

# Macro, Money and (International) Finance – Problem Set 2

February 15, 2019

Problem set prepared by Sebastian Merkel (smerkel@princeton.edu). Please let me know, if any tasks are unclear or you find mistakes in the problem descriptions. Questions about how to approach the problems are best directed to your local course TA.

Please submit to your local TA/coordinator by Thursday, February 21, before the lecture.<sup>1</sup> Do not submit your solutions to me.

## 1 PDE Review

Read the new sections in the updated differential equations notes.

## 2 The Kolmogorov Forward Equation for an Ornstein-Uhlenbeck Process

In this problem you are asked to write a solution algorithm for a relatively simple PDE, a (time-varying) Kolmogorov forward equation for a simple process. The generic form of the Kolmogorov forward equation for diffusion processes (Ito processes) is

$$\frac{\partial p}{\partial t}(x, t) = -\frac{\partial}{\partial x}(\mu(x, t)p(x, t)) + \frac{1}{2} \frac{\partial^2}{\partial x^2}(\sigma^2(x, t)p(x, t)). \quad (1)$$

If  $p_0$  is the probability density of the initial value  $X_0$  of a process  $X$  that follows the evolution

$$dX_t = \mu(X_t, t)dt + \sigma(X_t, t)dZ_t,$$

then  $p(\cdot, t)$  is the density of the random variable  $X_t$ , if the function  $p$  solves equation (1) and satisfies the initial condition  $p(x, 0) = p_0(x)$  for all  $x$ . Consider in this problem for  $X$  an Ornstein-Uhlenbeck process,

$$dX_t = \theta(\bar{x} - X_t)dt + \sigma dZ_t.$$

The Kolmogorov forward equation for this process is

$$\frac{\partial p}{\partial t}(x, t) = \theta(x - \bar{x})\frac{\partial}{\partial x}p(x, t) + \theta p(x, t) + \frac{\sigma^2}{2} \frac{\partial^2}{\partial x^2}p(x, t). \quad (2)$$

---

<sup>1</sup>Next Thursday's lecture may be moved (this is not clear yet and will be announced next Tuesday). If so, you have time until Friday.

Suppose the initial condition  $p_0$  is a normal density with mean  $m_0$  and variance  $v_0$ . This process is the continuous-time analog of a discrete-time AR(1) process with normal noise and we know that for a discrete-time AR(1) process  $x$ ,  $x_t$  is normally distributed for all  $t \geq 0$ , if the initial value  $x_0$  is normally distributed (potentially degenerate normal with zero variance). Indeed, the same is true for continuous time. The solution function is given by

$$p(x, t) = \frac{1}{\sqrt{v(t)}} \phi\left(\frac{x - m(t)}{\sqrt{v(t)}}\right) \quad (3)$$

with

$$\begin{aligned} v(t) &= v_0 e^{-2\theta t} + (1 - e^{-2\theta t}) \frac{\sigma^2}{2\theta} \\ m(t) &= m_0 e^{-\theta t} + (1 - e^{-\theta t}) \bar{x} \end{aligned}$$

where  $\phi(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2}$  is the standard normal density.<sup>2</sup> Use this PDE problem as a test problem to practice numerical solutions of PDEs:

1. Write a generic Kolmogorov forward equation solver for the Ornstein-Uhlenbeck process. The function should take the following inputs:
  - (a) equation parameters  $\theta$ ,  $\sigma$  and  $\bar{x}$ ;
  - (b) generic grid vectors along the  $x$  and  $t$  dimensions (not necessarily uniformly spaced);
  - (c) a vector of initial density values ( $p_0$ ) on the specified  $x$  grid (not necessarily a normal density);
  - (d) vectors of (artificial) boundary conditions  $p(\underline{x}, \cdot)$ ,  $p(\bar{x}, \cdot)$  on the specified  $t$  grid, where  $\underline{x}$  and  $\bar{x}$  are the left and right end points of the  $x$  grid;
  - (e) an option specifying the solution method (explicit or implicit Euler);
  - (f) an option specifying the computation of first derivatives (central differences, left differences, right differences or upwind)

Please add a header comment to the function that explains all inputs and their usage. The function is supposed to return a matrix that contains the solution values  $p(x, t)$  for all elements  $(x, t)$  on the specified grids.

2. Using your solver function, solve equation (2) for the simple case  $\bar{x} = 0, \theta = 0, \sigma = 1$  (then  $dX = dZ$ ). Let  $\underline{x} := -5, \bar{x} := 5, T := 2$  solve the equation on  $[\underline{x}, \bar{x}]$  in the time interval  $[0, T]$  taking as an initial value a normal density with variance  $v_0 = 0.1$  and mean  $m_0 = 0$ , and a uniform space and time discretization with  $\Delta x = 0.1$  and  $\Delta t = 0.005$ . Use central differences for the computation of first derivatives and compute a numerical solution using both the explicit and implicit method. For both numerical solutions  $\tilde{p}$  plot the absolute error  $\log_{10} |\tilde{p} - p|$  and the relative error  $\log_{10} \frac{|\tilde{p} - p|}{2(|\tilde{p}| + |p|)}$  as a two-dimensional plot, where  $p$  is the closed-form solution (3). Also plot the maximum relative error along the space dimension over time for both methods.

---

<sup>2</sup>If one wanted to derive this closed-form solution, one could start by guessing that the solution is a normal density, substituting it into the PDE and solving for  $v$  and  $m$ . Here, you can just take this solution as given.

3. Repeat the exercise of part 2, but this time use the finer space discretization  $\Delta x = 0.05$ . What happens?
4. Revert back to the values of  $\Delta x$ ,  $\Delta t$  of part 2, but now add mean reversion,  $\theta = 3$  and reduce the volatility to  $\sigma = 0.33$ . Also, let  $v_0 = 0.33$ ,  $m_0 = -3$  and  $T = 1$ . Solve the equation with all 8 possible method options (explicit/implicit, central/left/right/upwind). For each method check, whether it yields an acceptable result and report in 1-2 sentences why it does or what goes wrong. Add some plots to your answer to illustrate your point (this is not required for all 8 possibilities, 2-5 plots are sufficient).
5. Check that your function written in part 1 also works for nonuniformly spaced grids. Generate a random space grid with  $N = 200$  points and the restriction that  $x_0 = x_L = -5$ ,  $x_N = x_R = 5$  and solve the problem with parameters  $\bar{x} = 0$ ,  $\theta = 0.5$ ,  $\sigma = 0$  and initial conditions  $m_0 = 0$ ,  $v_0 = 0.1$  (how exactly you implement the random grid generation is your choice). Run it a number of times and verify that the numerical approximation quality is acceptable.

### 3 The Two-Sector Real Model with Recursive Utility

Solve the two-type example model of lecture 3, but instead of CRRA utility, assume that agents have isoelastic stochastic differential utility (“continuous-time Epstein-Zin preferences”) with risk aversion coefficients  $\gamma, \underline{\gamma} \neq 1$  and elasticity of intertemporal substitution  $\varphi = \underline{\varphi} = 1$ .<sup>3</sup> Optionally, you can also solve the model assuming general  $\varphi, \underline{\varphi}$ , but  $\varphi = \underline{\varphi} = 1$  simplifies some steps, so this may be easier to start with.

“Solve” includes writing a code that solves the model’s PDEs. Pick some parameter and compute a solution. Plot the price function ( $q$ ), the drift and volatility of the state ( $\mu^n \eta$  and  $\sigma^n \eta$ ) and whatever else you think is informative. If you are still motivated once you get there, do some comparative statics exercises similar to the ones at the end of lecture 3.

Some more details and hints:

- Stochastic differential utility is recursively defined by a backward stochastic differential equation<sup>4</sup>

$$U_t = E_t \left[ \int_t^\infty f(c_s, U_s) ds \right]$$

with

$$f(c, U) = \frac{1 - \gamma}{1 - \varphi^{-1}} \rho U \left( \left( \frac{c}{((1 - \gamma) U)^{\frac{1}{1-\gamma}}} \right)^{1-\varphi^{-1}} - 1 \right).$$

For  $\varphi = \gamma$  this simplifies to  $f(c, U) = \rho \frac{c^{1-\gamma}}{1-\gamma} - \rho U$  and thus generates the CRRA utility  $U_t =$

---

<sup>3</sup>The typical EIS symbol is  $\psi$ , but  $\psi$  is in the lecture the share of capital operated by experts, so for the purpose of this question  $\varphi$  is the EIS.

<sup>4</sup>It is not a typo that there is no discounting term  $e^{-\rho s}$ . If you are interested in how to interpret this infinite-horizon representation mathematically, check appendix C of Duffie and Epstein (1992).

$E_t \left[ \int_t^\infty \rho e^{-\rho(s-t)} \frac{c^{1-\gamma}}{1-\gamma} ds \right]$ .<sup>5</sup> For  $\varphi \rightarrow 1$  and  $\gamma \neq 1$ ,  $f$  converges to the log form

$$f(c, U) = (1 - \gamma) \rho U \log \left( \frac{c}{((1 - \gamma) U)^{\frac{1}{1-\gamma}}} \right) = (1 - \gamma) \rho U \left( \log(c) - \frac{1}{1 - \gamma} \log((1 - \gamma) U) \right).$$

Except for heavier notation, this type of utility is not really harder to deal with than CRRA utility (and for  $\varphi = 1$  in some ways simpler).

- You are free to follow either the martingale approach or the HJB approach.
  - If you choose the martingale approach, follow the steps in the lecture as closely as possible and make only adjustments, where necessary. Check my handout on the martingale pricing conditions for the correct representation of the SDF process  $\xi$ .
  - If you choose the HJB approach, skip step 1 in the lecture and start by deriving the HJB (you will recover the asset pricing/price-taking planner conditions by taking first-order conditions in the HJB plus imposing some market clearing conditions thereafter).
- In step 2 continue to conjecture that the value function has the representation  $V_t = \frac{(\omega_t n_t)^{1-\gamma}}{1-\gamma}$ .
- If you assume  $\varphi = \underline{\varphi} = 1$ , try to show that  $\frac{c}{n} = \rho$ ,  $\frac{\underline{c}}{\underline{n}} = \underline{\rho}$ . This will simplify the algebra.
- Some hints for the numerical solution procedure:
  - Do not put  $\eta = 0$  and  $\eta = 1$  into your grid for the state variable. Some variables have singularities at these boundary points. Instead let the state grid run from  $\eta = \epsilon$  to  $\eta = 1 - \epsilon$  with some small number  $\epsilon$ .
  - The time step for  $v$  and  $\underline{v}$  is basically a linear PDE problem (you should be experienced with these problems after having solved problem 2). Use an implicit method, upwind scheme and if you are sure the code is correct, but it seems unstable or does not converge, decrease the time step width.
  - The static step can lead to more problems and may require some experimentation until it works. Solve for the  $q(\eta)$  function from left to right, starting at your lowest  $\eta$  grid point  $\eta_0$ . To get an initial condition, just assume there  $\psi(\eta_0) = 0$  and derive  $q(\eta_0)$  from goods market clearing.

## References

DUFFIE, D., AND L. G. EPSTEIN (1992): “Stochastic differential utility,” *Econometrica: Journal of the Econometric Society*, pp. 353–394.

---

<sup>5</sup>Note that there is an additional factor  $\rho$  in the utility function. This is the reason why the value function guess below does not have the factor  $\frac{1}{\rho}$  that we had in the lecture.