

Problem Set 1 – Problem 4 (Stability of ODEs)

- Consider linear test problem (for $\lambda \in \mathbb{C}, \mathrm{Re}(\lambda) < 0$)
$$y' = \lambda y, \qquad y(0) = 1$$

- Solution $y(x) = e^{-\lambda x}$ is bounded, has strictly decreasing absolute value and converges to 0 for $x \to \infty$

- When do solutions based on explicit/implicit Euler methods have these properties?

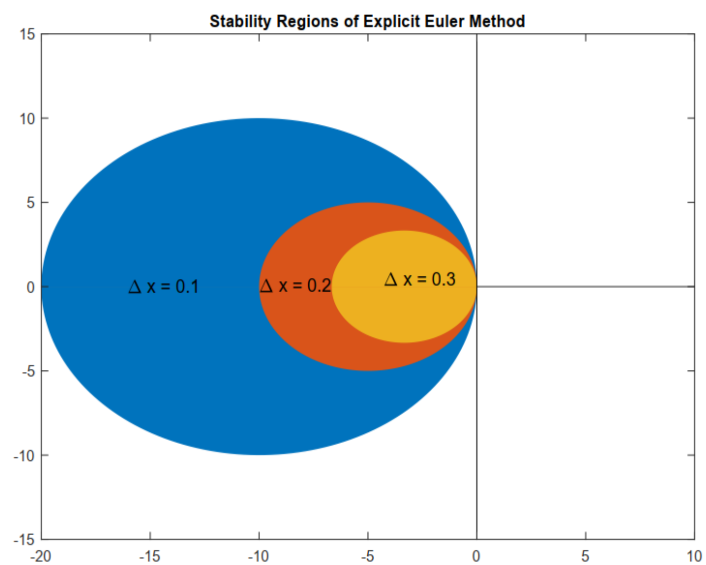
- Important for numerical stability: numerical errors in each step get dampened over time

Problem Set 1 – Problem 4 – Explicit Euler

$$y_i - y_{i-1} = \lambda y_{i-1} \Delta x \Rightarrow y_i = (1 + \lambda \Delta x)\, y_{i-1}.$$

$$|y_i| = |1 + \lambda \Delta x|^i\, |y_0| = |1 + \lambda \Delta x|^i\, .$$

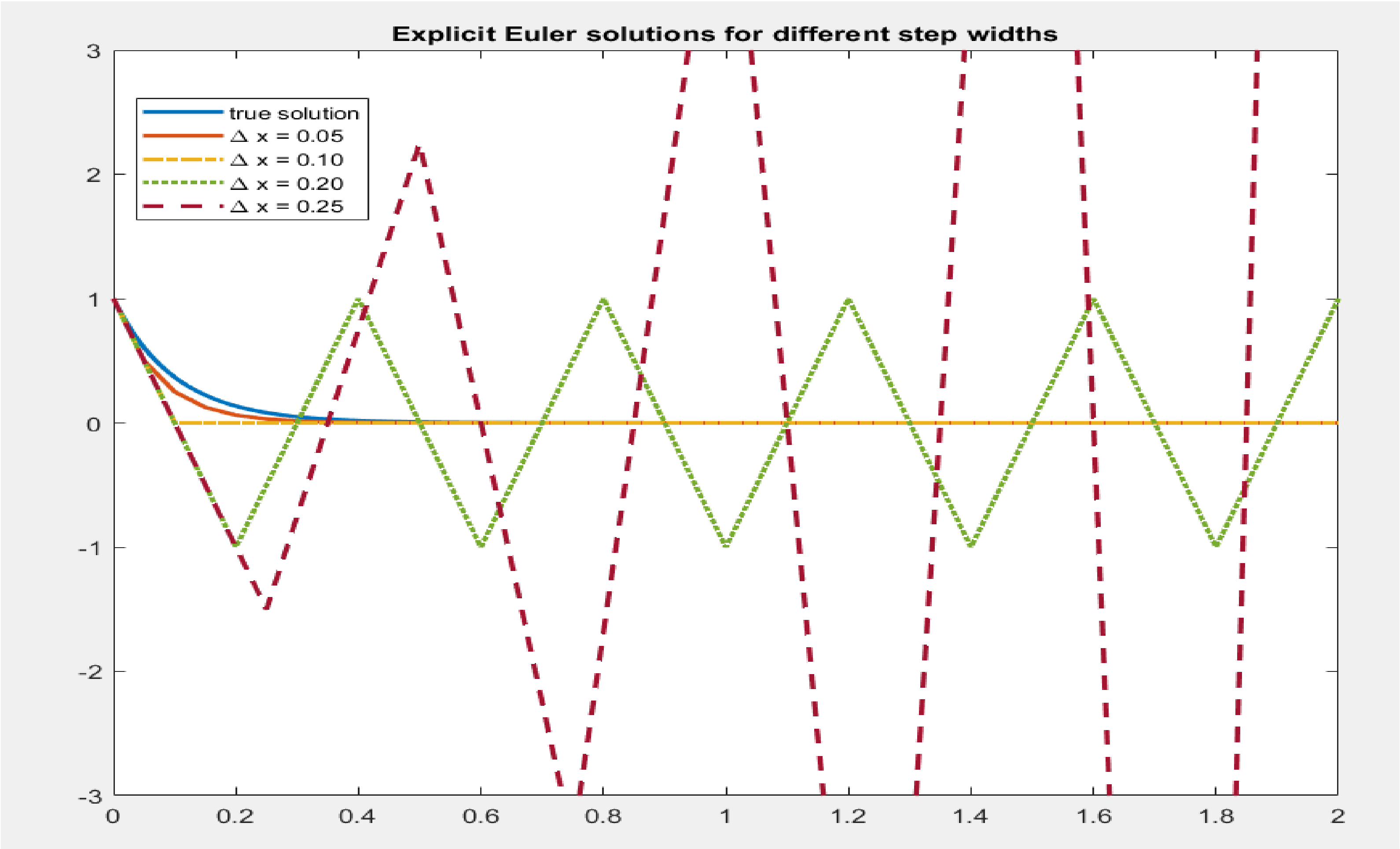
$$\left(\frac{1}{\Delta x} + \mathrm{Re}\lambda\right)^2 + (\mathrm{Im}\lambda)^2 < \frac{1}{\Delta x^2},$$



Problem Set 1 – Problem 4 – Explicit Euler

Numerical Example: $\lambda = -10$

$$y_i = (1 + \lambda \Delta x)\, y_{i-1}.$$



Problem Set 1 – Problem 4 – Implicit Euler

$$y_i - y_{i-1} = \lambda y_i \Delta x \Rightarrow y_i = \frac{1}{1 - \lambda \Delta x} y_{i-1}.$$

$$|y_i| = \frac{1}{|1 - \lambda \Delta x|^i}.$$

- $\mathrm{Re}(1 - \lambda\Delta x) > 1$, whenever $\mathrm{Re}(\lambda) < 0$

- hence $|1 - \lambda\Delta x| > 1$

$\Longrightarrow$ implicit Euler method unconditionally stable