Macro, Money and Finance

Lecture 03: Macro-Finance Solution Technique

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Spring 2019

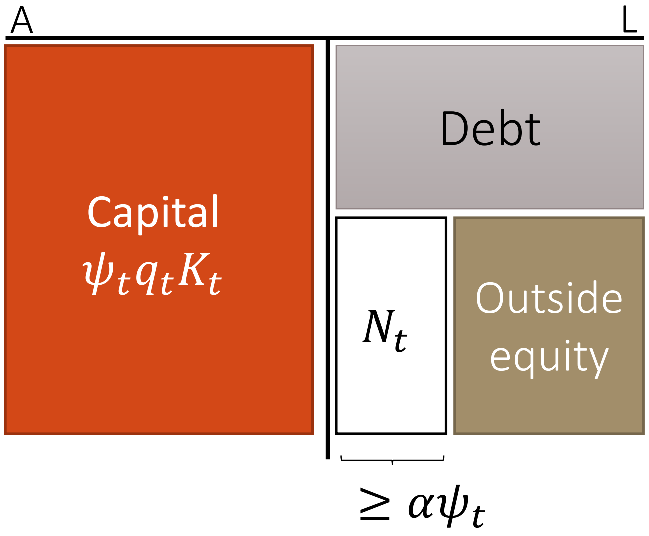
Desired Model Properties

- Normal regime: stable around steady state
  - Experts are adequately capitalized
  - Experts can absorb macro shock
- Endogenous risk
  - Fire-sales, liquidity spirals, fat tails
  - Spillovers across assets and agents
  - Market and funding liquidity connection
  - SDF vs. cash-flow news
- Volatility paradox
- Financial innovation less stable economy
- ("Net worth trap" double-humped stationary distribution)

Two Type/Sector Model with Outside Equity

BruSan 2017: Handbook of Macroeconomics, Chapter 18, Section 3

- Expert sector



A L

Capital $\psi_t q_t K_t$

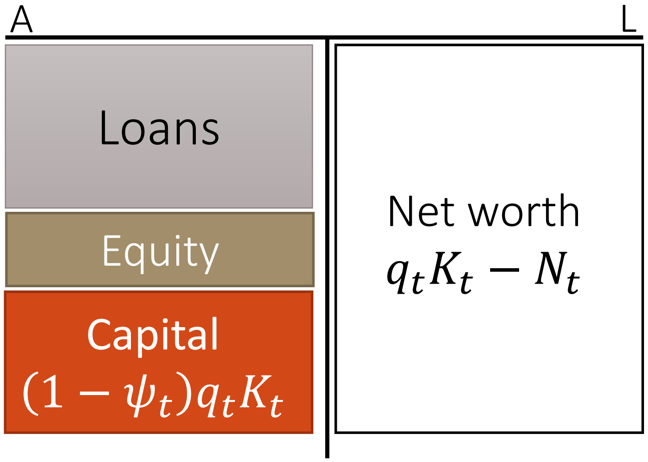
Debt

N_t

Outside equity

$\geq \alpha\psi_t$

Household sector



A L

Loans

Equity

Capital $(1-\psi_t)q_t K_t$

Net worth $q_t K_t - N_t$

- Experts must hold fraction $\chi_t \geq \alpha\psi_t$ (skin in the game constraint)
- Return on inside equity N_t can differ from outside equity
  - Issue outside equity at required return from HH
  - In related model, He and Krishnamurthy 2013 impose that inside and outside equity have same return

Two Type Model Setup

Expert sector

- Output: $y_t = a k_t$

$a \geq \underline{a}$

Household sector

- Output: $\underline{y}_t = \underline{a}\,\underline{k}_t$

$$A(\psi) = \psi a + (1 - \psi)\underline{a}$$

Capital share
of experts

Poll 4: Why is it important that households can hold capital?

a) to capture fire-sales
b) for households to speculate
c) to obtain stationary distribution

Two Type Model Setup

Expert sector

- Output: $y_t = ak_t$

$a \geq \underline{a}$

Household sector

- Output: $\underline{y}_t = \underline{a}k_t$

$$A(\psi) = \psi a + (1-\psi)\underline{a}$$

Capital share of experts

Poll 5: What are the modeling tricks to obtain stationary distribution?

a) switching types

b) agents die, OLG/perpetual youth models (without bequest motive)

c) different preference discount rates

Two Type Model Setup

Expert sector

- Output: $y_t = a k_t$
- Consumption rate: c_t
- Investment rate: ι_t

$$\frac{dk_t^{\tilde{\imath}}}{k_t^{\tilde{\imath}}} = (\Phi(\iota_t) - \delta)dt + \sigma dZ_t + \tilde{\sigma} d\tilde{Z}_t^{\tilde{\imath}}$$

$a \geq \underline{a}$

Household sector

- Output: $\underline{y}_t = \underline{a}\underline{k}_t$
- Consumption rate: $\underline{c}_t$
- Investment rate: $\underline{\iota}_t$

$$\frac{d\underline{k}_t^{\tilde{\imath}}}{\underline{k}_t^{\tilde{\imath}}} = \left(\Phi\left(\underline{\iota}_t\right) - \delta\right)dt + \sigma dZ_t + \tilde{\sigma} d\tilde{Z}_t^{\tilde{\imath}}$$

capital evolution absent market transactions/fire-sales

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- $E_0[\int_0^\infty e^{-\rho t} \frac{c_t^{1-\gamma}}{1-\gamma} dt]$

$a \geq \underline{a}$

$\rho \geq \underline{\rho}$

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- $E_0[\int_0^\infty e^{-\underline{\rho} t} \frac{\underline{c}_t^{1-\gamma}}{1-\gamma} dt]$

Friction: Can only issue

- Risk-free debt
- Equity, but most hold $\chi_t \geq \alpha \psi_t$

Solving MacroModels Step-by-Step

0. Postulate aggregates, price processes & obtain return processes
1. For given SDF processes *static*
   - a. Real investment ι, (portfolio $\boldsymbol{\theta}$, & consumption choice of *each agent*)
     - *Toolbox 1:* Martingale Approach
   - b. Asset/Risk Allocation *across types/sectors* & asset market clearing
     - *Toolbox 2:* "price-taking social planner approach" – Fisher separation theorem
2. Value functions *backward equation*
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3. Evolution of state variable η *forward equation*
   - *Toolbox 3:* Change in numeraire to total wealth (including SDF)
   - ("Money evaluation equation" μ^{ϑ})
4. Value function iteration & goods market clearing
   - a. PDE of de-scaled value fcn.
   - b. Value function iteration by solving PDE

0. Postulate Aggregates and processes

- Individual capital evolution:

$$\frac{dk_t^{\tilde{\imath},i}}{k_t^{\tilde{\imath},i}} = \left(\Phi\left(\iota^{\tilde{\imath},i}\right) - \delta\right)dt + \sigma dZ_t + d\Delta_t^{k,\tilde{\imath},i}$$

  - Where $\Delta_t^{k,\tilde{\imath},i}$ is the individual cumulative capital purchase process

(c is numeraire)

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- Capital aggregation:
  - Within sector i: $K_t^i \equiv \int k_t^{\tilde{\imath},i} d\tilde{\imath}$
  - Across sectors: $K_t \equiv \sum_i K_t^i$
  - Capital share: $\psi_t^i \equiv K_t^i / K_t$

$$\frac{dK_t}{K_t} = \int \left(\Phi\left(\iota^i\right) - \delta\right)di\, dt + \sigma dZ_t \quad \text{since } \Delta_t^{k,\tilde{\imath},i} \text{ add up to zero}$$

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- Value of capital stock: $q_t K_t$

Postulate $dq_t/q_t = \mu_t^q dt + \sigma_t^q dZ_t$

(c is numeraire)

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$$\frac{dK_t}{K_t} = \int \left(\Phi\left(\iota^i\right) - \delta\right) di \, dt + \sigma dZ_t$$

Poll 14: How many Brownian motions span prob. space?

a) one

b) two

c) one + number of sectors

d) two + number of sectors

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Postulate $dq_t / q_t = \mu_t^q dt + \sigma_t^q dZ_t$

Same Brownian

(c is numeraire)

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  Postulate $dq_t/q_t = {\color{red}\mu_t^q} dt + {\color{red}\sigma_t^q} dZ_t$

- Postulated SDF-process: $\frac{d\xi_t^i}{\xi_t^i} = \underbrace{\mu_t^\xi}_{\equiv -{\color{red}r_t}} + \underbrace{\sigma_t^{\xi^i}}_{\equiv -{\color{red}\varsigma_t^i}} dZ_t$ (c is numeraire)

0. Postulate Aggregates and Processes

- … from price processes to return processes (using Ito)
  - Use Ito product rule to obtain capital gain rate (in absence of purchases/sales)
    - Define $\check{k}_t^{\tilde{\imath}}$: $\frac{d\check{k}_t^{\tilde{\imath},i}}{\check{k}_t^{\tilde{\imath},i}} = \left(\Phi\left(\iota_t^{\tilde{\imath},i}\right) - \delta\right)dt + \sigma dZ_t \cancel{+ d\Delta_t^{k,\tilde{\imath},i}}$ without purchases/sales

$$dr_t^K\left(\iota_t^{\tilde{\imath},i}\right) = \left(\overbrace{\frac{A(\psi_t) - \iota_t^i}{q}}^{\text{Dividend yield}} + \overbrace{\Phi\left(\iota_t^i\right) - \delta + \mu_t^q + \sigma\sigma_t^q}^{\text{E[Capital gain rate]}= \frac{d(q_t\check{k}^i{}_t)}{(q_t\check{k}_t^i)}}\right)dt$$

$$+ \left(\sigma + \sigma_t^q\right)dZ_t$$

- Postulate SDF-process: (Example: $\xi_t^i = e^{-\rho t}u^{i\prime}(c_t)$)

$$\frac{d\xi_t^i}{\xi_t^i} = -r_t dt - \underset{\substack{\uparrow \\ \text{Price of risk}}}{\varsigma_t^i} dZ_t$$

0. Postulate Aggregates and Processes

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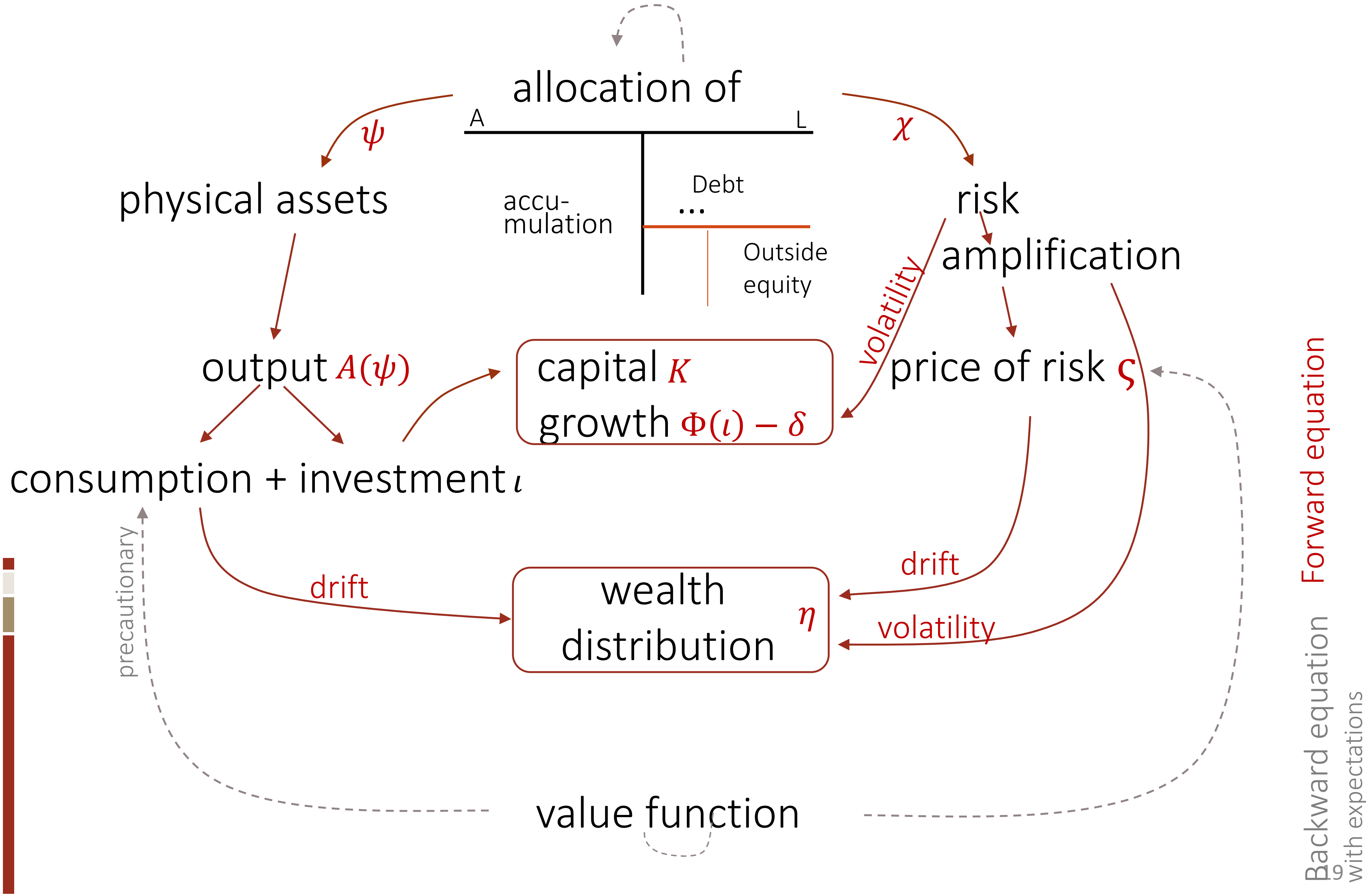
Poll 18: Why does drift of SDF equal risk-free rate

a) no idio risk

b) $e^{-r^F} = E[SDF]*1$

c) no jump in consumption

The Big Picture



allocation of

A

L

ψ

χ

physical assets

accu-
mulation

Debt

…

Outside
equity

risk

amplification

volatility

output $A(\psi)$

capital κ
growth $\Phi(\iota) - \delta$

price of risk ς

consumption + investment ι

precautionary

drift

wealth
distribution η

drift

volatility

value function

Forward equation

Backward equation
with expectations

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1. Individual Agent Choice of ι, θ, c

- Choice of ι is static problem (and separable) for each t

- $$\max_{\iota^i} dr^K(\iota^i) = \max_{\iota^i}\left(\frac{A(\psi)-\iota^i}{q}+\Phi(\iota^i)-\delta+\mu^q+\sigma\sigma^q\right)$$

- FOC: $\frac{1}{q}=\Phi'(\iota^i)$ Tobin's q
  - All agents $\iota^i=\iota$ $\Rightarrow \frac{dK_t}{K_t}=(\Phi(\iota)-\delta)\,dt+\sigma dZ_t$
  - Special functional form:
    - $\Phi(\iota)=\frac{1}{\kappa}\log(\kappa\iota+1)\Rightarrow\kappa\iota=q-1$

1a. Optimal Portfolio Choice

- Of experts with outside equity issuance (after plugging in households' outside equity choice)

$$\frac{a-\iota_t}{q_t} + \Phi(\iota_t) - \delta + \mu_t^q + \sigma\sigma_t^q - r_t =$$
$$= [\varsigma_t \chi_t/\psi_t + \underline{\varsigma}_t(1-\chi_t/\psi_t)](\sigma + \sigma^q)$$

- Of households' capital choice

$$\frac{\underline{a}-\iota_t}{q_t} + \Phi(\iota_t) - \delta + \mu_t^q + \sigma\sigma_t^q - r_t \leq \underline{\varsigma}_t(\sigma + \sigma^q)$$

with equality if $\psi_t < 1$

- New Approach replaces this step with Fisher Separation Social Planners' choice (see below)

1a. Individual Agent Choice of ι, θ, c

- Consumption Choice: Martingale Approach
  - Consider a self-financing trading strategy consisting of agent's networth *with consumption reinvested*.
  - $\frac{d(\xi_t^i n_t)}{\xi_t^i n_t} + \frac{c_t}{n_t} dt = \Big(\underbrace{-r_t + \mu_t^n - \varsigma_t^i \sigma_t^n + \frac{c_t}{n_t}}_{=0}\Big) dt + \sigma \dots$
  - $\frac{c_t}{n_t} = r_t - \mu_t^n + \varsigma_t^i \sigma_t^n$
  - (only) useful for steady state characterization

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1b. Asset/Risk Allocation across I Types

- Price-Taking Planner's Theorem:
  A social planner that takes prices as given chooses a real asset (capital/production) allocation, ψ_t, and risk allocation χ_t, that coincides with the choices implied by all individuals' portfolio choices.

$$\begin{aligned} \boldsymbol{\varsigma}_t &= (\varsigma_t^1, \dots, \varsigma_t^I) \\ \boldsymbol{\chi}_t &= (\chi_t^1, \dots, \chi_t^I) \\ \boldsymbol{\sigma}(\chi_t) &= \left(\sigma^1(\chi_t), \dots, \sigma^I(\chi_t)\right) \end{aligned}$$

- Planner's problem

$$\max_{\{\boldsymbol{\psi}_t, \boldsymbol{\chi}_t\}} E_t[dr_t^K(\psi_t)] - \boldsymbol{\varsigma}_t \sigma(\boldsymbol{\chi}_t) \quad = dr^F \text{ in equilibrium}$$

subject to friction: $F(\boldsymbol{\psi}_t, \boldsymbol{\chi}_t) \leq 0$

  - Examples:
    1. $\chi_t = \psi_t$ (if one holds capital, one has to hold risk)
    2. $\chi_t \geq \alpha \psi_t$ (skin in the game constraint)

1b. Asset/Risk Allocation across I Types

- Sketch of Proof of Theorem

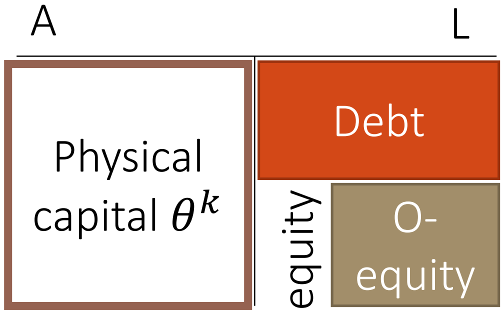
1. Fisher Separation Thm: (delegated portfolio choice by firm)
   - FOC yield the martingale approach solution
   - Each individual agent $(i, \tilde{\imath})$ portfolio maximization is equivalent to the maximization problem of a firm

$$\max_{\{\boldsymbol{\theta}^{j,i}\}} E_t\left[dr^{n^{(i,\tilde{\imath})}}\right] - \varsigma\sigma^{r^n}$$

   - $dr^{n^{(i,\tilde{\imath})}} = \sum_j \theta^{j,i} dr^j = \sum_j \theta^{j,i} E[dr^j] + \sum_j \theta^{j,i} \sigma^j dZ_t$ is linear in θs
     - Either bang-bang solution for θs s.t. portfolio constraints bind
     - Or prices/returns/risk premia are s.t. that firm is indifferent
2. Aggregate
   - Taking η-weighted sum to obtain return on aggregate wealth
3. Use market clearing to relate θs to ψs & χs (incl. θ-constraint

1b. Allocation of Capital/Risk: 2 Types

- Expert: $\boldsymbol{\theta} = (\theta^k, \theta^{oe}, \theta^d)$ for capital, outside equity, debt
  - Restrictions: $\theta^k \geq 0,$



$\theta^{oe} \leq 0,$ only issue outside equity

$\theta^{oe} \geq -(1-\alpha)\theta^k$ skin in the game

maximize

$$\theta^k E[dr^k] + \theta^{oe} E[dr^{oe}] + \theta^d r^f - \varsigma_t(\theta^k + \theta^{oe})\sigma^{r^k}$$

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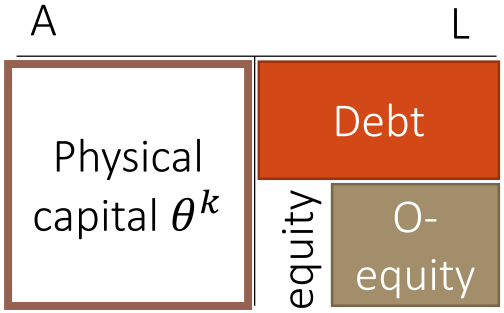

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maximize

$$\theta^k E[dr^k] + \theta^{oe} E[dr^{oe}] + \theta^d r^f - \varsigma_t(\theta^k + \theta^{oe})\sigma^{r^k}$$

- Household: $\underline{\boldsymbol{\theta}} = (\underline{\theta}^k, \underline{\theta}^{oe}, \underline{\theta}^d)$

maximize

$$\underline{\theta}^k E[d\underline{r}^k] + \underline{\theta}^{oe} E[dr^{oe}] + \underline{\theta}^d r^f - \underline{\varsigma}_t(\underline{\theta}^k + \underline{\theta}^{oe})\sigma^{r^k}$$

1b. Allocation of Capital/Risk: 2 Types

- Aggregate η-weighted sum of expert + HH max problem

$$\eta\{\dots\} + (1-\eta)\{\dots\}$$

- $\underbrace{\eta_t\theta_t^k}_{\psi_t :=} E[dr_t^k] + \underbrace{(1-\eta_t)\underline{\theta}_t{}^k}_{=:\underline{\psi}_t = 1-\psi_t} E[d\underline{r}_t{}^k] +$

$$\underbrace{(\eta_t\theta_t^{oe} + (1-\eta_t)\underline{\theta}_t^{oe})}_{=0} E[dr^{oe}] +$$

$$\underbrace{(\eta_t\theta_t^d + (1-\eta_t)\underline{\theta}_t^d)}_{=0} r_t^f$$

$$-\varsigma_t \underbrace{\eta_t\left(\theta_t^k + \theta_t^{oe}\right)}_{=:\chi_t} \sigma_t^{r^k} - \underline{\varsigma}_t \underbrace{(1-\eta_t)\left(\underline{\theta}_t^k + \underline{\theta}_t^{oe}\right)}_{=:\underline{\chi}_t} \sigma_t^{r^k} -$$

1b. Allocation of Capital/Risk: 2 Types

- Aggregate η-weighted sum of expert + HH max problem

$$\eta\{\ldots\} + (1-\eta)\{\ldots\}$$

- $$\underbrace{\eta_t\theta_t^k}_{\psi_t :=} E\left[dr_t^k\right] + \underbrace{(1-\eta_t)\underline{\theta}_t{}^k}_{=:\underline{\psi}_t = 1-\psi_t} E\left[d\underline{r}_t{}^k\right] +$$

$$\underbrace{\left(\eta_t\theta_t^{oe} + (1-\eta_t)\underline{\theta}_t^{oe}\right)}_{=0} E\left[dr^{oe}\right] +$$

$$\underbrace{\left(\eta_t\theta_t^d + (1-\eta_t)\underline{\theta}_t^d\right)}_{=0} r_t^f$$

$$-\varsigma_t \underbrace{\eta_t\left(\theta_t^k + \theta_t^{oe}\right)}_{=:\chi_t} \sigma_t^{r^k} - \underline{\varsigma}_t \underbrace{(1-\eta_t)\left(\underline{\theta}_t^k + \underline{\theta}_t^{oe}\right)}_{=:\underline{\chi}_t} \sigma_t^{r^k} -$$

Poll 33: Why $= 0$ *?*

a) because marginal benefits
= marginal costs at optimum
b) due to martingale behavior
c) because outside equity and
debt are in zero net supply

1b. Allocation of Capital/Risk: 2 Types

- Translate constraints:
  - $\chi_t \leq \psi_t$ experts cannot buy outside equity of others

    only important for the case with idio risk
  - $\chi_t = \underbrace{\eta_t \theta_t^k}_{\psi_t} + \underbrace{\eta_t \theta_t^{oe}}_{\geq -\psi_t(1-\alpha)} \geq \alpha \psi_t$
- Price-taking social planers problem

$$\max_{\{\psi_t, \underline{\psi}_t = 1-\psi_t, \chi_t \in [\alpha\psi_t, \psi_t], \underline{\chi}_t = 1-\chi_t\}} \frac{\psi_t a + \underline{\psi}_t \underline{a} - \iota_t}{q_t} + \Phi(\iota) - \delta - \varsigma \chi_t \sigma_t^{r^k} - \underline{\varsigma}_t \underline{\chi}_t \sigma_t^{r^k}$$

End of Proof. Q.E.D.

  - Linear objective (if frictions take form of constraints)
    - Price of risk adjust such that objective becomes flat *or*
    - Bang-bang solution hitting constraints

1b. Allocation of Capital/Risk: 2 Types

- Example: 2 Types + no outside equity $(\alpha = 1)$

$$\max_{\{\psi_t,\chi_t\}} \frac{\psi_t a + (1-\psi_t)\underline{a} - \iota_t}{q_t} - \chi_t \varsigma_t\left(\sigma + \sigma_t^q\right) - (1-\chi_t)\underline{\varsigma}_t\left(\sigma + \sigma_t^q\right)$$

s.t. friction $\chi_t = \psi_t$ if no outside equity can be issued

- FOC_χ: $\frac{a-\underline{a}}{q_t} = (\varsigma_t - \underline{\varsigma}_t)\left(\sigma + \sigma_t^q\right)$

- May hold only with inequality (≥), if at constraint $\psi_t = 1$

1b. Allocation of Capital/Risk: 2 Types

- Example: 2 Type + with outside equity

$$\max_{\{\psi_t,\chi_t\}} \frac{\psi_t a + (1-\psi_t)\underline{a} - \iota_t}{q_t} - \chi_t \varsigma_t\left(\sigma + \sigma_t^q\right) - (1-\chi_t)\underline{\varsigma}_t\left(\sigma + \sigma_t^q\right)$$

- FOC_χ: Case 1: $\varsigma_t\left(\sigma + \sigma_t^q\right) > \underline{\varsigma}_t\left(\sigma + \sigma_t^q\right) \Rightarrow \chi_t = \alpha\psi_t$

  Case 2: $=$ $\chi_t > \alpha\psi_t$

  - Case 1: plug $\chi_t = \alpha\psi_t$ in objective

    a. $FOC_\psi: \frac{a-\underline{a}}{q_t} \geq \alpha\left(\varsigma_t - \underline{\varsigma}_t\right)\left(\sigma + \sigma_t^q\right) \Rightarrow \psi_t = 1$

    b. $=$ $\Rightarrow \psi_t < 1$

  - Case 2:

    a. $FOC_\psi: \frac{a-\underline{a}}{q_t} > 0$ $\Rightarrow \psi_t = 1$

    b. $= 0 \Rightarrow \psi_t < 1$ impossible

1b. Allocation of Capital/Risk: 2 Types

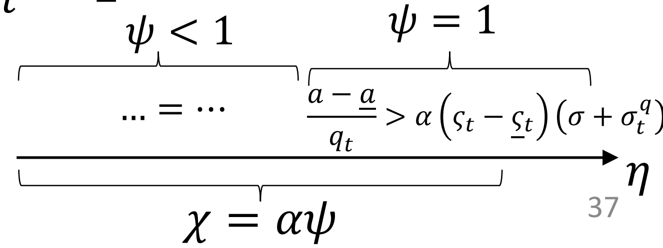
- Example: 2 Type + with outside equity

$$\max_{\{\psi_t,\chi_t\}} \frac{\psi_t a + (1-\psi_t)\underline{a} - \iota_t}{q_t} - \chi_t \varsigma_t (\sigma + \sigma_t^q) - (1-\chi_t)\underline{\varsigma}_t(\sigma + \sigma_t^q)$$

- FOC_χ: Case 1: $\varsigma_t(\sigma + \sigma_t^q) > \underline{\varsigma}_t(\sigma + \sigma_t^q) \Rightarrow \chi_t = \alpha\psi_t$

  Case 2: $=$ $\chi_t > \alpha\psi_t$

  - Case 1: plug $\chi_t = \alpha\psi_t$ in objective
    - a. FOC_ψ: $\frac{a-\underline{a}}{q_t} \geq \alpha\left(\varsigma_t - \underline{\varsigma}_t\right)(\sigma + \sigma_t^q) \Rightarrow \psi_t = 1$
    - b. $=$ $\Rightarrow \psi_t < 1$
  - Case 2:
    - a. FOC_ψ: $\frac{a-\underline{a}}{q_t} > 0$ $\Rightarrow \psi_t = 1$
    - b. $= 0 \Rightarrow \psi_t < 1$ impossible



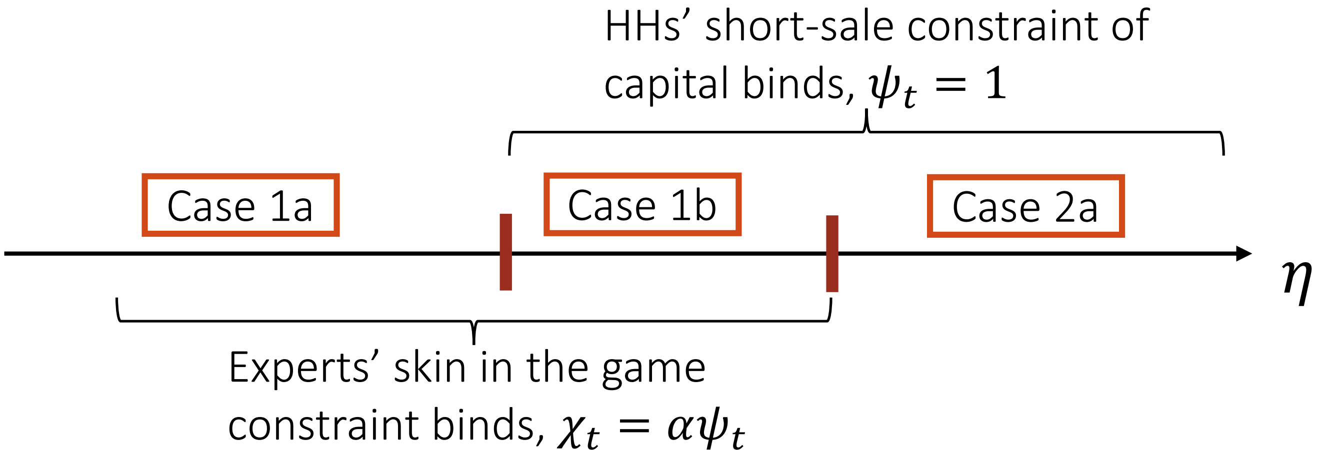
$$\overbrace{\dots = \cdots}^{\psi < 1} \quad \overbrace{\frac{a-\underline{a}}{q_t} > \alpha\left(\varsigma_t - \underline{\varsigma}_t\right)(\sigma + \sigma_t^q)}^{\psi = 1}$$

$$\underbrace{\xrightarrow{\hspace{6cm}}}_{\chi = \alpha\psi} \eta$$

1b. Allocation of Capital, ψ, and Risk, χ

- Summarizing previous slide (2 types with outside equity)

Cases	$\chi_t \geq \alpha\psi_t$	$\psi_t \leq 1$	$\frac{(a-\underline{a})}{q_t} \geq \alpha\left(\varsigma_t - \underline{\varsigma}_t\right)(\sigma+\sigma_t^q)$	$\left(\varsigma_t - \underline{\varsigma}_t\right)(\sigma+\sigma_t^q) \geq \mathbf{0}$
1a	=	<	=	>
1b	=	=	>	>
2a	>	=	>	=
impossible				



HHs' short-sale constraint of capital binds, $\psi_t = 1$

Case 1a

Case 1b

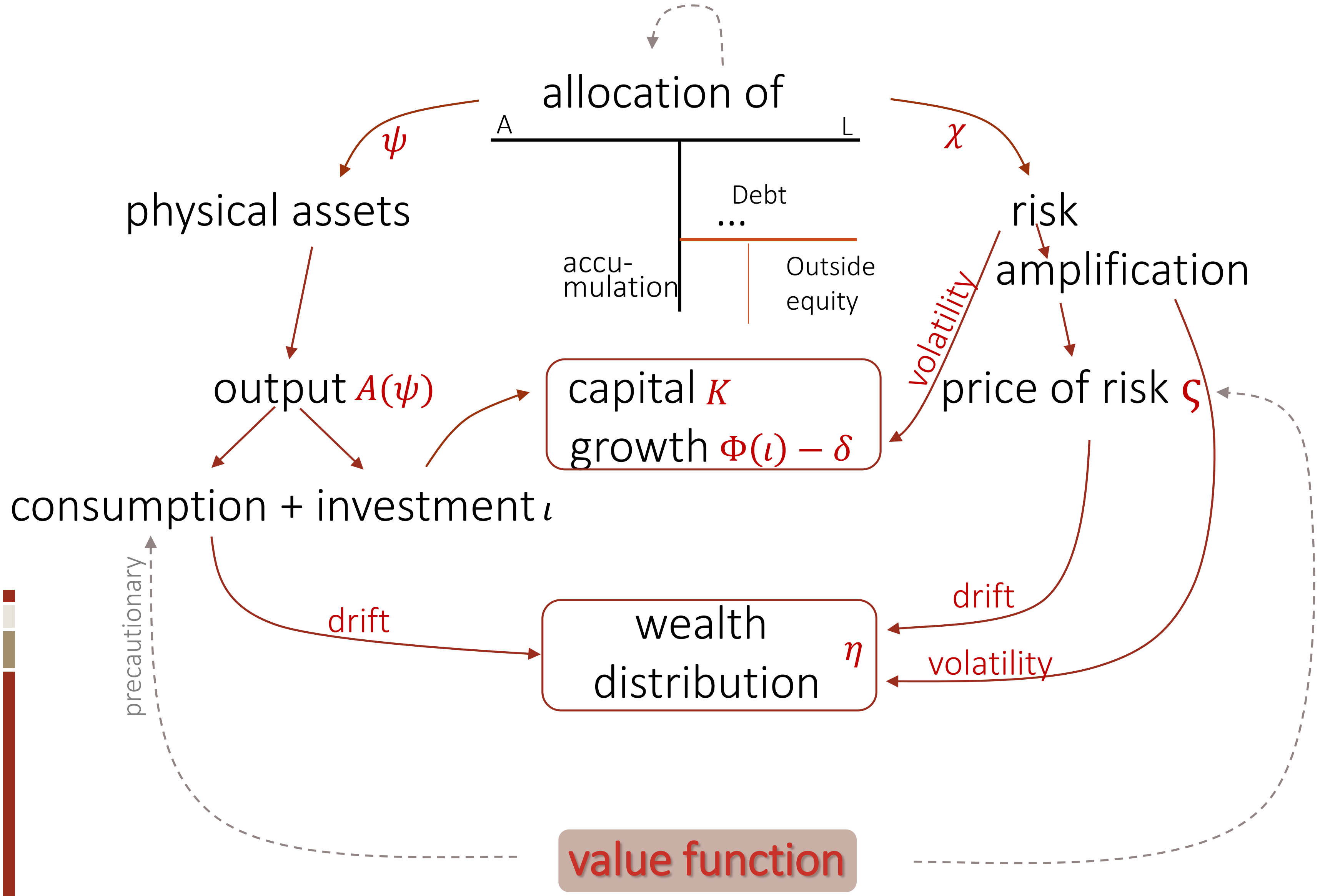
Case 2a

Experts' skin in the game constraint binds, $\chi_t = \alpha\psi_t$

Solving MacroModels Step-by-Step

0. Postulate aggregates, price processes & obtain return processes
1. For given SDF processes *static*
   a. Real investment ι, (portfolio $\boldsymbol{\theta}$, & consumption choice of *each agent*)
      - *Toolbox 1:* Martingale Approach
   b. Asset/Risk Allocation *across types/sectors* & asset market clearing
      - *Toolbox 2:* "price-taking social planner approach" – Fisher separation theorem
2. Value functions *backward equation*
   a. Value fcn. as fcn. of individual investment opportunities ω
      - *Special cases*
   b. De-scaled value fcn. as function of state variables η
      - *Digression:* HJB-approach (instead of martingale approach & envelop condition)
   c. Derive ς price of risk, C/N-ratio from value fcn. envelop condition
3. Evolution of state variable η *forward equation*
   - *Toolbox 3:* Change in numeraire to total wealth (including SDF)
   - ("Money evaluation equation" μ^{ϑ})
4. Value function iteration & goods market clearing
   a. PDE of de-scaled value fcn.
   b. Value function iteration by solving PDE

The Big Picture



allocation of

A

L

Debt

…

accu-
mulation

Outside
equity

ψ

χ

physical assets

risk

amplification

output $A(\psi)$

capital κ
growth $\Phi(\iota) - \delta$

volatility

price of risk ς

consumption + investment ι

precautionary

drift

wealth
distribution η

drift

volatility

value function

Forward equation

Backward equation
with expectations

2a. CRRA Value Function: relate to ω

Applies separately for each type of agent

- ω_t Investment opportunity/ “networth multiplier”
- CRRA/power utility $u(c) = \frac{c^{1-\gamma}-1}{1-\gamma}$

  $\Rightarrow$ increase networth by factor, optimal consumption for all future states increases by same factor

  $\Rightarrow \left(\frac{c}{n}\right)$-ratio is invariant in n
- $\Rightarrow$ value function can be written as $\frac{u(\omega_t n_t)}{\rho}$, that is

$$= \frac{1}{\rho}\frac{(\omega_t n_t)^{1-\gamma}-1}{1-\gamma} = \frac{1}{\rho}\frac{\omega_t^{1-\gamma} n_t^{1-\gamma}-1}{1-\gamma}$$

- $\frac{\partial V}{\partial n} = \frac{\partial u}{\partial c}$ by optimal consumption (if no corner solution)

$$\frac{\omega_t^{1-\gamma} n_t^{-\gamma}}{\rho} = c_t^{-\gamma} \Leftrightarrow \boxed{\frac{c_t}{n_t} = \rho^{1/\gamma}\omega_t^{1-1/\gamma}}$$

Next step:

a) Special simple cases

b) replace ω_t with something scale invariant

2a. CRRA Value Function: Special Cases

$$\boxed{\frac{c_t}{n_t} = \rho^{1/\gamma} \omega_t^{1-1/\gamma}}$$

- For log utility $\gamma = 1$:
  $\xi_t = e^{-\rho t}/c_t = e^{-\rho t}/(\rho n_t)$ for any $\omega_t \Rightarrow \sigma_t^n = \sigma_t^c = \varsigma_t$
  - Expected excess return: $\mu_t^A - r_t^F = \sigma_t^n \sigma_t^A$
  - Recall $\frac{dn_t}{n_t} = -\frac{c_t}{n_t} dt + (1-\theta) dr_t^K + \theta dr_t$

2a. CRRA Value Function: Special Cases

$$\boxed{\frac{c_t}{n_t} = \rho^{1/\gamma} \omega_t^{1-1/\gamma}}$$

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  - Expected excess return: $\mu_t^A - r_t^F = \sigma_t^n \sigma_t^A$
  - Recall $\frac{dn_t}{n_t} = -\frac{c_t}{n_t} dt + (1-\theta) dr_t^K + \theta dr_t$
- For constant investment opportunities $\omega_t = \omega$,
  $\Rightarrow \frac{c}{n}$ is constant and hence $\sigma_t^c = \sigma^n$
  - Expected excess return: $\mu_t^A - r_t^F = \gamma \sigma_t^n \sigma_t^A$

Poll 43: Which term refers to (dynamic/Mertonian) hedging demand?

a) γ
b) σ_t^n
c) hidden in risk-free rate
d) none of the above

2a. CRRA Value Function: Special Cases

$$\boxed{\frac{c_t}{n_t} = \rho^{1/\gamma}\omega_t^{1-1/\gamma}}$$

- For log utility $\gamma = 1$:
  $\xi_t = e^{-\rho t}/c_t = e^{-\rho t}/(\rho n_t)$ for any $\omega_t \Rightarrow \sigma_t^n = \sigma_t^c = \varsigma_t$
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- For constant investment opportunities $\omega_t = \omega$,
  $\Rightarrow \frac{c}{n}$ is constant and hence $\sigma_t^c = \sigma^n$
  - Expected excess return: $\mu_t^A - r_t^F = \gamma\sigma_t^n \sigma_t^A$
  - Now $\frac{dn_t}{n_t} = r^F dt + \frac{\varsigma^2}{\gamma}dt + \frac{\varsigma}{\gamma}dZ_t - \frac{c_t}{n_t}dt$
  - $r^F = \rho + \gamma\left(r^F + \frac{\varsigma^2}{\gamma} - \frac{c_t}{n_t}\right) - \frac{\gamma+1}{2}\frac{\varsigma^2}{\gamma}$

  $\Rightarrow \frac{c_t}{n_t} = \rho + \frac{\gamma-1}{\gamma}\left(r^F - \rho + \frac{\varsigma^2}{2\gamma}\right)$

Also holds if ω_t evolves deterministically

$= \rho + \gamma\left(r^F - \frac{c_t}{n_t}\right) + \frac{\gamma-1}{\gamma}\frac{\varsigma^2}{2}$

2a. CRRA Value Function: Special Cases

$$\boxed{\frac{c_t}{n_t} = \rho^{1/\gamma}\omega_t^{1-1/\gamma}}$$

- For log utility $\gamma = 1$:
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  $\Rightarrow \frac{c_t}{n_t} = \rho + \frac{\gamma-1}{\gamma}\left(r^F - \rho + \frac{\varsigma^2}{2\gamma}\right)$

Also holds if ω_t evolves deterministically

$$= \rho + \gamma\left(r^F - \frac{c_t}{n_t}\right) + \frac{\gamma-1}{\gamma}\frac{\varsigma^2}{2}$$

2a. CRRA Value Function: Special Cases

$$\boxed{\frac{c_t}{n_t} = \rho^{1/\gamma} \omega_t^{1-1/\gamma}}$$

- For log utility $\gamma = 1$:
  $\xi_t = e^{-\rho t}/c_t = e^{-\rho t}/(\rho n_t)$ for any $\omega_t \Rightarrow \sigma_t^n = \sigma_t^c = \varsigma_t$
  - Expected excess return: $\mu_t^A - r_t^F = \sigma_t^n \sigma_t^A$
  - Recall $\frac{dn_t}{n_t} = -\frac{c_t}{n_t} dt + (1-\theta) dr_t^K + \theta dr_t$
- For constant investment opportunities $\omega_t = \omega$,
  $\Rightarrow \frac{c}{n}$ is constant and hence $\sigma_t^c = \sigma^n$
  - Expected excess return: $\mu_t^A - r_t^F = \gamma \sigma_t^n \sigma_t^A$
  - Now $\frac{dn_t}{n_t} = r^F dt + \frac{\varsigma^2}{\gamma} dt + \frac{\varsigma}{\gamma} dZ_t - \frac{c_t}{n_t} dt$
  - $r^F = \rho + \gamma\left(r^F + \frac{\varsigma^2}{\gamma} - \frac{c_t}{n_t}\right) - \frac{\gamma+1}{2}\frac{\varsigma^2}{\gamma}$

$$= \rho + \gamma\left(r^F - \frac{c_t}{n_t}\right) + \frac{\gamma-1}{\gamma}\frac{\varsigma^2}{2}$$

$$\Rightarrow \frac{c_t}{n_t} = \rho + \frac{\gamma - 1}{\gamma}\left(r^F - \rho + \frac{\varsigma^2}{2\gamma}\right)$$

Also holds if ω_t evolves deterministically

Way to compute c_t/n_t if one can obtain from some other source r^F (omega can we avoided)

2b. CRRA Value Fcn. & State Variable η

- Recall Martingale approach: if x_t is the value of a portfolio with return $\frac{dn_t}{n_t} + \frac{c_t}{n_t}dt$, then $\xi_t x_t$must be a martingale

$$\frac{d(\xi_t n_t)}{\xi_t n_t} = -\frac{c_t}{n_t}dt + martingale$$

- Optimal consumption implies with CRRA-$V_t(n_t) = \frac{1}{\rho}\frac{(\omega_t n_t)^{1-\gamma}}{1-\gamma}$:

$$u'(c) = V_t'(n) \Leftrightarrow c_t^{-\gamma} = \frac{1}{\rho}\omega^{1-\gamma} n_t^{-\gamma} \Leftrightarrow e^{\rho t}\underbrace{e^{-\rho t} c_t^{-\gamma}}_{=\xi_t} n_t = \underbrace{\frac{1}{\rho}\omega^{1-\gamma} n_t^{1-\gamma}}_{(1-\gamma)V_t}$$

- Hence,

$$\frac{dV_t}{V_t} = \frac{d(e^{\rho t}\xi_t n_t)}{e^{\rho t}\xi_t n_t} = \left(\rho - \frac{c_t}{n_t}\right)dt + martingale$$

- Next, let's compute the drift of $\frac{dV_t}{V_t}$

2b. CRRA Value Fcn: De-scale by K_t

- Drift of $\frac{dV_t}{V_t}$, we could use Ito on $V_t = \frac{1}{\rho}\frac{(\omega_t n_t)^{1-\gamma}}{1-\gamma}$, but
  - *Poll 48: What could be the problem?*
    a. *Networth n_t is unbounded*
    b. *Networth $n_t(\eta_t)$ and N-multiplier $\omega_t(\eta_t)$ are not differentiable (if $q(\eta)$, $p(\eta)$ have a kink).*
    c. *N-multiplier is not scale invariant*

2b. CRRA Value Fcn: De-scale by K_t

- Drift of $\frac{dV_t}{V_t}$, we could use Ito on $V_t = \frac{1}{\rho}\frac{(\omega_t n_t)^{1-\gamma}}{1-\gamma}$, but
  - *Poll 49: What could be the problem?*
    a. *Networth n_t is unbounded*
    b. *Networth $n_t(\eta_t)$ and N-multiplier $\omega_t(\eta_t)$ are not differentiable (if $q(\eta), p(\eta)$ have a kink).*
    c. *N-multiplier is not scale invariant*
  - Answer: b.

- Let's de-scale the problem w.r.t. K_t

$$V_t = \frac{1}{\rho}\frac{(\omega_t n_t)^{1-\gamma}}{1-\gamma} = \underbrace{\frac{1}{\rho}\frac{\left(\omega_t \frac{n_t}{K_t}\right)^{1-\gamma}}{1-\gamma}}_{v_t :=} K_t^{1-\gamma}$$

and define v_t (which is twice differentiable in η_t)

- state variable K_t is easy to handle due to scale invariance

2b. CRRA Value Function

$$\frac{dV_t}{V_t} = \frac{d\left(v_t K_t^{1-\gamma}\right)}{v_t K_t^{1-\gamma}}$$

- By Ito's product rule

$$= \left(\mu_t^v + (1-\gamma)(\Phi(\iota)-\delta) - \frac{1}{2}\gamma(1-\gamma)(\sigma^2) + (1-\gamma)\sigma\sigma_t^v\right)dt$$
$$+ volatility\ terms$$

- Recall by consumption optimality $\frac{dV_t}{V_t} - \rho dt + \frac{c_t}{n_t} dt$ follows a martingale
- Hence, drift above $= \rho - \frac{c_t}{n_t}$

Still have to solve for μ_t^v, σ_t^v

Poll 51: Why martingale?

a) Because we can "price" networth with SDF

b) because ρ and c_t/n_t cancel out

2b. CRRA Value Fcn BSDE

- Only conceptual interim solution
    - We will transform it into a PDE in Step 4 below
- From last slide

$$\underbrace{\mu_t^v + (1-\gamma)(\Phi(\iota)-\delta) - \frac{1}{2}\gamma(1-\gamma)(\sigma^2) + (1-\gamma)\sigma\sigma_t^v}_{=:\mu_t^V} = \rho - \frac{c_t}{n_t}$$

- Can solve for μ_t^v, then v_t must follow

$$\frac{dv_t}{v_t} = f(\eta_t, v_t, \sigma_t^v)dt + \sigma_t^v dZ_t$$

with

$$f(\eta_t, v_t, \sigma_t^v) = \rho - \frac{c_t}{n_t} - (1-\gamma)(\Phi(\iota)-\delta) + \frac{1}{2}\gamma(1-\gamma)(\sigma^2) - (1-\gamma)\sigma\sigma_t^v$$

- Together with terminal condition v_T (possibly a constant for 1000 periods ahead), this is a backward stochastic differential equation (BSDE)
- A solution consists of processes v and σ^v
- Can use numerical BSDE solution methods (as random objects, so only get simulated paths)
- To solve this via a PDE we also need to get state evolution

Bellman Equation?

Poll 53: Where have we used the Bellman equation?

- *a) nowhere*
- *b) it is hidden in the Martingale Approach*
- *c) only needed in discrete time*

Digression: HJB Approach

- Alternative to Martingale Approach
- Start from continuous-time analogue of Bellman equation

$$V(n,\omega) = \max_{\{c,\boldsymbol{\theta}\}} E\left[\int_t^T e^{-\rho(s-t)}u(c_s)ds + e^{-\rho(T-t)}V(n_T,\omega_T) \mid n_t = n, \omega_t = \omega\right]$$

s.t. $\frac{dn_t}{n_t} = -\frac{c_t}{n_t}dt + \sum_j \theta_t^j dr_t^j$

- Subtract V

$$0 = \max_{\{c,\boldsymbol{\theta}\}} E_t\left[\int_t^T e^{-\rho(s-t)}u(c_s)ds + \underbrace{e^{-\rho(T-t)}V(n_T,\omega_T) - e^{-\rho(t-t)}V(n_t,\omega_t)}_{=\int_t^T d\left(e^{-\rho(s-t)}V(n_s,\omega_s)\right)}\right]$$

- Divide by $T-t$, take limit $T \to t$

$$\rho V(n_t,\omega_t)dt = \max_{\{c,\boldsymbol{\theta}\}} u(c_t)dt + E_t[dV(n_t,\omega_t)]$$

s.t. ....

Digression: HJB Approach – the HJB Equation

- What is $E_t[dV(n_t, \omega_t)]$? If V is differentiable in n, ω
  - Use Ito's Lemma

$$\frac{E_t[dV]}{dt} = \frac{\partial V}{\partial n}\mu^n n + \frac{\partial V}{\partial \omega}\mu^\omega \omega + \frac{1}{2}\frac{\partial^2 V}{\partial n^2}(\sigma^n n)^2 + \frac{1}{2}\frac{\partial^2 V}{\partial \omega^2}(\sigma^\omega \omega)^2 + \frac{\partial^2 V}{\partial \omega \partial n}\sigma^\omega \omega \sigma^n n$$

  - Plug in μ^n, σ^n

- Hence HJB equation becomes

$$\rho V = \max_{c,\theta}\left(u(c) + \frac{\partial V}{\partial n}\left(-c + \sum_j \theta^j \mu^{r^j} n\right) + \frac{1}{2}\frac{\partial^2 V}{\partial n^2}\left(\sum_j \theta^j \sigma^{r^j} n\right)^2 + \right.$$

Digression: HJB Approach – the Value Fcn BSDE

- Two possibilities to get backward equation
  1. Start from $\rho V(n_t, \omega_t)dt = u(c_t)dt + E_t[dV(n_t, \omega_t)]$
     - Can write this as

$$\mu_t^V = \rho - \frac{u(c_t)}{V(n_t, \omega_t)}$$

     - Plug in $V(n_t, \omega_t) = \frac{1}{\rho}\frac{(\omega_t n_t)^{1-\gamma}}{1-\gamma}$ and $\frac{c_t}{n_t} = \rho^{1/\gamma}\omega_t^{1-1/\gamma}$

$$\mu_t^V = \rho - \frac{c_t}{n_t}$$

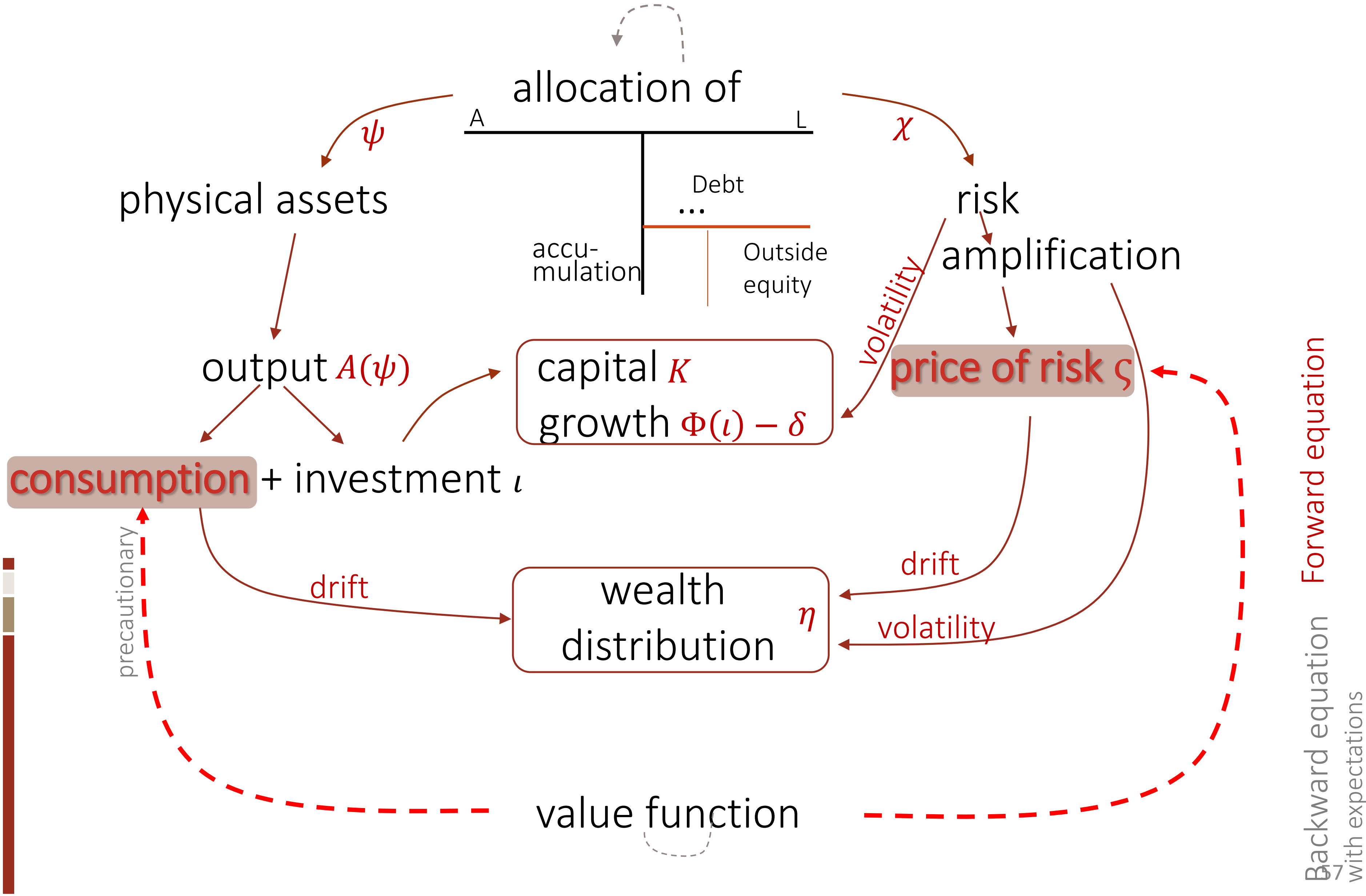
     - With $V = vK^{1-\gamma}$, get same BSDE for v as before

  2. Use HJB equation
     - Plug in $V(n_t, \omega_t) = \frac{1}{\rho}\frac{(\omega_t n_t)^{1-\gamma}}{1-\gamma}$ and its derivatives

$$\rho = \sum_j \theta^j \mu^{r^j} - \frac{\gamma}{2}\left(\sum_j \theta^j \sigma^{r^j}\right)^2 + (1-\gamma)\sigma^\omega \sum_j \theta^j \sigma^{r^j} + \mu^\omega - \frac{\gamma}{2}(\sigma^\omega)^2$$

     - This is a BSDE for ω (instead of v)

The Big Picture



2c. Get ςs from Value Function Envelop

- Experts value function

$$v_t \frac{K_t^{1-\gamma}}{1-\gamma}$$

HH's value function

$$\underline{v}_t \frac{K_t^{1-\underline{\gamma}}}{1-\underline{\gamma}}$$

- To obtain $\frac{\partial V_t(n)}{\partial n_t}$ use $K_t = \frac{N_t}{\eta_t q_t} = \frac{n_t}{\eta_t q_t}$

$$V_t(n) = v_t \frac{n_t^{1-\gamma}/(\eta_t q_t)^{1-\gamma}}{1-\gamma}$$

…

- Envelop condition $\frac{\partial V_t(n)}{\partial n_t} = \frac{\partial u(c_t)}{\partial c_t}$

$$v_t \frac{n_t^{-\gamma}}{(\eta_t q_t)^{1-\gamma}} = c_t^{-\gamma}$$

…

- Using $K_t = \frac{n_t}{\eta_t q_t}$, $C_t = c_t$

$$\frac{v_t}{\eta_t q_t} K_t^{-\gamma} = C_t^{-\gamma}$$

…

$$\sigma_t^v - \sigma_t^\eta - \sigma_t^q - \gamma\sigma = -\gamma\sigma_t^c, \qquad \sigma_t^{\underline{v}} - \sigma_t^{\underline{\eta}} - \sigma_t^q - \underline{\gamma}\sigma = -\underline{\gamma}\sigma_t^c$$

$$= -\varsigma_t \qquad\qquad = -\underline{\varsigma}_t$$

2c. Get $\frac{C_t}{N_t}, \frac{\underline{C}_t}{\underline{N}_t}$ from Value Function Envelop

- Experts

Households

- Recall $v_t \frac{n_t^{-\gamma}}{(\eta_t q_t)^{1-\gamma}} = c_t^{-\gamma}$

$$\frac{c_t}{n_t} = \frac{(\eta_t q_t)^{1/\gamma - 1}}{v_t^{1/\gamma}}$$

$$\frac{C_t}{N_t} = \frac{(\eta_t q_t)^{1/\gamma - 1}}{v_t^{1/\gamma}}$$

$$\frac{\underline{C}_t}{\underline{N}_t} = \frac{((1-\eta_t) q_t)^{1/\underline{\gamma} - 1}}{\underline{v}_t^{1/\underline{\gamma}}}$$

$$\frac{C_t + \underline{C}_t}{N_t + \underline{N}_t} = \eta_t \frac{C_t}{N_t} + (1 - \eta_t) \frac{\underline{C}_t}{\underline{N}_t}$$

Plug in from above

Solving MacroModels Step-by-Step

0. Postulate aggregates, price processes & obtain return processes
1. For given SDF processes *static*
   a. Real investment ι, (portfolio $\boldsymbol{\theta}$, & consumption choice of *each agent*)
      - *Toolbox 1:* Martingale Approach

   b. Asset/Risk Allocation *across types/sectors* & asset market clearing
      - *Toolbox 2:* "price-taking social planner approach" – Fisher separation theorem
2. Value functions *backward equation*
   a. Value fcn. as fcn. of individual investment opportunities ω
      - *Special cases*

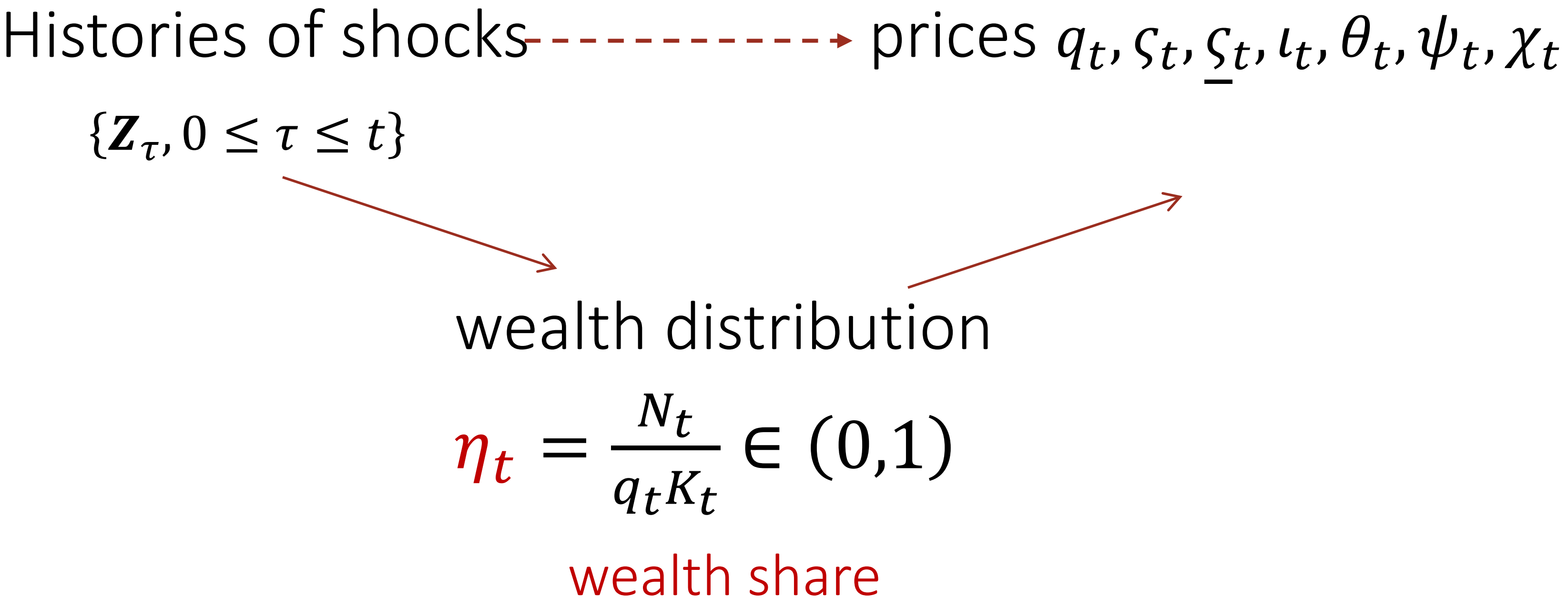
   b. De-scaled value fcn. as function of state variables η
      - *Digression:* HJB-approach (instead of martingale approach & envelop condition)

   c. Derive ς price of risk, C/N-ratio from value fcn. envelop condition
3. Evolution of state variable η *forward equation*
   - *Toolbox 3:* Change in numeraire to total wealth (including SDF)
   - ("Money evaluation equation" μ^{ϑ})
4. Value function iteration & goods market clearing
   a. PDE of de-scaled value fcn.

   b. Value function iteration by solving PDE

3. GE: Markov States and Equilibria

- Equilibrium is a map



Histories of shocks $\{\boldsymbol{Z}_\tau, 0 \leq \tau \leq t\}$

prices $q_t, \varsigma_t, \underline{\varsigma}_t, \iota_t, \theta_t, \psi_t, \chi_t$

wealth distribution

$$\eta_t = \frac{N_t}{q_t K_t} \in (0,1)$$

wealth share

3. Law of Motion of Wealth Share η_t

- **Method 1:** Using Ito's quotation rule $\eta_t = N_t/(q_t K_t)$
    - Recall

$$\frac{dN_t}{N_t} = r_t dt + \underbrace{\frac{\chi_t \psi_t}{\eta_t}(\sigma+\sigma_t^q)}_{risk} \underbrace{\varsigma_t}_{\substack{price\ of \\ risk}} dt$$

$$+\frac{\chi_t \psi_t}{\eta_t}(\sigma + \sigma_t^q) dZ_t - \frac{C_t}{N_t} dt$$

    - $\frac{d\eta_t}{\eta_t} = \ldots$ (lots of algebra)

- **Method 2:** Change of numeraire + Martingale Approach
    - New numeraire: Total wealth in the economy, N_t
    - Apply Martingale Approach for value of i's portfolio
        - Simple algebra to obtain drift of η_t: μ_t^η
          Note that change of numeraire does not affect ratio η!

3. Aside: Change of Numeraire

- x_t^A is a value of a self-financing strategy/asset in \$
- Y_t price of € in \$ (exchange rate)

$$\frac{dY_t}{Y_t} = \mu_t^Y dt + \sigma_t^Y dZ_t$$

- x_t^A / Y_t value of the self-financing strategy/asset in €

$$\underbrace{e^{-\rho t} u'(c_t)}_{=\xi_t} Y_t \frac{x_t^A}{Y_t} \text{ follows a martingale}$$

$$\text{Recall } \mu_t^A - \mu_t^B = \underbrace{(-\sigma_t^{\xi})}_{=\varsigma_t}\underbrace{(\sigma^A - \sigma_t^B)}_{risk}$$

$$\mu_t^{A/Y} - \mu_t^{B/Y} = \underbrace{(-\sigma_t^{\xi} - \sigma_t^Y)}_{price\ of\ risk}\underbrace{(\sigma^A - \cancel{\sigma_t^Y} - \sigma_t^B + \cancel{\sigma_t^Y})}_{risk}$$

- Price of risk $\varsigma^{€} = \varsigma^{\$} - \sigma^Y$

3. Aside: Change of Numeraire

- x_t^A is a value of a self-financing strategy/asset in \$
- Y_t price of € in \$ (exchange rate)

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$$\mu_t^{A/Y} - \mu_t^{B/Y} = \underbrace{(-\sigma_t^\xi - \sigma_t^Y)}_{price\ of\ risk} \underbrace{(\sigma^A - \cancel{\sigma_t^Y} - \sigma_t^B + \cancel{\sigma_t^Y})}_{risk}$$

- Price of risk $\varsigma^{\text{€}} = \varsigma^{\$} - \sigma^Y$

Poll 64: Why does the price of risk change, though real risk remains the same

a) b/c risk-free rate might not stay risk-free

b) b/c covariance structure changes

3. μ^{η} Drift of Wealth Share: Many Types

- New Numeraire
  - **"Total wealth"** in the economy, $\overline{N}_t$ (without superscript)
  - Type i's portfolio wealth = wealth share

- Martingale Approach with new numeraire
  - Asset A $= i$'s portfolio return in terms of total wealth,

$$\left(\underbrace{\frac{C_t^i}{N_t^i}}_{\text{Dividend yield}} + \underbrace{\mu_t^{\eta^i}}_{\text{E[capital gains] rate}}\right) dt + \sigma_t^{\eta^i} dZ_t$$

  - Asset B (benchmark asset that everyone can hold, e.g. risk-free asset or money (in terms of total economy wide wealth as numeraire))

$$r_t^M dt + \sigma_t^M dZ_t$$

  - Apply our martingale asset pricing formula

$$\mu_t^A - \mu_t^B = \varsigma_t^i(\sigma_t^A - \sigma_t^B)$$

3. μ^{η} Drift of Wealth Share: Many Types

- Asset pricing formula (relative to benchmark asset)

$$\mu_t^{\eta^i} + \frac{C_t^i}{N_t^i} - r_t^M = \left(\varsigma_t^i - \sigma_t^{\overline{N}}\right)\left(\sigma_t^{\eta^i} - \sigma_t^M\right)$$

due to change in numeraire

- Add up across types (weighted),

(capital letters with bar & without superscripts are aggregates for total economy)

$$\underbrace{\sum_{i'}^{I} \eta_t^{i'} \mu_t^{\eta^{i'}}}_{=0} + \frac{\bar{C}_t}{\overline{N}_t} - r_t^M = \sum_{i'} \eta_t^{i'} \left(\varsigma_t^{i'} - \sigma_t^{\overline{N}}\right)\left(\sigma_t^{\eta^{i'}} - \sigma_t^M\right)$$

Poll 66: Why $= 0$?

a) Because we have stationary distribution

b) Because ηs sum up to 1

c) Because ηs follow martingale

3. μ^{η} Drift of Wealth Share: Many Types

- Asset pricing formula (relative to benchmark asset)

$$\mu_t^{\eta^i} + \frac{C_t^i}{N_t^i} - r_t^M = \left(\varsigma_t^i - \sigma_t^{\bar{N}}\right)\left(\sigma_t^{\eta^i} - \sigma_t^M\right)$$

- Add up across types (weighted),

(capital letters with bar & without superscripts are aggregates for total economy)

$$\underbrace{\sum_{i'}^{I} \eta_t^{i'} \mu_t^{\eta^{i'}}}_{=0} + \frac{\bar{C}_t}{\bar{N}_t} - r_t^M = \sum_{i'} \eta_t^{i'} \left(\varsigma_t^{i'} - \sigma_t^{\bar{N}}\right)\left(\sigma_t^{\eta^{i'}} - \sigma_t^M\right)$$

- Subtract from each other yield wealth share dynamics

$$\mu_t^{\eta^i} = \left(\varsigma^i - \sigma^N\right)\left(\sigma^{\eta^i} - \sigma^M\right) - \sum_{i'} \eta_t^{i'}\left(\varsigma_t^{i'} - \sigma_t^N\right)\left(\sigma_t^{\eta^{i'}} - \sigma_t^M\right)$$

$$- \left(\frac{C_t^i}{N_t^i} - \frac{\bar{C}_t}{\bar{N}_t}\right)$$

3. μ^{η} Drift of Wealth Share: Two Types

- Asset pricing formula (relative to benchmark asset)

$$\mu_t^{\eta} + \frac{C_t}{N_t} - r_t^M = \left(\varsigma - \sigma_t^{\bar{N}}\right)(\sigma^{\eta} - \sigma^M)$$

- Add up across types (weighted),

(capital letters without superscripts are aggregates for total economy)

$$\underbrace{(\eta_t \mu_t^{\eta} + (1-\eta_t)\mu_t^{\underline{\eta}})}_{=0} + \frac{C_t}{N_t} - r_t^M =$$

$$\eta_t(\varsigma_t - \sigma_t^{\bar{N}})(\sigma_t^{\eta} - \sigma_t^M) - (1-\eta_t)\left(\underline{\varsigma}_t - \sigma_t^{\bar{N}}\right)\left(\sigma_t^{\underline{\eta}} - \sigma_t^M\right)$$

- Subtract from each other yields wealth share drift

$$\mu_t^{\eta} = (1-\eta_t)(\varsigma_t - \sigma_t^{\bar{N}})(\sigma_t^{\eta} - \sigma_t^M) - (1-\eta_t)\left(\underline{\varsigma}_t - \sigma_t^{\bar{N}}\right)\left(\sigma_t^{\underline{\eta}} - \sigma_t^M\right)$$

$$-\left(\frac{C_t}{N_t} - \frac{C_t + \underline{C}_t}{q_t K_t}\right)$$

3. σ^η Volatility of Wealth Share

- Since $\eta_t^i = N_t^i/\bar{N}_t$,

$$\sigma_t^{\eta^i} = \sigma_t^{N^i} - \sigma_t^{\bar{N}} = \sigma_t^{N^i} - \sum_{i'} \eta_t^{i'} \sigma_t^{N^{i'}}$$

$$= (1 - \eta_t^i)\sigma_t^{N^i} - \sum_{i^- \neq i} \eta_t^{i^-} \sigma_t^{N^{i^-}}$$

- Note for 2 types example

Change in notation in 2 type setting
Type-networth is $n = N^i$

$$\sigma_t^\eta = (1 - \eta_t)(\sigma_t^n - \sigma_t^{\underline{n}})$$

$$\sigma_t^n = \underbrace{\chi_t/\eta_t}_{=\theta^k + \theta^{oe}} (\sigma + \sigma_t^q) \qquad \sigma_t^{\underline{n}} = \frac{1-\chi_t}{1-\eta_t} (\sigma + \sigma_t^q)$$

  - Hence,

$$\sigma_t^\eta = \frac{\chi_t - \eta_t}{\eta_t} (\sigma + \sigma_t^q)$$

- Note also, $\eta_t \sigma_t^\eta + (1 - \eta_t)\sigma_t^{\underline{\eta}} = 0 \Rightarrow \sigma_t^{\underline{\eta}} = -\frac{\eta_t}{1-\eta_t} \sigma_t^\eta$

3. Amplification Formula: Loss Spiral

- Recall
- By Ito's Lemma on $q(\eta)$

$$\left.\begin{aligned} \sigma_t^\eta &= \underbrace{\frac{\chi_t-\eta_t}{\eta_t}}_{\text{leverage}}(\sigma+\sigma_t^q) \\ \sigma_t^q &= \frac{q'(\eta_t)}{q(\eta_t)}\eta_t\sigma_t^\eta \end{aligned}\right\}$$

$$\sigma_t^q = \underbrace{\frac{q'(\eta_t)}{q/\eta_t}}_{elasticity}\frac{\chi_t-\eta_t}{\eta_t}(\sigma+\sigma_t^q)$$

- Total volatility

$$\sigma+\sigma_t^q = \frac{\sigma}{1-\frac{q'(\eta_t)}{q/\eta_t}\frac{\chi_t-\eta_t}{\eta_t}}$$

- Loss spiral
  - Market illiquidity (price impact elasticity)

3. Amplification Formula: Loss Spiral

- Recall
- By Ito's Lemma on $q(\eta)$

$$\left.\begin{aligned} \sigma_t^\eta &= \underbrace{\frac{\chi_t-\eta_t}{\eta_t}}_{\text{leverage}}(\sigma+\sigma_t^q) \\ \sigma_t^q &= \frac{q'(\eta_t)}{q(\eta_t)}\eta_t\sigma_t^\eta \end{aligned}\right\}$$

$$\sigma_t^q = \frac{q'(\eta_t)}{\underbrace{q/\eta_t}_{elasticity}} \frac{\chi_t-\eta_t}{\eta_t}(\sigma+\sigma_t^q)$$

- Total volatility

$$\sigma+\sigma_t^q = \frac{\sigma}{1-\frac{q'(\eta_t)}{q/\eta_t}\frac{\chi_t-\eta_t}{\eta_t}}$$

Poll 71: Where is the spiral?
a) Sum of infinite geometric series (denominator)
b) in q', since with constant price, no spiral

- Loss spiral
  - Market illiquidity (price impact elasticity)

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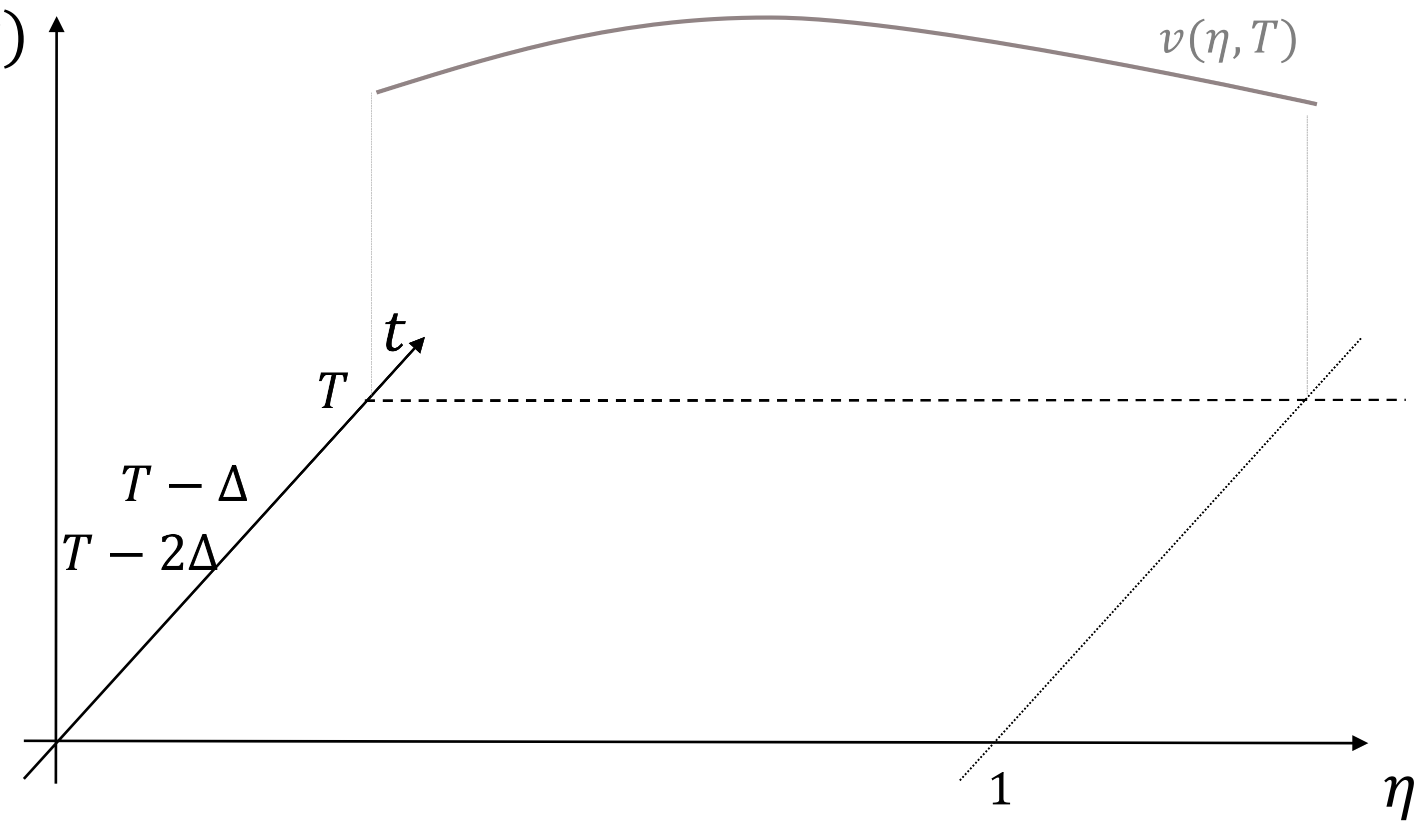
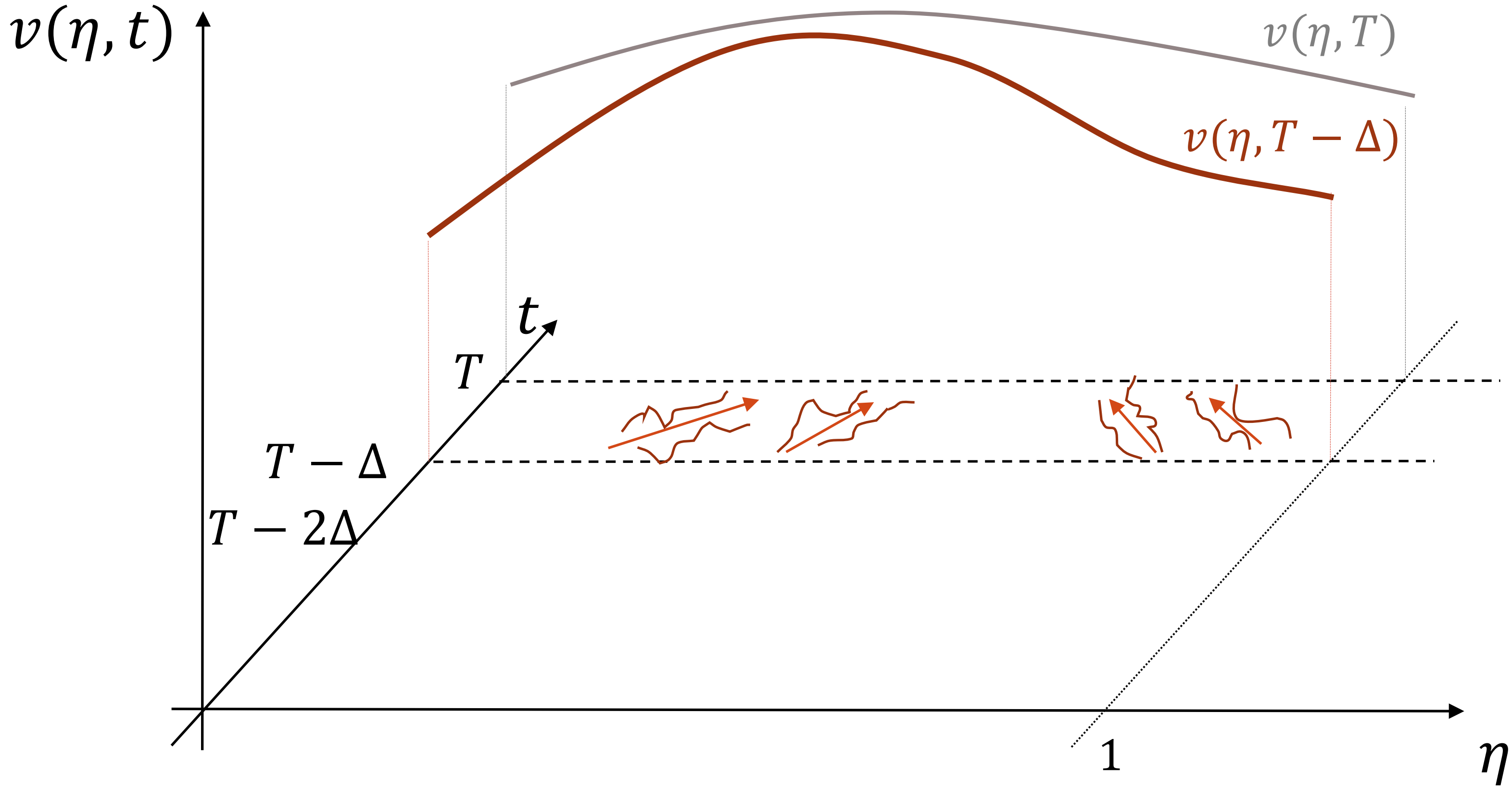
4. Value function Iteration - Big picture

- Add time, t, as an additional state variable $v(\eta, t)$, $\underline{v}(\eta, t)$
- Convert BSDE into PDE

4. Value Function Iteration – Big Picture



4. Value Function Iteration – Big Picture



4. Value function Iteration - Big picture

- Add time, t, as an additional state variable $v(\eta, t)$, $\underline{v}(\eta, t)$
- Convert BSDE into PDE using Ito's Lemma
  - $\mu_t^v v_t = \partial_t v_t + \eta_t \mu_t^\eta \partial_\eta v_t + \frac{1}{2}\left(\eta_t \sigma_t^\eta \right)^2 \partial_{\eta\eta} v_t$
  - $\mu_t^{\underline{v}} \underline{v}_t = \partial_t \underline{v}_t + \eta_t \mu_t^\eta \partial_\eta \underline{v}_t + \frac{1}{2}\left(\eta_t \sigma_t^\eta \right)^2 \partial_{\eta\eta} \underline{v}_t$

Short-hand notation:
$\partial_x f$ for $\partial f / \partial x$

- Propose any arbitrary value function $v(\eta, T)$ and $\underline{v}(\eta, T)$ (far in the future $t = T$)
- … and iterate back to $t = 0$
  - In each step use
    - From Step 2: $\mu_t^v v_t$, $\mu_t^{\underline{v}} \underline{v}_t$
    - From Step 3: $\eta_t \mu_t^\eta$ and $\eta_t \sigma_t^\eta$ (η-evolution)
    - Portfolio choice, planners' problem, (static conditions)
    - Market clearing
  - To calculate all terms in these $\mu_{t-\Delta}^v v_{t-\Delta}$, $\eta_{t-\Delta} \mu_{t-\Delta}^\eta$ and $\eta_{t-\Delta} \sigma_{t-\Delta}^\eta$

4a. PDE Expert Value Function Iteration

Short-hand notation:
$\partial_x f$ for $\partial f/\partial x$

- Postulate $v_t = v(\eta_t, t)$
- By Ito's Lemma
- $$\frac{dv_t}{v_t} = \underbrace{\frac{\partial_t v_t + \partial_\eta v_t \eta \mu_t^\eta + \frac{1}{2}\partial_{\eta\eta} v_t \left(\eta_t \sigma_t^\eta\right)^2}{v_t}}_{\mu_t^v} dt + \underbrace{\frac{\partial_\eta v_t \eta \sigma_t^\eta}{v_t}}_{\sigma_t^v} dZ_t$$
- That is,
- $\mu_t^v v_t = \partial_t v_t + \partial_\eta v_t \eta \mu_t^\eta + \frac{1}{2}\partial_{\eta\eta} v_t \left(\eta_t \sigma_t^\eta\right)^2$
- $\sigma_t^v v_t = \partial_\eta v_t \eta \sigma_t^\eta$
- Equating with Step 2 ⇒ "growth equation"

$$\boxed{\begin{aligned}&\partial_t v_t + \left(\eta \mu_t^\eta + (1-\gamma)\sigma \eta_t \sigma_t^\eta\right)\partial_\eta v_t + \frac{1}{2}\partial_{\eta\eta} v_t \left(\eta_t \sigma_t^\eta\right)^2 = \\ &= \left(\rho - (1-\gamma)(\Phi(\iota) - \delta) + \frac{1}{2}\gamma(1-\gamma)(\sigma^2)\right) v_t - \frac{c_t}{n_t} v_t\end{aligned}}$$

4a. PDE Expert Value Fcn: Replacing Terms

$$\partial_t v_t + \left(\eta_t \mu_t^\eta + (1-\gamma)\sigma\eta_t\sigma_t^\eta\right)\partial_\eta v_t + \frac{1}{2}\partial_{\eta\eta} v_t \left(\eta_t \sigma_t^\eta\right)^2 =$$

$$= \left(\rho - (1-\gamma)(\Phi(\iota) - \delta) + \frac{1}{2}\gamma(1-\gamma)(\sigma^2)\right) v_t - \frac{c_t}{n_t} v_t$$

1. Replace "blue terms" using results from Step 3.

$$\mu_t^\eta = (1-\eta_t)\left(\varsigma_t - \sigma_t^q - \sigma\right)\left(\sigma_t^\eta - \underbrace{\sigma_t^M}_{=0}\right)$$

$$-(1-\eta_t)\left(\underline{\varsigma}_t - \sigma_t^q - \sigma\right)\left(\underline{\sigma}_t^\eta - \underbrace{\sigma_t^M}_{=0}\right) - \left(\frac{C_t}{N_t} - \frac{C_t + \underline{C}_t}{q_t K_t}\right)$$

$$\sigma_t^\eta = \frac{\chi_t - \eta_t}{\eta_t}(\sigma + \sigma_t^q) \qquad \underline{\sigma}_t^\eta = -\frac{\eta_t}{1-\eta_t}\sigma_t^\eta$$

2. Replace "tanned terms" using results from Step 2c.

$$\varsigma_t = -\sigma_t^v + \sigma_t^\eta + \sigma_t^q + \gamma\sigma, \qquad \underline{\varsigma}_t = -\underline{\sigma}_t^v + \underline{\sigma}_t^\eta + \sigma_t^q + \underline{\gamma}\sigma$$

$$\frac{C_t}{N_t} = \frac{(\eta_t q_t)^{1/\gamma - 1}}{v_t^{1/\gamma}} \qquad \frac{\underline{C}_t}{\underline{N}_t} = \frac{((1-\eta_t)q_t)^{1/\underline{\gamma}-1}}{\underline{v}_t^{1/\underline{\gamma}}}$$

Recall from Ito's Lemma $\sigma_t^v v_t = \partial_\eta v_t \eta \sigma_t^\eta$

3. Replace "red terms" $\iota_t, \sigma_t^q, \chi_t, \underline{\chi}_t$ (see below)

4a. PDE HH Value Fcn: Replacing Terms

$$\partial_t \underline{v}_t + \left(\eta_t \mu_t^\eta + (1-\gamma)\sigma\eta_t\sigma_t^\eta\right)\partial_\eta \underline{v}_t + \frac{1}{2}\partial_{\eta\eta}\underline{v}_t\left(\eta_t\sigma_t^\eta\right)^2 =$$

$$= \left(\rho - \left(1-\underline{\gamma}\right)(\Phi(\iota)-\delta) + \frac{1}{2}\underline{\gamma}(1-\underline{\gamma})(\sigma^2)\right)\underline{v}_t - \frac{\underline{c}_t}{\underline{n}_t}\underline{v}_t$$

1. Replace “blue terms” using results from Step 3.

$$\mu_t^\eta = \cdots$$

$$\sigma_t^\eta = \frac{\chi_t - \eta_t}{\eta_t}(\sigma + \sigma_t^q) \qquad \sigma_t^{\underline{\eta}} = -\frac{\eta_t}{1-\eta_t}\sigma_t^\eta$$

2. Replace “tanned terms” using results from Step 2c.

$$\varsigma_t = -\sigma_t^v + \sigma_t^\eta + \sigma_t^q + \gamma\sigma, \qquad \underline{\varsigma}_t = -\sigma_t^{\underline{v}} + \sigma_t^{\underline{\eta}} + \sigma_t^q + \underline{\gamma}\sigma$$

$$\frac{C_t}{N_t} = \frac{(\eta_t q_t)^{1/\gamma - 1}}{v_t^{1/\gamma}} \qquad \frac{\underline{C}_t}{\underline{N}_t} = \frac{((1-\eta_t)q_t)^{1/\underline{\gamma}-1}}{\underline{v}_t^{1/\underline{\gamma}}}$$

Recall from Ito’s Lemma
$\sigma_t^v v_t = \partial_\eta v_t \eta \sigma_t^\eta$

3. Replace “red terms” $\iota_t, \sigma_t^q, \chi_t, \underline{\chi}_t$ (see below)

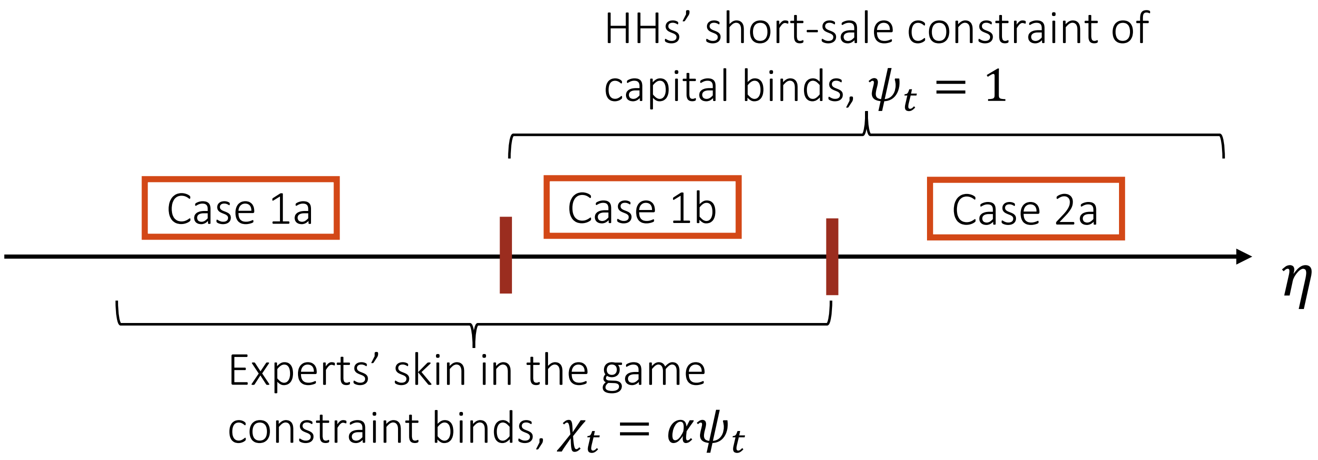
4a. Replacing ι_t

- Recall from optimal re-investment $\Phi'(\iota_t) = 1/q_t$
  - For $\Phi(\iota) = \frac{1}{\kappa}\log(\kappa\iota + 1) \Rightarrow \boxed{\kappa\iota = q - 1}$

4a. Replacing χ, obtain ψ for good mkt clearing

- Recall from planner's problem (Step 1b)

Cases	$\chi_t \geq \alpha\psi_t$	$\psi_t \leq 1$	$\frac{(a-\underline{a})}{q_t} \geq \alpha\left(\varsigma_t - \underline{\varsigma}_t\right)(\sigma + \sigma_t^q)$	$\left(\varsigma_t - \underline{\varsigma}_t\right)(\sigma + \sigma_t^q) \geq \mathbf{0}$
1a	=	<	=	>
1b	=	=	>	>
2a	>	=	>	=



HHs' short-sale constraint of capital binds, $\psi_t = 1$

Case 1a

Case 1b

Case 2a

η

Experts' skin in the game constraint binds, $\chi_t = \alpha\psi_t$

4a. Replacing χ, obtain ψ for good mkt clearing

- Need to determine diff in risk premia $(\varsigma_t - \underline{\varsigma}_t)(\sigma + \sigma_t^q)$:
- Recall
  - diff in price of risk: $\varsigma_t - \underline{\varsigma}_t = -\sigma_t^v + \sigma_t^{\underline{v}} + \frac{\sigma_t^\eta}{1-\eta_t}$
  - By Ito's lemma $\sigma_t^v = \frac{v'}{v}\eta_t\sigma_t^\eta$ and $\sigma_t^{\underline{v}} = \frac{\underline{v}'}{\underline{v}}\eta_t\sigma_t^\eta$

$$\Rightarrow (\varsigma_t - \underline{\varsigma}_t)(\sigma + \sigma_t^q) = \left(-\frac{v'}{v} + \frac{\underline{v}'}{\underline{v}} + \frac{1}{(1-\eta_t)\eta_t}\right)\eta_t\sigma_t^\eta(\sigma + \sigma_t^q)$$

$$= \left(-\frac{v'}{v} + \frac{\underline{v}'}{\underline{v}} + \frac{1}{(1-\eta_t)\eta_t}\right)(\chi_t - \eta_t)(\sigma + \sigma_t^q)^2$$

- Note, since $-\frac{v'}{v} + \frac{\underline{v}'}{\underline{v}} + \frac{1}{(1-\eta_t)\eta_t} > 0$,

$$(\varsigma_t - \underline{\varsigma}_t)(\sigma + \sigma_t^q) > 0 \Leftrightarrow \chi_t > \eta_t \Leftrightarrow \alpha > \eta_t$$

4a. Replacing χ, obtain ψ for good mkt clearing

- Determination of ψ_t

$$(a-\underline{a})/q_t \geq \underline{\alpha}\left(-\frac{v'}{v}+\frac{\underline{v}'}{\underline{v}}+\frac{1}{(1-\eta_t)\eta_t}\right)(\chi_t-\eta_t)(\sigma+\sigma_t^q)^2$$

with equality if $\psi_t < 1$

- Determination of χ_t

$$\chi_t = \max\{\alpha\psi_t, \eta_t\}$$

4a. Market Clearing

- Output good market

$$C_t = \left(\psi_t a + (1-\psi_t)\underline{a} - \iota_t\right)K_t$$

- … jointly restricts ψ_t and q_t

$$\boxed{\psi_t a + (1-\psi_t)\underline{a} - \iota(q) = \underbrace{\left(\frac{\eta_t q_t}{v_t}\right)^{1/\gamma}}_{C_t/K_t} + \underbrace{\left(\frac{(1-\eta_t)q_t}{\underline{v}_t}\right)^{1/\underline{\gamma}}}_{\underline{C}_t/K_t}}$$

4a. Markest Clearing

- Output good market

$$C_t = \left(\psi_t a + (1 - \psi_t)\underline{a} - \iota_t\right)K_t$$

- Capital market is taken care off by price taking social planner approach

$$1 - \theta_t = \frac{\psi_t q_t K_t}{\eta_t q_t K_t}$$

- Risk-free debt also taken care off by price taking social planner approach (would be cleared by Walras Law anyways)

4a. $\sigma^q(q, q')$

- Recall from “amplification slide” – Step 3

$$\sigma + \sigma_t^q = \frac{\sigma}{1 - \frac{q'(\eta_t)}{q/\eta_t} \frac{\chi_t - \eta_t}{\eta_t}}$$

$$\boxed{\sigma^q = \frac{q'(\eta_t)}{q(\eta_t)}(\chi_t - \eta_t)(\sigma + \sigma_t^q)}$$

- Now all red terms are replaced and we can solve …

4b. Algorithm – Static Step

- Suppose we know functions $v(\eta)$, $\underline{v}(\eta)$, have five static conditions:

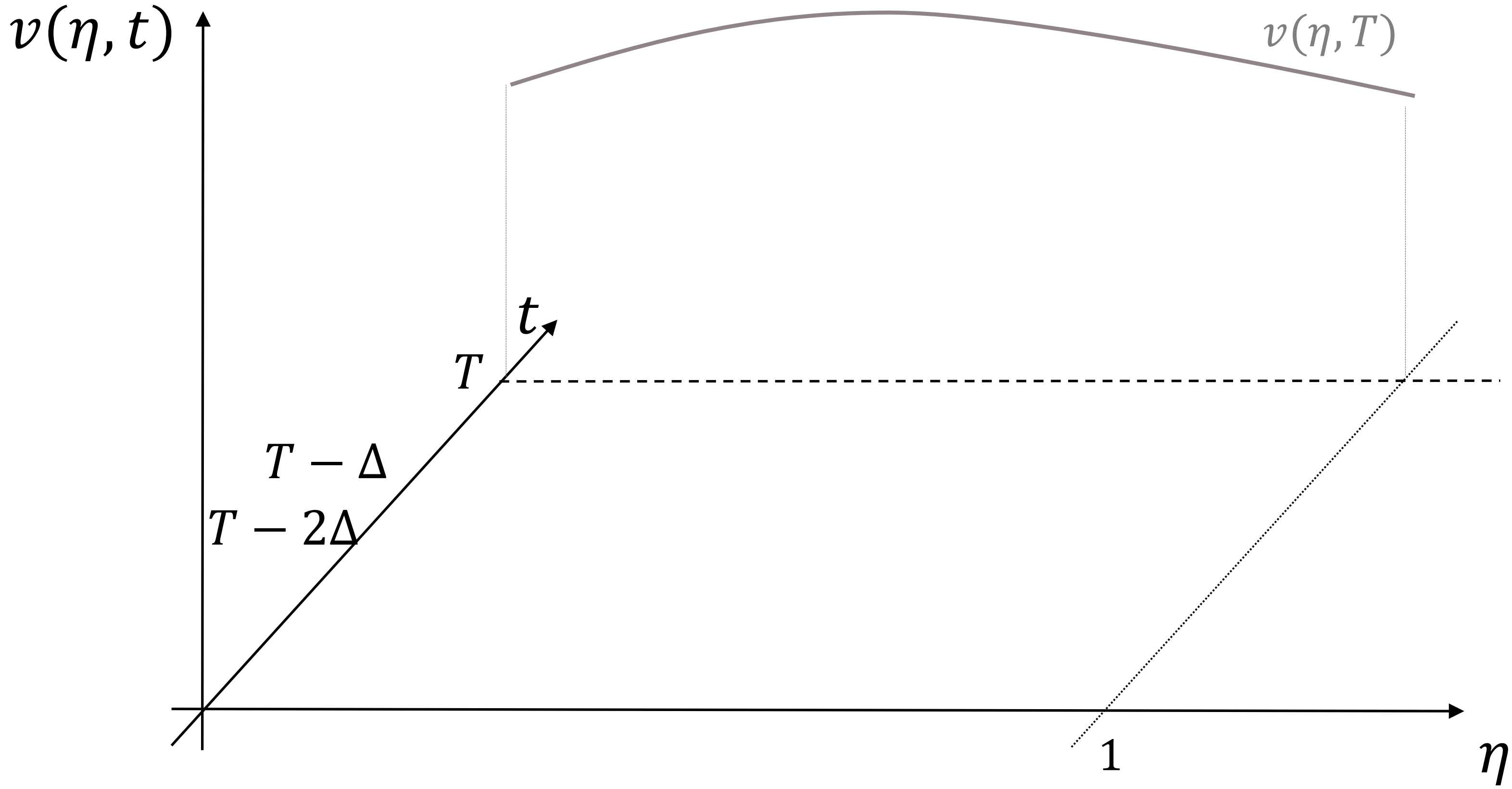
1. $\kappa\iota_t = q_t - 1$
2. Planner condition for ψ_t
3. Planner condition for χ_t
4. $\psi_t a + (1-\psi_t)\underline{a} - \iota(q) = \underbrace{\left(\frac{\eta_t q_t}{v_t}\right)^{1/\gamma}}_{C_t/K_t} + \underbrace{\left(\frac{(1-\eta_t)q_t}{\underline{v}_t}\right)^{1/\underline{\gamma}}}_{\underline{C}_t/K_t}$
5. $\sigma^q = \frac{q'(\eta_t)}{q(\eta_t)}(\chi_t - \eta_t)(\sigma + \sigma_t^q)$

⇒Get $q(\eta)$, $\psi(\eta)$, $\sigma^\eta(\eta)$

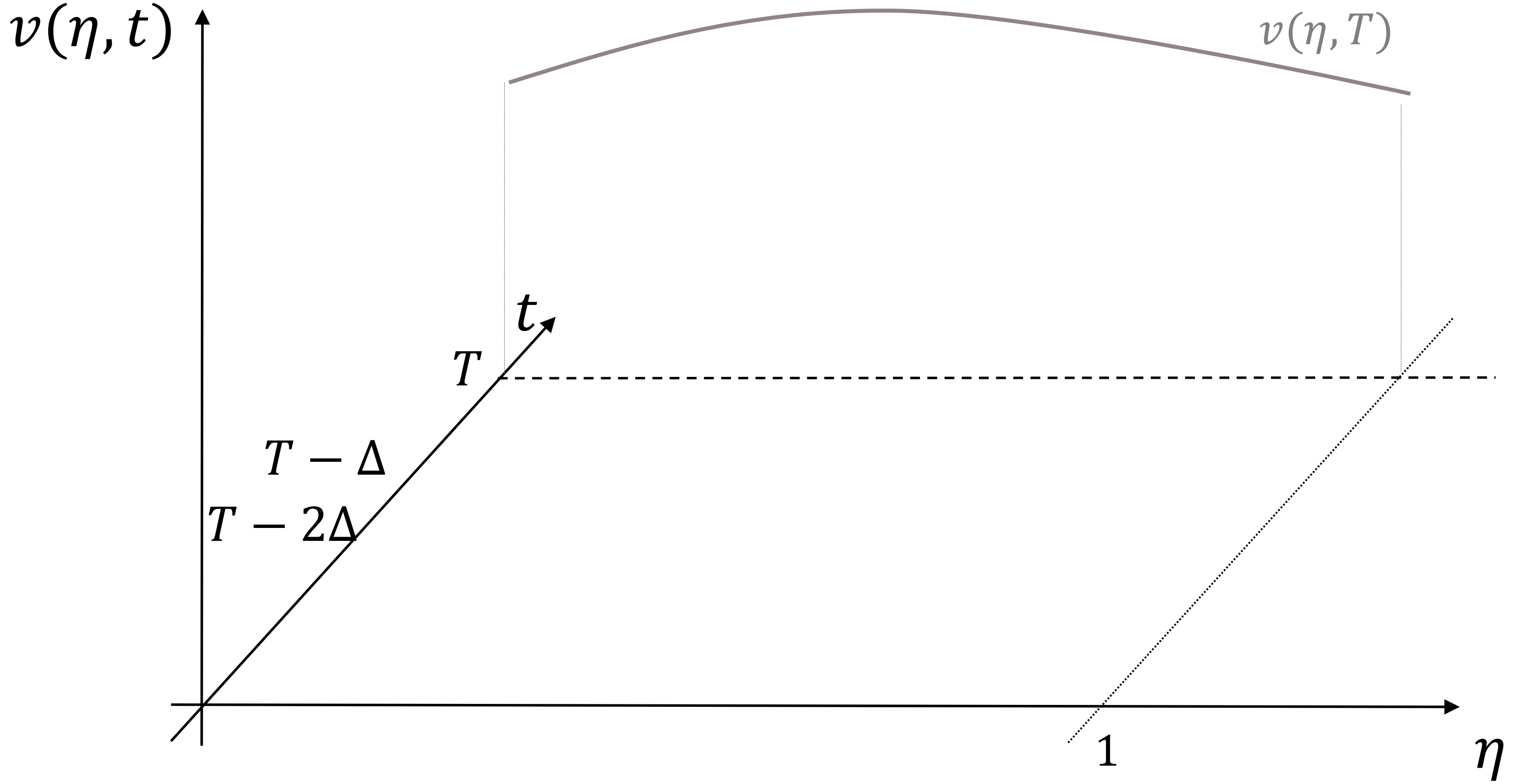
- Start at $q(0)$, solve to the right, use different procedure for two η regions depending on ψ:

1. While $\psi < 1$, solve ODE for $q(\eta)$:
    - For given $q(\eta)$, goods market clearing (4) and opt. investment (1) yield $\psi(\eta)$
    - Planner conditions (2) and (3) give $(\sigma + \sigma^q)(\eta)$
    - Risk equation (5) gives derivative $q'(\eta)$
2. When $\psi = 1$, (2) is no longer informative, solve (1) and (4) for $q(\eta)$

4b. Value Function Iteration



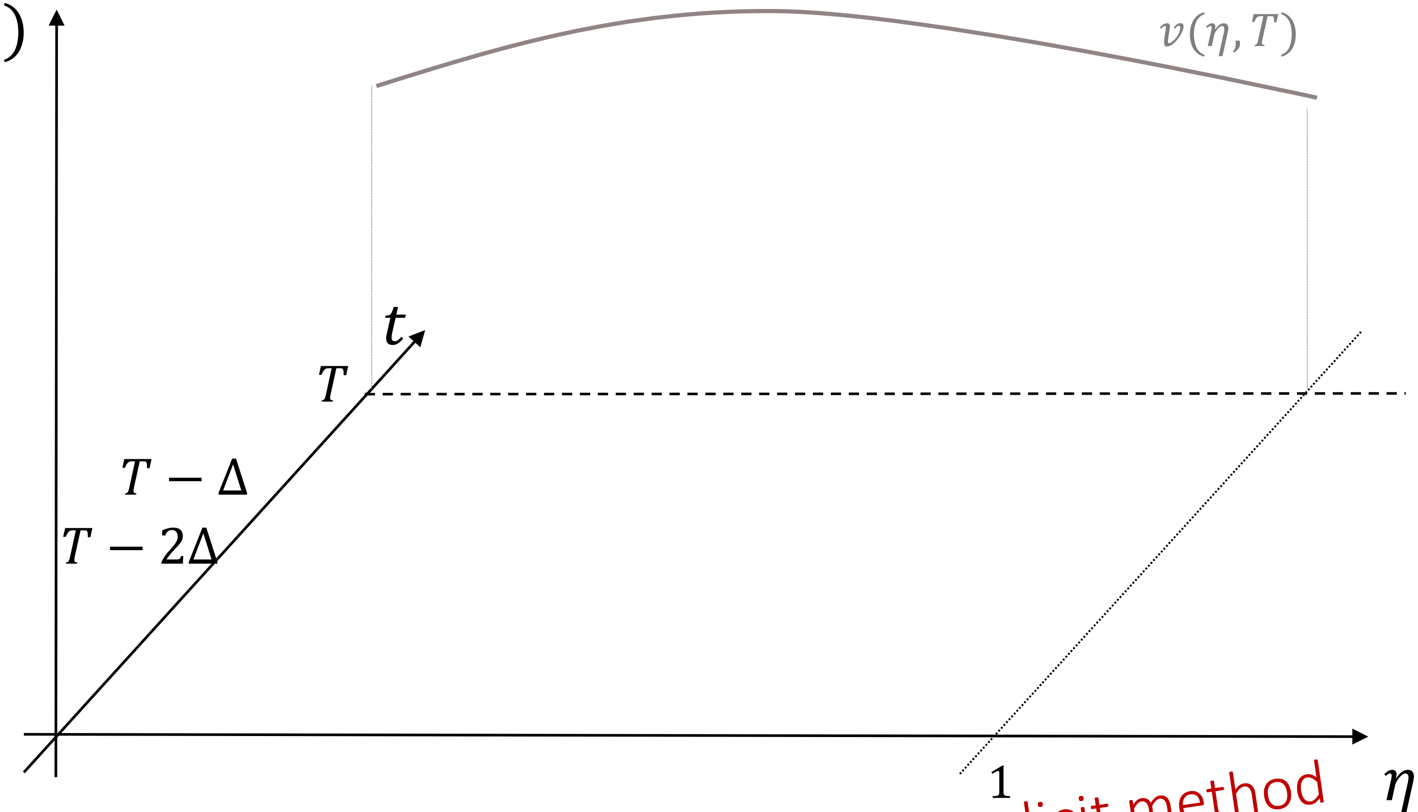
4b. Value Function Iteration



- For given $v(\eta, T)$, derive SDF ξ_T
- Optimal investment, portfolio, consumption, at T as fcn. of η

4. Market clearing at T obtain PDE coefficient at T
(pretend they are constant between T & $T-\Delta$)

4b. Value Function Iteration



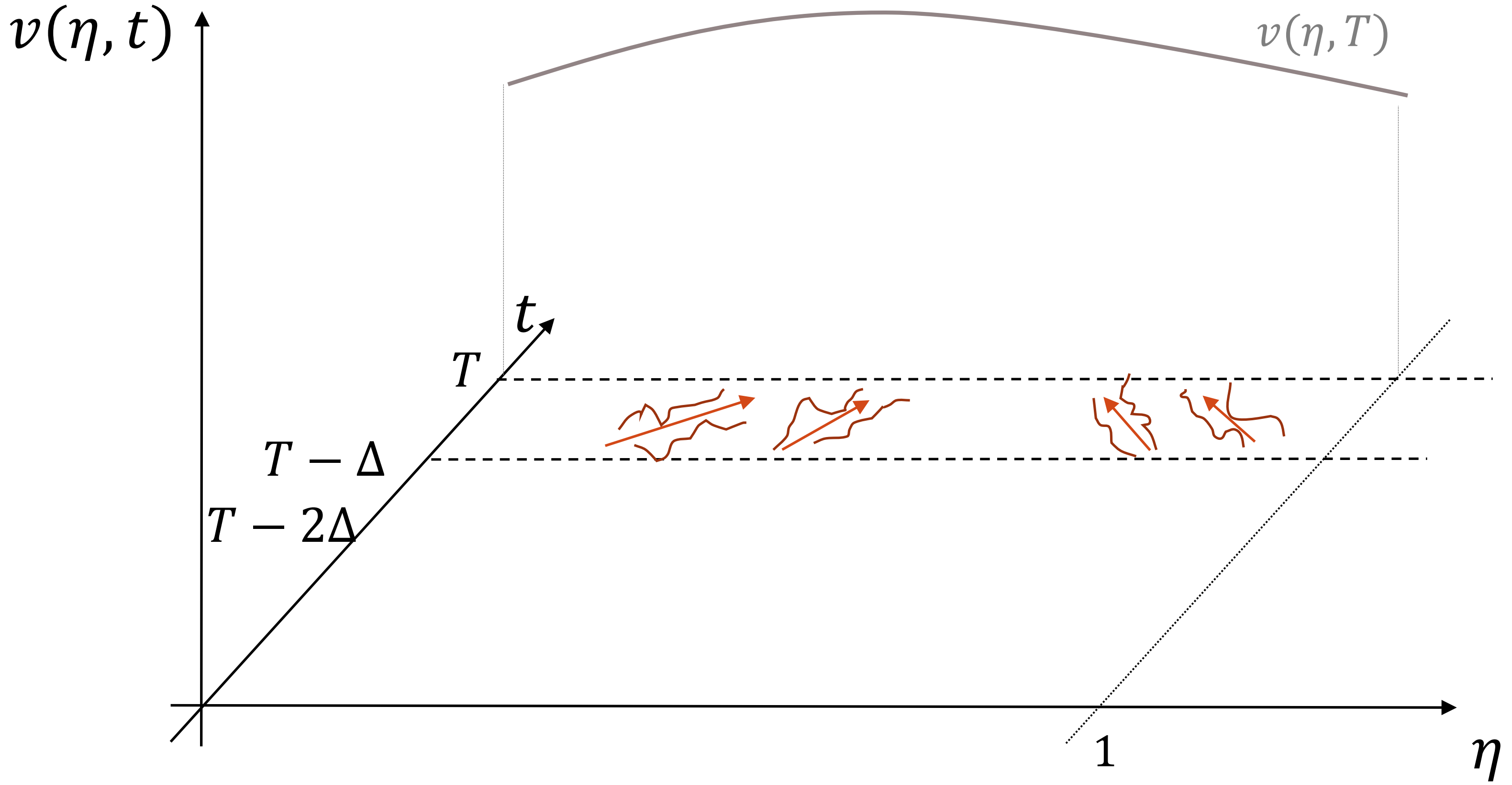
Explicit method

Implicit method uses $T-\Delta$

- For given $v(\eta,T)$, derive SDF ξ_T
- Optimal investment, portfolio, consumption, at T as fcn. of η

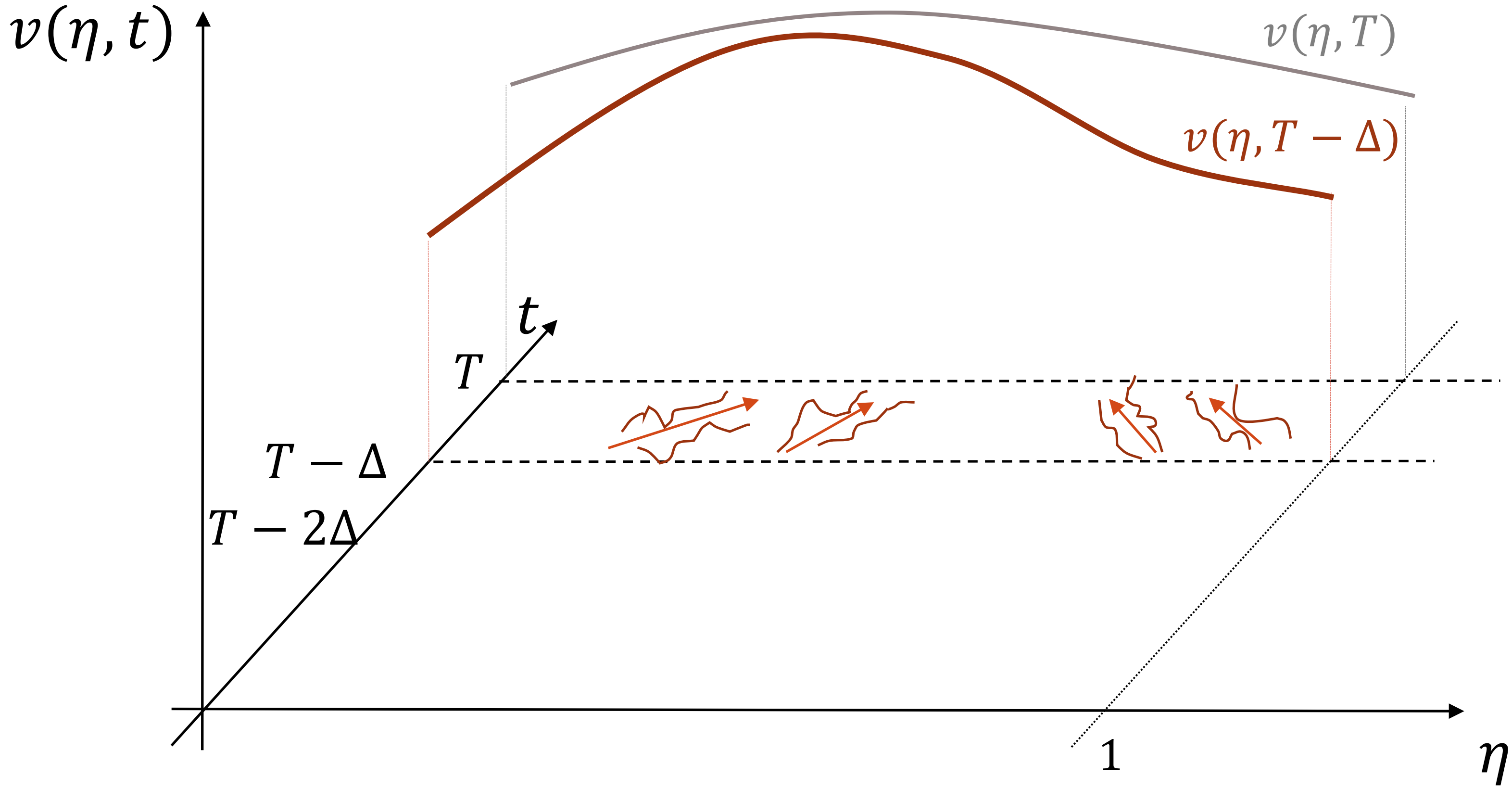
4. Market clearing at T obtain PDE coefficient at T (pretend they are constant between T & $T-\Delta$)

4b. Value Function Iteration



- Obtain descaled value function $v(\eta, T - \Delta)$
- Repeat previous steps

4b. Value Function Iteration

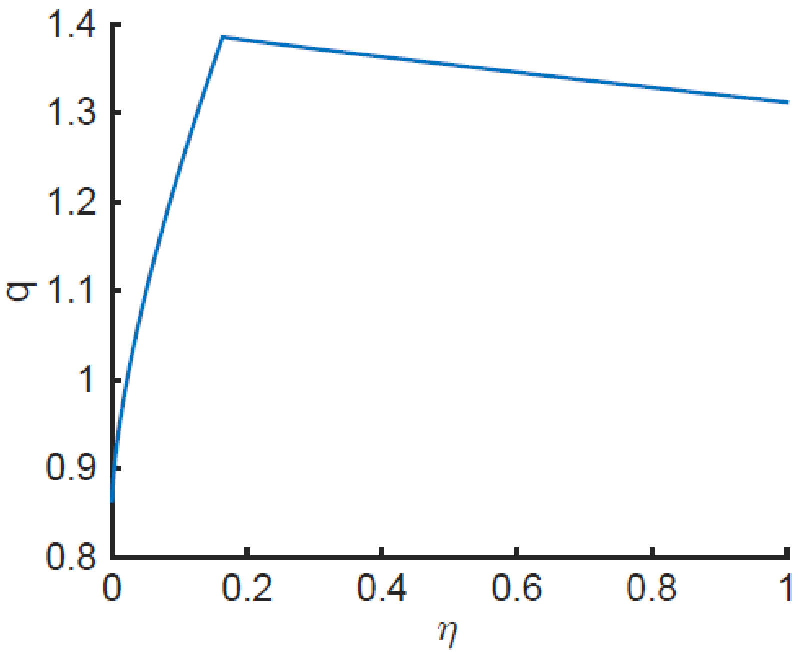


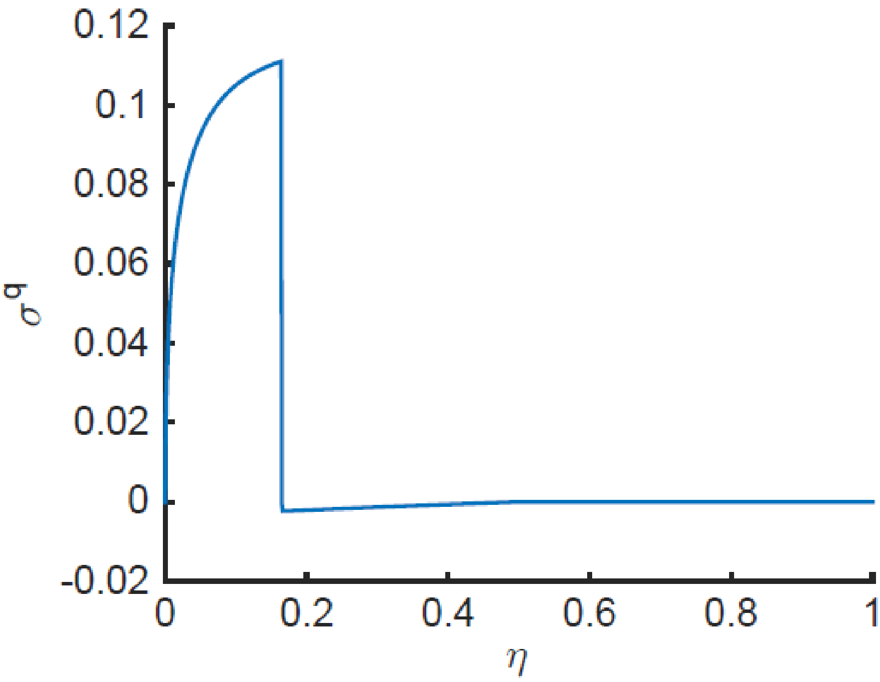
4b. Algo-code

- System of ODE (vector): solve backwards in time
- v is a vector on η-grid $[0, \Delta, 2\Delta, \dots 1]$ same for $\underline{v}$
- Use MatLab ODE solver to solve vector ODE
    - Sequence of steps to find $\mu^\eta, \sigma^\eta, \mu^v, \frac{C}{K}, \Phi(\iota) - \delta$
    - E.g. ODE45
    - Explicit vs. Implicit ODE solver
        - Explicit
            - Spatial step for ODE is Δ
            - To be stable time step $dt = o(\Delta^2)$ for explicit solver (1000 steps 10^6 time steps)
        - Implicit
- How to obtain v' and v'' from grid points
    - $v''(n) = \frac{v(n+1) - 2v(n) + v(n-1)}{\Delta^2}$
    - $v'(n) = \begin{cases} \frac{v(n+1)-v(n)}{\Delta} & \text{for} \quad \mu^\eta \eta > 0 \\ \frac{v(n)-v(n-1)}{\Delta} & \text{for} \quad \mu^\eta \eta < 0 \end{cases}$
- Solve $\mu_t^v v_t = v_t' \mu_t^\eta \eta_t + \frac{1}{2} v'' \left(\sigma_t^\eta \eta_t\right)^2$ using Yuliy's growth CODE

Solution

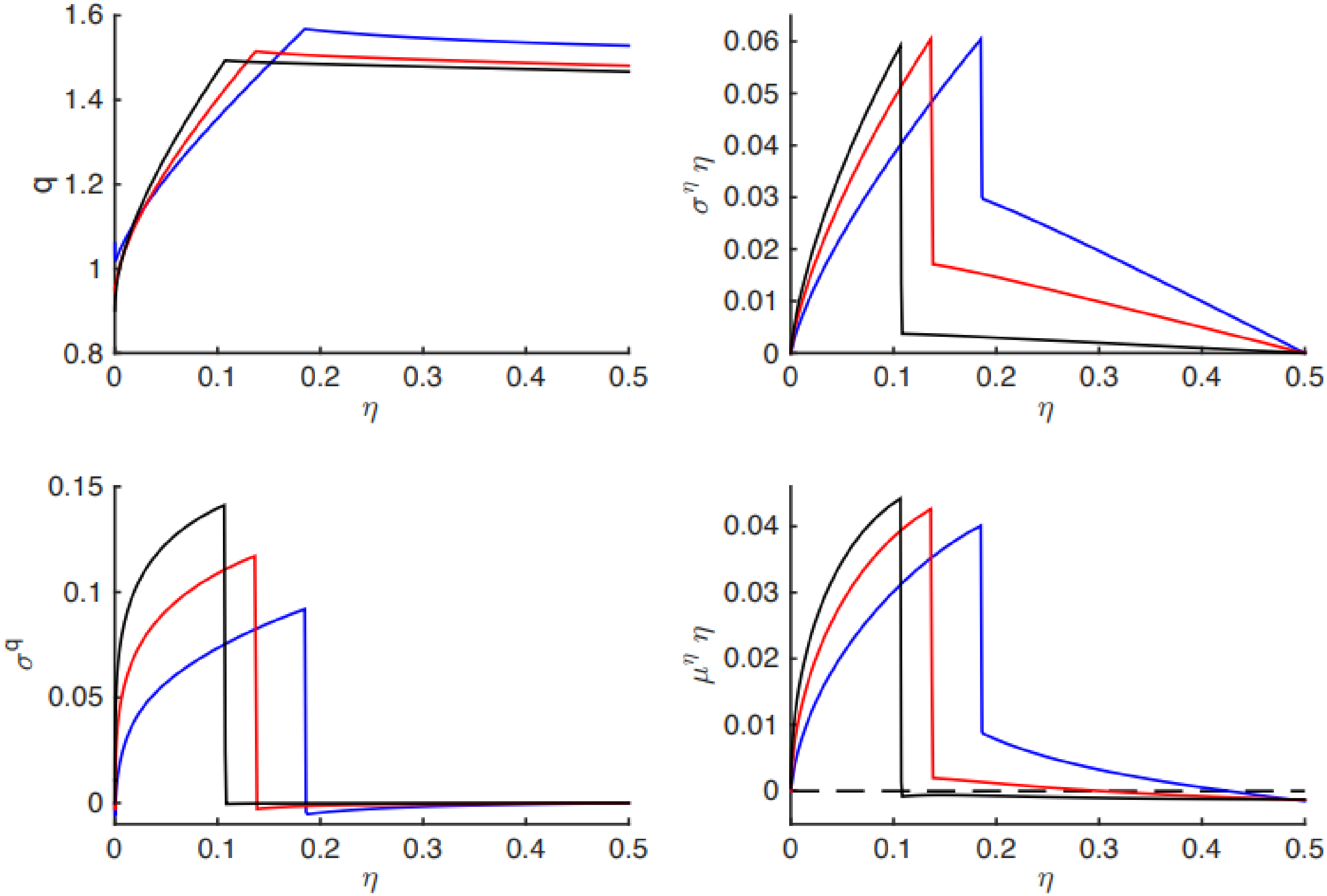
- Price of capital Amplification





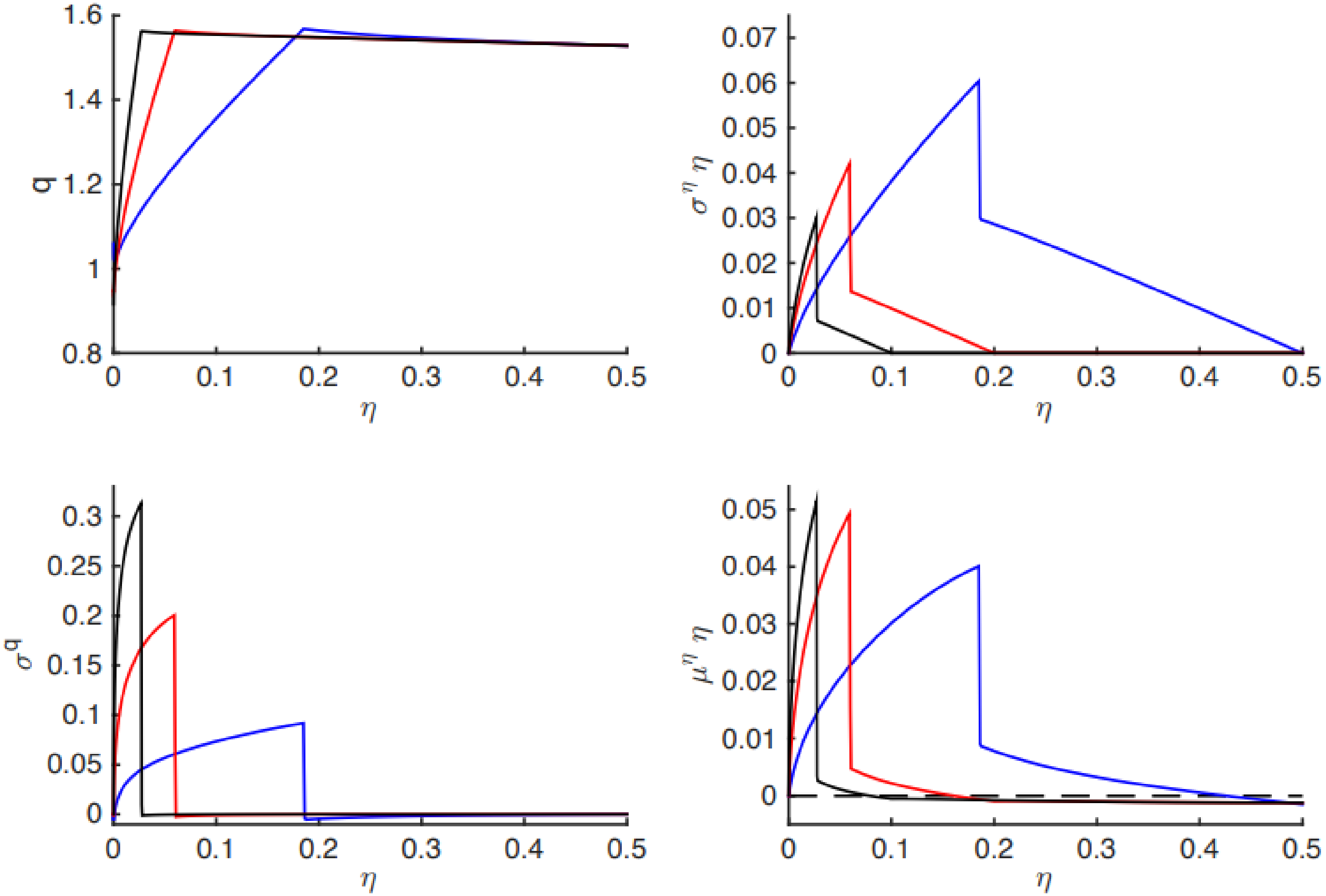
Volatility Paradox

- Comparative Static w.r.t. $\sigma = .1, .05, .01$



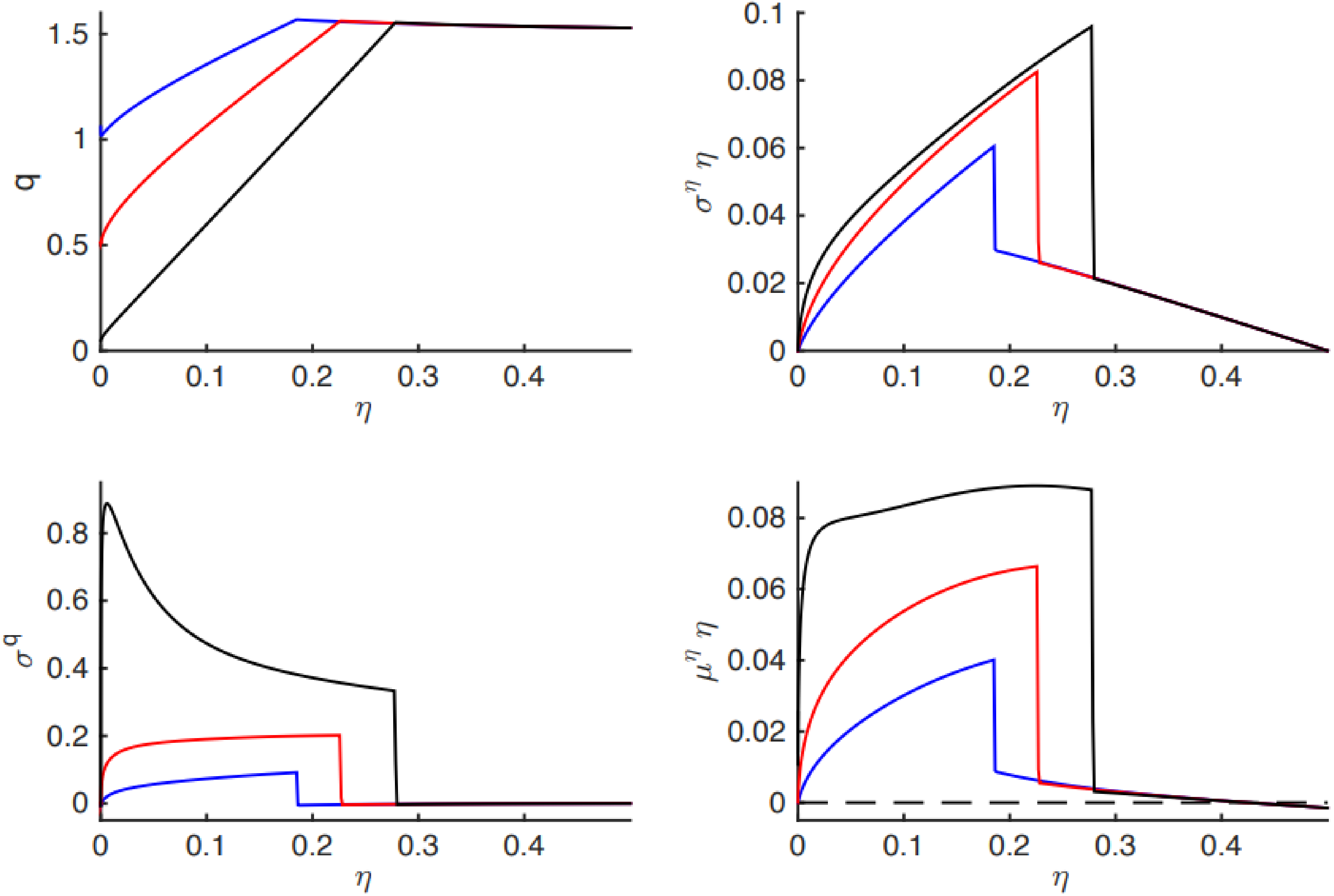
Risk Sharing via Outside Equity

- Comparative Static w.r.t. Risk sharing $\alpha = .5, .2, .1$ (skin the game constraint)



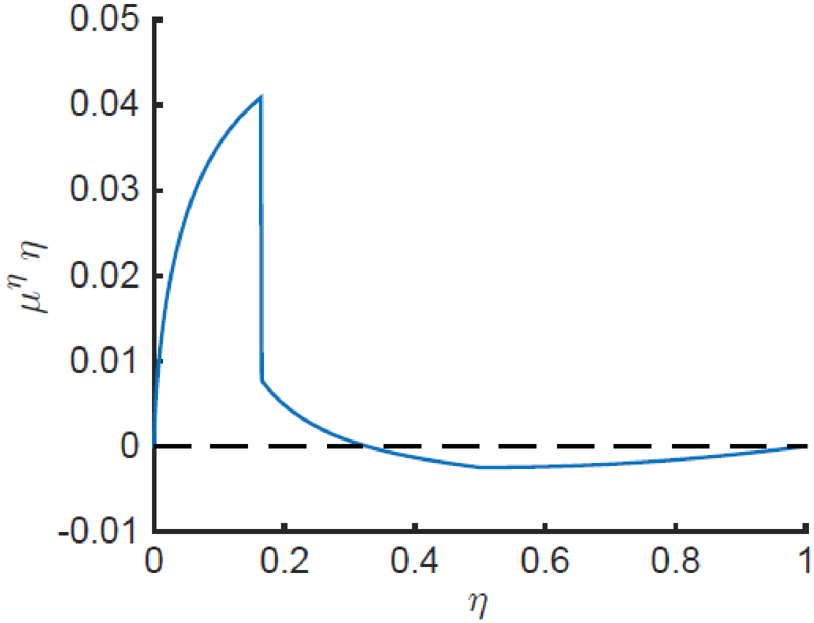
Market Liquidity

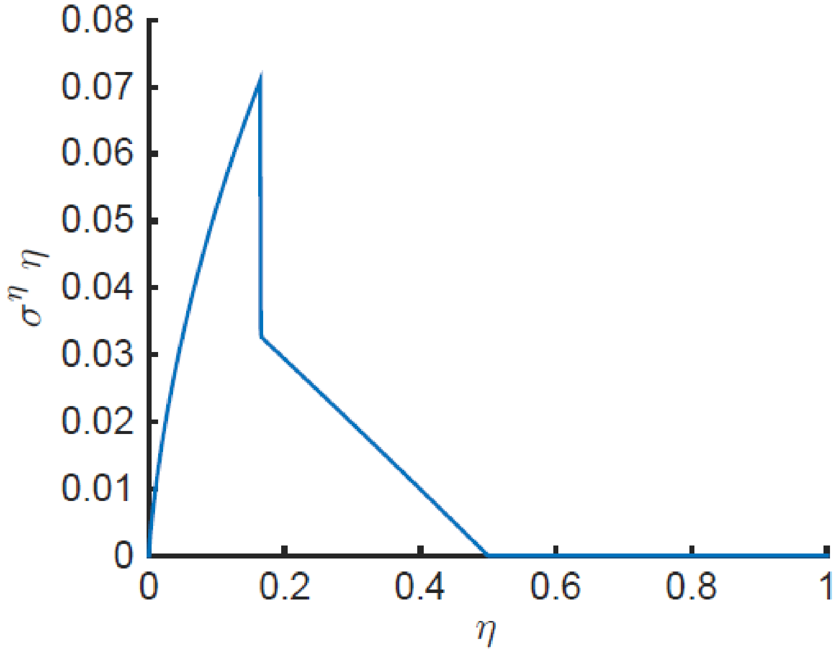
- Comparative static w.r.t. $\underline{a} = .03, -.03, -.09$



From $\mu^\eta(\eta)$ & $\sigma^\eta(\eta)$ to Stationary Distribution

- Kolmogorov forward equation





- Obtain stationary distribution

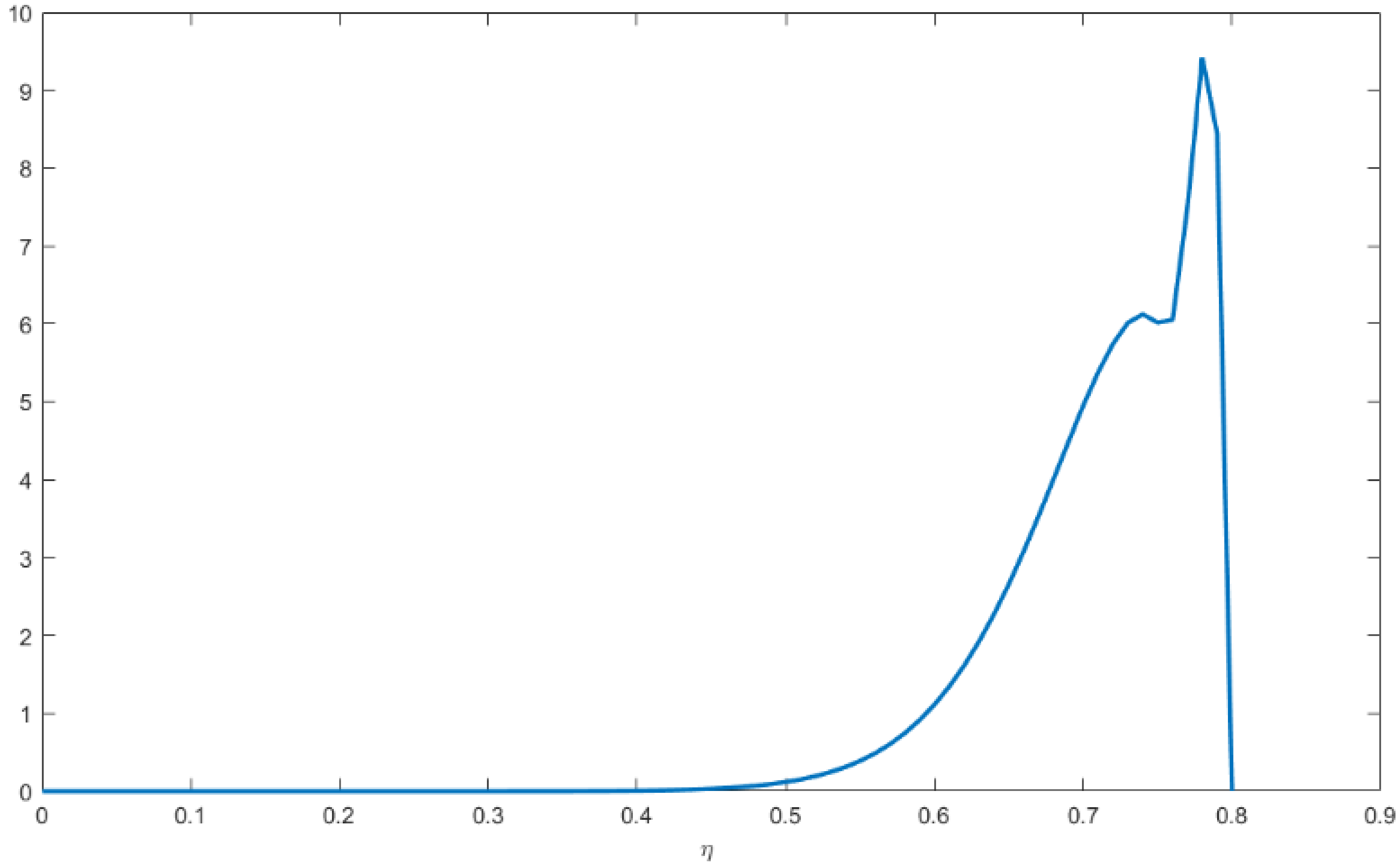
Kolmogorov Forward Equation

- Given an initial distribution $f(\eta,0)=f_0(\eta)$, $f(\eta,t)$ satisfies the following PDE
$$\frac{\partial f(\eta,t)}{\partial t}=\frac{\partial[f(\eta,t)\mu(\eta)]}{\partial\eta}+\frac{1}{2}\frac{\partial^2[f(\eta,t)\sigma^2(\eta)]}{\partial\eta^2}$$
- "Kolmogorov Forward Equation" is in physics referred to as "Fokker-Planck Equation"

- Corollary: if stationary distribution $f(\eta)$ exists, it satisfies the ODE
$$0=\frac{\partial[f(\eta,t)\mu(\eta)]}{\partial\eta}+\frac{1}{2}\frac{\partial^2[f(\eta,t)\sigma^2(\eta)]}{\partial\eta^2}$$

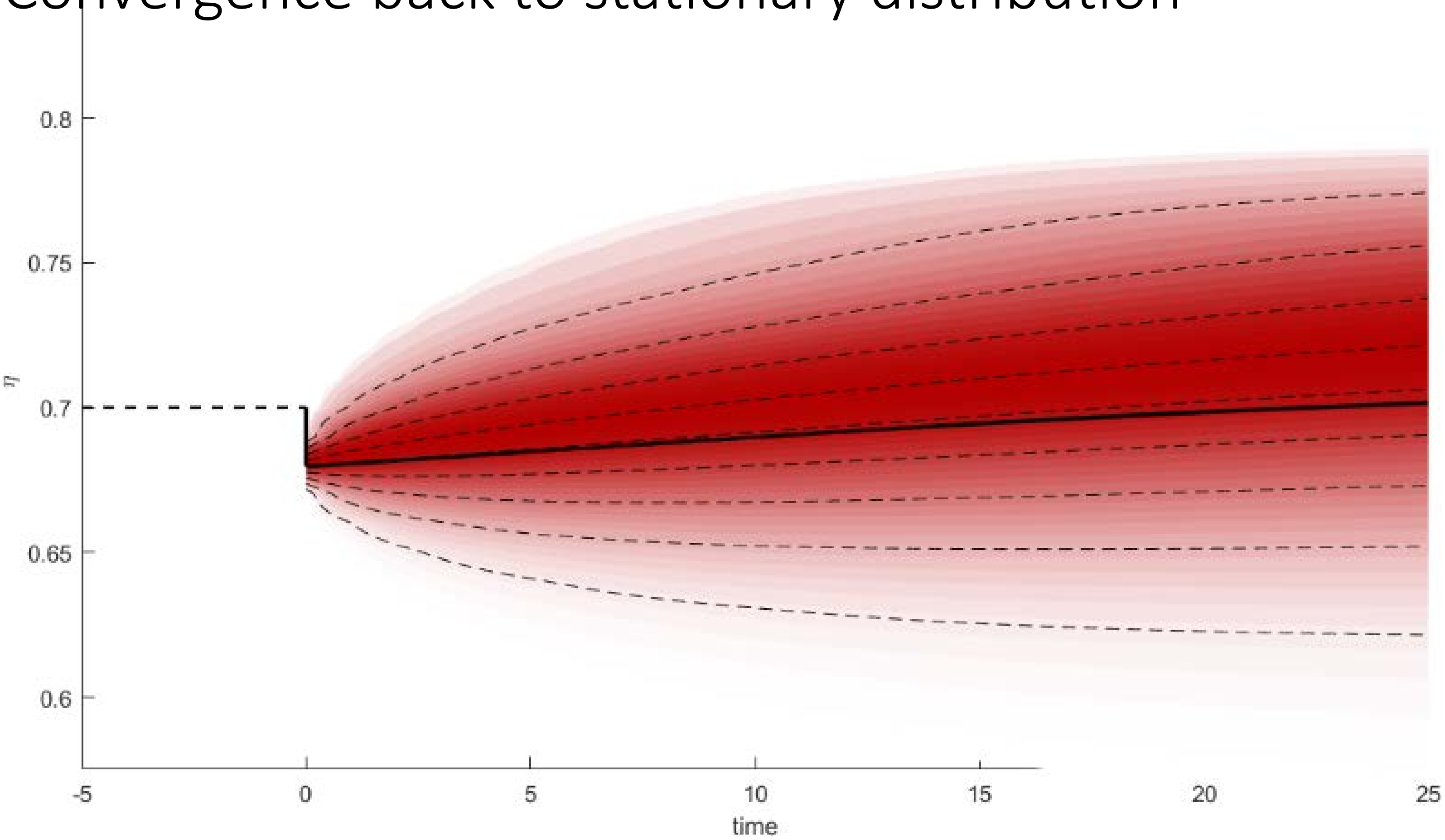
Stationary Distribution

- For different parameter settings



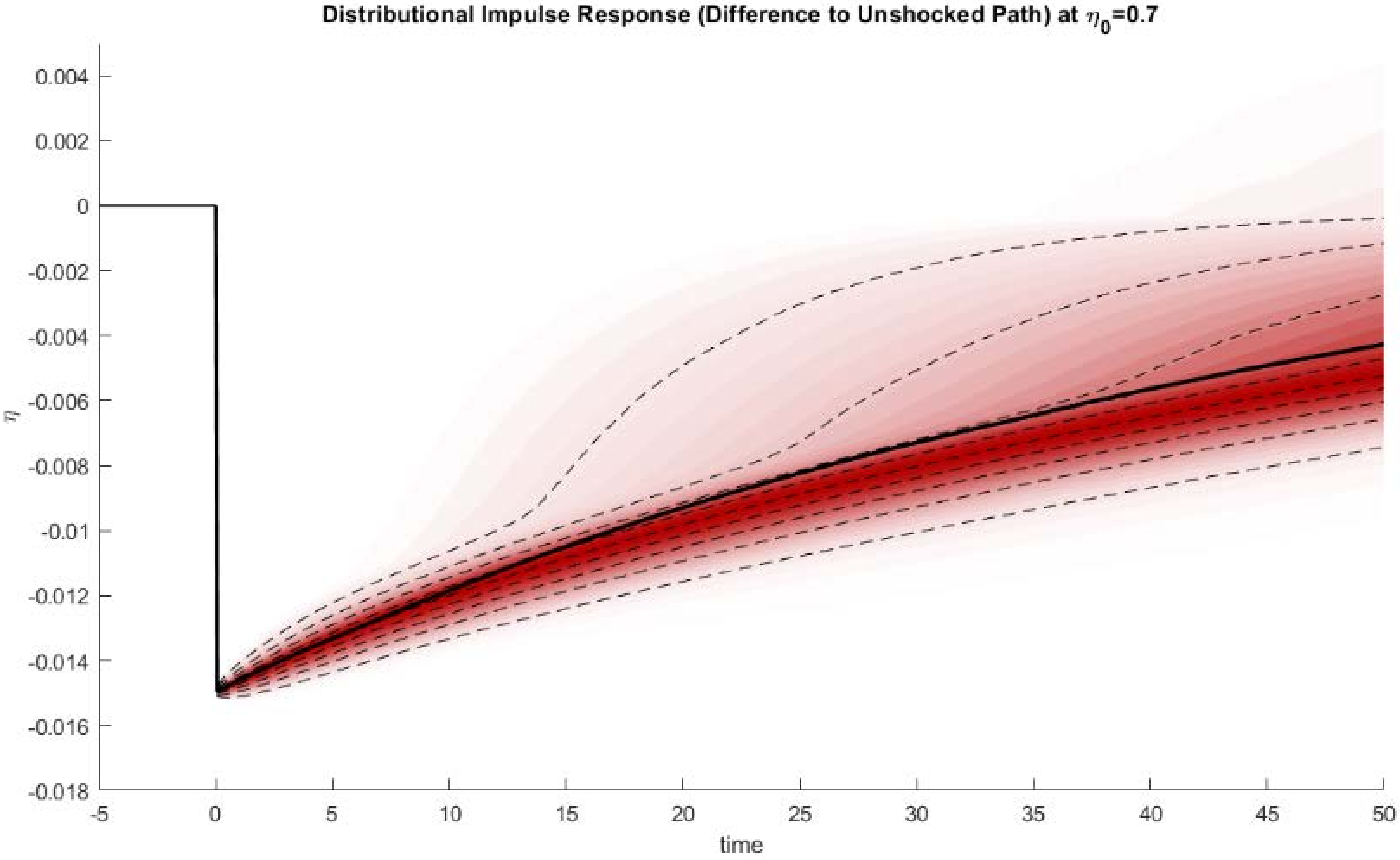
Fan chart

- … the theory to Bank of England’s empirical fan charts
- Start at $\eta_0 = .7$ and suffer a shock by one standard dev.
- Convergence back to stationary distribution



Distributional Impulse Response

- Difference between path with and without shock
- Difference converges to zero in the long-run



Solving MacroModels Step-by-Step

0. Postulate aggregates, price processes & obtain return processes
1. For given SDF processes *static*
   a. Real investment ι, (portfolio $\boldsymbol{\theta}$, & consumption choice of *each agent*)
      - *Toolbox 1:* Martingale Approach
   b. Asset/Risk Allocation *across types/sectors* & asset market clearing
      - *Toolbox 2:* "price-taking social planner approach" – Fisher separation theorem
2. Value functions *backward equation*
   a. Value fcn. as fcn. of individual investment opportunities ω
      - *Special cases*
   b. De-scaled value fcn. as function of state variables η
      - *Digression:* HJB-approach (instead of martingale approach & envelop condition)
   c. Derive ς price of risk, C/N-ratio from value fcn. envelop condition
3. Evolution of state variable η *forward equation*
   - *Toolbox 3:* Change in numeraire to total wealth (including SDF)
   - ("Money evaluation equation" μ^{ϑ})
4. Value function iteration & goods market clearing
   a. PDE of de-scaled value fcn.
   b. Value function iteration by solving PDE

Extra Slides

Recent Macro-finance Literature (in cts. time)

- Core
  - BrunSan (2014), Basak & Cuoco (1998) He & Krishnamurthy (2012,13), DiTella (2013), Isohätälä et al. (2014)
- Intermediation/shadow banking
  - Phelan (2014), Adrian & Boyarchenko (2012,13), Huang (2014), Moreira & Savov (2014), Klimenko & Rochet (2015)
- Quantification
  - He & Krishnamurthy (2014), Mittnik & Semmler (2013)
- International
  - BruSan (2015), Maggiori (2013)
- Monetary
  - "The I Theory of Money" (2012), Drechsler et al. (2014)
- ….