Macro, Money and (International) Finance – Problem Set 3

February 26, 2019

Problem set prepared by Sebastian Merkel (smerkel@princeton.edu). Please let me know, if any tasks are unclear or you find mistakes in the problem descriptions. Questions about how to approach the problems are best directed to your local course TA.

Please submit to your local TA/coordinator by Tueday, March 5, before the lecture. Do not submit your solutions to me.

1 The One-sector Money Model with a Cash-in-advance Constraint

Consider the one-sector money model from lecture 5 with log utility, no idiosyncratic risk and a fixed money stock ($\mu^M = 0$). However, assume now that agents have a cash-in-advance constraint. To make the transactions necessary to purchase c_t units of consumption goods, an agent has to hold money with a real value of at least c_t/ℓ, where $\ell > 0$ is some parameter. Assume the economy is in a steady state with time-invariant p and q.

1. Derive the HJB equation of an individual agent, let $\hat{\lambda}_t$ be the Lagrange multiplier on the cash-in-advance constraint, if written in the form $c_t \leq \ell\theta_t n_t$, and define $\lambda_t := \hat{\lambda}_t/V'(n_t)$, where V is the value function.[1] Derive the first-order conditions for c and θ and show that the latter is structurally identical to the portfolio choice condition on slide 39 and 40[2] of lecture 5. For this part of the question, do not use $u = \log$, but work with a general utility function u.

2. Solve the model for p, q, ι, λ and $\zeta = c/n$ (there is a closed-form solution). Does the cash-in-advance constraint bind in equilibrium? Is the (steady-state) equilibrium unique?

3. How do changes in ℓ affect the equilibrium (in a comparative statics sense). Does higher liquidity demand (i.e. lower velocity ℓ) crowd in or crowd out investment?

---

[1] This is a convenient normalization: $\hat{\lambda}$ measures the change in value V from giving the agent a marginal unit of extra liquidity $\ell\theta n$, $V'(n)$ is the change in value from giving the agent a marginal unit of extra wealth n. So λ is something like a ratio of marginal utilities and can thus be interpreted as a relative price of liquidity.

[2] Tuesday's lecture did not get there yet and it is likely that slides get revised, so the eventual page number may change.

2 Equilibrium Multiplicity in the One-sector Money Model

Consider the benchmark model of lecture 5, in which money is a pure bubble. We had assumed a steady state equilibrium (for p and q) and not admitted the possibility of changing prices over time. The aim of this problem is to characterize all equilibria of the model with deterministic price drift.[3] To simplify the analysis, assume that agents have log utility, that there is no aggregate risk ($\sigma = 0$) and that there is a fixed money supply ($\mu^M = 0$). For this problem you may assume $\tilde{\sigma}^2 > \rho$. This was the condition in the lecture for money to have positive value.

1. Find all steady state equilibria, that is equilibria with $dp_t = dq_t = 0$.

   Note: you will find two equilibria here,[4] which differ – among other things – in the price of capital q. In the following, $\underline{q}$ and $\bar{q}$ (with $\underline{q} < \bar{q}$) refer to the q values in these two steady state equilibria.

2. Now consider more generally all deterministically drifting equilibria, that is postulate the price processes

$$\frac{dp_t}{p_t} = \mu_t^p dt, \qquad \frac{dq_t}{q_t} = \mu_t^q dt. \tag{1}$$

   The goal is to characterize all equilibria with such price paths. Proceed as follows:

   (a) Use the equilibrium conditions to derive an ODE that describes the evolution of q, p or ϑ (choose one of the three) over time in any equilibrium satisfying (1). The right-hand side of the ODE (in explicit form) should only contain model parameters and q_t (or p_t or ϑ_t), but no other endogenous variables/unkowns (such as the other two of the three variables q, p or ϑ).

      Hint: you need to find a relationship between μ^p, μ^q and μ^ϑ or a subset of these variables; Ito's formula/time differentiation may be useful here.

   (b) For each of $\underline{q}$ and $\bar{q}$ found in part 1, choose at least three initial conditions q_{t_0} around those steady states and solve the ODE of part (a) numerically. Plot the resulting time paths for $\{q_t\}_{t=t_0}^T$ and for $\{p_t\}_{t=t_0}^T$. What are your observations? Write down whatever patterns you observe, but in particular with regard to the following two questions:

      i. Can all paths represent an actual equilibrium?
      ii. What can happen in the long run? Under which conditions and to where do paths converge/diverge?

      For your numerical experiments use the parameters $a = 1$, $\rho = 0.01$, $\tilde{\sigma} = 0.4$, $\kappa \to \infty$, take the initial time $t_0 = 0$ and plot at least $T = 50$ time periods.

   (c) Based on your observations from part (b), formulate a proposition and prove it. The proposition should make an exact characterization of the set of price pairs (q, p) that can be observed in at least one equilibrium at at least one point in time. Conditional on observing a pair (p_{t_0}, q_{t_0}) in this set, the proposition should make an assertion about the long-run behavior, $\lim_{t\to\infty}(p_t, q_t)$.

   (d) In which equilibria does money have a positive value? In which equilibria does it have a positive value even asymptotically (that is $\lim_{t\to\infty} p_t > 0$)?

---

[3]There are even more equilibria, if one allows for stochastic price changes.
[4]There is a third one that satisfies all equilibrium equations. But it violates the free disposal condition $q \geq 0$ and can therefore be ruled out.

3. **Tax backing of money and equilibrium uniqueness**

Now consider a small variation of the setting before. Suppose there is a government that taxes capital income/production. k_t units of capital still produce an output flow $ak_t dt$, but only $(1 - \tau)ak_t dt$ are kept by the owner of the capital, whereas $\tau ak_t dt$ is collected by the government. Aggregate tax revenues are thus $\tau a\bar{K} dt$. Suppose the government uses all tax revenues to back the value of money by either of the following two strategies

(i)   it pays a *real* dividend on money holdings, that is each household receives a fiscal transfer of consumption goods proportional to its money holdings

(ii)   it buys back money and destroys it to shrink the money supply[5]

Assume that the government keeps a balanced budget at all times. Assume further that p and q follow generic deterministic processes as in (1).

(a) Derive the processes for capital gains and dividend yields under both policies. Show that both policies generate the same return process for money.

(b) Modify your solution to part 2 (a) to reflect the new return process for money.[6] Derive again an ODE that describes the evolution of q (or p or ϑ) over time. Plot $\frac{dq}{dt}$ as a function of q for $\tau = 0$ and some small values $\tau > 0$.

As in part 2 (b), use $a = 1$, $\rho = 0.01$, $\tilde{\sigma} = 0.4$, $\kappa \to \infty$, in addition let $\tau \in \{0.1, 0.2, 0.4\}$ be the "small" positive tax rates.[7]

(c) Comparative statics: show that as τ is marginally increased at $\tau = 0$, the $\underline{q}$ steady state is reduced and the $\bar{q}$ steady state is increased. What does this mean for the associated p values?

Hint: do not actually solve for the steady states in the case $\tau > 0$ by solving the equation $\frac{dq}{dt} = 0$ for q, because this will be a third order polynomial equation. Instead use an argument based on the implicity function theorem.

(d) Show that (for arbitrarily small taxes $\tau > 0$) the tax backing eliminates all equilibria other than the steady state in which money has positive value.

Hint: show that all other potential equilibria eventually drift to a point that violates one of the free disposal conditions $q_t, p_t \geq 0$.

(e) Suppose a government wants to achieve equilibrium uniqueness, but does not want to distort the capital allocation by taxing capital.[8,9] The only policy tool of the government is still the capital income tax, but suppose the government can credibly commit to a state-contingent tax rate $\tau(p, q)$ as a function of p and q. Is it possible to design a tax policy that eliminates all but one equilibrium and satisfies $\tau = 0$ along the (unique) equilibrium path? Give an example or an impossibility argument.

---

[5] This is obviously equivalent to assuming that the government allows households to pay taxes with money instead of real goods and just disposes the collected tax revenues.

[6] Of course, also the return on capital changes due to the tax.

[7] Not exactly small tax rates, but we want to see something in the plot.

[8] Potentially, the government even wants to print money and/or subsidize capital for the same reasons as in Brunnermeier and Sannikov (2016) and Di Tella (2018).

[9] Alternatively, this desire can also come from outside the model by our preferences as model builders: we may not want the complication of a tax, but want to make an argument why our equilibrium is not only the limit of a sequence of unique equilibria in richer models, but is actually exactly the unique equilibrium of a slightly richer model.

References

BRUNNERMEIER, M. K., AND Y. SANNIKOV (2016): "On the optimal inflation rate," *American Economic Review*, 106(5), 484–89.

DI TELLA, S. (2018): "A Neoclassical Theory of Liquidity Traps," Working Paper.