Eco 529: Brunnermeier & Sannikov

Macro, Money and Finance

Problem Set 3 – Solutions (selective)

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Problem 1 – Cash-in-advance Constraint

- One sector money model of lecture 5 („I theory without I")
- No idiosyncratic risk
- No money growth
- CIA constraint: must hold a fraction $1/\ell$ of consumption expenditures in money

- Tasks:
  1. HJB of individual agent, derive liquidity-adjusted choice conditions
  2. Solve model. Does constraint bind? Equilibrium Uniqueness?
  3. How do velocity (ℓ) changes affect equilibrium allocation? Is there crowding in/out?

Problem 1 – Choice Conditions

- Take generic returns:

$$dr^k = \mu^{r,k}dt + \sigma^{r,k}dZ$$
$$dr^m = \mu^{r,m}dt + \sigma^{r,m}dZ$$

- HJB equation:

$$\rho V(n) = \max_{c,\theta}\left(u(c) + V'(n)\left(-c + \theta n \mu^{r,m} + (1-\theta)\, n\mu^{r,k}\right) + \frac{1}{2}V''(n)\left(\theta\sigma^{r,m} + (1-\theta)\,\sigma^{r,k}\right)^2 n^2\right)$$

- Constraint: $c \leq \ell\theta n$
- $\hat{\lambda}$ Lagrange multiplier and $\lambda \coloneqq \hat{\lambda}/V'(n)$ „price of liquidity“
- FOCs:

$$u'(c) = (1+\lambda)\, V'(n)$$

$$\mu^{r,k} - \mu^{r,m} = \underbrace{-\frac{V''(n)\, n}{V'(n)}}_{=\varsigma}\sigma^n\left(\sigma^{r,m} - \sigma^{r,k}\right) + \lambda\ell$$

Problem 1 – Model Solution

- Log utility, thus $V'(n) = \frac{1}{\rho n}$ and $-\frac{V''(n)n}{V'(n)} = 1$
- In addition:

$$\mu^{r,k} - \mu^{r,m} = \frac{a-\iota}{q} \text{ and } \sigma^{r,k} = \sigma^{r,m} = \sigma$$

- Substitute into consumption and portfolio condition

$$\frac{1}{c} = \frac{1+\lambda}{\rho n} \Rightarrow \zeta = \frac{\rho}{1+\lambda}$$

$$\frac{a-\iota}{q} = \sigma(\sigma - \sigma) + \lambda\ell = \lambda\ell$$

- Goods market clearing $\zeta(p+q) = a - \iota$
- Divide by q, combine with choice conditions

$$\frac{\rho}{1+\lambda}\frac{1}{1-\vartheta} = \lambda\ell \Leftrightarrow 1 - \vartheta = \frac{\rho}{(1+\lambda)\lambda\ell}$$

Problem 1 – Model Solution

- From last slide $$1-\vartheta = \frac{\rho}{(1+\lambda)\,\lambda\ell}$$
- Two possibilities:
  1. Constraint does not bind, then $\lambda = 0 \Rightarrow 1 - \vartheta = \infty$
     $\longrightarrow$ not possible
  2. Constraint binds, then $\zeta = \ell\vartheta$, thus
     $$\vartheta = \frac{\rho}{(1+\lambda)\,\ell}$$
- Combine two equations to get „price of liquidity“
  $$1 = \frac{\rho}{(1+\lambda)\,\ell}\left(1+\frac{1}{\lambda}\right) = \frac{\rho}{\ell\lambda} \Rightarrow \lambda = \frac{\rho}{\ell}$$
- Recover ϑ and ζ
  $$\vartheta = \frac{\rho}{\ell+\rho} \qquad \zeta = \frac{\rho\ell}{\ell+\rho}$$

Problem 1 – Model Solution

$$\vartheta = \frac{\rho}{\ell + \rho} \qquad \zeta = \frac{\rho \ell}{\ell + \rho} \qquad \lambda = \frac{\rho}{\ell}$$

- Remaining steps are standard (see lecture):

$$\iota = \frac{(1-\vartheta)\, a - \zeta}{1 - \vartheta + \kappa \zeta} = \frac{a - \rho}{1 + \kappa \rho}$$

$$q = (1-\vartheta)\, \frac{1 + \kappa a}{1 - \vartheta + \kappa \zeta} = \frac{1 + \kappa a}{1 + \kappa \rho}$$

$$p = \vartheta \frac{1 + \kappa a}{1 - \vartheta + \kappa \zeta} = \frac{\rho}{\ell} \frac{1 + \kappa a}{1 + \kappa \rho}$$

Problem 1 – How Does ℓ Affect Allocations?

- Model Solution

$$\iota = \frac{(1-\vartheta)\,a - \zeta}{1-\vartheta+\kappa\zeta} = \frac{a-\rho}{1+\kappa\rho}$$

$$q = (1-\vartheta)\,\frac{1+\kappa a}{1-\vartheta+\kappa\zeta} = \frac{1+\kappa a}{1+\kappa\rho}$$

$$p = \vartheta\frac{1+\kappa a}{1-\vartheta+\kappa\zeta} = \frac{\rho}{\ell}\frac{1+\kappa a}{1+\kappa\rho}$$

- Only p depends on ℓ, investment and consumption are unaffected
- If velocity doubles, price level doubles
- No crowding out/in of investment

Poll 7: When would this result change?

a) With sticky prices

b) With idiosyncratic risk

c) Never, with CIA money is always neutral

Problem 2 (Multiplicity in Money Model)

- Simple I Theory without I (+ simplifying assumption)

$$\frac{dk_t^i}{k_t^i} = \tilde{\sigma} d\tilde{Z}_t^i, \qquad A = 1, \qquad \tilde{\sigma}^2 > \rho, \qquad \kappa \to \infty$$

- Tasks

1. Steady states
2. All deterministic equilibria
3. Tax Backing of Money and Uniqueness

Problem 2
General Equilibrium Conditions

- Goods Market Clearing

$$p + q = \frac{1}{\rho}$$

- Capital Market Clearing

$$1 - \theta = \frac{q}{p+q} = \rho q$$

- Price of (idiosyncratic) Risk

$$\tilde{\varsigma} = \rho q \tilde{\sigma}$$

Problem 2 – Steady States

- One Steady State without Money

$$q = \frac{1}{\rho}, \qquad p = 0$$

- Steady States with Money
  - From portfolio choice and market clearing

$$\frac{1}{q} = \rho q \tilde{\sigma}^2 \Leftrightarrow q^2 = \frac{1}{\rho \tilde{\sigma}^2} \Leftrightarrow q = \pm \frac{1}{\sqrt{\rho}\tilde{\sigma}}.$$

  - Only positive solution is valid equilibrium

$$q = \frac{1}{\sqrt{\rho}\tilde{\sigma}}, \qquad p = \frac{\tilde{\sigma} - \sqrt{\rho}}{\rho}$$

Problem 2 – Deterministic Equilibria

- Postulate

$$\frac{dq_t}{q_t} = \mu_t^q dt, \qquad \frac{dp_t}{p_t} = \mu_t^p dt$$

- Return processes and portfolio choice

$$dr_t^k = \left(\frac{1}{q_t} + \mu_t^p\right) dt + \tilde{\sigma} d\tilde{Z}_t, \qquad dr_t^m = \mu_t^p dt$$

$$\frac{1}{q_t} + \mu_t^q - \mu_t^p = \rho q_t \tilde{\sigma}^2$$

- Differentiate goods market clearing ($p + q = \frac{1}{\rho}$)

$$0 = \dot{p}_t + \dot{q}_t \Rightarrow \underbrace{\frac{\dot{q}_t}{q_t} - \frac{\dot{p}_t}{p_t}}_{=\mu_t^q - \mu_t^p} = \left(1 + \frac{q_t}{p_t}\right) \frac{\dot{q}_t}{q_t} = \frac{1}{1 - \rho q_t} \underbrace{\frac{\dot{q}_t}{q_t}}_{=\mu_t^q}$$

Problem 2 – Deterministic Equilibria

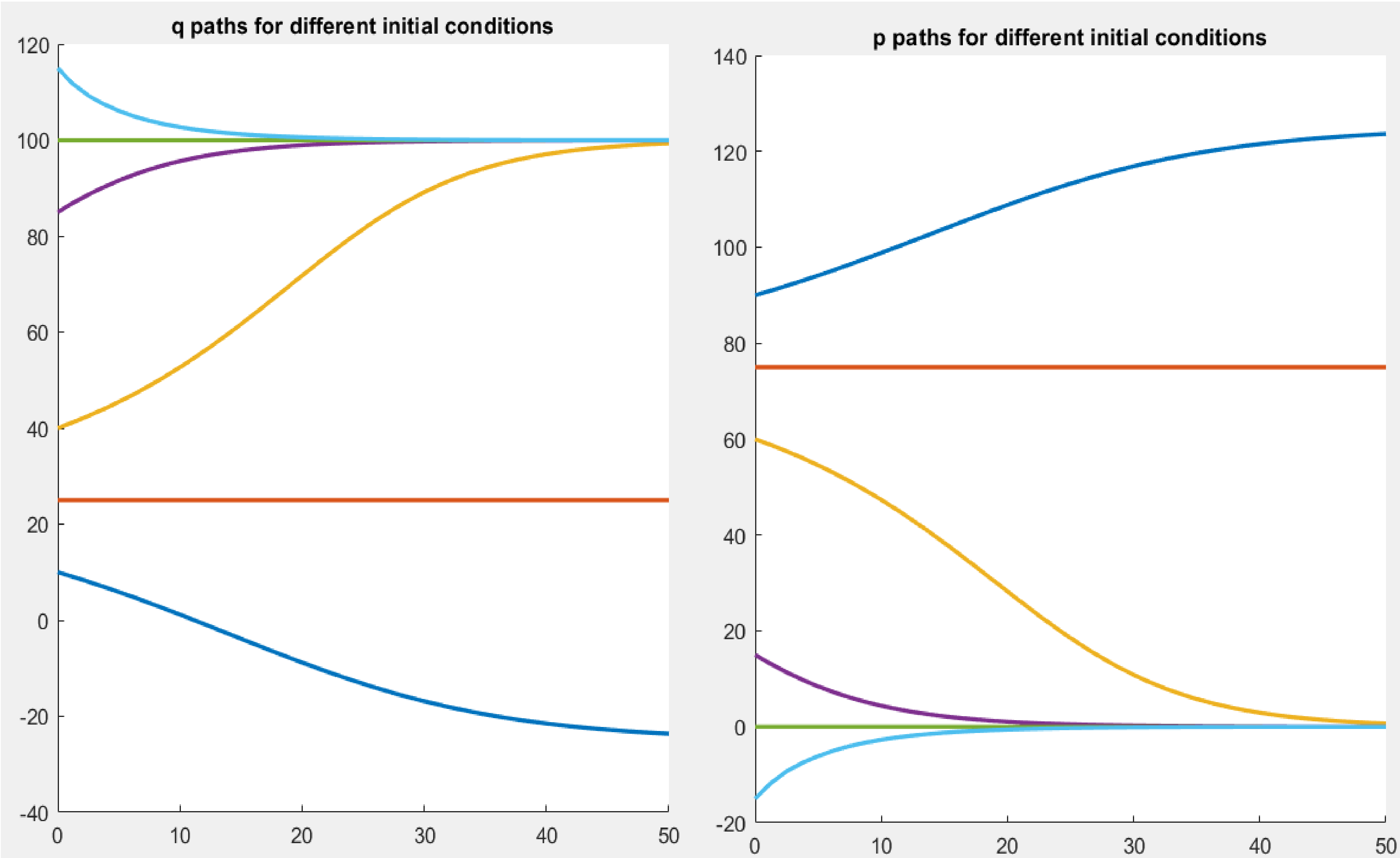
- Substitute into portfolio choice condition

$$\frac{1}{q_t} + \frac{\mu_t^q}{1 - \rho q_t} = \rho q_t \tilde{\sigma}^2$$

- … and rearrange

$$\dot{q}_t = \left(\rho q_t \tilde{\sigma}^2 - 1\right)\left(1 - \rho q_t\right) = \rho^2 \tilde{\sigma}^2 \left(q_t + \frac{1}{\sqrt{\rho}\tilde{\sigma}}\right)\left(q_t - \frac{1}{\sqrt{\rho}\tilde{\sigma}}\right)\left(\frac{1}{\rho} - q_t\right)$$

Problem 2 – Deterministic Equilibria



Problem 2 – Deterministic Equilibria

- Proposition
  - Set of possible initial conditions (q^0, p^0) is

$$\{(p,q) \mid q \in [\underline{q}, \overline{q}], p = \frac{1}{\rho} - q\}$$

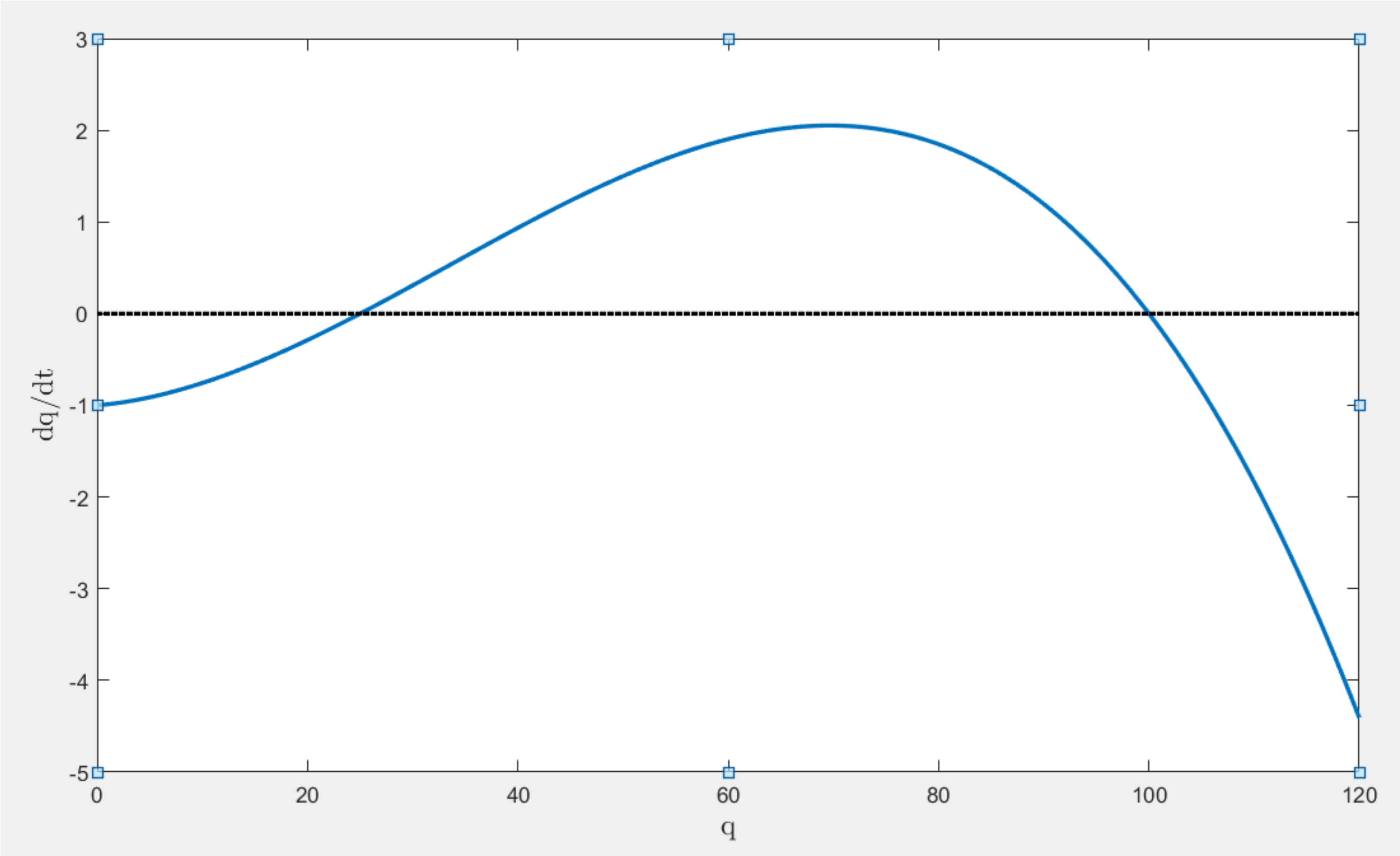
  - For each (q^0, p^0) there is exactly one equilibrium path with $q_{t_0} = q^0, p_{t_0} = p^0$

  - Asymptotic behavior

$$\lim_{t\to\infty} p_t = \begin{cases} p^*, & p_{t_0} = p^* \\ 0, & \text{otherwise} \end{cases}, \qquad \lim_{t\to\infty} q_t = \begin{cases} \underline{q}, & q_{t_0} = \underline{q} \\ \overline{q}, & \text{otherwise} \end{cases}$$

Problem 2 – Deterministic Equilibria

- Proof of Proposition (idea)



Problem 2 – Tax Backing

- Government imposes output tax, tax rate τ

- Subsidizes money by
  a) Real dividends to money holders
  b) Shrinking of money supply

- After-tax return on capital $$dr_t^k = \left(\frac{1-\tau}{q_t} + \mu_t^q\right) dt$$

- Return on money
  - Policy a)
    - dividend yield $\frac{\tau \bar{K}}{p_t \bar{K}} dt = \frac{\tau}{p_t} dt$
    - Capital gains $\frac{d(p_t K_t)}{p_t K_t} = \mu_t^p dt$

$$\Longrightarrow \quad dr_t^m = \left(\frac{\tau}{p_t} + \mu_t^p\right) dt$$

  - Policy b)
    - dividend yield 0
    - capital gains $\frac{dp_t^m}{p_t^m} = \frac{dp_t}{p_t} - \frac{dM_t}{M_t} = \left(\frac{\tau}{p_t} + \mu_t^p\right) dt$

$$dM_t = -\frac{\tau \bar{K}}{p_t^m} dt = -\frac{\tau}{p_t} M_t dt$$

Problem 2 – Tax Backing

- Asset Pricing Condition

$$\frac{1-\tau}{q_t} - \frac{\tau}{p_t} + \mu_t^q - \mu_t^p = \rho q_t \tilde{\sigma}$$

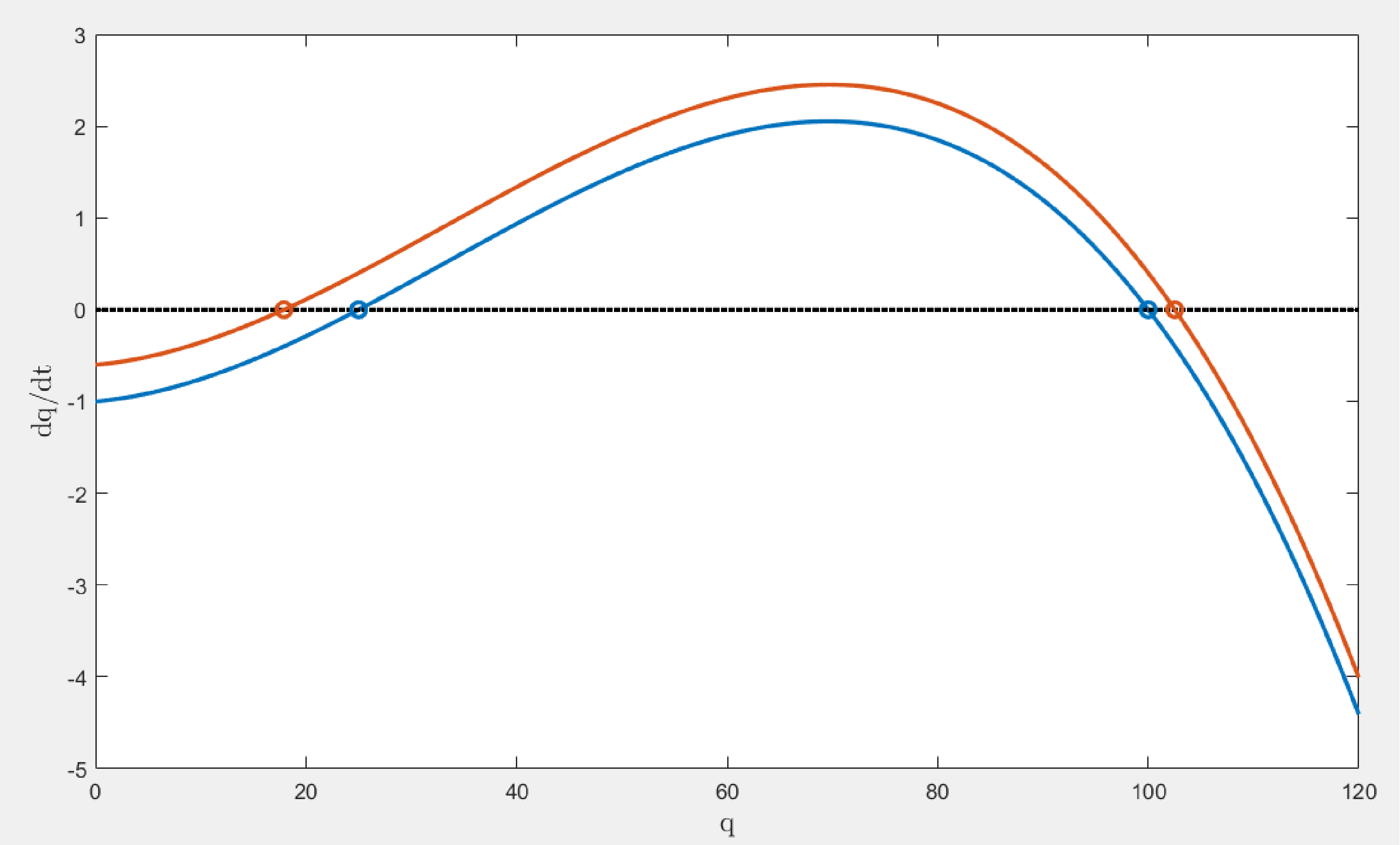
- Equilibrium ODE

$$\dot{q}_t = \rho^2 \tilde{\sigma}^2 \left(q_t + \frac{1}{\sqrt{\rho}\tilde{\sigma}}\right)\left(q_t - \frac{1}{\sqrt{\rho}\tilde{\sigma}}\right)\left(\frac{1}{\rho} - q_t\right) + \underbrace{\tau\left(1-\rho q_t\right) + \frac{\tau}{p_t} q_t \left(1-\rho q_t\right)}_{=\tau}$$

$$\tau\left(1-\rho q_t\right) + \frac{\tau}{p_t} q_t \left(1-\rho q_t\right) = \tau \frac{p_t}{p_t+q_t} + \frac{\tau}{\cancel{p_t}} q_t \frac{\cancel{p_t}}{p_t+q_t} = \tau$$

Problem 2 – Tax Backing
Uniqueness of Money Steady State

$$\dot{q}_t = \rho^2 \tilde{\sigma}^2 \left(q_t + \frac{1}{\sqrt{\rho}\tilde{\sigma}} \right) \left(q_t - \frac{1}{\sqrt{\rho}\tilde{\sigma}} \right) \left(\frac{1}{\rho} - q_t \right) + \tau$$



Problem 2 – Can Tax Backing Achieve Uniqueness without ever Taxing?

- Capital taxation to back the money stock may be undesirable from a welfare perspective
(in this model for high idiosyncratic risk want to inflate/subsidize capital, compare Brunnermeier, Sannikov 2016)

- If government can credibly commit to future taxation:
  - Start taxing as soon as value of money falls below some threshold $\hat{p} \in (0, p^*)$:

$$\tau(p,q) = \begin{cases} 0, & p \geq \hat{p} \\ \bar{\tau}, & p < \hat{p} \end{cases}$$

  - Eliminates all equilibrium paths with $p < \hat{p}$
(same argument as before)
  - Sufficient to eliminate all equilibria other than $\left(\underline{q}, p^*\right)$