Macro, Money and Finance

Problem Set 2 – Solutions (selective)

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Problem Set 2 – Problem 1 (KFE OU Process)

- General Kolmogorov Forward Equation (KFE)

$$\frac{\partial p}{\partial t}(x,t) = -\frac{\partial}{\partial x}\left(\mu(x,t)p(x,t)\right) + \frac{1}{2}\frac{\partial^2}{\partial x^2}\left(\sigma^2(x,t)p(x,t)\right)$$

- Describes density evolution of process X with

$$dX_t = \mu(X_t,t)dt + \sigma(X_t,t)dZ_t$$

- For this problem: $X =$ Ornstein-Uhlenbeck process

  (continuous-time AR(1))

$$dX_t = \theta(\bar{x} - X_t)dt + \sigma dZ_t$$

- Get then special KFE

$$\frac{\partial p}{\partial t}(x,t) = \theta(x-\bar{x})\frac{\partial}{\partial x}p(x,t) + \theta p(x,t) + \frac{\sigma^2}{2}\frac{\partial^2}{\partial x^2}p(x,t)$$

Problem Set 2 – Problem 1 (KFE OU Process)

- For this problem: $X =$ Ornstein-Uhlenbeck process (continuous-time AR(1))

$$dX_t = \mu(X_t, t)dt + \sigma(X_t, t)dZ_t$$

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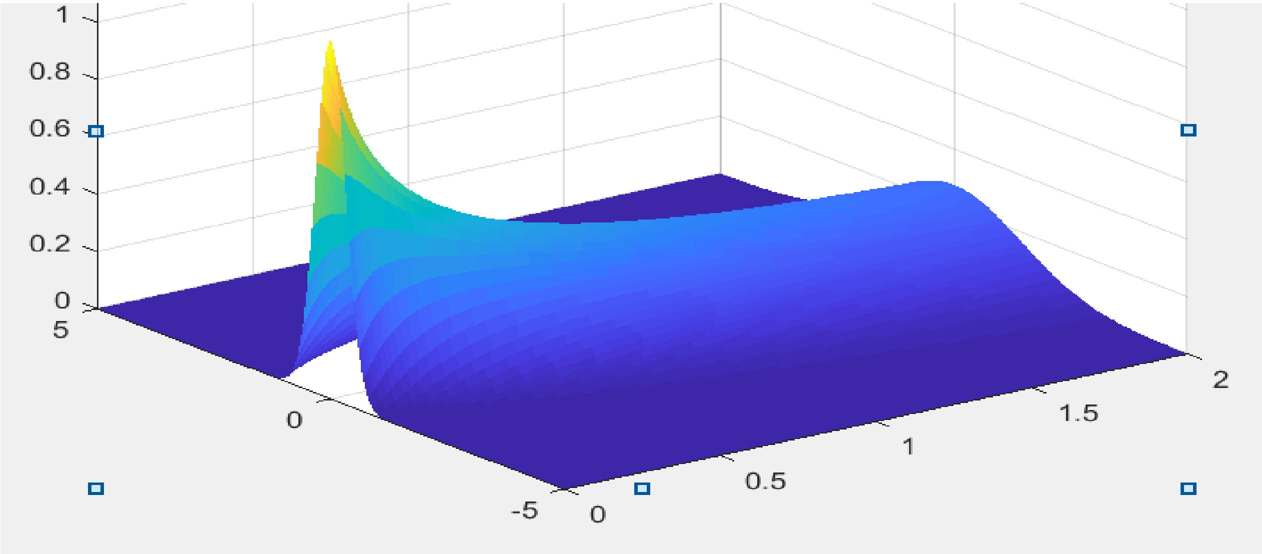
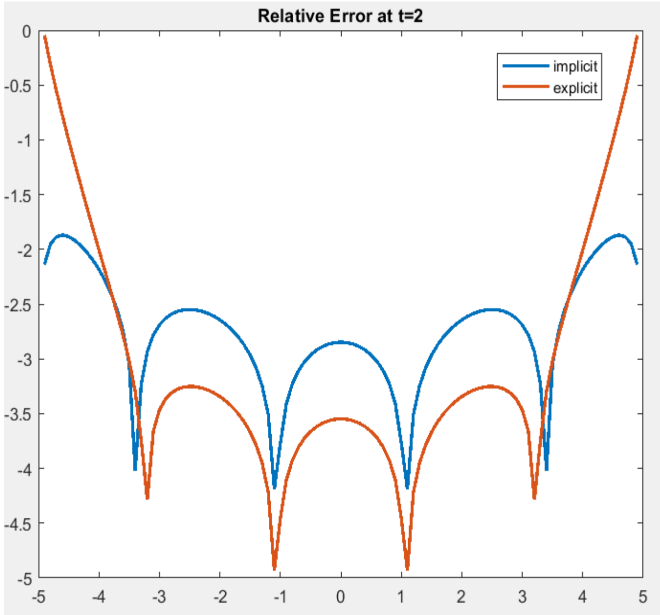
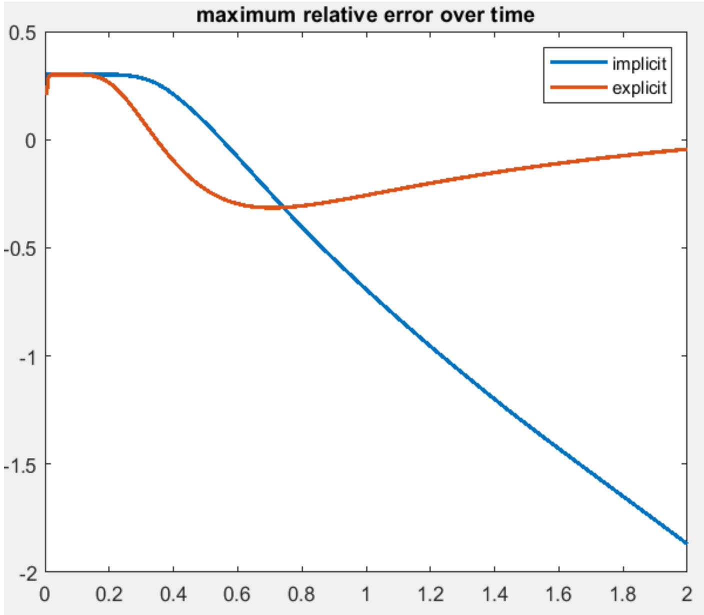
$$\frac{\partial p}{\partial t}(x,t) = \theta(x - \bar{x})\frac{\partial}{\partial x}p(x,t) + \theta p(x,t) + \frac{\sigma^2}{2}\frac{\partial^2}{\partial x^2}p(x,t)$$

- Tasks:
  - Solve equation numerically using different schemes and parameters
  - Compare with known closed-form solution
  - Identify problems with some schemes

Solution for $\theta = 0$



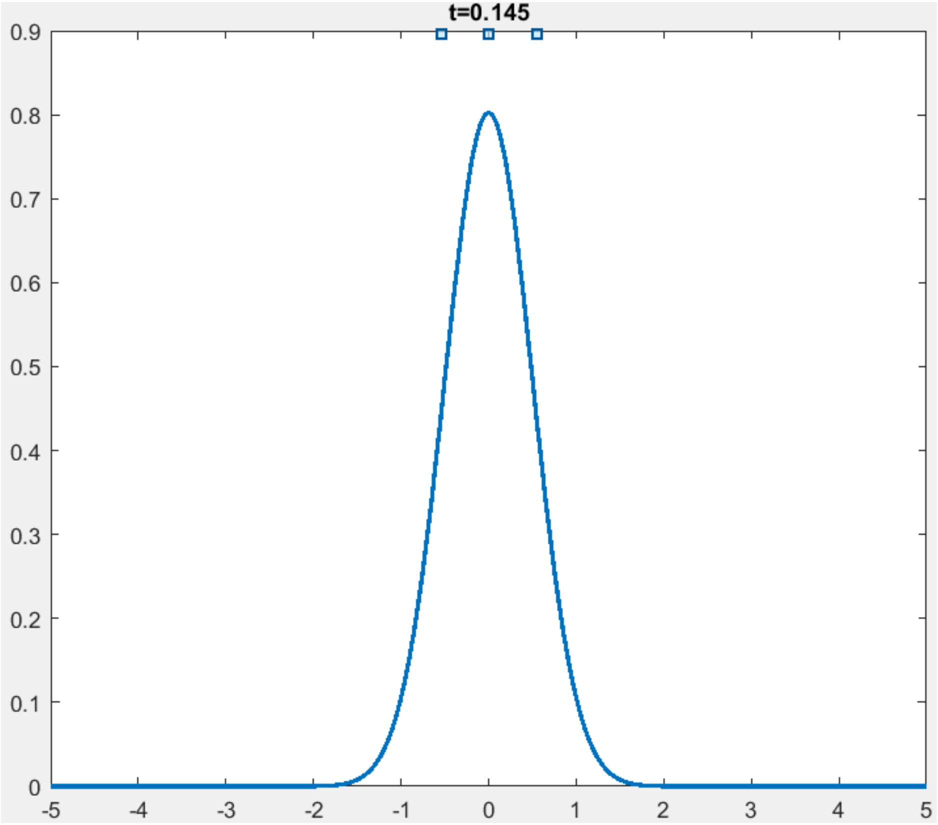




What Happens for Finer Space Grid?

- Implicit Method: errors slightly smaller

- Explicit Method:



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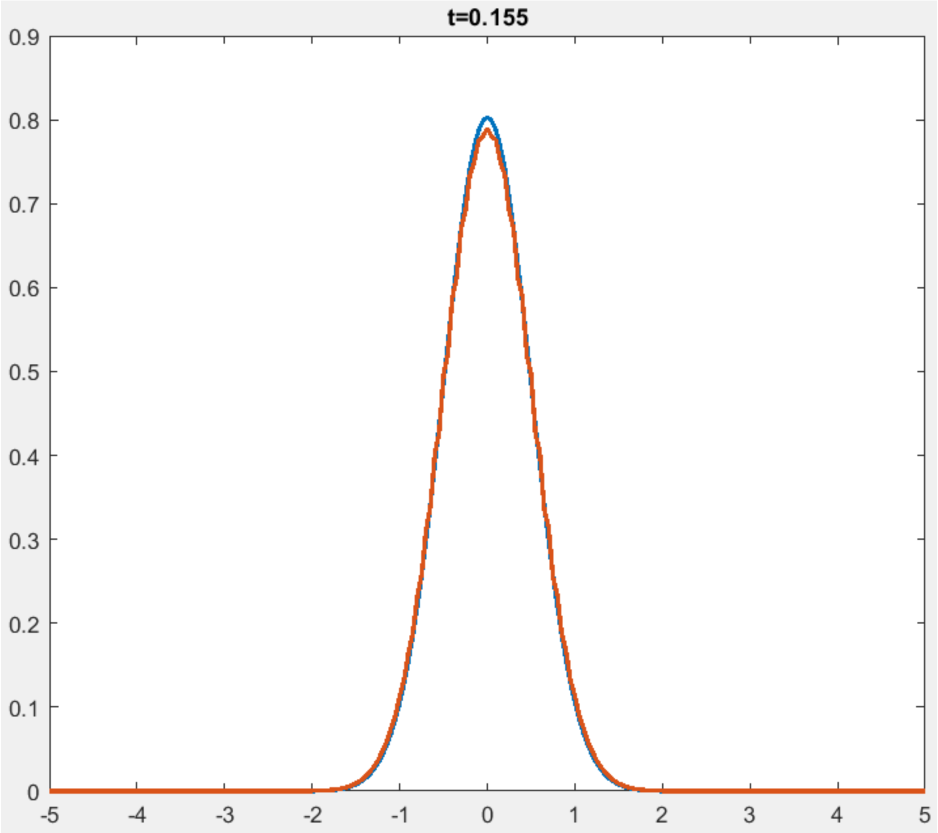

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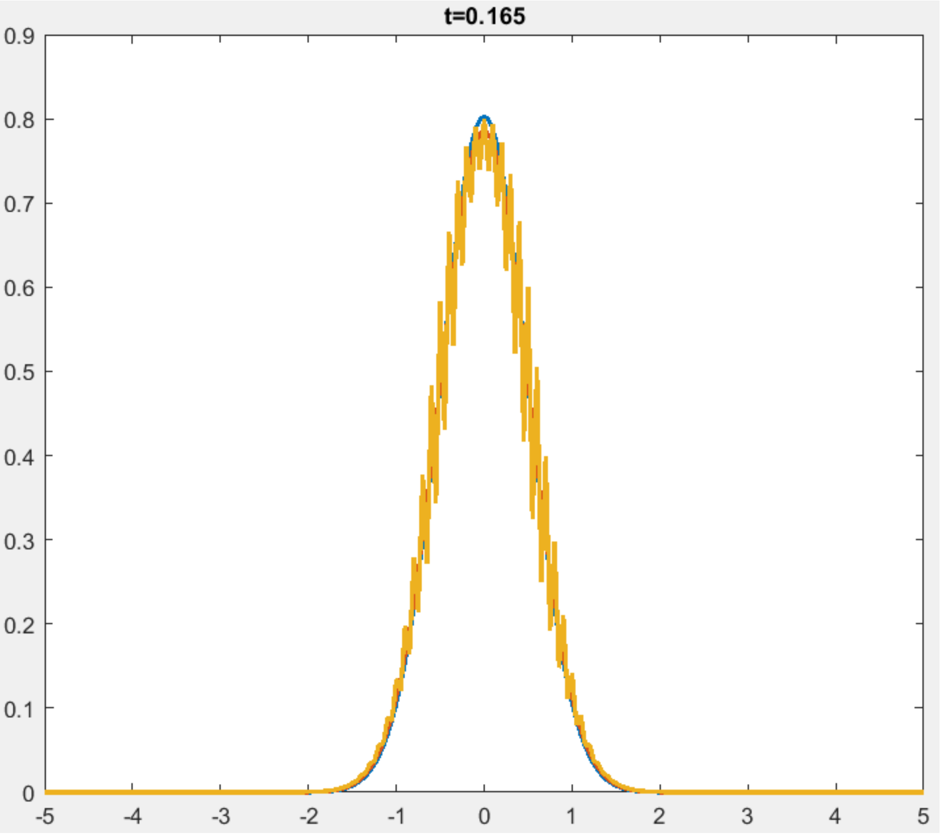

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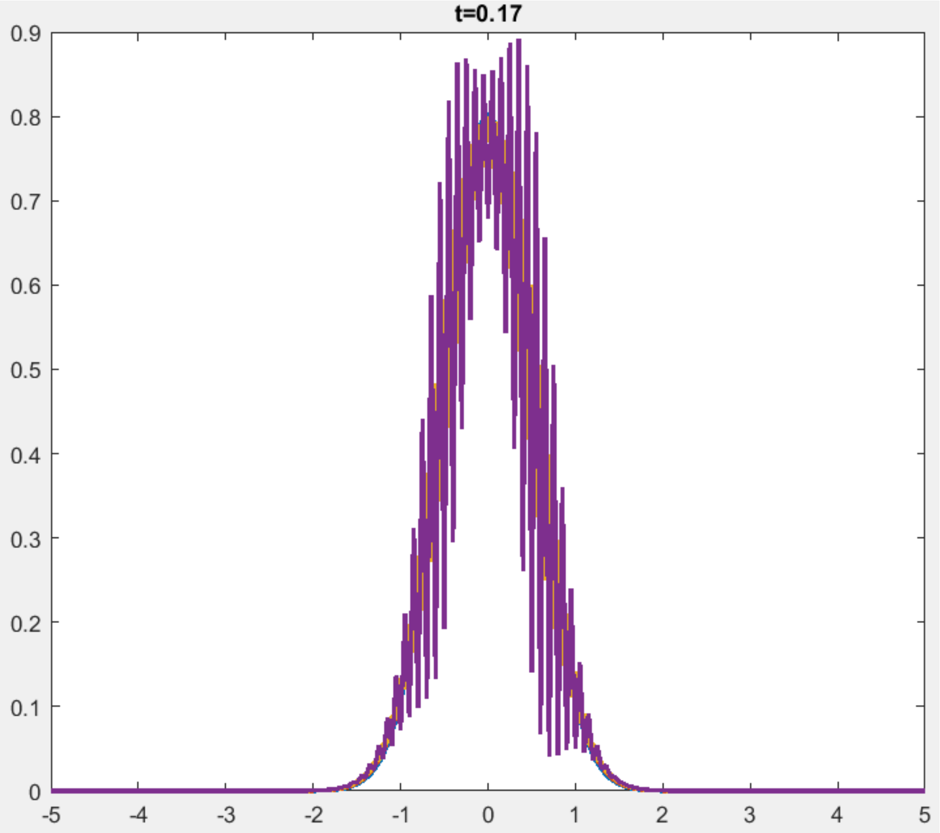

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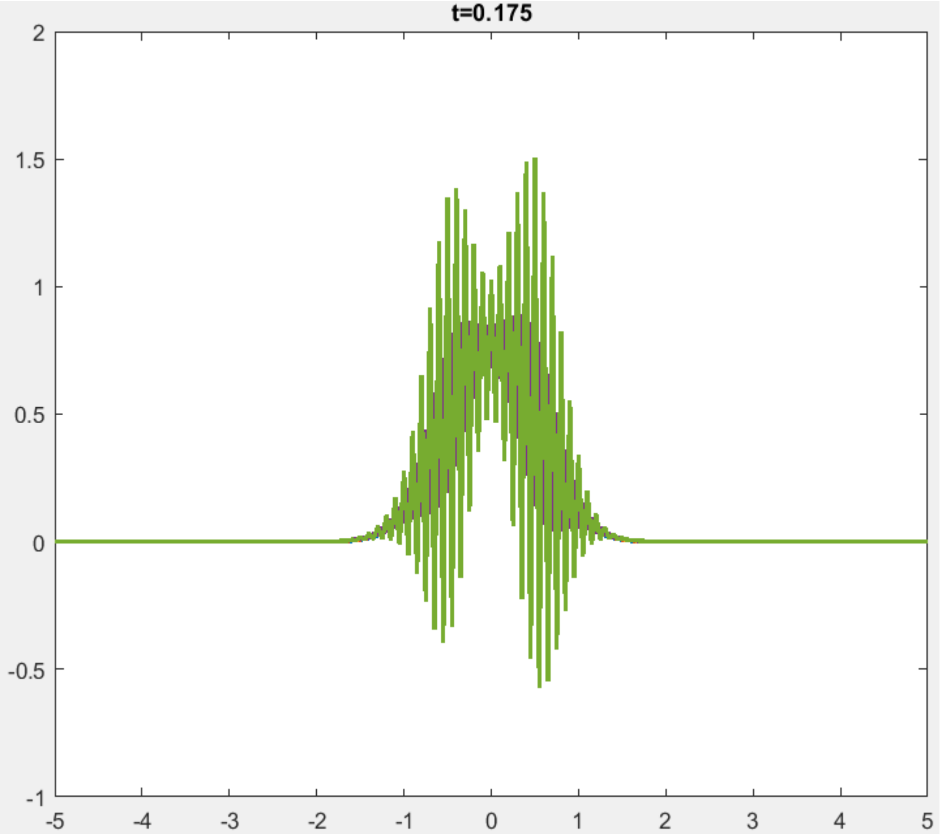
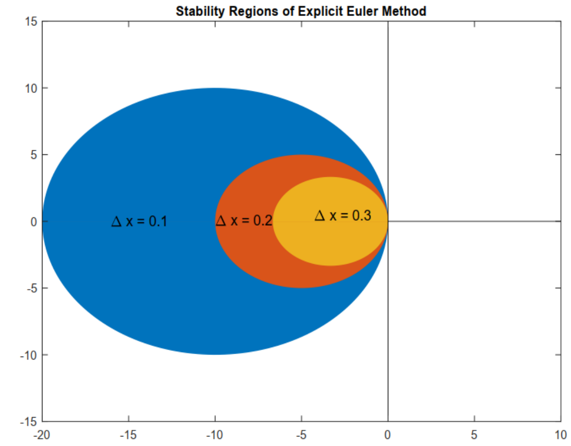
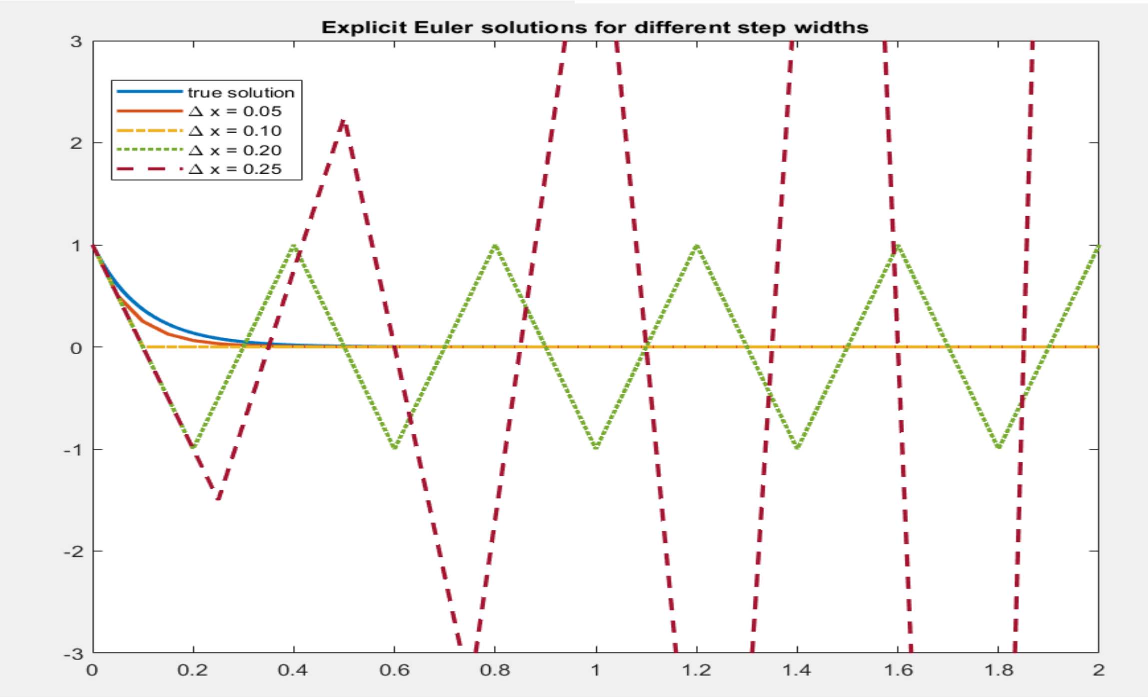


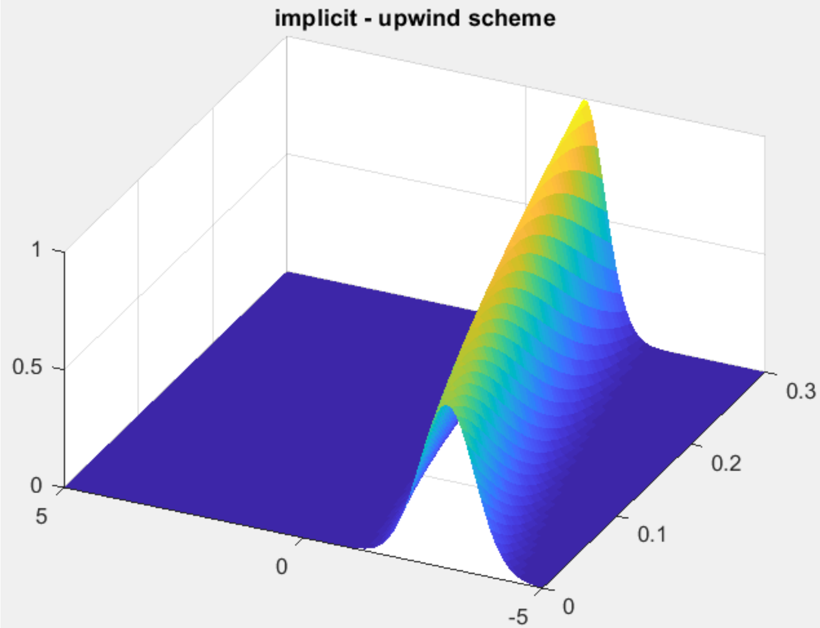
What Goes Wrong with the Explicit Method?

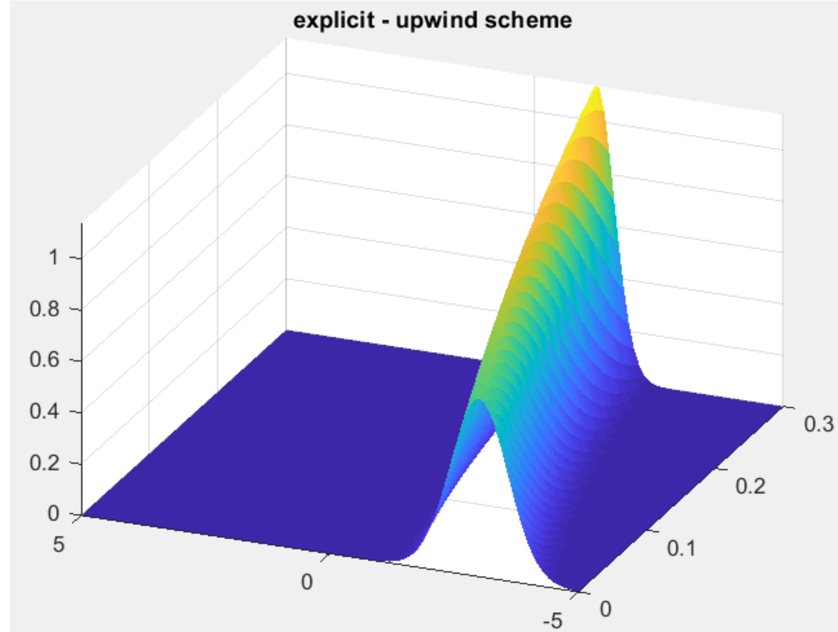
- Recall, Stability of ODEs:
  - Stable, if eigenvalue in stability region
  - Smallest eigenvalue of heat equation space discretization $\lambda \approx -\frac{1}{\Delta x^2}$
  - If too small, explicit method becomes unstable:

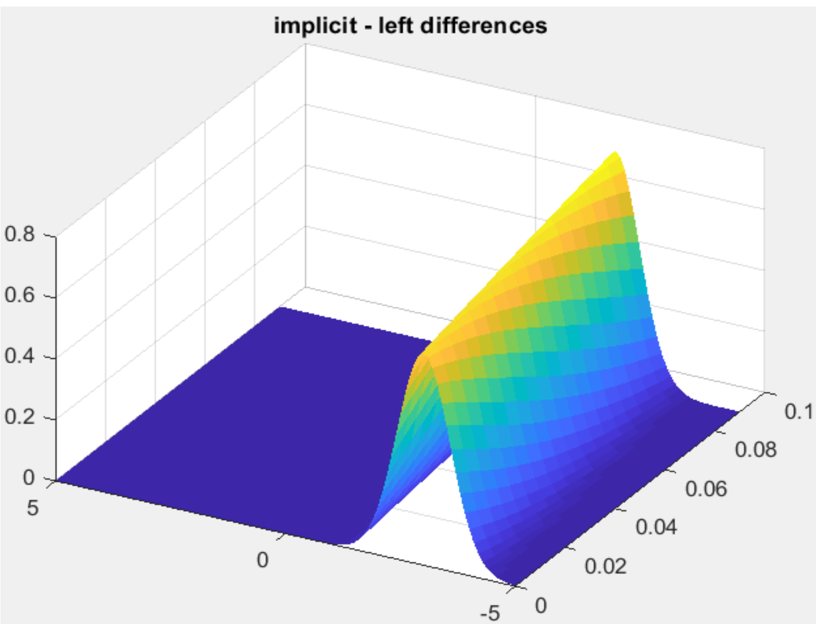


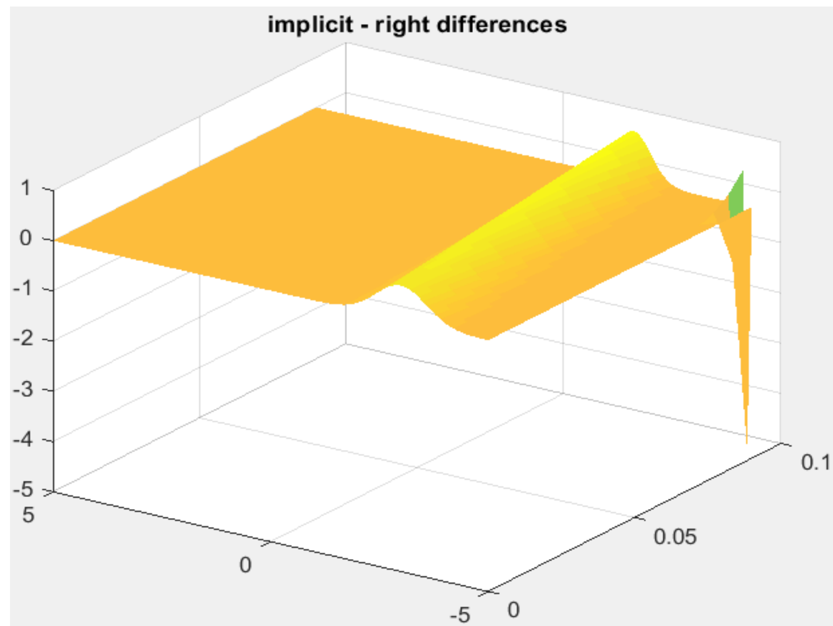


Solution for $\theta = 3$



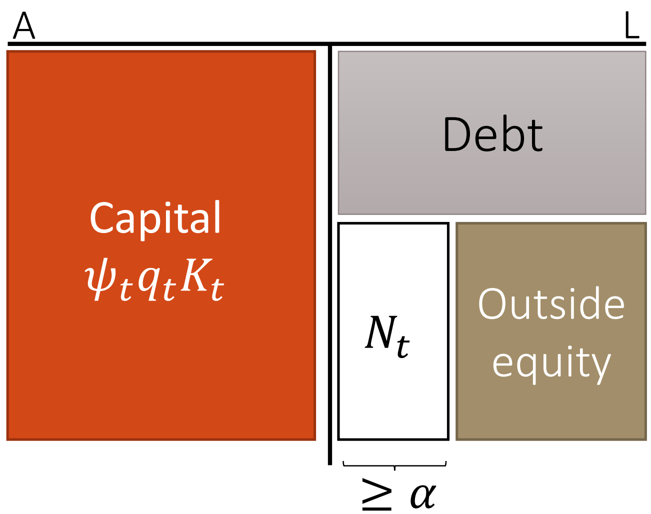




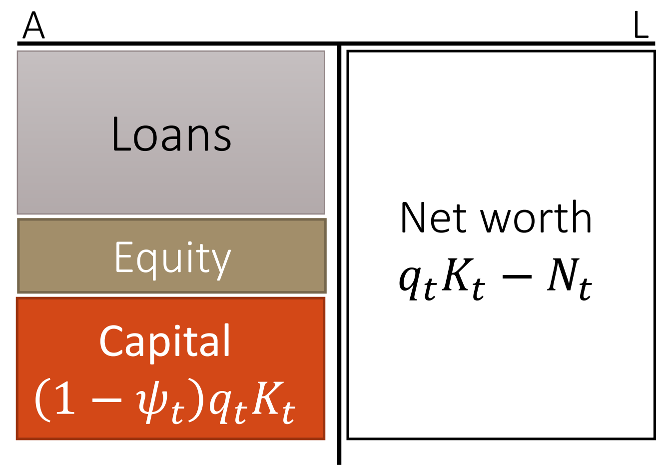


Problem Set 2 – Problem 2 (Lecture 3 Model with EZ Utility)

- Expert sector



Household sector



- Experts must hold fraction $\chi_t \geq \alpha\psi_t$ (skin in the game constraint)
- But now recursive utility

$$U_t = E_t\left[\int_t^\infty f(c_s, U_s)\, ds\right]$$

$$f(c,U) = (1-\gamma)\rho U \left(\log(c) - \frac{1}{1-\gamma}\log((1-\gamma)U)\right)$$

Solving MacroModels Step-by-Step

0. Postulate aggregates, price processes & obtain return processes
1. For given SDF processes *static*
   a. Real investment ι, (portfolio $\boldsymbol{\theta}$, & consumption choice of *each agent*)
      - *Toolbox 1:* Martingale Approach

   b. Asset/Risk Allocation *across types/sectors* & asset market clearing
      - *Toolbox 2:* "price-taking social planner approach" – Fisher separation theorem
2. Value functions *backward equation*
   a. Value fcn. as fcn. of individual investment opportunities ω
      - *Special cases*

   b. De-scaled value fcn. as function of state variables η
      - *Digression:* HJB-approach (instead of martingale approach & envelop condition)

   c. Derive ς-risk premia, C/N-ratio from value fcn. envelop condition
3. Evolution of state variable η *forward equation*
   - *Toolbox 3:* Change in numeraire to total wealth (including SDF)
   - ("Money evaluation equation" μ^{ϑ})
4. Value function iteration & goods market clearing
   a. PDE of de-scaled value fcn.
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- **Generic Steps, no assumptions about specific preferences.**
- **Here: physical and contract environment identical**

$\Longrightarrow$ no changes

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2a. CRRA Value Function: relate to ω

Applies separately for each type of agent

- ω_t Investment opportunity/ "networth multiplier"
- CRRA/power utility $u(c) = \frac{c^{1-\gamma}-1}{1-\gamma}$

  $\Rightarrow$ increase networth by factor, optimal consumption for all future states increases by same factor

  $\Rightarrow \left(\frac{c}{n}\right)$-ratio is invariant in n
- $\Rightarrow$ value function can be written as $\frac{u(\omega_t n_t)}{\rho}$, that is

$$= \frac{1}{\rho}\frac{(\omega_t n_t)^{1-\gamma}-1}{1-\gamma} = \frac{1}{\rho}\frac{\omega_t^{1-\gamma} n_t^{1-\gamma}-1}{1-\gamma}$$

- $\frac{\partial V}{\partial n} = u'(c)$ by optimal consumption (if no corner solution)

$$\frac{\omega_t^{1-\gamma} n_t^{-\gamma}}{\rho} = c_t^{-\gamma} \Leftrightarrow \boxed{\frac{c_t}{n_t} = \rho^{1/\gamma}\omega_t^{1-1/\gamma}}$$

2a. CRRA Value Function: relate to ω

Applies separately for each type of agent

- ω_t Investment opportunity/ "networth multiplier"
- recursive utility $f(c,U) = (1-\gamma)\rho U \left(\log(c) - \frac{1}{1-\gamma}\log((1-\gamma)U)\right)$

  $\Rightarrow$ increase networth by factor, optimal consumption for all future states increases by same factor

  $\Rightarrow \left(\frac{c}{n}\right)$-ratio is invariant in n

- $\Rightarrow$ value function can be written as $\frac{1}{\rho}\frac{(\omega_t n_t)^{1-\gamma}}{1-\gamma}$, that is

- $\frac{\partial V}{\partial n} = u'(c)$ by optimal consumption (if no corner solution)

Optimal consumption is different:

$$\omega^{1-\gamma} n^{-\gamma} = \frac{\partial V}{\partial n} = \frac{\partial f}{\partial c} = \rho(\omega n)^{1-\gamma}\frac{1}{c}$$

$$\Longrightarrow \frac{c}{n} = \rho$$

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2b. CRRA Value Fcn. & State Variable η

- Recall Martingale approach: if x_t is the value of a portfolio with return $\frac{dn_t}{n_t} + \frac{c_t}{n_t}dt$, then $\xi_t x_t$must be a martingale

$$\frac{d(\xi_t n_t)}{\xi_t n_t} = -\frac{c_t}{n_t}dt + martingale$$

- Optimal consumption implies with CRRA-$V_t(n_t) = \frac{1}{\rho}\frac{(\omega_t n_t)^{1-\gamma}}{1-\gamma}$:

$$u'(c) = V_t'(n) \Leftrightarrow c_t^{-\gamma} = \frac{1}{\rho}\omega^{1-\gamma}n_t^{-\gamma} \Leftrightarrow \underbrace{e^{-\rho t}c_t^{-\gamma}}_{=\xi_t} n_t = e^{-\rho t}\underbrace{\frac{1}{\rho}\omega^{1-\gamma}n_t^{1-\gamma}}_{(1-\gamma)V_t}$$

- Hence,

$$\frac{dV_t}{V_t} = \frac{d(e^{\rho t}\xi_t n_t)}{e^{\rho t}\xi_t n_t} = \left(\rho - \frac{c_t}{n_t}\right)dt + martingale$$

- Next, let’s compute the drift of $\frac{dV_t}{V_t}$

2b. ~~CRRA~~ EZ Value Fcn. & State Variable η

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Generic martingale argument ⇒ unchanged

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- SDF now

$$\xi_t = e^{\int_0^t \frac{\partial f}{\partial V}(c_s, V_s)ds}\frac{\partial V}{\partial n} = e^{\int_0^t \frac{\partial f}{\partial V}(c_s, V_s)ds}\omega_t^{1-\gamma}n_t^{-\gamma}$$

- Get new discounting term

$$e^{-\int_0^t \frac{\partial f}{\partial V}(c_s, V_s)ds}\xi_t n_t = (1-\gamma)V_t$$

$$\Longrightarrow E_t[dV_t]/V_t = (-\partial f/\partial V_t - c_t/n_t)dt$$

2c. EZ Value Fcn BSDE

- Other Manipulations (descaling, introducing v) identical to lectu
- Computing the discounting term

$$\frac{\partial f}{\partial V} = (1-\gamma)\rho(\log c - \log \omega n) - \rho$$

- Use

$$\omega = v^{\frac{1}{1-\gamma}}/\eta q$$

- Obtain the Value Function BSDE

$$\mu_t^v + (1-\gamma)(\Phi(\iota) - \delta) - \frac{1}{2}\gamma(1-\gamma)(\sigma^2) + (1-\gamma)\sigma\sigma_t^v$$
$$= (\gamma - 1)\rho(\log \rho - \log(\eta q)) + \log v$$

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   a. ...
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- Risk premia: similar steps as in lecture, same result

$$\varsigma = -\sigma^{v} + \sigma^{\eta} + \sigma^{q} + \gamma\sigma, \qquad \underline{\varsigma} = \cdots$$

- C/N already known $(= \rho)$

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generic step, no changes

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- Market clearing and collecting equations:
  - Mostly identical to lecture
  - Goods market clearing simpler (due to $c/n = \rho$)
- PDEs: identical up to additional term (see above)
- Algorithm: nothing changes

4. Value function iteration & goods market clearing
   a. PDE of de-scaled value fcn.
   b. Value function iteration by solving PDE