Comparative Valuation Dynamics in Models with Financing Frictions

II. Models with Frictions

Today's Lecture:
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Recap of Lars' Lecture

1. Continuous-time recursive utility (Duffie-Epstein-Zin)
2. Model with production and adjustment costs
3. "Shock Elasticities" as model diagnostics
4. Illustration of how RRA and IES affect shock-exposure and shock-price elasticities, with and without production

Today's Plan

1. Add heterogeneity and frictions to the frictionless continuous-time model
    - Heterogeneity in productivity, preferences, frictions
    - Theoretical solution method

2. Numerical solution method
    - PDEs solved using finite-differences
    - Computational considerations

Part I

Model

Notation differences from Markus

Variable	Markus	Us
Expert capital share	ψ	κ
Risk price (SDF) loading on shocks	ς	π
Capital price	q	Q
Investment opportunities	ω	$\exp(\xi)$
Discount rate	ρ	δ
Value function	V	$\hat{U} = U^{1-\gamma}$
SDF	ξ	S
Consumption-wealth ratio	ζ	$c^* := C/N$
Brownian motions	dZ	dB

Preferences

Recursive utility with small time-step ϵ,

$$U_t = \left[(1 - \exp(-\delta\epsilon))(C_t)^{1-\rho} + \exp(-\delta\epsilon)\mathcal{R}_t(U_{t+\epsilon})^{1-\rho}\right]^{\frac{1}{1-\rho}}$$

where

$$\mathcal{R}_t(U_{t+\epsilon}) = \mathbb{E}\left[U_{t+\epsilon}^{1-\gamma} \mid \mathcal{F}_t\right]^{\frac{1}{1-\gamma}}$$

- δ – rate of time preference
- $1/\rho$ – intertemporal elasticity of substitution (IES)
- γ – relative risk aversion (RRA)

Experts and households can have different preferences:

$$\delta_e \text{ vs } \delta_h \quad \rho_e \text{ vs } \rho_h \quad \gamma_e \text{ vs } \gamma_h$$

Technology

Agent $j \in [0, 1]$ within agent group $g \in \{e, h\}$ (experts versus households) holds capital $K_{g,t}^{(j)}$.

Production with differential productivity:

$$a_g K_{g,t}^{(j)} \quad a_e \geq a_h$$

Capital evolution:

$$\frac{dK_{g,t}^{(j)}}{K_{g,t}^{(j)}} = \Big[\underbrace{\Phi(I_{g,t}^{(j)}/K_{g,t}^{(j)})}_{\text{endogenous growth}} + \underbrace{Z_t - \alpha_k}_{\text{exogenous growth}}\Big]dt + \underbrace{\sqrt{V_t}\sigma_k \cdot dB_t}_{\text{aggregate shocks}} + \underbrace{\sqrt{\tilde{V}_t}\tilde{\sigma}_k d\tilde{B}_t^{(j)}}_{\text{idiosyncratic shocks}}$$

note: $\int_0^1 K_{g,t}^{(j)} d\tilde{B}_t^{(j)} dj = 0$

Exogenous states

$$\frac{dK_{g,t}^{(j)}}{K_{g,t}^{(j)}} = \Big[\underbrace{\Phi(I_{g,t}^{(j)}/K_{g,t}^{(j)})}_{\text{endogenous growth}} + \underbrace{Z_t - \alpha_k}_{\text{exogenous growth}}\Big]dt + \underbrace{\sqrt{V_t}\sigma_k \cdot dB_t}_{\text{aggregate shocks}} + \underbrace{\sqrt{\tilde{V}_t}\tilde{\sigma}_k d\tilde{B}_t^{(j)}}_{\text{idiosyncratic shocks}}$$

where

(exogenous growth) $\quad dZ_t = -\lambda_z Z_t dt + \sqrt{V_t}\sigma_z \cdot dB_t$

(aggregate variance) $\quad dV_t = -\lambda_v (V_t - 1)dt + \sqrt{V_t}\sigma_v \cdot dB_t$

(idiosyncratic variance) $\quad d\tilde{V}_t = -\lambda_{\tilde{v}} (\tilde{V}_t - 1)dt + \sqrt{\tilde{V}_t}\sigma_{\tilde{v}} \cdot dB_t$

Financial markets and constraints

- Frictionless capital market, with single price Q_t
- Frictionless short-term risk-free debt market, with return r_t

  SDF drifts: $\frac{1}{dt}\mathbb{E}_t[dS^{(j)}_{e,t}/S^{(j)}_{e,t}] = \frac{1}{dt}\mathbb{E}_t[dS^{(j)}_{h,t}/S^{(j)}_{h,t}] = -r_t$

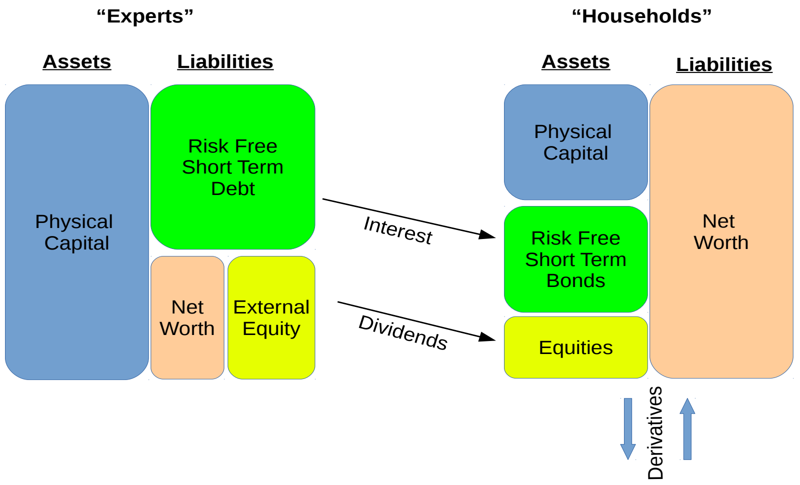
- Expert equity market (when is this a restriction?), delivering market risk-price π_t

  Skin-in-the-game constraint: Experts can issue equity, subject to retaining a fraction $\chi^{(j)}_t \geq \underline{\chi} \in [0, 1]$ of their capital risk

- Arrow-Debreu markets on the aggregate shocks dB_t, delivering market risk prices π_t

  Restriction: Only households can trade in this market, so $\frac{1}{dt}\text{Cov}_t[dS^{(j)}_{h,t}/S^{(j)}_{h,t}, dB_t] := \pi^{(j)}_{h,t} = \pi_t$ but $\frac{1}{dt}\text{Cov}_t[dS^{(j)}_{e,t}/S^{(j)}_{e,t}, dB_t] := \pi^{(j)}_{e,t} \neq \pi_t$

Balance sheets and flows of funds



Net worth evolution

$$\frac{dN_{g,t}^{(j)}}{N_{g,t}^{(j)}} = \left(\mu_{n,g,t}^{(j)} - C_{g,t}^{(j)}/N_{g,t}^{(j)}\right)dt + \sigma_{n,g,t}^{(j)} \cdot dB_t + \tilde{\sigma}_{n,g,t}^{(j)} d\tilde{B}_t^{(j)},$$

where drifts and diffusions are

$$\mu_{n,g,t}^{(j)} = r_t \quad + \quad \underbrace{\beta_{g,t}^{(j)} \left[\mu_{R,g,t} - r_t\right]}_{\text{expected excess ret-on-capital}} \quad + \quad \underbrace{\theta_{g,t}^{(j)} \cdot \pi_{h,t} + \tilde{\theta}_{g,t}^{(j)} \cdot 0}_{\text{market compensation/payments}}$$

$$\sigma_{n,g,t}^{(j)} = \beta_{g,t}^{(j)} \sigma_{R,t} + \theta_{g,t}^{(j)}$$

$$\tilde{\sigma}_{n,g,t}^{(j)} = \beta_{g,t}^{(j)} \tilde{\sigma}_{R,t} + \tilde{\theta}_{g,t}^{(j)},$$

$\beta_{g,t}^{(j)} := \frac{Q_t K_{g,t}^{(j)}}{N_{g,t}^{(j)}} \geq 0$, and trading constraints are given by

$$\theta_{h,t}^{(j)} \in \mathbb{R}^d \quad \text{and} \quad \theta_{e,t}^{(j)} \in \left\{\theta \in \mathbb{R}^d \,:\, \theta = (\chi_t^{(j)} - 1)\beta_{e,t}^{(j)} \sigma_{R,t};\ \chi_t^{(j)} \geq \underline{\chi}\right\}$$

$$\tilde{\theta}_{h,t}^{(j)} = 0 \quad \text{and} \quad \tilde{\theta}_{e,t}^{(j)} \in \left\{\theta \in \mathbb{R}^1 \,:\, \theta = (\chi_t^{(j)} - 1)\beta_{e,t}^{(j)} \tilde{\sigma}_{R,t};\ \chi_t^{(j)} \geq \underline{\chi}\right\}$$

Homogeneity property

Assumptions so far:

- Utility recursion is homogeneous of degree 1 in $(C_t, U_{t+\epsilon})$
- Budget set is homogeneous of degree 1 in N_t (i.e., net worth evolutions are linear and trading constraints are homogeneous)

Common result:

- Utility separability:

$$\underbrace{\log U_{g,t}^{(j)}}_{\substack{\text{continuation} \\ \text{utility}}} = \underbrace{\log N_{g,t}^{(j)}}_{\text{net worth}} + \underbrace{\xi_{g,t}}_{\substack{\text{investment} \\ \text{opportunities}}}$$

- All appropriately-scaled choices $I_{g,t}^{(j)}/K_{g,t}^{(j)}$, $K_{g,t}^{(j)}/N_{g,t}^{(j)}$, $C_{g,}^{(j)}/N_{g,t}^{(j)}$, $\theta_{g,t}^{(j)}$ are independent of j

Inhomogeneous examples from canonical macro models

Example 1.

$$\frac{dN^{(j)}_{g,t}}{N^{(j)}_{g,t}} = \left(\mu^{(j)}_{n,g,t} - C^{(j)}_{g,t}/N^{(j)}_{g,t} + \omega_t Y^{(j)}_{g,t}/N^{(j)}_{g,t}\right)dt + \sigma^{(j)}_{n,g,t}\cdot dB_t,$$

where idiosyncratic labor productivity follows a (stationary) diffusion

$$dY^{(j)}_{g,t} = \mu_{y,g}(Y^{(j)}_{g,t})dt + \sigma_{y,g}(Y^{(j)}_{g,t})\cdot dB_t + \underbrace{\tilde{\sigma}_{y,g}(Y^{(j)}_{g,t})d\tilde{B}^{(j)}_t}_{\text{non-tradable piece}}$$

e.g., Aiyagari-Bewley-Huggett models, recently analyzed in continuous time by Achdou-Han-Lasry-Lions-Moll

Example 2.

Think about what happens if $K^{(j)}_{g,t}$ is not tradable and production exhibits decreasing returns-to-scale.

Market clearing

- Goods market:

$$a_e \int_0^1 K_{e,t}^{(j)} dj + a_h \int_0^1 K_{h,t}^{(j)} dj = \int_0^1 C_{e,t}^{(j)} dj + \int_0^1 C_{h,t}^{(j)} dj + \int_0^1 I_{e,t}^{(j)} dj + \int_0^1 I_{h,t}^{(j)} dj$$

- Capital market:

$$K_t = \int_0^1 K_{e,t}^{(j)} dj + \int_0^1 K_{h,t}^{(j)} dj$$

- Aggregate risk markets:

$$0 = \int_0^1 \theta_{e,t}^{(j)} N_{e,t}^{(j)} dj + \int_0^1 \theta_{h,t}^{(j)} N_{h,t}^{(j)} dj$$

(recall: zero-net supply of equity and Arrow-Debreu securities)

Market clearing

Using the homogeneity properties, we can aggregate to representative expert and household.

- Goods market:

$$a_e K_{e,t} + a_h K_{h,t} = C_{e,t} + C_{h,t} + I_{e,t} + I_{h,t}$$

- Capital market:

$$K_t = K_{e,t} + K_{h,t}$$

- Aggregate risk markets:

$$0 = \theta_{e,t} N_{e,t} + \theta_{h,t} N_{h,t}$$

(recall: zero-net supply of equity and Arrow-Debreu securities)

HJB equations of representative agents

Last time, Lars showed that with recursive utility (limit as $\epsilon \to 0$):

$$0 = \sup\left\{ \underbrace{\delta \frac{(C_t/U_t)^{1-\rho} - 1}{1-\rho}}_{\text{flow payoff}} + \underbrace{\mu_{u,t} - \frac{\gamma}{2}|\sigma_{u,t}|^2}_{(1-\gamma)^{-1}\mathbb{E}_t[dU_t^{1-\gamma}]/U_t^{1-\gamma}} \right\}$$

where

$$dU_t = U_t[\mu_{u,t} dt + \sigma_{u,t} \cdot dB_t]$$

Digression: sometimes people will instead write an equivalent "integral representation" for $\hat{U}_t := U_t^{1-\gamma}$, i.e.

$$\hat{U}_t = \mathbb{E}_t\Big[\int_t^\infty f(C_s, \hat{U}_s) ds\Big], \quad \text{where} \quad f(c, \hat{u}) := \delta \frac{1-\gamma}{1-\rho}[c^{1-\rho}\hat{u}^{\frac{\rho-\gamma}{1-\gamma}} - \hat{u}].$$

HJB equations of representative agents

Using $\log U_{g,t} = \log N_{g,t} + \xi_{g,t}$, and defining dynamics

$$d\xi_{g,t} = \mu_{\xi,g,t}dt + \sigma_{\xi,g,t} \cdot dB_t,$$

we have

$$\begin{aligned}
0 = \sup\Big\{ & \delta_g \frac{[\exp(-\xi_{g,t})C_{g,t}/N_{g,t}]^{1-\rho_g} - 1}{1-\rho_g} - C_{g,t}/N_{g,t} \\
& + \mu_{n,g,t} - \frac{\gamma_g}{2}|\sigma_{n,g,t}|^2 - \frac{\gamma_g}{2}\tilde{\sigma}^2_{n,g,t} - (\gamma_g - 1)\sigma_{n,g,t} \cdot \sigma_{\xi,g,t} \\
& + \mu_{\xi,g,t} - \frac{\gamma_g - 1}{2}|\sigma_{\xi,g,t}|^2 \Big\}
\end{aligned}$$

HJB EQUATIONS OF REPRESENTATIVE AGENTS

1. Consumption-savings

$$
\begin{aligned}
0 = \sup \Big\{ & \delta_g \frac{[\exp(-\xi_{g,t}) C_{g,t}/N_{g,t}]^{1-\rho_g} - 1}{1-\rho_g} - C_{g,t}/N_{g,t} \\
& + \mu_{n,g,t} - \frac{\gamma_g}{2}|\sigma_{n,g,t}|^2 - \frac{\gamma_g}{2}\tilde{\sigma}^2_{n,g,t} - (\gamma_g - 1)\sigma_{n,g,t} \cdot \sigma_{\xi,g,t} \\
& + \mu_{\xi,g,t} - \frac{\gamma_g - 1}{2}|\sigma_{\xi,g,t}|^2 \Big\}
\end{aligned}
$$

so

$$
c^*_{g,t} := C_{g,t}/N_{g,t} = \delta_g^{1/\rho_g} \exp[(1 - 1/\rho_g)\xi_{g,t}]
$$

- $(\rho_g = 1)$ $\quad c^*_g = \delta_g$
- $(\rho_g > 1)$ $\quad c^*_g$ increasing in ξ_g
- $(\rho_g < 1)$ $\quad c^*_g$ decreasing in ξ_g

2. Portfolio-choice

$$0 = \sup \left\{ \delta_g \frac{[\exp(-\xi_{g,t}) C_{g,t}/N_{g,t}]^{1-\rho_g} - 1}{1-\rho_g} - C_{g,t}/N_{g,t} \right.$$
$$+ \mu_{n,g,t} - \frac{\gamma_g}{2} |\sigma_{n,g,t}|^2 - \frac{\gamma_g}{2} \tilde{\sigma}^2_{n,g,t} - (\gamma_g - 1) \sigma_{n,g,t} \cdot \sigma_{\xi,g,t}$$
$$\left. + \mu_{\xi,g,t} - \frac{\gamma_g - 1}{2} |\sigma_{\xi,g,t}|^2 \right\}$$

so

$$(\beta_{g,t}, \theta_{g,t}) \in \arg\max \left\{ \underbrace{\mu_{n,g,t} - \frac{\gamma_g}{2} |\sigma_{n,g,t}|^2 - \frac{\gamma_g}{2} \tilde{\sigma}^2_{n,g,t}}_{\text{mean-variance}} - \underbrace{(\gamma_g - 1) \sigma_{n,g,t} \cdot \sigma_{\xi,g,t}}_{\text{hedging-demand}} \right\}$$

2a. Expert portfolio-choice

$$(\beta_e, \theta_e) \in \arg\max \left\{ \mu_{n,e} - \frac{\gamma_e}{2} |\sigma_{n,e}|^2 - \frac{\gamma_e}{2} \tilde{\sigma}^2_{n,e} - (\gamma_e - 1) \sigma_{n,e} \cdot \sigma_{\xi,e} \right\}$$

Define expert bonus risk premium:

$$\Delta_e := \underline{\chi}^{-1} [\mu_{R,e} - r - \sigma_R \cdot \pi_h].$$

Optimality conditions:

$$[\theta_e, \tilde{\theta}_e, \chi] : \quad 0 = \min(\chi - \underline{\chi},\, \Delta_e)$$

and

$$[\beta_e] : \quad \Delta_e + \sigma_R \cdot \pi_h = \gamma_e [\sigma_R \cdot \sigma_{n,e} + \tilde{\sigma}_R \tilde{\sigma}_{n,e}] + (\gamma_e - 1) \sigma_R \cdot \sigma_{\xi,e}$$

2b. Household portfolio-choice

$$(\beta_h, \theta_h) \in \arg\max \left\{ \mu_{n,h} - \frac{\gamma_h}{2}|\sigma_{n,h}|^2 - \frac{\gamma_h}{2}\tilde{\sigma}_{n,h}^2 - (\gamma_h - 1)\sigma_{n,h} \cdot \sigma_{\xi,h} \right\}$$

Define household bonus risk premium:

$$\Delta_h := \mu_{R,h} - r - \sigma_R \cdot \pi_h.$$

Optimality conditions:

$$[\beta_h] : \quad 0 = \min(\beta_h,\, \gamma_h \tilde{\sigma}_R^2 \beta_h - \Delta_h)$$

and

$$[\theta_h] : \quad \pi_h = \gamma_h \sigma_{n,h} + (\gamma_h - 1)\sigma_{\xi,h}$$

3. Continuation-utility dynamics

$$
\begin{aligned}
0 = \sup \Big\{ & \delta_g \frac{[\exp(-\xi_{g,t}) C_{g,t}/N_{g,t}]^{1-\rho_g} - 1}{1-\rho_g} - C_{g,t}/N_{g,t} \\
& + \mu_{n,g,t} - \frac{\gamma_g}{2}|\sigma_{n,g,t}|^2 - \frac{\gamma_g}{2}\tilde{\sigma}^2_{n,g,t} - (\gamma_g - 1)\sigma_{n,g,t}\cdot\sigma_{\xi,g,t} \\
& + \mu_{\xi,g,t} - \frac{\gamma_g - 1}{2}|\sigma_{\xi,g,t}|^2 \Big\}
\end{aligned}
$$

so we can iterate backward (like value-function-iteration) as follows:

(a) Given $\xi_{g,t} = \xi_g(X_t)$ as a function of "state variables" X_t, use Itô's formula to get $\mu_{\xi,g,t} = \mu_x(X_t)\partial_x\xi_g(X_t) + \frac{1}{2}\mathrm{tr}[\sigma_x(X_t)\sigma_x(X_t)'\partial_{xx'}\xi_g(X_t)]$ and $\sigma_{\xi,g,t} = \sigma_x(X_t)\partial_x\xi_g(X_t)$;

(b) Plug into the HJB equation above to obtain a PDE for ξ_g.

Markov equilibrium: state variables X_t

Exogenous states:

$$\hat{X}_t := (Z_t, V_t, \tilde{V}_t)'$$

Endogenous state:

$$W_t := \frac{N_{e,t}}{N_{e,t} + N_{h,t}}$$

Stack:

$$X_t := (W_t, \hat{X}_t')'$$

$$dX_t = \mu_x(X_t)dt + \sigma_x(X_t)dB_t$$

$$\text{where} \quad \underbrace{\mu_x(x) := \begin{pmatrix} \mu_w(x) \\ \mu_{\hat{x}}(\hat{x}) \end{pmatrix}}_{\text{dim } 4\times 1} \quad \underbrace{\sigma_x(x) := \begin{pmatrix} \sigma_w(x) \\ \sigma_{\hat{x}}(\hat{x})' \end{pmatrix}}_{\text{dim } 4\times d}$$

Next step: derive μ_w, σ_w

OLG for stationarity of W_t

- Idiosyncratic Poisson birth/death at rate λ_d
- Fraction of newborns (population shares): ν experts; $1-\nu$ households
- No bequest motive
- Preferences only altered by the discount rate, i.e., $\delta \mapsto \delta + \lambda_d$ [see Appendix D of Gârleanu-Panageas (2015)]
- Given absence of labor income, assume no "insurance company" offering life insurance [unlike Blanchard (1985) and Gârleanu-Panageas (2015)]
- Dying agents' wealth redistributed equally to newborns

Wealth share dynamics

Aggregate net worth dynamics:

$$\frac{dN_{h,t}}{N_{h,t}} = \left[r_t - c^*_{h,t} + \sigma_{n,h,t} \cdot \pi_{h,t} + \beta_{h,t}\Delta_{h,t} - \lambda_d + \frac{(1-\nu)\lambda_d}{1-W_t}\right]dt + \sigma_{n,h,t} \cdot dB_t$$
$$\frac{dN_{e,t}}{N_{e,t}} = \left[r_t - c^*_{e,t} + \sigma_{n,e,t} \cdot \pi_{h,t} + \chi_t\beta_{e,t}\Delta_{e,t} - \lambda_d + \frac{\nu\lambda_d}{W_t}\right]dt + \sigma_{n,e,t} \cdot dB_t,$$

where $\kappa := K_e/K$ and

$$\sigma_{n,h} = \frac{1-\chi\kappa}{1-w}\sigma_R$$
$$\sigma_{n,e} = \frac{\chi\kappa}{w}\sigma_R.$$

Use Itô's formula on $W_t := N_{e,t}/(N_{e,t} + N_{h,t})$ to get

$$\mu_w = w(1-w)\left[c^*_h - c^*_e + \chi\beta_e\Delta_e - \beta_h\Delta_h\right] + \sigma_w \cdot (\pi_h - \sigma_R) + \lambda_d(\nu - w)$$
$$\sigma_w = (\chi\kappa - w)\sigma_R.$$

Capital price and "amplification"

In Markov equilibrium, $Q_t = q(X_t)$, which solves the goods market clearing condition (given knowledge of κ):

$$q[(1-w)c_h^* + wc_e^*] + i^*(q) = (1-\kappa)a_h + \kappa a_e.$$

q can decrease for 3 reasons:

1. $\kappa \downarrow$ [e.g., Brunnermeier-Sannikov 2014]
2. $c_h^*, c_e^* \uparrow$ [e.g., Bansal-Yaron 2004]
3. $w \downarrow$ and $c_h^* > c_e^*$ [e.g., Gârleanu-Panageas 2015]

Plugging in $\sigma_q = \sigma_x' \partial_x \log q$ and using the previous result for σ_w:

$$\sigma_R = \sqrt{v}\sigma_k + \sigma_q = \frac{\sqrt{v}\sigma_k + \sigma_{\hat{x}}' \partial_{\hat{x}} \log q}{1 - (\chi\kappa - w)\partial_w \log q}.$$

κ still endogenous...

Solving for key equilibrium shares (χ, κ)

Recall FOCs for χ and β_h:

$$0 = \min(\chi - \underline{\chi}, \Delta_e)$$
$$0 = \min(\beta_h, \gamma_h \tilde{\sigma}_R^2 \beta_h - \Delta_h)$$

SOLVING FOR KEY EQUILIBRIUM SHARES (χ, κ)

Substitute $\beta_h = (1-\kappa)/(1-w)$:

$$0 = \min(\chi - \underline{\chi},\, \Delta_e)$$
$$0 = \min\left(1-\kappa,\, \gamma_h \tilde{\sigma}_R^2 \frac{1-\kappa}{1-w} - \Delta_h\right)$$

Solving for key equilibrium shares (χ, κ)

$$0 = \min(\chi - \underline{\chi},\ \Delta_e)$$
$$0 = \min(1 - \kappa,\ \gamma_h \tilde{\sigma}_R^2 \frac{1-\kappa}{1-w} - \Delta_h)$$

Recall

$$\begin{aligned}\Delta_h &:= \mu_{R,h} - r - \sigma_R \cdot \pi_h \\ &= \mu_{R,e} - r - \sigma_R \cdot \pi_h - (\mu_{R,e} - \mu_{R,h}) \\ &= \underline{\chi}\Delta_e - q^{-1}(a_e - a_h)\end{aligned}$$

Solving for key equilibrium shares (χ, κ)

$$0 = \min(\chi - \underline{\chi},\ \Delta_e)$$
$$0 = \min(1 - \kappa,\ \gamma_h \tilde{\sigma}_R^2 \frac{1-\kappa}{1-w} - \Delta_h)$$

In addition, we have equations for (Δ_e, π_h) from the other portfolio FOCs:

$$\Delta_h = \underline{\chi}\Delta_e - q^{-1}(a_e - a_h)$$
$$\Delta_e = -\sigma_R \cdot \pi_h + \gamma_e[\sigma_R \cdot \sigma_{n,e} + \tilde{\sigma}_R \tilde{\sigma}_{n,e}] + (\gamma_e - 1)\sigma_R \cdot \sigma_{\xi,e}$$
$$\pi_h = \gamma_h \sigma_{n,h} + (\gamma_h - 1)\sigma_{\xi,h}$$

Solving for key equilibrium shares (χ, κ)

$$0 = \min(\chi - \underline{\chi},\ \Delta_e)$$
$$0 = \min\left(1 - \kappa,\ \gamma_h \tilde{\sigma}_R^2 \frac{1-\kappa}{1-w} - \Delta_h\right)$$

Plug π_h into Δ_e and plug Δ_e into Δ_h:

$$\Delta_h = -q^{-1}(a_e - a_h) + \underline{\chi}\Big\{\sigma_R \cdot \Big[\gamma_e \sigma_{n,e} - \gamma_h \sigma_{n,h}\Big] + \gamma_e \tilde{\sigma}_R \tilde{\sigma}_{n,e} + \sigma_R \cdot \Big[(\gamma_e - 1)\sigma_{\xi,e} - (\gamma_h - 1)\sigma_{\xi,h}\Big]\Big\}$$
$$\Delta_e = \sigma_R \cdot \Big[\gamma_e \sigma_{n,e} - \gamma_h \sigma_{n,h}\Big] + \gamma_e \tilde{\sigma}_R \tilde{\sigma}_{n,e} + \sigma_R \cdot \Big[(\gamma_e - 1)\sigma_{\xi,e} - (\gamma_h - 1)\sigma_{\xi,h}\Big]$$

Solving for key equilibrium shares (χ, κ)

$$0 = \min(\chi - \underline{\chi},\ \Delta_e)$$
$$0 = \min(1 - \kappa,\ \gamma_h \tilde{\sigma}_R^2 \frac{1-\kappa}{1-w} - \Delta_h)$$

If $\chi > \underline{\chi}$, then $\Delta_h < \Delta_e = 0$, implying $\kappa = 1$. Thus, we may substitute

- $\chi = \underline{\chi}$ into the expression for Δ_h
- $\kappa = 1$ into the expression for Δ_e

These equations become decoupled.

Solving for key equilibrium shares (χ, κ)

$$0 = \min(\chi - \underline{\chi},\, \Delta_e^{\kappa=1})$$
$$0 = \min(1 - \kappa,\, \gamma_h \tilde{\sigma}_R^2 \frac{1-\kappa}{1-w} - \Delta_h^{\chi=\underline{\chi}})$$

$$\begin{aligned}
\Delta_h^{\chi=\underline{\chi}} &= -q^{-1}(a_e - a_h) \\
&\quad + \underline{\chi}\Big\{|\sigma_R^{\chi=\underline{\chi}}|^2 \Big[\gamma_e \frac{\underline{\chi}\kappa}{w} - \gamma_h \frac{1-\underline{\chi}\kappa}{1-w}\Big] + \gamma_e \tilde{\sigma}_R^2 \frac{\underline{\chi}\kappa}{w} \\
&\quad + \sigma_R^{\chi=\underline{\chi}} \cdot \Big[(\gamma_e - 1)\sigma_{\xi,e}^{\chi=\underline{\chi}} - (\gamma_h - 1)\sigma_{\xi,h}^{\chi=\underline{\chi}}\Big]\Big\}
\end{aligned}$$

$$\begin{aligned}
\Delta_e^{\kappa=1} &= |\sigma_R^{\kappa=1}|^2 \Big[\gamma_e \frac{\chi}{w} - \gamma_h \frac{1-\chi}{1-w}\Big] + \gamma_e \tilde{\sigma}_R^2 \frac{\chi}{w} \\
&\quad + \sigma_R^{\kappa=1} \cdot \Big[(\gamma_e - 1)\sigma_{\xi,e}^{\kappa=1} - (\gamma_h - 1)\sigma_{\xi,h}^{\kappa=1}\Big]
\end{aligned}$$

Solving for key equilibrium shares (χ, κ)

$$0 = \min(\chi - \underline{\chi},\ \Delta_e^{\kappa=1})$$
$$0 = \min(1 - \kappa,\ \gamma_h \tilde{\sigma}_R^2 \frac{1-\kappa}{1-w} - \Delta_h^{\chi=\underline{\chi}})$$

Finally, recall:

$$q[(1-w)c_h^* + wc_e^*] + i^*(q) = (1-\kappa)a_h + \kappa a_e$$
$$\sigma_R = \frac{\sqrt{v}\sigma_k + \sigma_{\hat{x}}' \partial_{\hat{x}} \log q}{1 - (\chi\kappa - w)\partial_w \log q}$$
$$\sigma_{\xi,g} = \sigma_x \cdot \partial_x \xi_g, \quad g \in \{e, h\}.$$

- If $\kappa = 1$, then $q(x; \xi_e, \xi_h)$ is known, so the equation for χ is algebraic.
- The equation for κ is differential.

Part II

Numerical Solution Method

Value function iteration

Statement of the problem. Scaled value functions $\{\xi_g\}_{g=e,h}$ solve PDEs like

$$0 = F_g + A_g \xi_g + B_g \cdot \partial_x \xi_g + \frac{1}{2}\text{tr}[C_g C_g' \partial_{xx'} \xi_g], \quad x = (w, z, v, \tilde{v}),$$

where the coefficients are:

$$\begin{aligned}
F_g &= F_g(x, \xi_e, \xi_h, \partial_x \xi_e, \partial_x \xi_h) \\
A_g &= A_g(x, \xi_e, \xi_h, \partial_x \xi_e, \partial_x \xi_h) \\
B_g &= B_g(x, \xi_e, \xi_h, \partial_x \xi_e, \partial_x \xi_h) \\
C_g &= C_g(x, \xi_e, \xi_h, \partial_x \xi_e, \partial_x \xi_h)
\end{aligned}$$

The dependence of A, B, C on (ξ_e, ξ_h) arises due to the preferences and general equilibrium. Solve this PDE system with a back-and-forth iteration:

1. given coefficients, solve the linear PDE system and obtain $\{\xi_g\}_{g=e,h}$
2. given PDE solution $\{\xi_g\}_{g=e,h}$, update coefficients

Value function iteration

Step 1. Augment the PDE with a "false transient," which is an artificial time-derivative $\partial_t \xi_g$ (Itô with time t, $\mu_{\xi,g} = \partial_t \xi_g + \mu_x' \partial_x \xi_g + \frac{1}{2}\text{tr}[\sigma_x \sigma_x' \partial_{xx'} \xi_g]$):

$$0 = F_g + \partial_t \xi_g + A_g \xi_g + B_g \cdot \partial_x \xi_g + \frac{1}{2}\text{tr}[C_g C_g' \partial_{xx'} \xi_g],$$

where

$$\begin{aligned} F_g &= F_g(x, \xi_e, \xi_h, \partial_x \xi_e, \partial_x \xi_h) \\ A_g &= A_g(x, \xi_e, \xi_h, \partial_x \xi_e, \partial_x \xi_h) \\ B_g &= B_g(x, \xi_e, \xi_h, \partial_x \xi_e, \partial_x \xi_h) \\ C_g &= C_g(x, \xi_e, \xi_h, \partial_x \xi_e, \partial_x \xi_h) \end{aligned}$$

We will use this to "work backward" from the distant future (T), just as in discrete-time value function iteration (may set terminal condition $\xi_g^{(T)}$ to anything in a stationary environment).

Stop iterating when reaching fixed point, i.e., $\partial_t \xi_g \approx 0$.

Value function iteration

Step 2. Given an iterant or guess $(\xi_e^{(t)}, \xi_h^{(t)})$, we substitute the coefficients $(F_g^{(t)}, A_g^{(t)}, B_g^{(t)}, C_g^{(t)})$.

$$0 = F_g^{(t)} + \partial_t \xi_g + A_g^{(t)} \xi_g + B_g^{(t)} \cdot \partial_x \xi_g + \frac{1}{2} \mathrm{tr}[C_g^{(t)} C_g^{(t)\prime} \partial_{xx'} \xi_g],$$

where

$$\begin{aligned}
F_g^{(t)} &:= F_g(x, \xi_e^{(t)}, \xi_h^{(t)}, \partial_x \xi_e^{(t)}, \partial_x \xi_h^{(t)}) \\
A_g^{(t)} &:= A_g(x, \xi_e^{(t)}, \xi_h^{(t)}, \partial_x \xi_e^{(t)}, \partial_x \xi_h^{(t)}) \\
B_g^{(t)} &:= B_g(x, \xi_e^{(t)}, \xi_h^{(t)}, \partial_x \xi_e^{(t)}, \partial_x \xi_h^{(t)}) \\
C_g^{(t)} &:= C_g(x, \xi_e^{(t)}, \xi_h^{(t)}, \partial_x \xi_e^{(t)}, \partial_x \xi_h^{(t)})
\end{aligned}$$

Value function iteration

Step 3. Discretize the time derivatives and write all spatial derivatives in terms of $\xi_g^{(t-\Delta)}$ ("implicit" finite differences, as opposed to "explicit"), i.e.,

$$\frac{\xi_g^{(t-\Delta)} - \xi_g^{(t)}}{\Delta} = F_g^{(t)} + A_g^{(t)} \xi_g^{(t-\Delta)} + B_g^{(t)} \cdot \partial_x \xi_g^{(t-\Delta)} + \frac{1}{2} \text{tr}[C_g^{(t)} C_g^{(t)\prime} \partial_{xx'} \xi_g^{(t-\Delta)}],$$

where

$$\begin{aligned}
F_g^{(t)} &= F_g(x, \xi_e^{(t)}, \xi_h^{(t)}, \partial_x \xi_e^{(t)}, \partial_x \xi_h^{(t)}) \\
A_g^{(t)} &= A_g(x, \xi_e^{(t)}, \xi_h^{(t)}, \partial_x \xi_e^{(t)}, \partial_x \xi_h^{(t)}) \\
B_g^{(t)} &= B_g(x, \xi_e^{(t)}, \xi_h^{(t)}, \partial_x \xi_e^{(t)}, \partial_x \xi_h^{(t)}) \\
C_g^{(t)} &= C_g(x, \xi_e^{(t)}, \xi_h^{(t)}, \partial_x \xi_e^{(t)}, \partial_x \xi_h^{(t)})
\end{aligned}$$

To hope for scheme "monotonicity" [i.e., Barles-Souganidis (1991)]:

- "Upwinding" for discretization of $\partial_x \xi_g^{(t-\Delta)}$;
- Cross-partial derivatives computed using $\xi_g^{(t)}$ and added into $F_g^{(t)}$

Step 3-alt. Discretize the time derivatives and write all spatial derivatives in terms of $\xi_g^{(t)}$ ("explicit" finite differences, as opposed to "implicit"), i.e.,

$$\frac{\xi_g^{(t-\Delta)} - \xi_g^{(t)}}{\Delta} = F_g^{(t)} + A_g^{(t)} \xi_g^{(t)} + B_g^{(t)} \cdot \partial_x \xi_g^{(t)} + \frac{1}{2} \mathrm{tr}[C_g^{(t)} C_g^{(t)\prime} \partial_{xx'} \xi_g^{(t)}],$$

where

$$\begin{aligned} F_g^{(t)} &= F_g(x, \xi_e^{(t)}, \xi_h^{(t)}, \partial_x \xi_e^{(t)}, \partial_x \xi_h^{(t)}) \\ A_g^{(t)} &= A_g(x, \xi_e^{(t)}, \xi_h^{(t)}, \partial_x \xi_e^{(t)}, \partial_x \xi_h^{(t)}) \\ B_g^{(t)} &= B_g(x, \xi_e^{(t)}, \xi_h^{(t)}, \partial_x \xi_e^{(t)}, \partial_x \xi_h^{(t)}) \\ C_g^{(t)} &= C_g(x, \xi_e^{(t)}, \xi_h^{(t)}, \partial_x \xi_e^{(t)}, \partial_x \xi_h^{(t)}) \end{aligned}$$

With explicit schemes, often a smaller Δ is required (e.g., CFL condition). We use implicit schemes.

Step 4. By discretizing the spatial derivatives $\partial_x \xi_g^{(t-\Delta)}$ and $\partial_{xx'} \xi_g^{(t-\Delta)}$, the PDE becomes a system of linear equations in the unknown value function at the discretization points:

$$\xi_g^{(t-\Delta)} = \xi_g^{(t)} + \Delta F_g^{(t)} + \Delta L_g^{(t)} \xi_g^{(t-\Delta)},$$

where $L_g^{(t)} \xi_g^{(t-\Delta)}$ is the discretized version of

$$A_g^{(t)} \xi_g^{(t-\Delta)} + B_g^{(t)} \cdot \partial_x \xi_g^{(t-\Delta)} + \frac{1}{2} \mathrm{tr}[C_g^{(t)} C_g^{(t)\prime} \partial_{xx'} \xi_g^{(t-\Delta)}].$$

Solve this system for $(\xi_e^{(t-\Delta)}, \xi_h^{(t-\Delta)})$.

Value function iteration

Implicit FD example. Suppose spatial variable x is one-dimensional:

$$0 = F + \partial_t \xi + A\xi + B\partial_x \xi + \frac{1}{2} C^2 \partial_{xx} \xi.$$

Discretization with space step "dx":

$$\frac{\xi^{(t-\Delta)}(x) - \xi^{(t)}(x)}{\Delta} = F^{(t)}(x) + A^{(t)}(x)\xi^{(t-\Delta)}(x)$$

$$+ B^{(t)}(x) \underbrace{\left[\mathbf{1}_{\{B^{(t)}(x)>0\}} \frac{\xi^{(t-\Delta)}(x+dx) - \xi^{(t-\Delta)}(x)}{dx} + \mathbf{1}_{\{B^{(t)}(x)<0\}} \frac{\xi^{(t-\Delta)}(x) - \xi^{(t-\Delta)}(x-dx)}{dx} \right]}_{\text{“upwinding” for first derivative}}$$

$$+ \frac{1}{2}(C^{(t)}(x))^2 \underbrace{\frac{\xi^{(t-\Delta)}(x+dx) - 2\xi^{(t-\Delta)}(x) + \xi^{(t-\Delta)}(x-dx)}{dx^2}}_{\text{second derivative approximation}}$$

Value function iteration

Implicit FD example continued. Write the system as

$$\frac{\xi^{(t-\Delta)} - \xi^{(t)}}{\Delta} = F^{(t)} + L^{(t)}\xi^{(t-\Delta)} \;\Rightarrow\; [I - \Delta L^{(t)}]\xi^{(t-\Delta)} = \xi^{(t)} + \Delta F^{(t)}.$$

The row for x has $L^{(t)}(x,:)$ constructed as...

$$L^{(t)}(x,x) = A^{(t)}(x) - \frac{|B^{(t)}(x)|}{dx} - \frac{(C^{(t)}(x))^2}{dx^2} < 0 \quad \text{if} \quad A^{(t)}(x) < 0$$

$$L^{(t)}(x,x+dx) = \frac{|\max[0, B^{(t)}(x)]|}{dx} + \frac{1}{2}\frac{(C^{(t)}(x))^2}{dx^2} > 0$$

$$L^{(t)}(x,x-dx) = \frac{|\min[0, B^{(t)}(x)]|}{dx} + \frac{1}{2}\frac{(C^{(t)}(x))^2}{dx^2} > 0$$

$$L^{(t)}(x,y) = 0 \quad \text{for} \quad y \notin \{x-dx, x, x+dx\}$$

Sparsity: $I - \Delta L^{(t)}$ is a highly-sparse (tri-diagonal) matrix.

Monotonicity: Opposing signs of diagonal $I - \Delta L^{(t)}(x,x) > 0$ and off-diagonal elements $I - \Delta L^{(t)}(x,y) \leq 0$ for $y \neq x$.

Value function iteration

Computational considerations. Solving $\left[I - \Delta L_g^{(t)}\right] \xi_g^{(t-\Delta)} = \xi_g^{(t)} + \Delta F_g^{(t)}$.

Direct approach: essentially "invert" $I - \Delta L_g^{(t)}$ to the other side (technically, solve system using LU decomposition)

- Upside: generates exact solution for $\xi_g^{(t-\Delta)}$
- Downside: each iteration (t), the problem of inverting $I - \Delta L_g^{(t)}$ changes

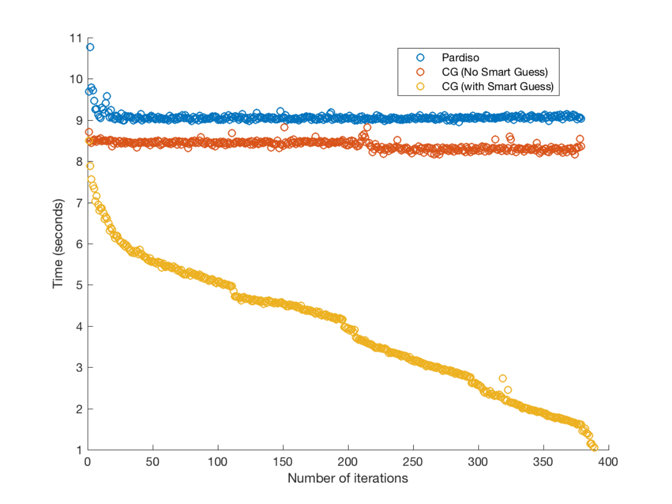
Iterative approach: solve (using "conjugate gradient" algorithm)

$$\xi_g^{(t-\Delta)} = \underbrace{\arg\min_{v} \left\{ \frac{1}{2} v' [I - \Delta L_g^{(t)}]' [I - \Delta L_g^{(t)}] v - v' [I - \Delta L_g^{(t)}]' [\xi_g^{(t)} + \Delta F_g^{(t)}] \right\}}_{\substack{\text{any equation } Ax=b \text{ can be solved for } x \\ \text{using } \min_x \frac{1}{2} x'A'Ax - x'b \text{ as long as } A'A \text{ is positive definite}}} .$$

- Upside: can provide "smart guess" based on $\xi_g^{(t)}$
- Downside: only an approximate solution for $\xi_g^{(t-\Delta)}$

Value function iteration

LU versus CG. Solving $\left[I - \Delta L_g^{(t)}\right] \xi_g^{(t-\Delta)} = \xi_g^{(t)} + \Delta F_g^{(t)}$.



Boundary conditions

So far, I said nothing about boundary conditions! These models usually have sensitive boundaries (example: $\pi_e(0+) = +\infty$ is possible)

But the boundaries are unattainable in the sense of zero-probability events (example: $\pi_e(0+) = +\infty$ implies $\mu_w(0+) = +\infty$)

Therefore, we need not provide any special boundary conditions!

Heuristic idea: if (F, A, B, C) are known functions in the PDE

$$0 = F + \partial_t \xi + A\xi + B\partial_x \xi + \frac{1}{2}\mathrm{tr}[CC'\partial_{xx'}\xi],$$

then the solution can be written (Feynman-Kac theorem)

$$\xi(x) = \mathbb{E}\Big[\int_0^\infty e^{\int_0^t A(s,X_s)ds} F(t, X_t)dt \mid X_0 = x\Big]$$

$$\text{subject to} \quad dX_t = B(t, X_t)dt + C(t, X_t) \cdot \underbrace{dZ_t}_{\substack{\text{Brownian}\\\text{motion}}}$$

CONSTRAINTS AND (χ, κ)

Statement of the problem. Capital distribution $\kappa \in [0, 1]$ and expert equity-retention $\chi \in [\underline{\chi}, 1]$ determine occasionally-binding constraints

$$0 = \min(1 - \kappa, G_h)$$
$$0 = \min(\chi - \underline{\chi}, G_e)$$

where we showed theoretically that

$$G_h = G_h(x, \kappa, \partial_x \kappa; \xi_e, \xi_h)$$
$$G_e = G_e(x, \chi; \xi_e, \xi_h).$$

These are sometimes called variational inequalities.

Constraints and (χ, κ)

Solution method.

1. Given an iterant $\xi_e^{(t)}, \xi_h^{(t)}$, construct

$$G_h^{(t)}(x, \kappa, \partial_x \kappa) := G_h(x, \kappa, \partial_x \kappa; \xi_e^{(t)}, \xi_h^{(t)})$$
$$G_e^{(t)}(x, \chi) := G_e(x, \chi; \xi_e^{(t)}, \xi_h^{(t)})$$

2. Since $0 = \min(\chi - \underline{\chi}, G_e^{(t)})$ is a univariate algebraic equation in χ, simply use nonlinear solver to obtain solution $\chi^{(t)}$
3. Since $0 = \min(1 - \kappa, G_h^{(t)})$ is a univariate differential equation in κ, use explicit finite-difference scheme with false transient, i.e.,

$$\frac{\tilde{\kappa}^{(\tau+\tilde{\Delta})} - \tilde{\kappa}^{(\tau)}}{\tilde{\Delta}} = \min\left(1 - \tilde{\kappa}^{(\tau)}, G_h^{(t)}(x, \tilde{\kappa}^{(\tau)}, \partial_x \tilde{\kappa}^{(\tau)})\right), \quad \tilde{\kappa}^{(0)} = \kappa^{(t+\Delta)}.$$

If LHS becomes small at τ, put $\kappa^{(t)} := \tilde{\kappa}^{(\tau)}$.
[See Oberman (2006) for nonlinear first-order PDE schemes of this type; small enough $\tilde{\Delta}$ is required.]

Stationary density

Step 1. After solving for all value functions and equilibrium objects, we have the equilibrium state dynamics μ_x and σ_x.

Recall the "transition operator" associated with the Kolmogorov Backward Equation (also called the "generator" of a diffusion):

$$\mathcal{P}f := \mu_x' \partial_x f + \frac{1}{2}\mathrm{tr}[\sigma_x \sigma_x' \partial_{xx'} f]$$

Discretize this linear operator with a matrix P, e.g. in 1D example:

$$P(x,x) = -\frac{|\mu_x(x)|}{dx} - \frac{(\sigma_x(x))^2}{dx^2}$$
$$P(x,x+dx) = \frac{|\max[0,\mu_x(x)]|}{dx} + \frac{1}{2}\frac{(\sigma_x(x))^2}{dx^2}$$
$$P(x,x-dx) = \frac{|\min[0,\mu_x(x)]|}{dx} + \frac{1}{2}\frac{(\sigma_x(x))^2}{dx^2}$$

Notice that P is a transition matrix for a continuous-time Markov chain (e.g., row-sums are 0).

Stationary density

Step 2. Can obtain stationary density approximation ω by solving (as in CTMC theory)

$$P'\omega = 0.$$

- **Alternative 1:** Recall that the Kolmogorov Forward Equation is the adjoint equation to the backward equation, and since adjoints in finite-dimensional space are matrix transposes, $P'\omega = 0$ is the discretized adjoint equation to $Pf = 0$.
- **Alternative 2:** $I + \Delta P$ is a discrete-time Markov matrix, for small time-step Δ, so just solve $\omega'(I + \Delta P) = \omega'$.

Just an eigenvector-eigenvalue problem.

Next time

Fabrice will talk about:

- Evaluating this class of models
- Comparing different models

  Example: what is similar and different about models in which the "experts" are more productive (i.e., $a_e > a_h$) versus more risk-tolerant (i.e., $\gamma_e < \gamma_h$)
- Show everything with a user-friendly web application to solve models, downloadable at `https://larspeterhansen.org/mfr-suite/`