Macro, Money and (International) Finance – Problem Set 1

February 8, 2019 – *updated February 10, 2019*

Problem set prepared by Sebastian Merkel (smerkel@princeton.edu). Please let me know, if any tasks are unclear or you find mistakes in the problem descriptions. Questions about how to approach the problems are best directed to your local course TA.

Please submit to your local TA/coordinator by Friday, February 15. Do not submit your solutions to me.

Please also prepare and submit a (preliminary) solution to Problem 2, parts 1 and 2 until Tuesday, February 12, 3 pm New York time.

1 Capital (Quality) and Technology Shocks

Consider the simple model from lecture 2. There, we assumed that expert capital follows

$$\frac{dk_t}{k_t} = (\Phi(\iota_t) - \delta)\, dt + \sigma dZ_t \tag{1}$$

and produces an output flow[1]

$$y_t = \bar{a} k_t. \tag{2}$$

In this problem set you are asked to consider a different specification with consumption-specific technology shocks instead of capital shocks. Specifically, suppose instead of equations (1) and (2) that capital evolves according to

$$\frac{dk_t}{k_t} = \left(\Phi\left(\frac{\bar{a}}{a_t}\iota_t\right) - \delta\right) dt \tag{3}$$

and produces an output flow

$$y_t = a_t k_t \tag{4}$$

where a_t is now a stochastic process given by[2,3]

$$\frac{da_t}{a_t} = \phi\left(\log \bar{a} - \log a_t\right) dt + \sigma dZ_t. \tag{5}$$

[1] I use here $\bar{a}$ instead of a from the lecture to distinguish this more clearly from the process a defined below.

[2] One gets to this equation by imposing that $\log a_t$ follows an Ornstein-Uhlenbeck process, the continuous-time equivalent of a discrete-time AR(1) process, and correcting by a deterministic time drift, such that the long-run mean of A_t is not growing/shrinking over time. The equivalent in discrete time is often taken as a productivity process in standard macro models.

[3] *In an earlier version, the following equation had a $-$ in front of ϕ. This was a mistake.*

For $\phi = 0$ this specification implies a geometric Brownian motion for productivity, for $\phi > 0$, a_t mean-reverts to the level $\bar{a}$ in the long run. The additional term $\bar{a}/a_t$ in the Φ function implies that only consumption production is impacted by changes of a_t.

1. Show that without productivity mean reversion ($\phi = 0$), the model with capital shocks (evolution (1) and output (2)) and the model with consumption-specific technology shocks (capital evolution (3), output (4) and productivity process (5)) are isomorphic in the sense that they imply the same dynamics for output, consumption, net worth, the expert wealth share η and the risk-free rate.

2. How are the two models related, if $\phi > 0$?

3. Explain economically, why the two shock types are not equivalent for neutral technology shocks (i.e. if the investment technology is $\Phi(\iota_t)$) and a nonconstant Φ function (1-2 sentences are sufficient).

2 The Basak-Cuoco Model with Heterogeneous Discount Rates

Consider the model from lecture 2 (now again with capital shocks), but unlike there assume that households are more patient than experts, i.e. they have a discount rate $\underline{\rho} < \rho$. This is the simplest way to generate both a nondegenerate stationary distribution and some endogenous capital price dynamics.

1. Derive closed-form expressions for ι, q, σ^q, μ^η and σ^η as a function of η and model parameters.[4] You do not actually have to follow the order of steps in the lecture. In this simple model it pays off to start with goods market clearing.

2. Replicate the figures from slide 34 (δ is not stated there, so choose just some parameter that generates similar numbers for the risk-free rate), then add to each plot the corresponding line for the model with $\underline{\rho} < \rho$ using $\underline{\rho} = 2\%$ (and all other parameters as before).

3. Assume $\kappa > 0$. Show that in this model asset price movements mitigate exogenous risk. Explain economically, why this happens.

4. Argue that the model must have a nondegenerate stationary distribution (just give some intuition, not a fully spelled-out formal proof). Compute the stationary density of η by numerically solving the ODE stated on page 16 of Yuliy's stochastic calculus notes using the same parameters as in part 2.[5] What is the stationary density of q?

3 ODE Review

1. Read the section on ODEs in the differential equation document distributed with this problem set.

2. Solve the simple ODEs

$$y' = y \tag{6}$$

$$y' = x \cos\left(x^2\right) y^2 \tag{7}$$

$$y'' = -y \tag{8}$$

[4] As in the lecture, assume the specific functional form $\Phi(\iota) = \frac{1}{\kappa}\log(1 + \kappa\iota)$ for Φ.

[5] Have a look at Problem 3 before you do this.

on the interval $[0, 10]$ with the initial condition $y(0) = 1$ for all three equations and the additional initial condition $y'(0) = 0$ for equation (8) using the following three methods:

(a) explicit Euler method;

(b) implicit Euler method;

(c) a build-in solver of your numerical software or your favorite numerical library for your programming language.

For the first two methods, make a suitable discretization/step width choice yourself. For each of the three equations plot the results from the three approximation methods together with the respective true solution. These are given by (please verify):

$$y(x) = \exp(x), \qquad y(x) = \frac{1}{1 - \sin x^2/2}, \qquad y(x) = \cos(x).$$

4 Stability of Euler Methods

In this problem we study stability properties of Euler methods, both theoretically and numerically. This is useful for two reasons. First, as you will see in the numerical part of this problem, unstable methods have very bad error-propagation properties and are thus sometimes unsuitable (or require very small step widths and thus a long time to solve). Second, correct long-run convergence behavior will turn out to be important for the "iterative method" that we will use later in this course to solve continuous-time macro-finance models.

1. Consider the linear test equation

$$y' = \lambda y, \qquad y(0) = 1 \tag{9}$$

with a (complex) parameter $\lambda \in \mathbb{C}$. Its solution is $y(x) = e^{\lambda x}$ and if the real part of λ is negative, then $|y(x)|$ is bounded, strictly decreasing and converges to 0 as $x \to 0$. A numerical solution method (for a fixed step width) is called A-stable, if the numerical approximation to y has the same properties for all λ with negative real part.

   (a) Consider the explicit Euler method with a fixed step width Δx (that is $x_i - x_{i-1} = \Delta x$ for all i) for the test problem (9). Find an explicit formula for y_i and determine when $\{|y_i|\}_{i=1}^N$ is strictly decreasing and when it remains bounded and/or converges to 0 as $N \to \infty$. Conclude that the region of parameters λ, for which all three are satisfied is decreasing in Δx, but even for small Δx never includes all λ with negative real part – in particular, the explicit Euler method is not A-stable.

   (b) Do the same analysis for the implicit Euler method for a fixed step width Δx. For which $\Delta x > 0$ is the implicit Euler method A-stable?

2. Choose $\lambda = -10$ and solve the test problem (9) using the explicit Euler method for $\Delta x = 0.05, 0.1, 0.19, 0.2, 0.21, 0.25$ over the time interval $[0, 2]$, plot the results together with the true solution. Repeat the same exercise with the the implicit method. Explain why your results are expected based on the analysis in part 1.