Lecture 05

One Sector Money Model with Idio Risk

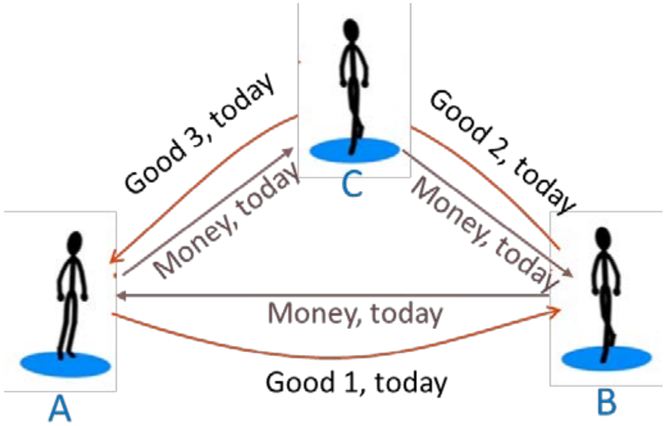
Markus Brunnermeier & Yuliy Sannikov

Eco 529: Financial and Monetary Economics Princeton, Feb. 2019

The 4 Roles of Money

- Store of value
  - "I Theory of Money without I"
    Less risky than other "capital" – no idiosyncratic risk
  - Fiscal theory of the price level

- Medium of exchange
  - Overcome double-coincidence of wants problem



- Unit of account

- Record keeping device
  - Virtual ledger

Models on Money as Store of Value

\Friction	OLG	Incomplete Markets + idiosyncratic risk	
Risk	deterministic	endowment risk borrowing constraint	investment risk
Only money	Samuelson	Bewley	
With capital	Diamond	Aiyagari	Angeletos
		Risk tied up with individual	capital

Models on Money as Store of Value

\Friction	OLG	Incomplete Markets + idiosyncratic risk	
Risk	deterministic	endowment risk borrowing constraint	investment risk
Only money	Samuelson	Bewley	
			"I Theory without I"
With capital	Diamond	Aiyagari	**cash flow shock**
	$f'(k^*) = r^*$, Dynamic inefficiency $r < r^*, K > K^*$	Inefficiency $r < r^*, K > K^*$	Pecuniary externality Inefficiency $r > r^*, K < K^*$
	(money) bubbles if $r < g$ Abel et al. vs. Geerolf		$r^M = g$

One Sector Model with Money

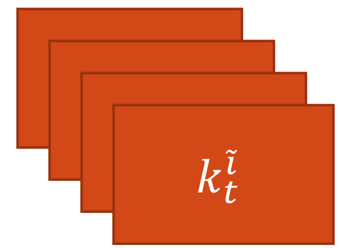
See Brunnermeier Sannikov (2016)
"On Optimal Inflation Rate" AER PP

- Agent $\tilde{\imath}$'s preferences

$$E\left[\int_0^{\infty} e^{-\rho t} \frac{\left(c_t^{\tilde{\imath}}\right)^{1-\gamma}}{1-\gamma} dt\right]$$

- Each agent operates one firm
  - Output

$$y_t^{\tilde{\imath}} = a k_t^{\tilde{\imath}}$$



  - Physical capital k

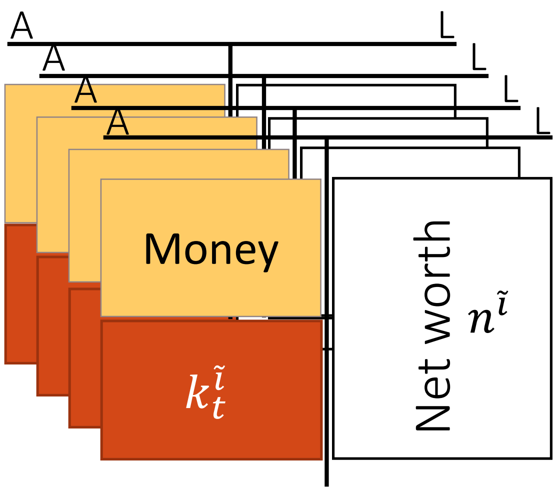
$$\frac{dk_t^{\tilde{\imath}}}{k_t^{\tilde{\imath}}} = \left(\Phi\left(\iota_t^{\tilde{\imath}}\right) - \delta\right)dt + \sigma dZ_t + \tilde{\sigma} d\tilde{Z}_t^{\tilde{\imath}}$$

- Financial Friction: Incomplete markets: Agents cannot share $d\tilde{Z}_t^{\tilde{\imath}}$

One Sector Model with Money

- Agent $\tilde{\imath}$'s preferences

$$E\left[\int_0^\infty e^{-\rho t}\frac{(c_t^{\tilde{\imath}})^{1-\gamma}}{1-\gamma}dt\right]$$



- Each agent operates one firm
  - Output

$$y_t^{\tilde{\imath}} = a k_t^{\tilde{\imath}}$$

  - Physical capital k

$$\frac{dk_t^{\tilde{\imath}}}{k_t^{\tilde{\imath}}} = (\Phi(\iota_t^{i}) - \delta)dt + \sigma dZ_t + \tilde{\sigma} d\tilde{Z}_t^{\tilde{\imath}}$$

- Financial Friction: Incomplete markets: Agents cannot share $d\tilde{Z}_t^{\tilde{\imath}}$
- Outside money
  - Money supply growth rate $(\mu^M + \mu^{Mi})$
    - μ^{Mi} used to pay interest on money (reserves)
    - μ^M generates seignorage
      $\Rightarrow$ transfers to agents proportional to networth $n^{\tilde{\imath}}$

Postulate Aggregates and Processes

- $q_t K_t$ value of physical capital
- $p_t K_t$ value of nominal capital/outside money
  - $\frac{p_t K_t}{M_t}$ value of one unit of (outside) money
- $\vartheta_t = \frac{p_t}{q_t + p_t}$ fraction of nominal wealth

Postulate Aggregates and Processes

- $q_t K_t$ value of physical capital
- $p_t K_t$ value of nominal capital/outside money
  - $\frac{p_t K_t}{M_t}$ value of one unit of (outside) money
- $\vartheta_t = \frac{p_t}{q_t + p_t}$ fraction of nominal wealth

0. Postulate
   - q-price process $\quad dq_t/q_t = \mu_t^q dt + \sigma_t^q dZ_t$
   - p-price process $\quad dp_t/p_t = \mu_t^p dt + \sigma_t^p dZ_t,$
   - SDF for each $\tilde{\iota}$ agent $\quad d\xi_t^{\tilde{\iota}}/\xi_t^{\tilde{\iota}} = -r_t^{f,\tilde{\iota}} dt - \varsigma_t^{\tilde{\iota}} dZ_t - \tilde{\varsigma}_t^{\tilde{\iota}} d\tilde{Z}_t^{\tilde{\iota}}$

0. Return processes

$$dr_t^{K,\tilde{\iota}} = \left(\frac{a - \iota_t^{\tilde{\iota}}}{q_t} + \Phi(\iota_t^{\tilde{\iota}}) - \delta + \mu_t^q + \sigma \sigma_t^q \right) dt \; + \; (\sigma + \sigma_t^q) dZ_t + \tilde{\sigma} d\tilde{Z}_t^{\tilde{\iota}}$$

$$dr_t^M = \left(\Phi(\iota_t) - \delta + \mu_t^p + \sigma \sigma_t^p - \mu^M \underbrace{- \mu^{Mi} + r^{\mu Mi}}_{=0} \right) dt \; + \; (\sigma + \sigma_t^p) dZ_t$$

One Sector Model with Money

1b. Optimal Choices

- Optimal investment rate

$$\kappa\iota_t = q_t - 1$$

$\frac{1}{q_t} = \Phi'(\iota_t^{\tilde{i}})$ Tobin's q

All agents $\iota_t^{\tilde{i}} = \iota_t$

Special functional form:

$$\Phi(\iota_t) = \frac{1}{\kappa}\log(\kappa\iota_t + 1) \Rightarrow \kappa\iota_t = q - 1$$

One Sector Model with Money

1b. Optimal Choices

- Optimal investment rate

$$\kappa\iota_t = q_t - 1$$

- Optimal portfolio

$$1-\theta = \frac{(a-\iota)/q}{\gamma\tilde{\sigma}^2} + \frac{\mu^M}{\gamma\tilde{\sigma}^2}$$

$$E[dr_t^{K,\tilde{\iota}}]/dt = \frac{a-\iota_t}{q_t} + \Phi(\iota_t) - \delta + \mu_t^q + \sigma\sigma_t^q \qquad = r_t^f + \varsigma_t(\sigma+\sigma_t^q) + \tilde{\varsigma}_t\tilde{\sigma}$$

$$E[dr_t^M]/dt \;=\; \uparrow \qquad \Phi(\iota_t) - \delta + \mu_t^p + \sigma\sigma_t^p - \mu^M = r_t^f + \varsigma_t(\sigma+\sigma_t^p) \qquad \uparrow$$

$$\frac{a-\iota_t}{q_t} + \mu_t^q + \sigma(\sigma_t^q - \sigma_t^p) + \mu^M = \varsigma_t(\sigma_t^q - \sigma_t^p) + \tilde{\varsigma}_t\tilde{\sigma}$$

One Sector Model with Money

1b. Optimal Choices

- Optimal investment rate

$$\kappa\iota_t = q_t - 1$$

- Optimal portfolio

$$1-\theta = \frac{(a-\iota)/q}{\gamma\tilde{\sigma}^2} + \frac{\mu^M}{\gamma\tilde{\sigma}^2}$$

$$E[dr_t^{K,\tilde{\imath}}]/dt = \frac{a-\iota_t}{q_t} + \Phi(\iota_t) - \delta + \mu_t^q + \sigma\sigma_t^q \qquad = r_t^f + \varsigma_t(\sigma + \sigma_t^q) + \tilde{\varsigma}_t\tilde{\sigma}$$

$$E[dr_t^M]/dt = \qquad \Phi(\iota_t) - \delta + \mu_t^p + \sigma\sigma_t^p - \mu^M = r_t^f + \varsigma_t(\sigma + \sigma_t^p)$$

$$\frac{a-\iota_t}{q_t} + \mu_t^q - \mu_t^p + \sigma(\sigma_t^q - \sigma_t^p) + \mu^M = \varsigma_t(\sigma_t^q - \sigma_t^p) + \tilde{\varsigma}_t\tilde{\sigma}$$

Price of Risk: $\varsigma_t = -\sigma_t^v + \sigma_t^{p+q} + \gamma\sigma, \qquad \tilde{\varsigma}_t = \gamma\tilde{\sigma}_t^n = \gamma(1-\theta_t)\tilde{\sigma}$

One Sector Model with Money

1b. Optimal Choices

- Optimal investment rate

$$\kappa\iota_t = q_t - 1$$

- Optimal portfolio

$$1-\theta = \frac{(a-\iota)/q}{\gamma\tilde{\sigma}^2} + \frac{\mu^M}{\gamma\tilde{\sigma}^2}$$

$$E[dr_t^{K,\tilde{\iota}}]/dt = \frac{a-\iota_t}{q_t} + \Phi(\iota_t) - \delta + \mu_t^q + \sigma\sigma_t^q \qquad = r_t^f + \varsigma_t(\sigma+\sigma_t^q) + \tilde{\varsigma}_t\tilde{\sigma}$$

$$E[dr_t^M]/dt \; = \qquad \Phi(\iota_t) - \delta + \mu_t^p + \sigma\sigma_t^p - \mu^M = r_t^f + \varsigma_t(\sigma+\sigma_t^p)$$

In Steady State constant q, p

$$\frac{a-\iota_{\cancel{t}}}{q_{\cancel{t}}} + \cancel{\mu_t^q - \mu_t^p} + \cancel{\sigma(\sigma_t^q - \sigma_t^p)} + \mu^M = \cancel{\varsigma_t(\sigma_t^q - \sigma_t^p)} + \tilde{\varsigma}_{\cancel{t}}\tilde{\sigma}$$

yields $1-\theta = ..$

Price of Risk: $\varsigma_{\cancel{t}} = \cancel{-\sigma_t^v + \sigma_t^{p+q}} + \gamma\sigma,$ $\quad \tilde{\varsigma}_{\cancel{t}} = \gamma\tilde{\sigma}^n_{\cancel{t}} = \gamma(1-\theta_{\cancel{t}})\tilde{\sigma}$

One Sector Model with Money

1b. Optimal Choices

- Optimal investment rate

$$\kappa\iota_t = q_t - 1$$

- Optimal portfolio

$$1-\theta = \frac{(a-\iota)/q}{\gamma\tilde{\sigma}^2} + \frac{\mu^M}{\gamma\tilde{\sigma}^2}$$

$$E[dr_t^{K,\tilde{i}}]/dt = \frac{a-\iota_t}{q_t} + \Phi(\iota_t) - \delta + \mu_t^q + \sigma\sigma_t^q \quad = r_t^f + \varsigma_t(\sigma + \sigma_t^q) + \tilde{\varsigma}_t\tilde{\sigma}$$

$$E[dr_t^M]/dt = \Phi(\iota_t) - \delta + \mu_t^p + \sigma\sigma_t^p - \mu^M = r_t^f + \varsigma_t(\sigma + \sigma_t^p)$$

In Steady State constant q, p

$$\frac{a-\iota}{q} + \mu^M = \tilde{\varsigma}_t\tilde{\sigma}$$

Price of Risk: $\varsigma = \gamma\sigma, \quad \tilde{\varsigma} = \gamma\tilde{\sigma}^n = \gamma(1-\theta)\tilde{\sigma}$

yields $1-\theta = ..$

One Sector Model with Money

1b. Optimal Choices

- Optimal investment rate

$$\kappa\iota_t = q_t - 1$$

- Optimal portfolio

$$1-\theta = \frac{(a-\iota)/q}{\gamma\tilde{\sigma}^2} + \frac{\mu^M}{\gamma\tilde{\sigma}^2}$$

$$E[dr_t^{K,\tilde{\imath}}]/dt = \frac{a-\iota_t}{q_t} + \Phi(\iota_t) - \delta + \mu_t^q + \sigma\sigma_t^q = r_t^f + \varsigma_t(\sigma + \sigma_t^q) + \tilde{\varsigma}_t\tilde{\sigma}$$

$$E[dr_t^M]/dt = \Phi(\iota_t) - \delta + \mu_t^p + \sigma\sigma_t^p - \mu^M = r_t^f + \varsigma_t(\sigma + \sigma_t^p)$$

In Steady State constant q, p

$$\frac{a-\iota}{q} + \mu^M = \tilde{\varsigma}_t\tilde{\sigma}$$

Price of Risk: $\varsigma = \gamma\sigma, \quad \tilde{\varsigma} = \gamma\tilde{\sigma}^n = \gamma(1-\theta)\tilde{\sigma}$

yields $1-\theta = ..$

Risk-free rate: $r^f = \Phi(\iota) - \delta - \mu^M - \gamma\sigma^2$

One Sector Model with Money

1b. Optimal Choices

- Optimal investment rate

$$\kappa\iota_t = q_t - 1$$

- Optimal portfolio

$$1-\theta = \frac{(a-\iota)/q}{\gamma\tilde{\sigma}^2} + \frac{\mu^M}{\gamma\tilde{\sigma}^2}$$

Poll 18: r^f is

a) Risk-free rate

b) Shadow risk-free rate

c) Differs across individuals

$$E[dr_t^{K,\tilde{\iota}}]/dt = \frac{a-\iota_t}{q_t} + \Phi(\iota_t) - \delta + \mu_t^q + \sigma\sigma_t^q = r_t^f + \varsigma_t(\sigma+\sigma_t^q) + \tilde{\varsigma}_t\tilde{\sigma}$$

$$E[dr_t^M]/dt = \Phi(\iota_t) - \delta + \mu_t^p + \sigma\sigma_t^p - \mu^M = r_t^f + \varsigma_t(\sigma+\sigma_t^p)$$

In Steady State constant q, p

$$\frac{a-\iota}{q} + \mu^M = \tilde{\varsigma}_t\tilde{\sigma}$$

Price of Risk: $\varsigma = \gamma\sigma, \quad \tilde{\varsigma} = \gamma\tilde{\sigma}^n = \gamma(1-\theta)\tilde{\sigma}$

Risk-free rate: $r^f = \Phi(\iota) - \delta - \mu^M - \gamma\sigma^2$

yields $1-\theta = ..$

One Sector Model with Money

1b. Optimal Choices

- Optimal investment rate

$$\kappa\iota_t = q_t - 1$$

- Optimal portfolio

$$1-\theta = \frac{(a-\iota)/q}{\gamma\tilde{\sigma}^2} + \frac{\mu^M}{\gamma\tilde{\sigma}^2}$$

Poll 18: why does real r^f decline with μ^M

a) Because investment rate ι changes

b) Insurance via money becomes more costly

c) Prices are not sticky, money is neutral, and hence the real rate should not be affected

$$E[dr_t^{K,\tilde{\iota}}]/dt = \frac{a-\iota_t}{q_t} + \Phi(\iota_t) - \delta + \mu_t^q + \sigma\sigma_t^q \quad = r_t^f + \varsigma_t(\sigma + \sigma_t^q) + \tilde{\varsigma}_t\tilde{\sigma}$$

$$E[dr_t^M]/dt = \Phi(\iota_t) - \delta + \mu_t^p + \sigma\sigma_t^p - \mu^M = r_t^f + \varsigma_t(\sigma + \sigma_t^p)$$

In Steady State constant q, p

$$\frac{a-\iota}{q} + \mu^M = \tilde{\varsigma}_t\tilde{\sigma}$$

Price of Risk: $\varsigma = \gamma\sigma, \quad \tilde{\varsigma} = \gamma\tilde{\sigma}^n = \gamma(1-\theta)\tilde{\sigma}$

yields $1-\theta = ..$

Risk-free rate: $r^f = \Phi(\iota) - \delta - \mu^M - \gamma\sigma^2$

One Sector Model with Money

1b. Optimal Choices

- Optimal investment rate

$$\kappa\iota_t = q_t - 1$$

- Optimal portfolio

$$1-\theta = \frac{(a-\iota)/q}{\gamma\tilde{\sigma}^2} + \frac{\mu^M}{\gamma\tilde{\sigma}^2}$$

Poll 17: the real r^f does not depend on $\tilde{\sigma}$

a) Because determined by growth rate of K

b) Because it is a shadow price/rate

c) Because return on money $E[dr_t^M]/dt$ doesn't

$$E[dr_t^{K,\tilde{\iota}}]/dt = \frac{a-\iota_t}{q_t} + \Phi(\iota_t) - \delta + \mu_t^q + \sigma\sigma_t^q \quad = r_t^f + \varsigma_t(\sigma+\sigma_t^q) + \tilde{\varsigma}_t\tilde{\sigma}$$

$$E[dr_t^M]/dt \quad = \quad \Phi(\iota_t) - \delta + \mu_t^p + \sigma\sigma_t^p - \mu^M = r_t^f + \varsigma_t(\sigma+\sigma_t^p)$$

In Steady State
constant q, p

$$\frac{a-\iota}{q} + \mu^M = \tilde{\varsigma}_t\tilde{\sigma}$$

Price of Risk: $\varsigma = \gamma\sigma, \quad \tilde{\varsigma} = \gamma\tilde{\sigma}^n = \gamma(1-\theta)\tilde{\sigma}$

yields
$1-\theta = ..$

Risk-free rate: $r^f = \Phi(\iota) - \delta - \mu^M - \gamma\sigma^2$

One Sector Model with Money

1b. Optimal Choices

- Optimal investment rate

$$\kappa\iota_t = q_t - 1$$

- Optimal portfolio

$$1 - \theta = \frac{(a-\iota)/q}{\gamma\tilde{\sigma}^2} + \frac{\mu^M}{\gamma\tilde{\sigma}^2}$$

- Optimal consumption

$\frac{c}{n} =: \zeta$ is a constant

  - Why a constant?
    Recall $\frac{c}{n} = \rho^{1/\gamma}\omega^{1-1/\gamma}$ and investment opportunity/networth multiplier is constant over time in steady state

One Sector Model with Money

1b. Optimal Choices

- Optimal investment rate

$$\kappa\iota_t = q_t - 1$$

- Optimal portfolio

$$1 - \theta = \frac{(a-\iota)/q}{\gamma\tilde{\sigma}^2} + \frac{\mu^M}{\gamma\tilde{\sigma}^2}$$

- Optimal consumption

$$\frac{c}{n} =: \zeta \Rightarrow C = \zeta(p+q)K_t$$

4. Market Clearing

$$= 1 - \vartheta = \frac{qK_t}{(p+q)K_t}$$

$$= (a - \iota)K_t$$

One Sector Model with Money

1b. Optimal Choices

- Optimal investment rate

$$\kappa\iota_t = q_t - 1$$

- Optimal portfolio

$$1-\theta = \frac{(a-\iota)/q}{\gamma\tilde{\sigma}^2} + \frac{\mu^M}{\gamma\tilde{\sigma}^2}$$

- Optimal consumption

$$\frac{c}{n} =: \zeta \Rightarrow C = \zeta \underbrace{\frac{(p+q)}{q}}_{1/(1-\vartheta)} \cancel{K}_t$$

4. Market Clearing

$$= 1 - \vartheta = \frac{qK_t}{(p+q)K_t}$$

$$= \frac{(a-\iota)}{q}\cancel{K}_t$$

One Sector Model with Money

1b. Optimal Choices

- Optimal investment rate

$$\kappa\iota_t = q_t - 1$$

- Optimal portfolio

$$1-\theta = \frac{(a-\iota)/q}{\gamma\tilde{\sigma}^2} + \frac{\mu^M}{\gamma\tilde{\sigma}^2}$$

- Optimal consumption

$$\frac{c}{n} =: \zeta \Rightarrow C = \zeta(p+q)$$

$$\underbrace{\frac{p+q}{q}}_{1/(1-\vartheta)}$$

4. Market Clearing

$$= 1-\vartheta = \frac{qK_t}{(p+q)K_t}$$

$$= \frac{(a-\iota)}{q}$$

$$\Rightarrow \iota = \frac{(1-\vartheta)a-\zeta}{1-\vartheta+\kappa\zeta}$$

One Sector Model with Money

1b. Optimal Choices

- Optimal investment rate

$$\kappa\iota_t = q_t - 1$$

- Optimal portfolio

$$1 - \theta = \frac{(a-\iota)/q}{\gamma\tilde{\sigma}^2} + \frac{\mu^M}{\gamma\tilde{\sigma}^2}$$

- Optimal consumption

$$\frac{c}{n} =: \zeta \Rightarrow C = \zeta\underbrace{\frac{(p+q)}{q}}_{1/(1-\vartheta)}$$

4. Market Clearing

$$= 1 - \vartheta = \frac{qK_t}{(p+q)K_t}$$

$$= \frac{(a-\iota)}{q}$$

$$\Rightarrow \iota = \frac{(1-\vartheta)a - \zeta}{1 - \vartheta + \kappa\zeta}$$

$$q = (1-\vartheta)\frac{1+\kappa a}{1-\vartheta+\kappa\zeta}$$

One Sector Model with Money

1b. Optimal Choices

4. Market Clearing

- Optimal investment rate

$$\kappa\iota_t = q_t - 1$$

Let $\hat{\mu}^M := (1-\vartheta)\mu^M$ (monotone transformation)

- Optimal portfolio

$$1-\theta = \frac{(a-\iota)/q}{\gamma\tilde{\sigma}^2} + \frac{\mu^M}{\gamma\tilde{\sigma}^2} = 1-\vartheta = \frac{qK_t}{(p+q)K_t}$$

- Optimal consumption

$$\frac{c}{n} =: \zeta \Rightarrow C = \zeta\underbrace{\frac{(p+q)}{q}}_{1/(1-\vartheta)} = \frac{(a-\iota)}{q}$$

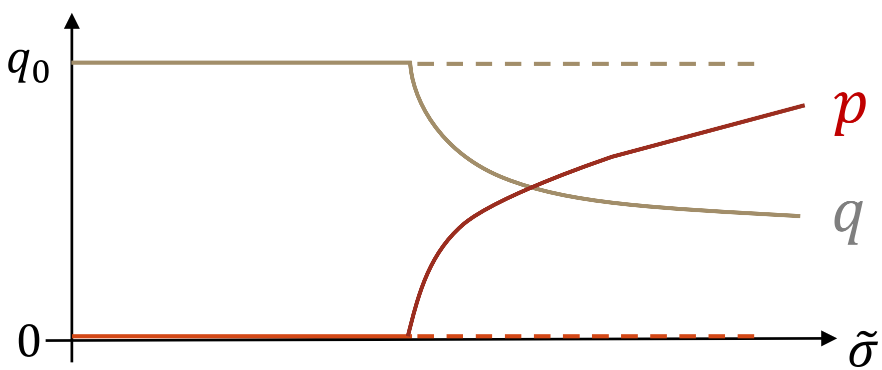
$$\Rightarrow \iota = \frac{(1-\vartheta)a-\zeta}{1-\vartheta+\kappa\zeta}$$

$$(1-\vartheta) = \sqrt{\frac{\zeta+\hat{\mu}^M}{\gamma\tilde{\sigma}^2}} = \frac{q}{q+p}$$

$$q = (1-\vartheta)\frac{1+\kappa a}{1-\vartheta+\kappa\zeta}$$

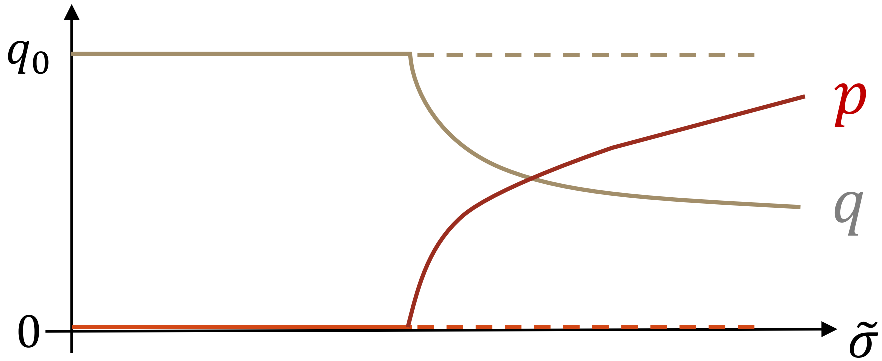
Two Stationary Equilibria

Moneyless equilibrium	Money equilibrium
$p_0 = 0$	$p = \dfrac{(1+\kappa a)\left(\sqrt{\gamma}\tilde{\sigma} - \sqrt{\zeta + \hat{\mu}^M}\right)}{\sqrt{\zeta + \hat{\mu}^M} + \kappa\sqrt{\gamma}\tilde{\sigma}\zeta}$
$q_0 = \dfrac{1+\kappa a}{1+\kappa\zeta}$	$q = \dfrac{(1+\kappa a)\sqrt{\zeta + \hat{\mu}^M}}{\sqrt{\zeta + \hat{\mu}^M} + \kappa\sqrt{\gamma}\tilde{\sigma}\zeta}$
$\iota = \dfrac{a-\zeta}{1+\kappa\zeta}$	$\iota = \dfrac{\sqrt{\zeta + \hat{\mu}^M}\,a - \sqrt{\gamma}\tilde{\sigma}\zeta}{\sqrt{\zeta + \hat{\mu}^M} + \kappa\sqrt{\gamma}\tilde{\sigma}\zeta}$



Two Stationary Equilibria

Moneyless equilibrium	Money equilibrium
$p_0 = 0$	$p = \dfrac{(1+\kappa a)\left(\sqrt{\gamma}\tilde{\sigma} - \sqrt{\zeta + \hat{\mu}^M}\right)}{\sqrt{\zeta + \hat{\mu}^M} + \kappa\sqrt{\gamma}\tilde{\sigma}\zeta}$
$q_0 = \dfrac{1+\kappa a}{1+\kappa\zeta}$	$q = \dfrac{(1+\kappa a)\sqrt{\zeta + \hat{\mu}^M}}{\sqrt{\zeta + \hat{\mu}^M} + \kappa\sqrt{\gamma}\tilde{\sigma}\zeta}$
$\iota = \dfrac{a-\zeta}{1+\kappa\zeta}$	$\iota = \dfrac{\sqrt{\zeta + \hat{\mu}^M}\,a - \sqrt{\gamma}\tilde{\sigma}\zeta}{\sqrt{\zeta + \hat{\mu}^M} + \kappa\sqrt{\gamma}\tilde{\sigma}\zeta}$



Poll 25: Why does aggregate risk σ not show up in solution

a) We had to set it to zero to solve

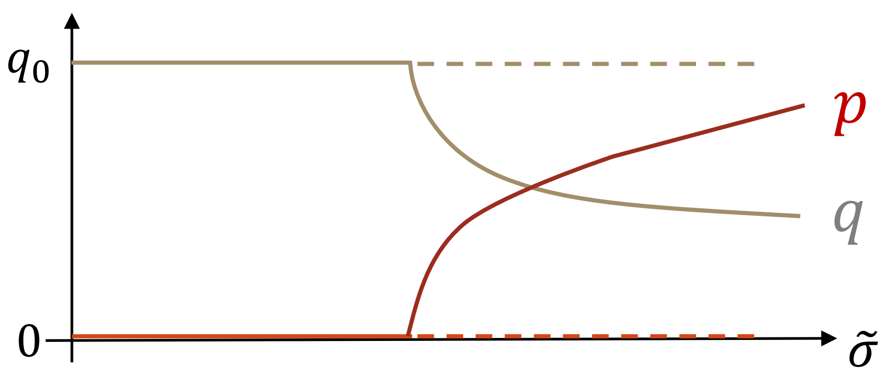
b) It scales everything in AK

c) It is hidden in ζ

d) It is hidden in $\hat{\mu}^M$

Two Stationary Equilibria

Moneyless equilibrium	Money equilibrium
$p_0 = 0$	$p = \dfrac{(1+\kappa a)\left(\sqrt{\gamma}\tilde{\sigma} - \sqrt{\zeta + \hat{\mu}^M}\right)}{\sqrt{\zeta + \hat{\mu}^M} + \kappa\sqrt{\gamma}\tilde{\sigma}\zeta}$
$q_0 = \dfrac{1+\kappa a}{1+\kappa\zeta}$	$q = \dfrac{(1+\kappa a)\sqrt{\zeta + \hat{\mu}^M}}{\sqrt{\zeta + \hat{\mu}^M} + \kappa\sqrt{\gamma}\tilde{\sigma}\zeta}$
$\iota = \dfrac{a-\zeta}{1+\kappa\zeta}$	$\iota = \dfrac{\sqrt{\zeta + \hat{\mu}^M}\,a - \sqrt{\gamma}\tilde{\sigma}\zeta}{\sqrt{\zeta + \hat{\mu}^M} + \kappa\sqrt{\gamma}\tilde{\sigma}\zeta}$



Poll 26: Why is p moving in the opposite direction to q in $\tilde{\sigma}$?

a) Flight to safety

b) With high $\tilde{\sigma}$ insurance role of money is more important

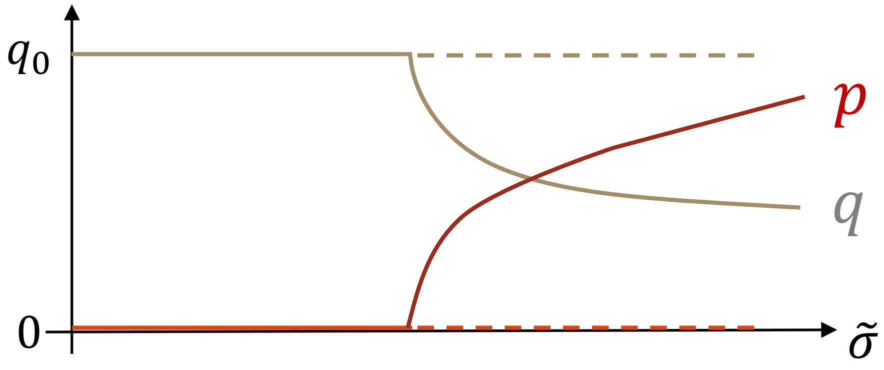
Equilibrium consumption/networth ratio ζ

- Recall $\zeta = \rho + \frac{\gamma-1}{\gamma}\left(r_t^f - \rho + \frac{\gamma\left(\sigma^2+\left((1-\vartheta)\tilde{\sigma}\right)^2\right)}{2}\right)$
  and using $\tilde{\varsigma} = \gamma(1-\vartheta)\tilde{\sigma}$
  - Since $r_t^f = \Phi(\iota_t) - \delta - \mu^M - \gamma\sigma^2$ (from previous slide above)
    and using $\iota = \frac{\sqrt{\zeta + \hat{\mu}^M} a - \sqrt{\gamma}\tilde{\sigma}\zeta}{\sqrt{\zeta + \hat{\mu}^M} + \kappa\sqrt{\gamma}\tilde{\sigma}\zeta}$
  - ... we obtain ζ
  - Of course for log utility ($\gamma = 1$), simply $\zeta = \rho$

Poll 27: precautionary savings for $\gamma > 1$
a) *Consumption-wealth ratio* ζ *decreases in* σ, *only for* $\gamma < 1$
b) *Risk* σ *affects* r^f
c) *Risk* $\tilde{\sigma}$ *affects* r^f
d) *Precautionary savings only exists with borrowing constraints*

Two Stationary Equilibria for $\gamma = 1, \mu^M = 0$

Moneyless equilibrium	Money equilibrium
$p_0 = 0$	$p = \frac{(1+\kappa a)(\tilde{\sigma} - \sqrt{\zeta})}{\sqrt{\zeta} + \kappa\tilde{\sigma}\zeta}$
$q_0 = \frac{1+\kappa a}{1+\kappa\zeta}$	$q = \frac{(1+\kappa a)\sqrt{\zeta}}{\sqrt{\zeta} + \kappa\tilde{\sigma}\zeta}$
$\iota = \frac{a-\zeta}{1+\kappa\zeta}$	$\iota = \frac{\sqrt{\zeta}a - \tilde{\sigma}\zeta}{\sqrt{\zeta} + \kappa\tilde{\sigma}\zeta}$



where $\zeta = \rho$

Welfare

- Value function for log utility

$$V = \int_0^\infty e^{-\rho t} E[\log c_t] dt = \frac{1}{\rho}\log\rho + \int_0^\infty e^{-\rho t} E[\log n_t] dt$$

- By Ito:

$$\log n_t = \log n_0 + \int_0^t \left(\frac{dn_s}{n_s} - \frac{1}{2}\frac{d<n>_s}{n_s^2}\right)$$
$$= \log n_0 + \int_0^t \left(\mu_s^n - \frac{1}{2}(\sigma_s^n)^2 - \frac{1}{2}(\tilde{\sigma}_s^n)^2\right) ds + \int_0^t \sigma_s^n dZ_s + \int_0^t \tilde{\sigma}_s^n d\tilde{Z}_s$$

- $V = \frac{\log\rho}{\rho} + \frac{\log n_0}{\rho} + \int_0^\infty e^{-\rho t} \int_0^t E\left[\mu_s^n - \frac{1}{2}(\sigma_s^n)^2 - \frac{1}{2}(\tilde{\sigma}_s^n)^2\right] ds\, dt$
- in steady state $\mu_s^n = \mu^n = \Phi(\iota) - \delta$, $\sigma_s^n = \sigma^n = \sigma$, $\tilde{\sigma}_s^n = \tilde{\sigma}^n = (1-\vartheta)\tilde{\sigma}$
- $\int_0^t E[\dots] ds = \left(\mu_s^n - \frac{1}{2}(\sigma_s^n)^2 - \frac{1}{2}(\tilde{\sigma}_s^n)^2\right) t = \left(\Phi(\iota) - \delta - \frac{1}{2}\sigma^2 - \frac{1}{2}(1-\vartheta)^2\tilde{\sigma}^2\right) t$

- Hence, $\int_0^\infty e^{-\rho t} \int_0^t E[\dots] ds\, dt = \int_0^\infty e^{-\rho t} \left(\Phi(\iota) - \delta - \frac{1}{2}\sigma^2 - \frac{1}{2}(1-\vartheta)^2\tilde{\sigma}^2\right) t\, dt$
- $= \frac{1}{\rho}\int_0^\infty e^{-\rho t} dt \left(\Phi(\iota) - \delta - \frac{1}{2}\sigma^2 - \frac{1}{2}(1-\vartheta)^2\tilde{\sigma}^2\right)$ (integration by parts)

$$= \frac{1}{\rho^2}\left(\Phi(\iota) - \delta - \frac{1}{2}\sigma^2 - \frac{1}{2}(1-\vartheta)^2\tilde{\sigma}^2\right)$$

Welfare

- Value function
- $$V = \underbrace{\frac{\log\rho}{\rho} - \frac{\delta + \frac{1}{2}\sigma^2}{\rho^2} + \frac{\log K_0}{\rho}}_{\substack{V_0 := \\ \text{(does not depend on } \hat{\mu}^M\text{)}}} + \underbrace{\frac{\log(p+q)}{\rho}}_{\substack{\text{Effect of } \hat{\mu}^M \text{ on} \\ \text{total (initial) wealth}}} + \underbrace{\frac{\Phi(\iota) - \frac{1}{2}(1-\vartheta)^2\tilde{\sigma}^2}{\rho^2}}_{\text{Growth-risk trade-off}}$$
- Plug in model solution for $p+q, \Phi(\iota)$, and ϑ

$$\boxed{V = V_0 + \frac{1}{\rho}\left(\frac{1}{\kappa}\log\frac{(1+\kappa a)\tilde{\sigma}}{\kappa\rho\tilde{\sigma} + \sqrt{\rho + \hat{\mu}^M}}\right) + \frac{1}{\rho^2}\left(\frac{1}{\kappa}\log\frac{(1+\kappa a)\sqrt{\rho + \hat{\mu}^M}}{\kappa\rho\tilde{\sigma} + \sqrt{\rho + \hat{\mu}^M}}\right) - \frac{1}{2}(\rho + \mu^M)}$$

Closed form!
(up to $\hat{\mu}^M$-transformation)

Optimal Inflation Rate

- Money growth μ^M affects
  - Shadow risk-free rate
  - (Steady state) inflation in two ways

$$\pi = \mu^M + \mu^{Mi} - \underbrace{\left(\Phi\big(\iota(\mu^M)\big) - \delta\right)}_{g}$$

- Proposition:
  - For sufficiently large $\tilde{\sigma}$ and $\kappa < \infty$ welfare maximizing $\mu^{M^*} > 0$.
    - Laissez-faire Market outcome is not even constrained Pareto efficient
    - Economic growth rate g is also higher
  - Growth maximizing $\mu^{g*} \geq \mu^{M*}$, s.t. $p^{g*} = 0$, Tobin (1965)

- Corollary: No super-neutrality of money
  - μ^{Mi}: Super-neutrality only w.r.t. part of money growth rate that is used to pay interest on money
  - μ^M: Nominal money growth rate affects real economic growth by distorting portfolio choice if $\kappa < \infty$
    - No price/wage rigidity, no monopolistic competition

Optimal Inflation Rate

- Pecuniary Externalities
  - Individual agent takes prices, including interest rate as given

  - Tilt portfolio towards (physical capital)
  - $\Rightarrow q$ rises
    - Investment rate ι rises, growth rate is higher increases r^M
    - Idiosyncratic risk increases reduces welfare
      - After negative shock, replacing lost capital is cheaper

      - due to "capital shocks"
      - Not with "cash flow shock" (in consumption units) as in Brunnermeier & Sannikov (2016) AER P&P

Optimal Inflation Rate: Emerging Markets

- Proposition: (comparative static)
  μ^{M*} and optimal inflation target
  - does not depend on depreciation rate δ, but inflation does
  - is strictly increasing in idiosyncratic risk $\tilde{\sigma}$
    "Emerging markets should have higher inflation target"

In sum..

- What should the (long-run) optimal inflation rate be?
  - Competitive market outcome is constrained Pareto inefficient.
  - Inflation is Pigouvian & internalizes pecuniary externality!
    - HH take real interest rate as given, but
    - Portfolio choice affects economic growth and real interest rate

- What role do financial frictions play?
  - incomplete markets ⇒ no superneutrality of money
    - No price/wage rigidity needed

- Emerging markets, with less developed financial markets, should have higher inflation rate/target
  - Higher idiosyncratic risk ⇒ higher pecuniary externality

The 4 Roles of Money

- Store of value
  - "I Theory of Money without I"
    Less risky than other "capital" – no idiosyncratic risk
  - Fiscal theory of the price level
- Medium of exchange
  - Overcome double-coincidence
    of wants problem
- Unit of account
- Record keeping device
  - Virtual ledger

Fiscal Theory of the Price Level

- Money in a broad sense (includes government debt)
  - store of value emphasis!
- Suppose one can pay taxes with money (fiscal backing)
  - HH can pay with money instead of real goods
- Central bank might "print money" to pay expenditures and dilute real value of government debt
- FTPL equation: What is the real value of government debt
  - Like asset pricing equation (in discrete time)

$$\frac{M_t + B_t}{\wp_t} = E\left[\sum_{\tau=t}^{\infty} \frac{\xi_\tau}{\xi_t} s_\tau K_\tau\right]$$

  - B_t all nominal government debt (long-term government bond $B_t = 0$
  - $s_\tau K_\tau$ is primary surplus
    (tax revenue minus government expenditure (without interest payments))
  - $\wp_t = M_t / p_t K_t$ price level (inverse of "value of money")

FTPL Equation

- Fiscal budget   with $B_t = 0\ \forall t$

$$\frac{p_t K_t}{M_t} \mu^M M_t dt + \tau a K_t = g K_t$$

  - $p_t K_t \mu^M dt$ seignorage   (Recall μ^M is money growth rate that excludes the part used to pay interest)
  - τ tax minus transfers per unit of output
  - g government expenditures per unit of K_t (totally wasted)
  - If $g = 0$, then $\tau a K_t$ is primary surplus, denoted by $s K_t$

- FTPL equation: $\frac{M_t + B_t}{\wp_t} = E\left[\sum_{\tau=t}^{\infty} \frac{\xi_\tau}{\xi_t} s_\tau K_\tau\right]$

$$p_t K_t = \lim_{T\to\infty} \int_t^T E_t\left[\frac{\xi_\tau}{\xi_t} s_\tau K_\tau\right] d\tau + \underbrace{\lim_{T\to\infty} E_t\left[\frac{\xi_T}{\xi_t} p_T K_T\right]}_{\text{Bubble}}$$

FTPL and Money Bubbles

- FTPL equation:

$$p_t K_t = \lim_{T\to\infty} \int_t^T E_t\left[\frac{\xi_\tau}{\xi_t} s_\tau K_\tau\right] d\tau + \underbrace{\lim_{T\to\infty} E_t\left[\frac{\xi_T}{\xi_t} p_T K_T\right]}_{\text{Bubble}}$$

- w/o aggregate risk, $\sigma = 0$:

$\Rightarrow \frac{\xi_\tau}{\xi_t} = e^{-r^f(\tau - t)}$ and $r^f = \underbrace{(\Phi(\iota) - \delta)}_{=g} - \mu^M$ growth rate, not g

  - If $\mu^M = 0$ $\Rightarrow s = 0$ $r^f = g$, bubble can exist

FTPL and Money Bubbles

- FTPL equation:

$$p_t K_t = \lim_{T\to\infty} \int_t^T E_t\left[\frac{\xi_\tau}{\xi_t} s_\tau K_\tau\right] d\tau + \underbrace{\lim_{T\to\infty} E_t\left[\frac{\xi_T}{\xi_t} p_T K_T\right]}_{\text{Bubble}}$$

  - w/o aggregate risk, $\sigma = 0$:

    $\Rightarrow \frac{\xi_\tau}{\xi_t} = e^{-r^f(\tau-t)}$ and $r^f = \underbrace{(\Phi(\iota) - \delta)}_{=g} - \mu^M$ growth rate, not $\cancel{g}$

    - If $\mu^M = 0$ $\Rightarrow s = 0$ $r^f = g$, bubble can exist

Poll 39: What pins down the size of the money bubble?
a) For $r^f = g$ bubble can take on any size
b) Asset pricing/Euler equation
c) Output good market clearing equation

FTPL and Money Bubbles

- FTPL equation:

$$p_t K_t = \lim_{T\to\infty}\int_t^T E_t\left[\frac{\xi_\tau}{\xi_t}s_\tau K_\tau\right]d\tau + \underbrace{\lim_{T\to\infty} E_t\left[\frac{\xi_T}{\xi_t}p_T K_T\right]}_{\text{Bubble}}$$

  - w/o aggregate risk, $\sigma = 0$:

    $\Rightarrow \frac{\xi_\tau}{\xi_t} = e^{-r^f(\tau-t)}$ and $r^f = \underbrace{(\Phi(\iota)-\delta)}_{=g} - \mu^M$ growth rate, not g

    - If $\mu^M = 0$ $\Rightarrow s = 0$ $r^f = g$, bubble can exist
    - If $\mu^M > 0 \Rightarrow$ transfers $\Rightarrow s < 0$ $r^f < g$, fundamtl<0, bubble>0
    - If $\mu^M < 0 \Rightarrow$ taxes $\Rightarrow s > 0$ $r^f > g$, fundamental only

  - w/ aggregate risk similar
    - homework

FTPL and Money Bubbles

- FTPL equation:

$$p_t K_t = \lim_{T\to\infty} \int_t^T E_t\left[\frac{\xi_\tau}{\xi_t} s_\tau K_\tau\right] d\tau + \underbrace{\lim_{T\to\infty} E_t\left[\frac{\xi_T}{\xi_t} p_T K_T\right]}_{\text{Bubble}}$$

- w/o aggregate risk, $\sigma = 0$:

$\Rightarrow \frac{\xi_\tau}{\xi_t} = e^{-r^f(\tau-t)}$ and $r^f = \underbrace{(\Phi(\iota) - \delta)}_{=g} - \mu^M$ growth rate, not $\mathfrak{g}$

  - If $\mu^M = 0$ $\Rightarrow s = 0$ $r^f = g$, bubble can exist
  - If $\mu^M > 0 \Rightarrow$ transfers $\Rightarrow s < 0$ $r^f < g$, fundamtl<0, bubble>0
  - If $\mu^M < 0 \Rightarrow$ taxes $\Rightarrow s > 0$ $r^f > g$, fundamental only

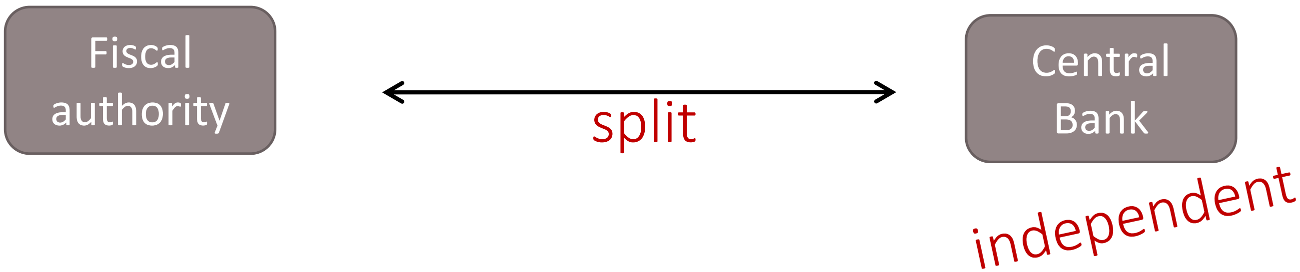
- w/ aggregate risk similar
  - homework

Poll 41: Suppose gov. $\mathfrak{g} > 0$ (and wasted)
a) Analysis doesn't change
b) Only goods market clearing changes
c) SDF ξ_t is different, and so is r^f

FTPL: Resolving Equilibrium Multiplicity

- Equilibria
  - Moneyless steady state with $p^0 = 0$
  - Price p_t converges over time to zero (hyperinflation)
- With $\varepsilon > 0$ fiscal backing $p_t > \varepsilon$,
  these equilibria are eliminated
  $\Rightarrow$ only steady state money equilibrium remains
- Off equilibrium fiscal backing suffices
  to rule out moneyless and hyperinflation equilibria
  - If after a hypothetical jump into the moneyless equilibrium,
    one can pay (a small amount) of taxes with money.
    Hence, money is not worthless and
    the moneyless equilibrium does not exist.

FTPL: Who controls inflation?



- Monetary dominance
  - Fiscal authority is forced to adjust budget deficits
- Fiscal dominance
  - Inability or unwillingness of fiscal authorities to control long-run expenditure/GDP ratio
  - Limits monetary authority to raise interest rates
- 0/1 Dominance vs. battle: "dynamic game of chicken"

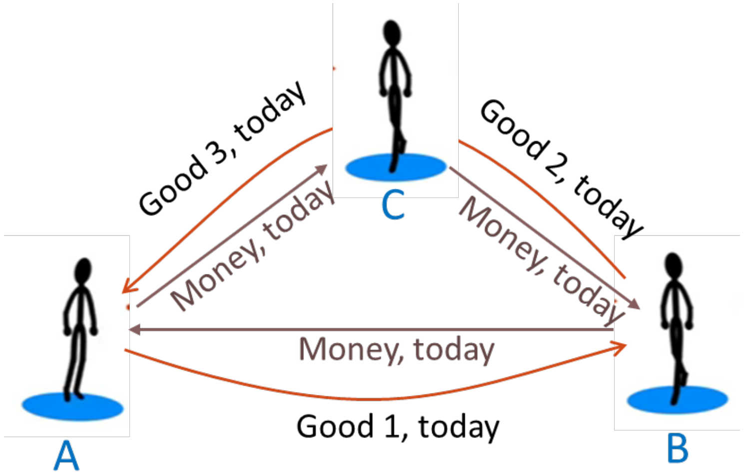
See YouTube video 4, minute 4:15

The 4 Roles of Money

- Store of value
  - "I Theory of Money without I"
    Less risky than other "capital" – no idiosyncratic risk
  - Fiscal theory of the price level
- Medium of exchange
  - Overcome double-coincidence
    of wants problem
- Unit of account
- Record keeping device
  - Virtual ledger

Medium of Exchange – Transaction Role

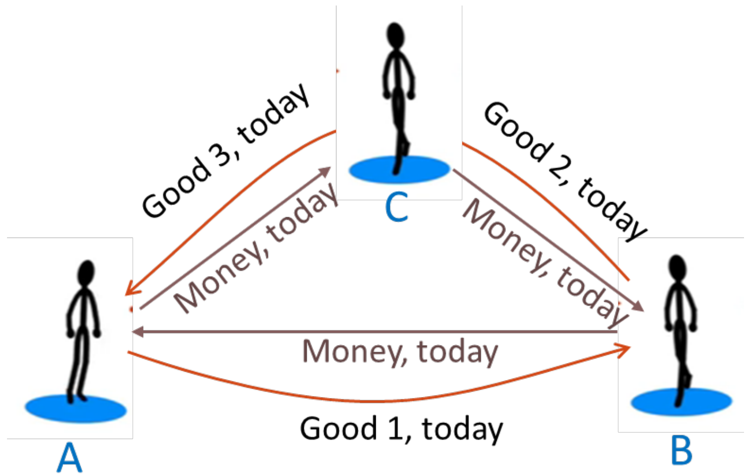
- Overcome double-coincidence of wants



- Quantity equation: $\boxed{\wp_t T_t = \nu M_t}$

Medium of Exchange – Transaction Role

- Overcome double-coincidence of wants

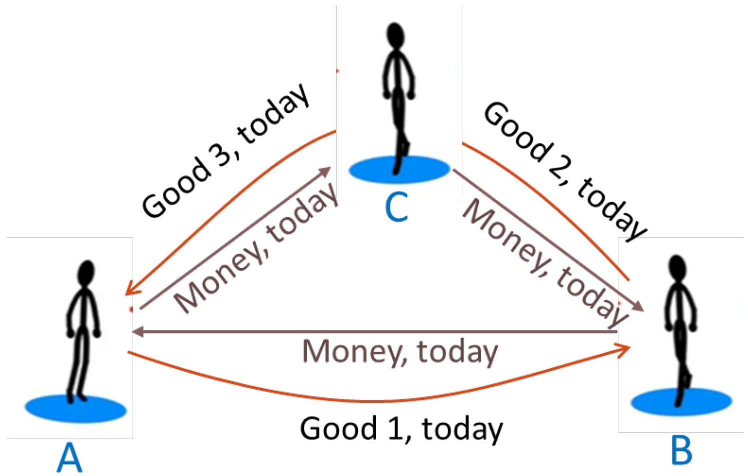


- Quantity equation: $\boxed{\wp_t T = \nu M_t}$
  - ν (nu) is velocity (Monetarism: ν exogenous, constant)
  - T transactions
    - Consumption C
    - New investment production ιK
    - Transaction of physical capital $d\Delta^k$
    - Transaction of financial claims $\boldsymbol{d\theta^{j\notin M}}$

$$\left.\begin{array}{l} C \\ \iota K \end{array}\right\} Y$$

Medium of Exchange – Transaction Role

- Overcome double-coincidence of wants



- Quantity equation: $\boxed{\wp_t T_t = \nu M_t}$
  - ν (nu) is velocity (Monetarism: ν exogenous, constant)
  - T transactions
    - Consumption C
    - New investment production ιK produce own machines
    - Transaction of physical capital $d\Delta^k$ infinite velocity
    - Transaction of financial claims $\boldsymbol{d\theta}^{j\notin M}$ infinite velocity

Models of Medium of Exchange

- Reduced form models
  - Cash in advance $T_t = \nu \frac{M_t}{\wp_t}$ Only asset with money-like features

    $c_t \leq \sum_{j \in M} \nu^j \theta^j n_t$
  - Shopping time models $c = (c^c, l)$
  - Money in the utility function consume money CES
    - New Keynesian Models
    - No satiation point
  - New Monetary Economics

For general setting:
see Brunnermeier-Niepelt 2018

Cash in Advance

- Liquidity/cash in advance constraint
  - $c_t \leq \sum_{j \in M} \nu^j \theta^j n_t$ Lagrange multiplier $\hat{\lambda}_t$
  - Asset $j \in M$ which relaxes liquidity/CIA constraint

Money yields extra “liquidity service” (relaxes constraint)

- Price of liquid/money asset

$$p_t^{j\in M} = E_t\left[\frac{\xi_{t+\Delta}}{\xi_t}(x_{t+\Delta} + p_{t+\Delta})\right] - \hat{\lambda}_t \nu^j p_t^{j\in M}$$

$$p_t^{j\in M} = E_t\left[\frac{\xi_{t+\Delta}}{\xi_t} \underbrace{\frac{1}{1+\hat{\lambda}_t \nu^j}}_{\Lambda^j_{t+\Delta}/\Lambda^j_t :=} (x_{t+\Delta} + p_{t+\Delta})\right]$$

$$p_t^{j\in M} = \lim_{T\to\infty} E_t\left[\sum_{\tau=1}^{(T-t)/\Delta} \frac{\xi_{t+\tau\Delta}}{\xi_t} \frac{\Lambda^j{}_{t+\tau\Delta}}{\Lambda^j{}_t} x_{t+\tau\Delta}\right] + \lim_{T\to\infty} E_t\left[\frac{\xi_T}{\xi_t}\frac{\Lambda^j{}_T}{\Lambda^j{}_t} p_T\right]$$

As if SDF is multiplied by “liquidity multiplier” (Brunnermeier Niepelt)

Cash in Advance

- Liquidity/cash in advance constraint
  - $c_t \leq \sum_{j\in M} \nu^j \theta^j n_t$ Lagrange multiplier $\hat{\lambda}_t$
  - Asset $j \in M$ which relaxes liquidity/CIA constraint

$$p_t^{j\in M} = \lim_{T\to\infty} E_t\left[\int_t^T \frac{\xi_\tau}{\xi_t}\frac{\Lambda^j{}_\tau}{\Lambda^j{}_t} x_\tau d\tau\right] + \underbrace{\lim_{T\to\infty} E_t\left[\frac{\xi_T}{\xi_t}\frac{\Lambda^j{}_T}{\Lambda^j{}_t} p_T\right]}_{\text{Bubble}}$$

  - "Money bubble" easier to obtain due to liquidity service
    - Condition absent aggregate risk: $r^M < g$ easier to obtain since $r^M < r^f$
- HJB approach (Problem Set #3)

$$\mu_t^{r,j} = \underbrace{r_t^f}_{\text{(Shadow) risk-free rate of illiquid asset}} + \varsigma_t \sigma_t^{r,j} + \tilde{\varsigma}_t \tilde{\sigma}^{r,j} - \lambda_t \nu^j$$

where $\lambda_t = \hat{\lambda}_t / V'(n_t)$

Add Cash in Advance to BruSan Model

- Return on money
  - Store of value – as before
  - Liquidity service

$$\frac{E[dr_t^M]}{dt} = \Phi(\iota_t) - \delta + \mu_t^p + \sigma\sigma_t^p - \mu^M = r_t^f + \varsigma_t(\sigma + \sigma_t^p) - \lambda_t \nu^M$$

- In steady state

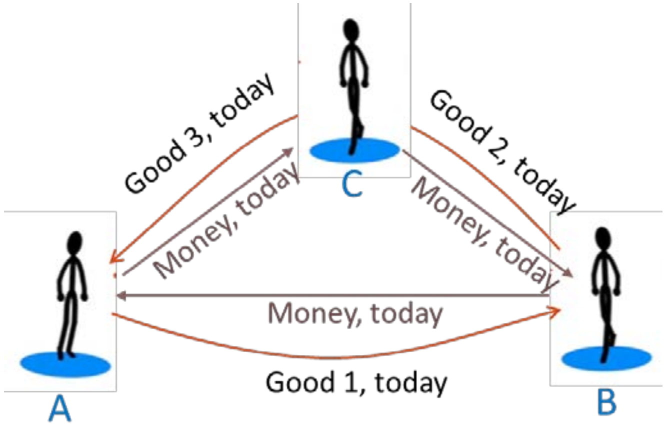
$$\Phi(\iota) - \delta - \underbrace{(\mu^M - \lambda\nu^M)}_{\breve{\mu}^M :=} = r^f + \varsigma\sigma$$

- Solving the model as before …
  - By simply replace μ^M with $\mu^M - \lambda_t \nu_t^M$
  - Special case: $\breve{\mu}^M = 0$, i.e. $\mu^M = \lambda\nu^M$, $\gamma = 1 \Rightarrow$ explicit solution as fcn of ζ
    - Same q and p as a function of ζ,
    - But $\zeta \neq \rho$ if CIA constraint binds in steady state
      - Check:
        1. Assume it binds, i.e. $\zeta = \nu\vartheta$
        2. Recall from slide 21 for $\hat{\mu}^M = 0$ and $\gamma = 1$, $\vartheta = \frac{\tilde{\sigma} - \sqrt{\zeta}}{\tilde{\sigma}}$
        3. Equate 1. and 2. to obtain quadratic solution for ζ
           1. If $< \rho$, then solution equals ζ
           2. If $> \rho$, then $\zeta = \rho$ and hence CIA doesn't bind, $\lambda = 0$, above solution
- "Occasionally" binding CIA constraint (outside of steady state) since for sufficiently high $\tilde{\sigma}$ agents hold money as store of value (insurance motive) $\Rightarrow \lambda_t = 0$
- Money in the utility function is as if constraint always binds, see DiTella (2018)

The 4 Roles of Money

- Store of value
  - "I Theory of Money without I"
    Less risky than other "capital" – no idiosyncratic risk
  - Fiscal theory of the price level

- Medium of exchange
  - Overcome double-coincidence of wants problem



- Unit of account
  - Benchmark price to have agreed upon/fewer relative prices
  - Price stickiness in New Keynesian Models

- Record keeping device
  - Virtual ledger

Extra Slides