Brunnermeier & Sannikov

Lecture 04

A Symmetric International Model with Runs/Sudden Stops

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Overview

- So far
  - Multi/two-type/sector model asymmetric
    - Experts: more productive ak_t less patient ρ
    - Households: less productive $\underline{a}k_t$ more patient $\underline{\rho} < \rho$
  - Focus on equilibrium without jumps/runs

- Now
  - Two type/sector model symmetric
    - Symmetric productivity – 2 goods, one country more productive in one
    - Same preference discount rates
  - 2 Brownian shocks
  - $A(\psi)$ is micro-founded with 2 good economy
  - Asset price run ≠ depositor run a la Diamond-Dybvig

Role of (international) financial markets

1. Better allocation of physical capital/resources
   - ψ
2. Better allocation of risk (sharing)
   - χ

depends on future

- Complete markets
  - (1) and (2) can be controlled separately
  - Pecuniary externalities have 2nd order w-effects
- Frictions/incomplete markets $F(\psi, \chi) \leq 0$, e.g. $\psi = \chi$
  - (1) and (2) are interlinked
  - Pecuniary externalities have welfare effects

First Best

Second Best

Pecuniary Externalities: Some Literature

- Constrained inefficiency, pecuniary/firesale externalities
  - Incomplete markets:
    - Stiglitz 1982, Newsbury & Stiglitz 1984, Geanakoplos & Polemarchakis 1986, He & Kondor 2013
  - Debt collateral constraint (that depends on price):
    - Stiglitz & Greenwald, Lorenzoni 2005, Bianchi 2011, Bianchi & Mendozza 2012, Jeanne & Korinek 2012, Stein 2012,
    - Davilia & Korinek 2017, …
- "terms of trade hedge"
    - Cole & Obstfeld 1991, Martin 2010

International Macro Interpretation

- Old "Washington consensus" in decline
  - Free trade: flow of goods/services intratemporal
  - Free finance: flow of capital intertemporal
- When does full capital account liberalization reduce (capital controls/macropru regulation improve) welfare?

Brunnermeier, Markus & Yuliy Sannikov.
"International Credit Flows and Pecuniary Externalities".
American Economic Journal: Macroeconomics 71 (2015): , 7, 1, 297-338.

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1. Sudden stop including runs due to liquidity mismatch
   - Technological illiquidity: irreversibility (adjustment costs)
   - Market illiquidity: redeployability/specificity – not this paper
   - Funding illiquidity: short-term debt, "hot money"
     - Type of capital flow matters: FDI, portfolio flows (equity), long-term debt

Asset side

Liability

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     - Type of capital flow matters: FDI, portfolio flows (equity), long-term debt
2. "Terms of trade hedge" (Cole-Obstfeld) can be undermined when
   - Industry's output is not easily substitutable. Consumers cannot easily find substitutes
   - No strong competitors in other countries
     - Natural resources: oil, copper for Chile,
     - Hard drives in Thailand, Bananas in Ecuador

Model setup - symmetric

- Preferences

$$E\left[\int_0^\infty e^{-\rho t}\frac{c_t^{1-\gamma}}{1-\gamma}dt\right]$$

  - Same preference discount rate ρ – "saving out of constraint"
- Two output goods y^a and y^b - imperfect substitutes

$$y_t=\left[\frac{1}{2}(y_t^a)^{\frac{s-1}{s}}+\frac{1}{2}(y_t^b)^{\frac{s-1}{s}}\right]^{s/(s-1)}$$

- (Comparative) advantages:

	Good a	Good b
Country A	$\bar{a}k_t$	$\underline{a}k_t$
Country B	$\underline{a}k_t$	$\bar{a}k_t$

Two country/sector/type model

- World capital shares:

$$\psi_t^{Aa} + \psi_t^{Ab} + \psi_t^{Ba} + \psi_t^{Bb} = 1$$

- World supply of (output) goods:

$$Y_t^a = \left(\psi_t^{Aa}\overline{a} + \psi_t^{Ba}\underline{a}\right)K_t \quad Y_t^b = \left(\psi_t^{Bb}\overline{a} + \psi_t^{Ab}\underline{a}\right)K_t$$

  - Plug into $y_t = \left[\frac{1}{2}(y_t^a)^{\frac{s-1}{s}} + \frac{1}{2}\left(y_t^b\right)^{\frac{s-1}{s}}\right]^{s/(s-1)}$ to get $A(\boldsymbol{\psi}_t)$

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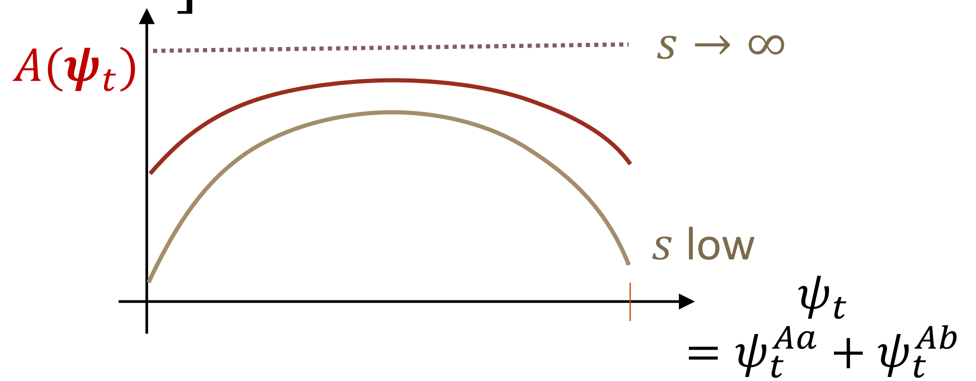

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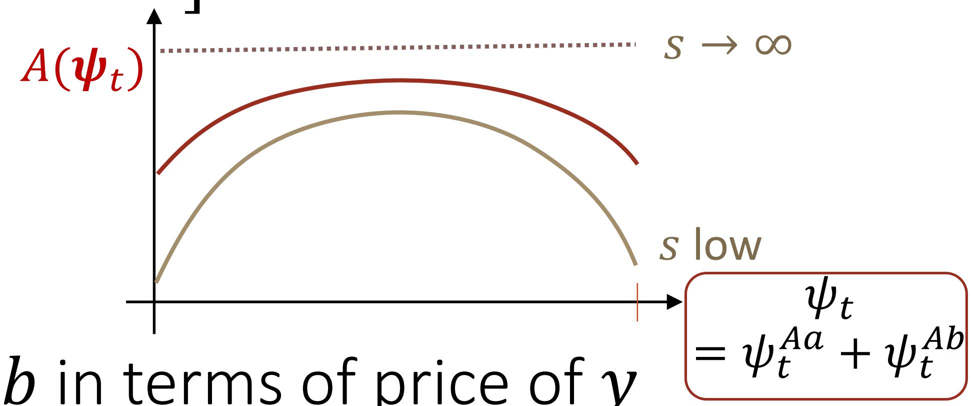

- Price of output goods a and b in terms of price of y

$$P_t^a = \frac{1}{2}\left(\frac{Y_t}{Y_t^a}\right)^{1/s} \text{ and } P_t^b = \frac{1}{2}\left(\frac{Y_t}{Y_t^b}\right)^{1/s}$$

  - Terms of trade P_t^a / P_t^b

Two country/sector/type model

- Capital evolution for
  - $dk_t = (\Phi(\iota_t) - \delta)k_t dt + \sigma^A k_t dZ_t^A$ in country A
  - $dk_t = (\Phi(\iota_t) - \delta)k_t dt + \sigma^B k_t dZ_t^B$ in country B
    - Φ concavity – technological illiquidity
    - Single type of capital
    - Investment in composite good

Now there are 2 independent Brownian

- Shocks are
  - Two dimensional
    - Affect global capital stock $dZ_t^A + dZ_t^B$
    - Redistributive (initial shock + amplification) $\Rightarrow$ affects wealth share, η_t
      - Example: Apple vs. Samsung lawsuit

Market structures

	Trade		Finance	
Markets	Output y^a, y^b	Physical capital K	Debt	Equity
Complete Markets Full integration/First Best	X	X	X	X
Open credit account (equity home bias)	X	X	X	
Closed credit account	X	X		
	intratemporal		intertemporal	

Add taxes/capital controls

Solving MacroModels Step-by-Step

0. Postulate aggregates, price processes & obtain return processes
1. For given SDF processes *static*
   a. Real investment ι, (portfolio $\boldsymbol{\theta}$, & consumption choice of *each agent*)
      - *Toolbox 1:* Martingale Approach
   b. Asset/Risk Allocation *across types/sectors* & asset market clearing
      - *Toolbox 2:* "price-taking social planner approach" – Fisher separation theorem
2. Value functions *backward equation*
   a. Value fcn. as fcn. of individual investment opportunities ω
      - *Special cases*
   b. De-scaled value fcn. as function of state variables η
      - *Digression:* HJB-approach (instead of martingale approach & envelop condition)
   c. Derive ς-risk premia, C/N-ratio from value fcn. envelop condition
3. Evolution of state variable η *forward equation*
   - *Toolbox 3:* Change in numeraire to total wealth (including SDF)
   - ("Money evaluation equation" μ^{ϑ})
4. Value function iteration & goods market clearing
   a. PDE of de-scaled value fcn.
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0. Postulate price process & derive returns

- $dk_t/k_t = (\Phi(\iota_t) - \delta)dt + \sigma^A dZ_t^A$

- Postulate
  - $dq_t/q_t \ =$
  - $d\xi_t^A/\xi_t^A =$
  - $d\xi_t^B/\xi_t^B =$

Poll 15: Do these postulated processes depend on

a) *For* $\frac{d\xi_t^A}{\xi_t^A}$ *on* dZ^A

b) *For* $\frac{d\xi_t^B}{\xi_t^B}$ *on* dZ^B

c) *On both Brownians*

0. Postulate price process & derive returns

- $dk_t/k_t = (\Phi(\iota_t) - \delta)dt + \sigma^A dZ_t^A$

- Postulate
  - $dq_t/q_t \ = \mu_t^q dt + \sigma_t^{qA} dZ_t^A + \sigma_t^{qB} dZ_t^B$
  - $d\xi_t^A/\xi_t^A = -r_t^{A,F} dt - \varsigma_t^{AA} dZ_t^A - \varsigma_t^{AB} dZ_t^B$
  - $d\xi_t^B/\xi_t^B = -r_t^{B,F} dt - \varsigma_t^{BA} dZ_t^A - \varsigma_t^{BB} dZ_t^B$

0. Postulate price process & derive returns

- $dk_t/k_t = (\Phi(\iota_t) - \delta)dt + \sigma^A dZ_t^A$

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  - $d\xi_t^B/\xi_t^B = -r_t^{B,F} dt + \varsigma_t^{BA} dZ_t^A + \varsigma_t^{BB} dZ_t^B$

- Returns from holding physical capital
  - $dr_t^{Aa} = \left(\frac{\overline{a}P_t^a - \iota_t}{q_t} + \mu_t^q + \Phi(\iota_t) - \delta + \sigma^A \sigma_t^{qA}\right) dt +$

$$+(\sigma^A + \sigma_t^{qA})dZ_t^A + \sigma_t^{qB} dZ_t^B$$

  - $dr_t^{Ab} = \left(\frac{\underline{a}P_t^b - \iota_t}{q_t} + \mu_t^q + \Phi(\iota_t) - \delta + \sigma^A \sigma_t^{qA}\right) dt +$

$$+(\sigma^A + \sigma_t^{qA})dZ_t^A + \sigma_t^{qB} dZ_t^B$$

Ito product rule:
$d(X_t Y_t) = dX_t Y_t + X_t dY_t + \sigma_X \sigma_Y dt$

Step 1a. Optimal Reinvestment Rate ι

- Tobin's q -- as before a simple static problem

$$\Phi'(\iota_t) = 1/q_t$$

- All agents $\iota^i = \iota$
- Special functional form:
  - Quadratic adjustment cost
  - Investment rate of $\iota = \Phi + \frac{1}{\kappa}\Phi^2$ generates new capital at rate Φ
  - $\Phi(\iota) = \frac{1}{\kappa}\left(\sqrt{1+2\kappa\iota} - 1\right)$
  - Alternative specification: $\Phi(\iota) = \frac{1}{\kappa}\log(\kappa\iota + 1) \Rightarrow \kappa\iota = q - 1$

Step 1a. Asset pricing equations

- Martingale approach
  - recall discrete time analog
  - $\xi_t^A p_t = E_t[\xi_{t+s}^A(p_{t+s}+d_{t+s})]$ follows a martingale
- Pricing of self-financing asset X:
  If wealth ϵ_t is invested in X, s.t. $\frac{d\epsilon_t}{\epsilon_t}=dr_t^X$, $\xi_t^A\epsilon_t$ must follow a
  - Martingale $E_t[\xi_{t+s}^A\epsilon_{t+s}] = \xi_t^A\epsilon_t$ if portfolio position > 0
  - Supermartingale $E_t[\xi_{t+s}^A\epsilon_{t+s}] < \xi_t^A\epsilon_t$ if portfolio position = 0
- Risk premium
  - $\frac{E[dr_t^{Aa}]}{dt} - r_t^{F,A} = \varsigma_t^{AA}(\sigma_t^{qA} + \sigma^A) + \varsigma_t^{AB}\sigma_t^{qB}$
  - $\frac{E[dr_t^{Ab}]}{dt} - r_t^{F,A} \leq \varsigma_t^{AA}(\sigma_t^{qA} + \sigma^A) + \varsigma_t^{AB}\sigma_t^{qB}$ (equality if $\psi^{Ab} > 0$)
  - Analog for citizens in country B
- Risk free rate
  - Drift of $d\xi_t/\xi_t$

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Market structures

1. Complete markets ⇒ First best
2. Incomplete markets (equity home bias)
   - Levered short-term debt financing
   - Sudden stops: (varying technological illiquidity)
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Step 1b. First Best: Planner's allocation

- First-best:
  - Capital allocation ψs and
  - Risk allocation χs

  can be chosen independently

1. Complete markets: First Best

1. Perfect specialization
   - Investment rate equalization $\iota_t^A = \iota_t^B$
   - Full specialization $\psi_t^{Aa} = \psi_t^{Bb} = 1/2$
   - Output equalization $y_t^a = y_t^b \qquad Y_t = \overline{a}\frac{K_t}{2}$
2. Perfect risk sharing
   - Consumption (intensity) shares where λ^A and λ^B are Pareto weights $C_t^A = \lambda^A \overline{C}_t,\ C_t^B = \lambda^B \overline{C}_t$
   - $\frac{dZ_t^A + dZ_t^B}{\sqrt{2}} \equiv dZ_t$ (standard Brownian)
   - Global capital evolution

$$dK_t = [\Phi(\iota_t) - \delta]K_t dt + \frac{\sigma}{\sqrt{2}} K_t \underbrace{\frac{dZ_t^A + dZ_t^B}{\sqrt{2}}}_{:= dZ_t}$$

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1. First Best Prices

- SDF: $\xi_t^A = e^{-\rho t}\left(\frac{C_t^A}{C_0^A}\right)^{-\gamma} = e^{-\rho t}\left(\frac{K_t}{K_0}\right)^{-\gamma} = e^{-\rho t}\left(\frac{C_t^B}{C_0^B}\right)^{-\gamma}$

Since C_t^A/K_t is a constant

$$\frac{d\xi_t^A}{\xi_t^A} = \frac{d\xi_t^B}{\xi_t^B} = \underbrace{\left\{-\rho - \gamma[\Phi(\iota_t) - \delta] + \frac{\gamma(\gamma+1)\sigma^2}{4}\right\}}_{=E\left[\frac{d\xi_t}{\xi_t dt}\right]} dt - \frac{\gamma\sigma}{\sqrt{2}} dZ_t$$

- *Poll 25: Where does the 4 in the denominator come from?*

a) From $\left(\frac{1}{2}\right)^2$

b) From 4 parts of ψ^{Aa}, …

c) From $\left(\frac{1}{\sqrt{2}}\right)^2 * \frac{1}{2}$, *where second term comes from Ito's lemma*

1. First Best Prices

- SDF: $\xi_t^A = e^{-\rho t}\left(\frac{C_t^A}{C_0^A}\right)^{-\gamma} = e^{-\rho t}\left(\frac{K_t}{K_0}\right)^{-\gamma} = e^{-\rho t}\left(\frac{C_t^B}{C_0^B}\right)^{-\gamma}$

$$\frac{d\xi_t^A}{\xi_t^A} = \frac{d\xi_t^B}{\xi_t^B} = \underbrace{\left\{-\rho - \gamma[\Phi(\iota_t) - \delta] + \frac{\gamma(\gamma+1)\sigma^2}{4}\right\}}_{=E\left[\frac{d\xi_t}{\xi_t dt}\right]} dt - \frac{\gamma\sigma}{\sqrt{2}} dZ_t$$

- Risk-free rate: $r^F = \rho + \gamma[\Phi(\iota) - \delta] - \frac{\gamma(\gamma+1)\sigma^2}{4}$

- *Poll 26: If we had one country in autarky, how would r^F change?*
  a) Not at all
  b) 4 is replaced by 2 since we can't diversify
  c) None of the above

1. First Best Prices

- SDF: $\xi_t^A = e^{-\rho t}\left(\frac{c_t^A}{c_0^A}\right)^{-\gamma} = e^{-\rho t}\left(\frac{K_t}{K_0}\right)^{-\gamma} = e^{-\rho t}\left(\frac{c_t^B}{c_0^B}\right)^{-\gamma}$

$$\frac{d\xi_t^A}{\xi_t^A} = \frac{d\xi_t^B}{\xi_t^B} = \underbrace{\left\{-\rho - \gamma[\Phi(\iota_t) - \delta] + \frac{\gamma(\gamma+1)\sigma^2}{4}\right\}}_{=E\left[\frac{d\xi_t}{\xi_t dt}\right]} dt - \frac{\gamma\sigma}{\sqrt{2}} dZ_t$$

- Risk-free rate: $r^F = \rho + \gamma[\Phi(\iota) - \delta] - \frac{\gamma(\gamma+1)\sigma^2}{4}$
- Price of capital:
  - Since we know r^F and have constant investment opportunities, we can use $\frac{c_t}{n_t} = \rho + \frac{\gamma-1}{\gamma}\left(r^F - \rho + \frac{\varsigma^2}{2\gamma}\right)$ (from slide 45 in lecture 03).
  - Together with goods market clearing condition yields

Gordon Growth Formula $\frac{d}{r-g}$

$$q = \frac{\bar{a} - \iota_t}{r_t^F + \frac{\gamma}{2}\sigma^2 - [\Phi(\iota) - \delta]}$$

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Gordon Growth Formula $\frac{d}{r-g}$

$$q = \frac{\bar{a} - \iota_t}{r_t^F + \frac{\gamma}{2}\sigma^2 - [\Phi(\iota) - \delta]}$$

Poll 28: Why $\frac{\gamma}{2}\sigma^2$*-term?*

a) Risk adjustment for risky capital K

b) Term reflects $\varsigma\sigma$

1. Complete markets: First Best Remarks

- Perfect capital allocation + perfect risk sharing

- Prices are constant and independent of shocks
- Economy shrinks/expands with (multiplicative) shocks
- Elasticity of substitution, s, has no impact on prices

Market structures

1. Complete markets ⇒ First best
2. Incomplete markets (equity home bias)
   - Levered (short-term) debt financing
   - Sudden stops: (varying technological illiquidity, irreversibility)
     - Amplification
     - Runs due to sunspots
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Step 1b. Price Taking Planners Problem

- max

$$E\left[dr_t^{\bar{N}}\right](\boldsymbol{\psi}_t) \overbrace{- (\varsigma_t^{AA}\chi_t^{AA} + \varsigma_t^{BA}\chi_t^{BA})\sigma_t^{\bar{N}A}(\boldsymbol{\psi}_t)}^{A\text{-risk premia}} \underbrace{- (\varsigma_t^{AB}\chi_t^{AB} + \varsigma_t^{BB}\chi_t^{BB})\sigma_t^{\bar{N}B}(\boldsymbol{\psi}_t)}_{B\text{-risk premia}}$$

  - Where $\sigma_t^{\bar{N}A}(\boldsymbol{\psi}_t) = \psi_t^A\sigma^A + \sigma_t^{qA}, \sigma_t^{\bar{N}B}(\boldsymbol{\psi}_t) = \psi_t^B\sigma^B + \sigma_t^{qB}$

- Frictions: no outside equity issuance
  - $\chi_t^{AA} = \frac{\psi_t^A\left(\sigma^A + \sigma_t^{qA}\right)}{\sigma_t^{\bar{N}A}(\boldsymbol{\psi}_t)}$ $\quad\quad$ $\chi_t^{BB} = \frac{\psi_t^B\left(\sigma^B + \sigma_t^{qB}\right)}{\sigma_t^{\bar{N}B}(\boldsymbol{\psi}_t)}$
  - $\chi_t^{AB} = \frac{\psi_t^A\sigma_t^{qB}}{\sigma_t^{\bar{N}A}(\boldsymbol{\psi}_t)}$ $\quad\quad$ $\chi_t^{BA} = \frac{\psi_t^B\sigma_t^{qA}}{\sigma_t^{\bar{N}B}(\boldsymbol{\psi}_t)}$

Step 2. Get ςs from Value Function Envelop

As in previous lecture, but with 2 Brownian

- A's value function

B's value function

$$v_t \frac{K_t^{1-\gamma}}{1-\gamma}$$

- To obtain $\frac{\partial V_t^A(n^A)}{\partial n_t^A}$ use $K_t = \frac{N_t^A}{\eta_t q_t} = \frac{n_t^A}{\eta_t q_t}$

$$V_t^A(n_t^A) = v_t^A \frac{\left(n_t^A\right)^{1-\gamma}/(\eta_t q_t)^{1-\gamma}}{1-\gamma}$$

…

- Envelop condition $\frac{\partial V_t^A(n^A)}{\partial n_t^A} = \frac{\partial u(c_t^A)}{\partial c_t^A}$

Analogous for B

$$v_t^A \frac{\left(n_t^A\right)^{-\gamma}}{(\eta_t q_t)^{1-\gamma}} = (c_t^A)^{-\gamma}$$

…

- Using $K_t = \frac{n_t^A}{\eta_t q_t}$, $C_t^A = c_t^A$

$$\frac{v_t^A}{\eta_t q_t}(K_t^A)^{-\gamma} = (C_t^A)^{-\gamma}$$

…

$$\sigma_t^{vA} - \sigma_t^{\eta A} - \sigma_t^{qA} - \gamma \psi_t^A \sigma^A = -\gamma \sigma_t^{cA} = -\varsigma_t^{AA}$$

$$\sigma_t^{vB} - \sigma_t^{\eta B} - \sigma_t^{qB} - \gamma \psi_t^B \sigma^B = -\gamma \sigma_t^{cB} = -\varsigma_t^{AB}$$

Markov equilibrium

- Equilibrium is a map

Histories of shocks $\{Z_s^A, Z_s^B, s \leq t\}$ → prices allocation $q_t, \psi_t^{Aa} \dots, \iota_t^A, \iota_t^B, \zeta_t^A, \zeta_t^B$

wealth distribution

$$\eta_t = \frac{N_t^A}{q_t K_t} \in (0,1)$$ A's wealth share

Step 3. μ^η Drift of Wealth Share

- Martingale condition (relative to benchmark asset)

$$\mu_t^\eta + \frac{C_t^A}{N_t^A} - r_t^M = (\varsigma_t^{AA} - \sigma_t^{\bar{N}A})(\sigma_t^{\eta A} - \underbrace{\sigma_t^{MA}}_{=0}) + (\varsigma_t^{AB} - \sigma_t^{\bar{N}B})(\sigma_t^{\eta B} - \sigma_t^{MB})$$

- Add up across types (weighted),
  (capital letters with bars are aggregates for total world economy)

$$\underbrace{(\eta_t \mu_t^\eta + (1-\eta_t)\mu_t^{1-\eta})}_{=0} + \frac{\bar{C}_t}{\bar{N}_t} - r_t^M = \eta_t(\varsigma_t^{AA} - \sigma_t^{\bar{N}A})\sigma_t^{\eta A}$$

$$+(1-\eta_t)\left(\varsigma_t^{BA} - \sigma_t^{\bar{N}A}\right)\sigma_t^{1-\eta,A} + \eta_t(\varsigma_t^{AB} - \sigma_t^{\bar{N}B})\sigma_t^{\eta B} + (1-\eta_t)\left(\varsigma_t^{BB} - \sigma_t^{\bar{N}B}\right)\sigma_t^{1-\eta,B}$$

- Subtract from each other yields wealth share drift

$$\mu_t^\eta = (1-\eta_t)\left(\varsigma_t^{AA} - \sigma_t^{\bar{N}A}\right)\sigma_t^{\eta A}$$
$$-(1-\eta_t)\left(\varsigma_t^{BA} - \sigma_t^{\bar{N}A}\right)\sigma_t^{1-\eta,A}$$
$$+(1-\eta_t)\left(\varsigma_t^{AB} - \sigma_t^{\bar{N}B}\right)\sigma_t^{\eta B}$$
$$-(1-\eta_t)\left(\varsigma_t^{BB} - \sigma_t^{\bar{N}B}\right)\sigma_t^{1-\eta,B} - \left(\frac{C_t}{N_t} - \frac{C_t + \underline{C}_t}{q_t K_t}\right)$$

Step 3. $\sigma^{\eta A}, \sigma^{\eta B}$ Volatility of Wealth Share

- In general for multi-sector models, since $\eta_t^i = N_t^i / \bar{N}_t$,

$$\sigma_t^{\eta^i A} = \sigma_t^{N^i A} - \sigma_t^{\bar{N} A} = \sigma_t^{N^i A} - \sum_{i'} \eta_t^{i'} \sigma_t^{N^{i'} A} = \left(1 - \eta_t^i\right) \sigma_t^{N^i A} - \sum_{i^- \neq i} \eta_t^{i^-} \sigma_t^{N^{i^-} A}$$

$$\sigma_t^{\eta^i B} = \cdots$$

- Recall notation in our setting: $\eta_t = \eta_t^A$ and $1 - \eta_t = \eta_t^B$

$$\sigma_t^{\eta A} = (1 - \eta_t)(\sigma_t^{n^A A} - \sigma_t^{n^B A})$$

$$\sigma_t^{n^A A} = \frac{(\psi_t^{Aa} + \psi_t^{Ab})}{\eta_t} (\sigma^A + \sigma_t^{qA}) \qquad \sigma_t^{n^B A} = \frac{1 - (\psi_t^{Aa} + \psi_t^{Ab})}{1 - \eta_t} (\sigma_t^{qA})$$

  - Hence, $\sigma_t^{\eta A} = \frac{1}{\eta_t}[(1 - \eta_t)\left(\psi_t^{Aa} + \psi_t^{Ab}\right)\sigma^A + \left(\left(\psi_t^{Aa} + \psi_t^{Ab}\right) - \eta_t\right)\sigma_t^{qA}]$

- Similarly,

$$\sigma_t^{\eta B} = (1 - \eta_t)(\sigma_t^{n^A B} - \sigma_t^{n^B B})$$

$$\sigma_t^{n^A B} = \frac{(\psi_t^{Aa} + \psi_t^{Ab})}{\eta_t} (\sigma_t^{qB}) \qquad \sigma_t^{n^B B} = \frac{1 - (\psi_t^{Aa} + \psi_t^{Ab})}{1 - \eta_t} (\sigma^B + \sigma_t^{qB})$$

  - Hence, $\sigma_t^{\eta B} = \cdots$

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Step 3. $\sigma^{\eta A}, \sigma^{\eta B}$ Volatility of Wealth Share

- From previous slide
- $\sigma_t^{\eta A} = \frac{1}{\eta_t}\left[(1-\eta_t)\left(\psi_t^{Aa}+\psi_t^{Ab}\right)\sigma^A + \left(\left(\psi_t^{Aa}+\psi_t^{Ab}\right)-\eta_t\right)\sigma_t^{qA}\right]$
- $\sigma_t^{\eta B} = \cdots$

- Note also,

$$\eta_t\sigma_t^{\eta A} + (1-\eta_t)\sigma_t^{1-\eta,A} = 0 \Rightarrow \sigma_t^{1-\eta,A} = -\frac{\eta_t}{1-\eta_t}\sigma_t^{\eta A}$$

2. Three regions of state variable η

- Wealth share η
  - Three regions

		Full specialization	
A produces	a	a	a, b
B produces	a, b	b	b

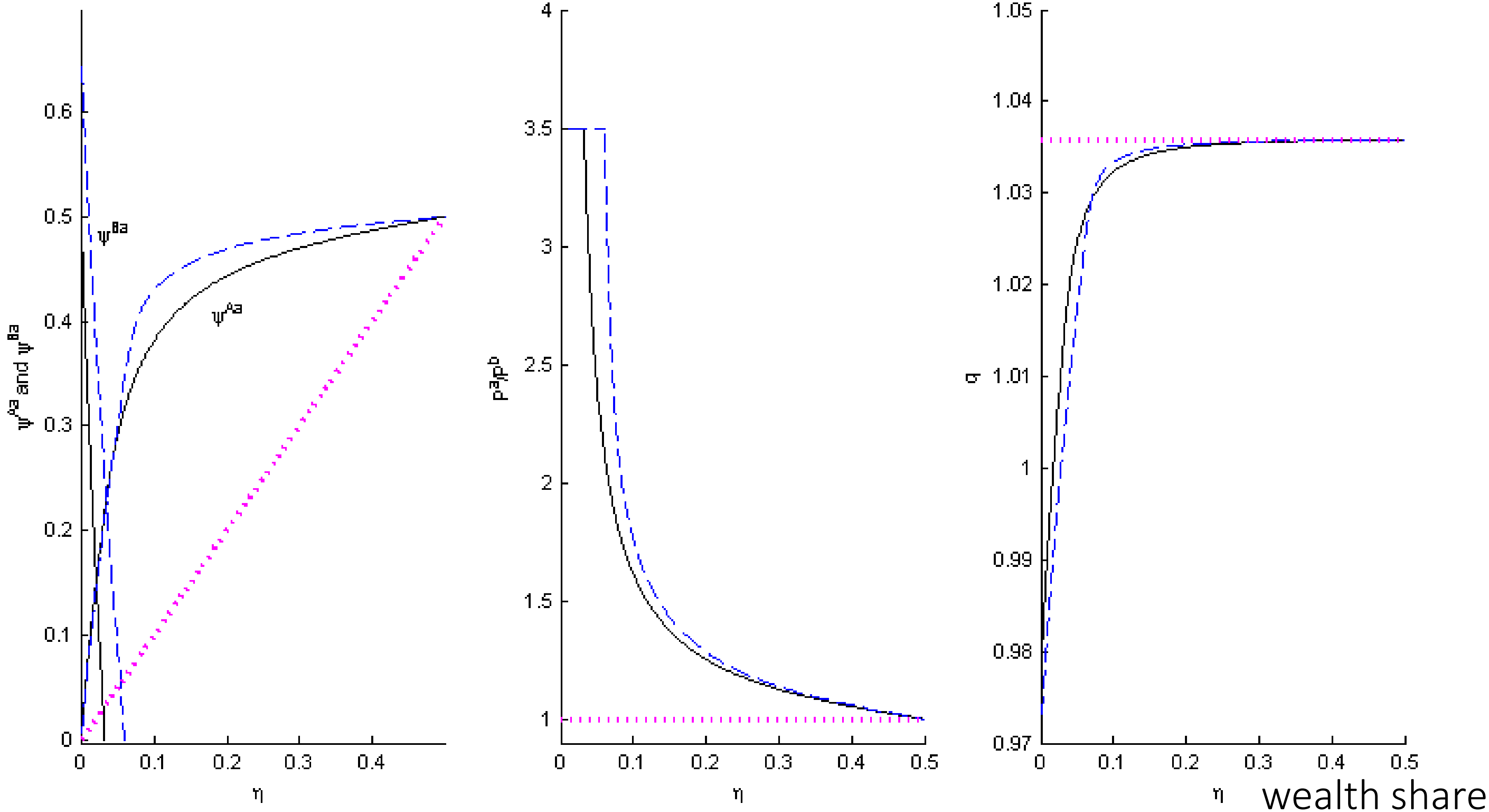
0 $\quad$ $1/2$ $\quad$ 1 $\quad$ η

$$\psi_t^{Aa} = \eta_t$$
$$\psi_t^{Bb} = 1 - \eta_t$$
$$\psi_t^{Ba} = \psi_t^{Ab} = 0$$

  - Symmetric

2. Capital share, terms of trade, price of capital

- Numerical: $\rho = 5\%, \gamma = 1, \overline{a} = 14\%, \underline{a} = 4\%, \delta = 5\%, \kappa = 2, \sigma^A = \sigma^B = 10\%$



- Three different elasticities of substitution: $s = \{.5, 1, \infty\}$

2. TOT: Supply vs. demand shock

- Supply versus demand shock

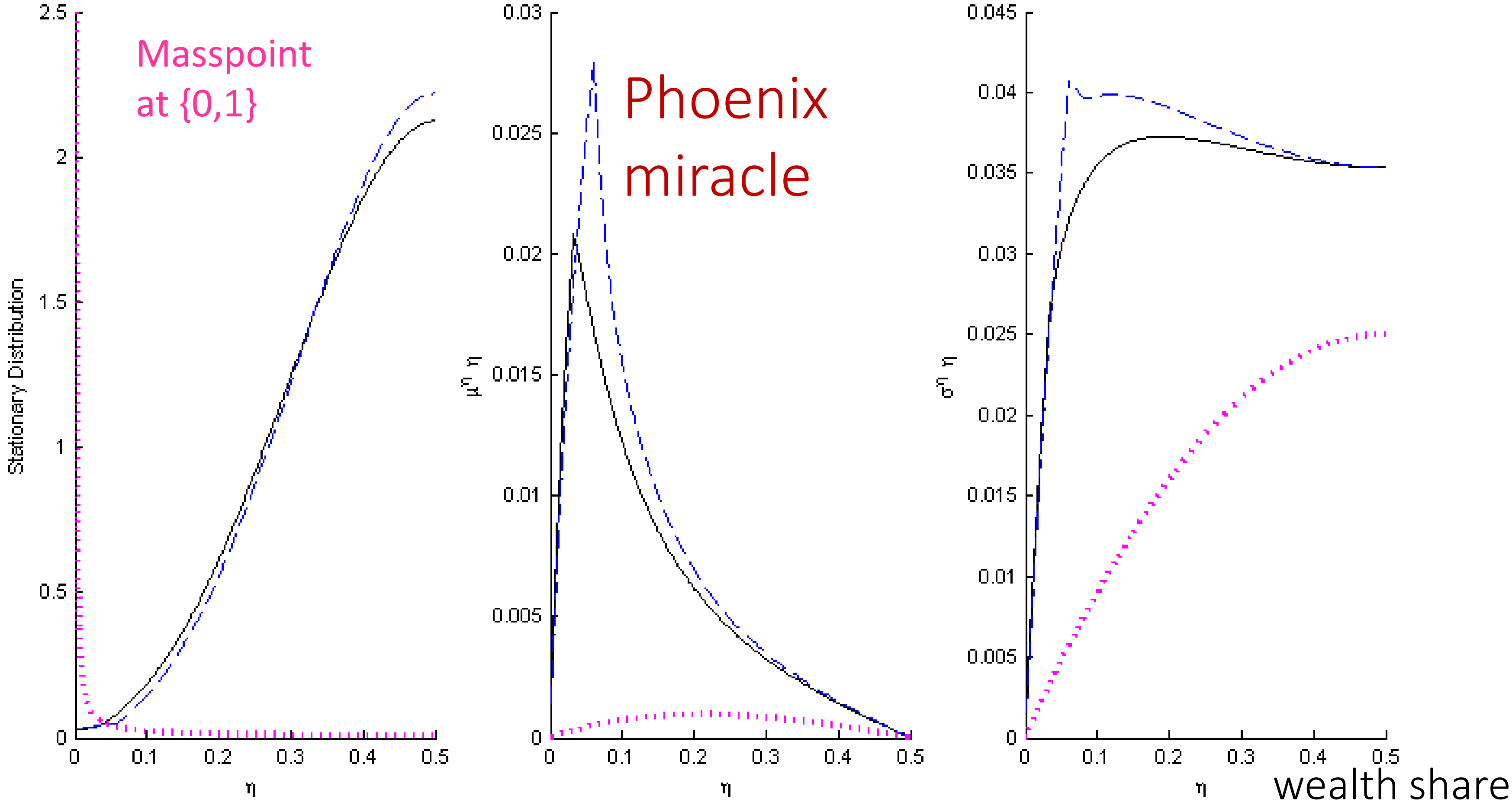
  TOT improve for A as η_t declines for $\eta_t \in [\overline{\eta}, .5)$
  can be due to
  - $dZ^A < 0$: Negative <u>supply shock</u> World recession
  - $dZ^B > 0$: Positive <u>demand shock</u> World boom

- TOT: Output price

- …but fire-sale of (physical) capital stock k_t

2. Stability, Phoenix Miracle for different s

- Stationary distribution drift volatility



- Three different elasticities of substitution: $s = \{.5, 1, \infty\}$
- Difference to Cole & Obstfeld 1991: persistence of capital, $\delta < \infty$

Overview

1. Complete markets ⇒ First best
2. Incomplete markets (equity home bias)
   - Levered short-term debt financing
   - Sudden stops: (varying technological illiquidity)
     - Amplification
     - Runs due to sunspots
3. No equity, no debt
   - Closed capital account: capital controls
4. Welfare analysis

2. Amplification

asset-equity ratio

$$\sigma_t^{\eta A} = \frac{\frac{\psi_t^{Aa}}{\eta_t}(1-\eta_t)}{1 - \frac{\psi_t^{Aa} - \eta_t}{\eta_t} \frac{q'(\eta_t)}{q(\eta_t)/\eta_t}} \sigma^A$$

Market illiquidity (price impact elasticity)

Leverage: debt-equity ratio

- Leverage effect ψ_t^{Aa}/η_t, $(\psi_t^{Aa} - \eta_t)/\eta_t$
-

2. Amplification

asset-equity ratio

$$\sigma_t^{\eta A} = \frac{\frac{\psi_t^{Aa}}{\eta_t}(1-\eta_t)}{1 - \frac{\psi_t^{Aa}-\eta_t}{\eta_t}\frac{q'(\eta_t)}{q(\eta_t)/\eta_t}}\sigma^A$$

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- Leverage effect ψ_t^{Aa}/η_t, $(\psi_t^{Aa}-\eta_t)/\eta_t$
- Loss spiral $1/\{1 - \frac{\psi_t^{Aa}-\eta_t}{\eta_t}\frac{q'(\eta_t)}{q(\eta_t)/\eta_t}\}$ (infinite sum)

2. Amplification

$$\sigma_t^{\eta A} = \frac{\frac{\psi_t^{Aa}}{\eta_t}(1-\eta_t)}{1 - \frac{\psi_t^{Aa} - \eta_t}{\eta_t} \frac{q'(\eta_t)}{q(\eta_t)/\eta_t}} \sigma^A$$

asset-equity ratio

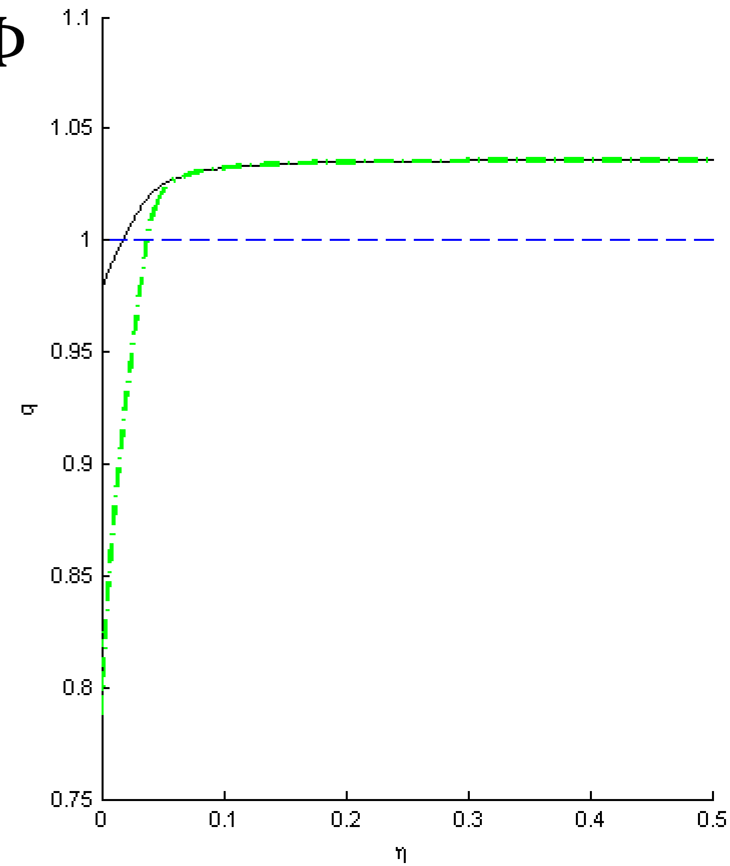
Market illiquidity (price impact elasticity)

Leverage: debt-equity ratio

- Leverage effect ψ_t^{Aa}/η_t, $(\psi_t^{Aa} - \eta_t)/\eta_t$
- Loss spiral $1/\{1 - \frac{\psi_t^{Aa}-\eta_t}{\eta_t} \frac{q'(\eta_t)}{q(\eta_t)/\eta_t}\}$ (infinite sum)
- Technological illiquidity $(\kappa, \delta) \Rightarrow$ market illiquidity $q'(\eta)$
  - (dis)investment adjustment cost

2. Technological $(\kappa, \delta) \Rightarrow$ market illiquidity $q'(\eta)$

- Quadratic adjustment cost
- Investment rate of $\iota = \Phi + \frac{1}{\kappa}\Phi^2$ generates new capital at rate Φ
- $\Phi(\iota) = \frac{1}{\kappa}\left(\sqrt{1 + 2\kappa\iota} - 1\right)$
- Three cases
  - $\kappa = 0 \Rightarrow q = 1$
  - $\kappa = 2$
  - $\kappa_{\iota<0} = 100$ and $\kappa_{\iota>0} = 2$



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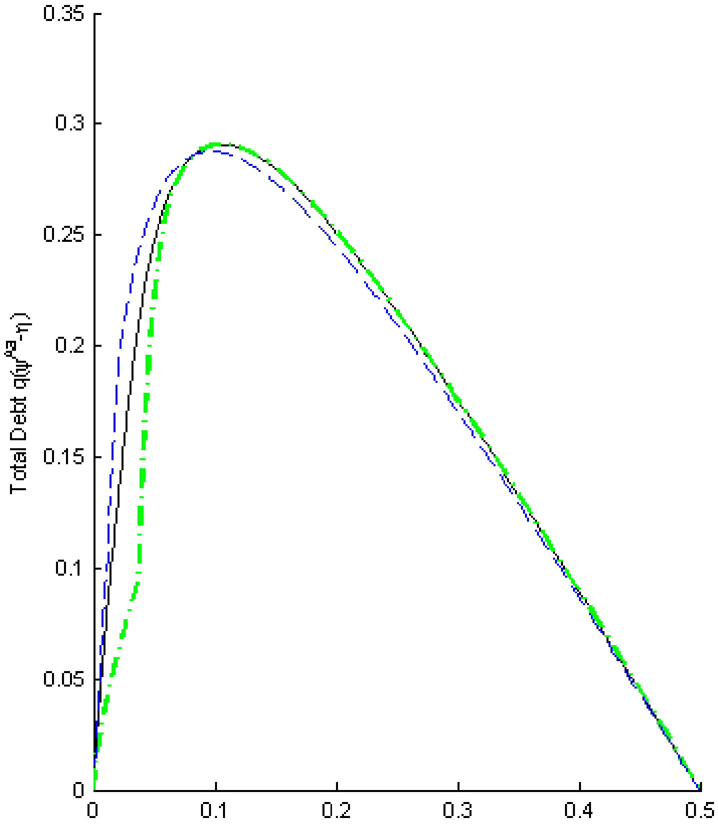
2. Sudden stops: amplification & runs

- Sudden stop
  - Adverse fundamental triggers %-decline in debt that exceeds %-decline in net worth; $\frac{\partial(\psi^{Aa}-\eta)}{\partial\eta}\frac{\eta}{\psi^{Aa}-\eta} > 1 \Leftrightarrow \frac{\partial\psi^{Aa}}{\partial\eta} > \frac{\psi^{Aa}}{\eta}$ $\Leftrightarrow$pro-cyclical leverage



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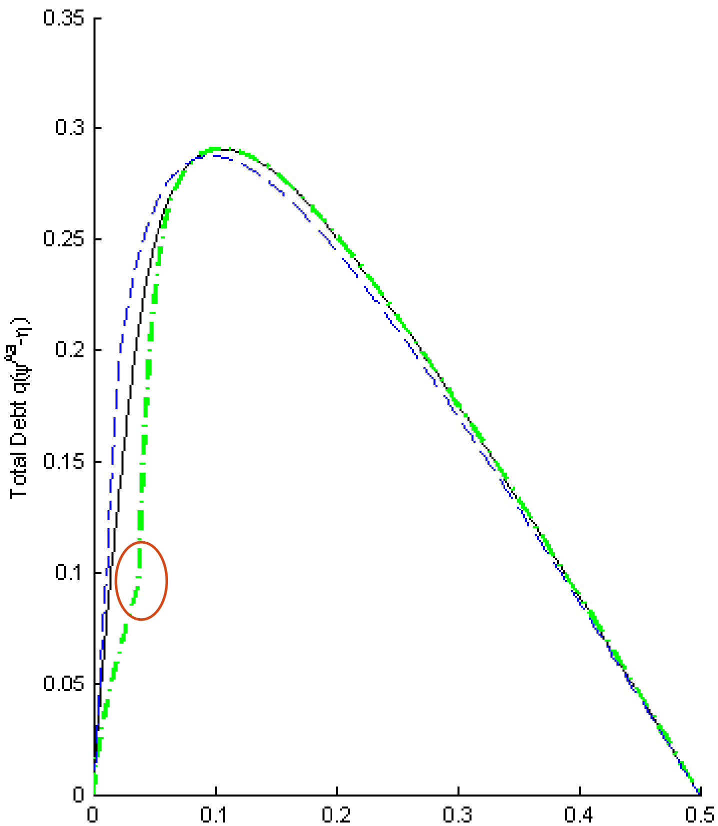
    $\Leftrightarrow$pro-cyclical leverage

Slope of tangent vs. secant



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2. Sudden stops: amplification & runs

Sudden stop

- Adverse fundamental triggers %-decline in debt that exceeds %-decline in net worth; $\frac{\partial(\psi^{Aa}-\eta)}{\partial\eta}\frac{\eta}{\psi^{Aa}-\eta} > 1 \Leftrightarrow \frac{\partial\psi^{Aa}}{\partial\eta} > \frac{\psi^{Aa}}{\eta}$ $\Leftrightarrow$ pro-cyclical leverage

- An unanticipated sunspot triggers a sudden capital price drop from q to $\tilde{q}$, accompanied by a drop in η to $\tilde{\eta}$.

$$\tilde{q}\tilde{\eta} = \max\{\eta q + \psi^{Aa}(\tilde{q} - q), 0\}$$

2. Sudden stops: amplification & runs

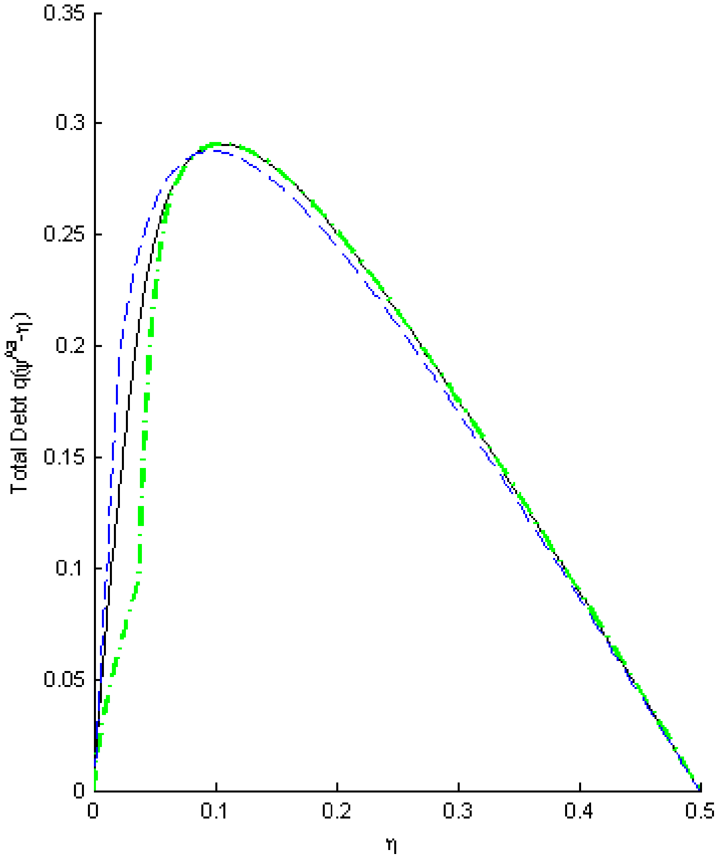
- Sudden stop
  - Adverse fundamental triggers %-decline in debt that exceeds %-decline in net worth; $\frac{\partial(\psi^{Aa}-\eta)}{\partial\eta}\frac{\eta}{\psi^{Aa}-\eta} > 1 \Leftrightarrow \frac{\partial\psi^{Aa}}{\partial\eta} > \frac{\psi^{Aa}}{\eta}$ $\Leftrightarrow$ pro-cyclical leverage

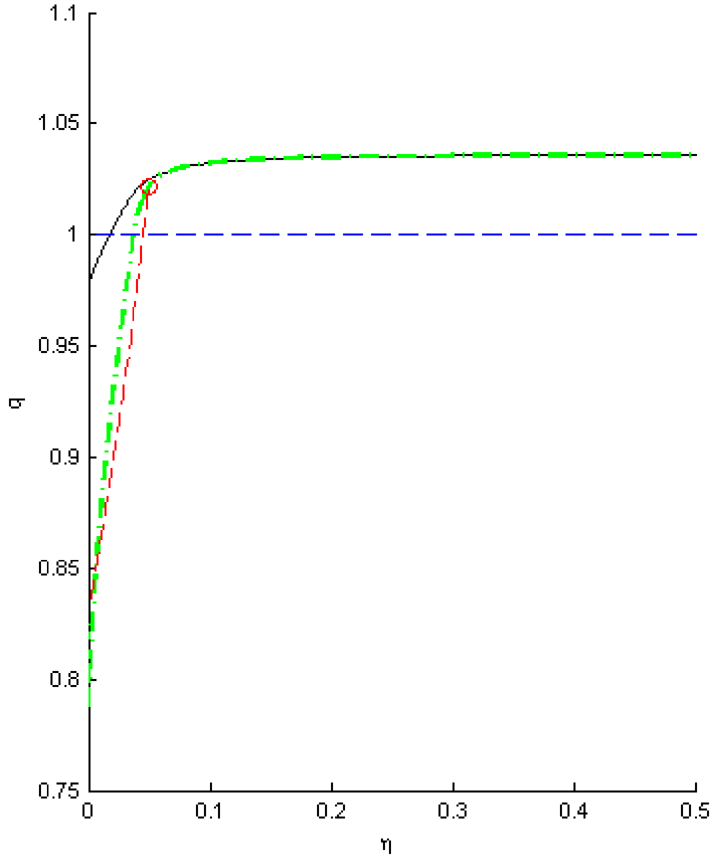
  - An unanticipated sunspot triggers a sudden capital price drop from q to $\tilde{q}$, accompanied by a drop in η to $\tilde{\eta}$.

$$\tilde{q} = \frac{\max\{\eta q + \psi^{Aa}(\tilde{q} - q), 0\}}{\tilde{\eta}}$$

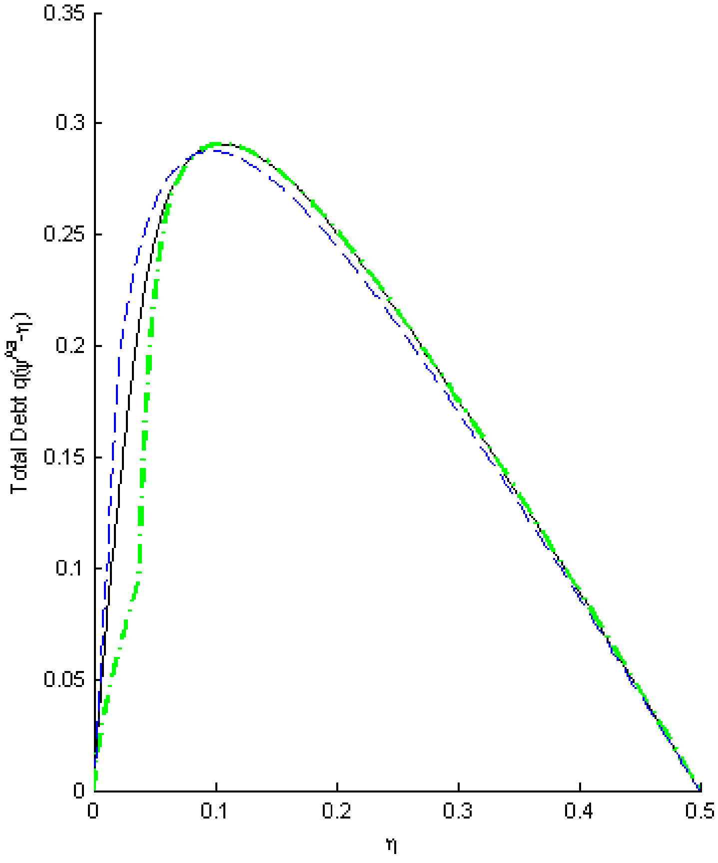
hyperbola

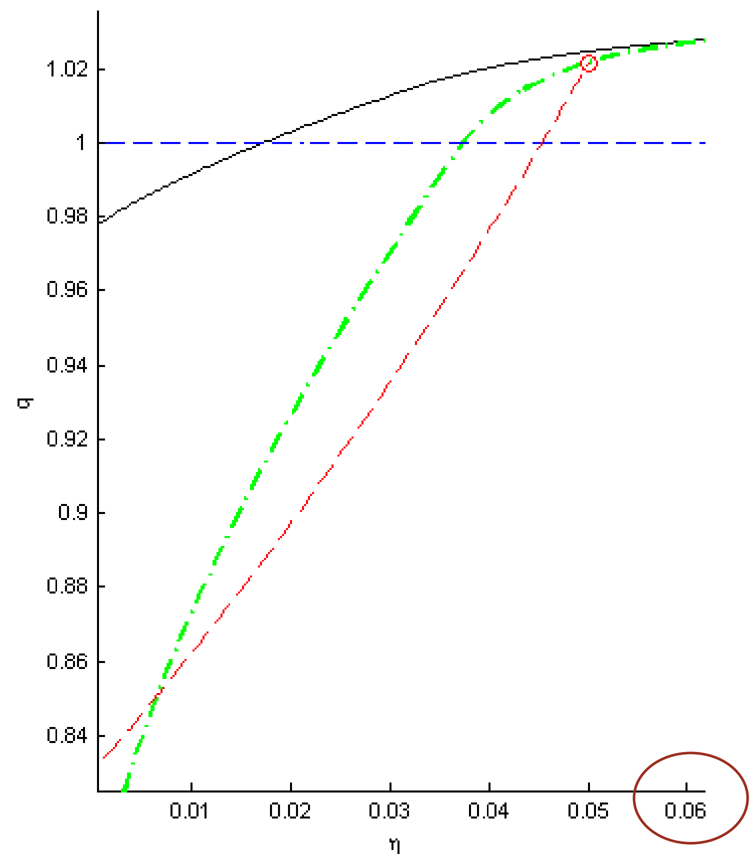
2. Sudden stops: amplification & runs





2. Sudden stop due to run: Zoomed in





Overview

1. Complete markets ⇒ First best
2. Incomplete markets (equity home bias)
   - Levered short-term debt financing
   - Sudden stops: (varying technological illiquidity)
     - Amplification
     - Runs due to sunspots
3. No equity, no debt
   - Closed capital account: capital controls
4. Welfare analysis

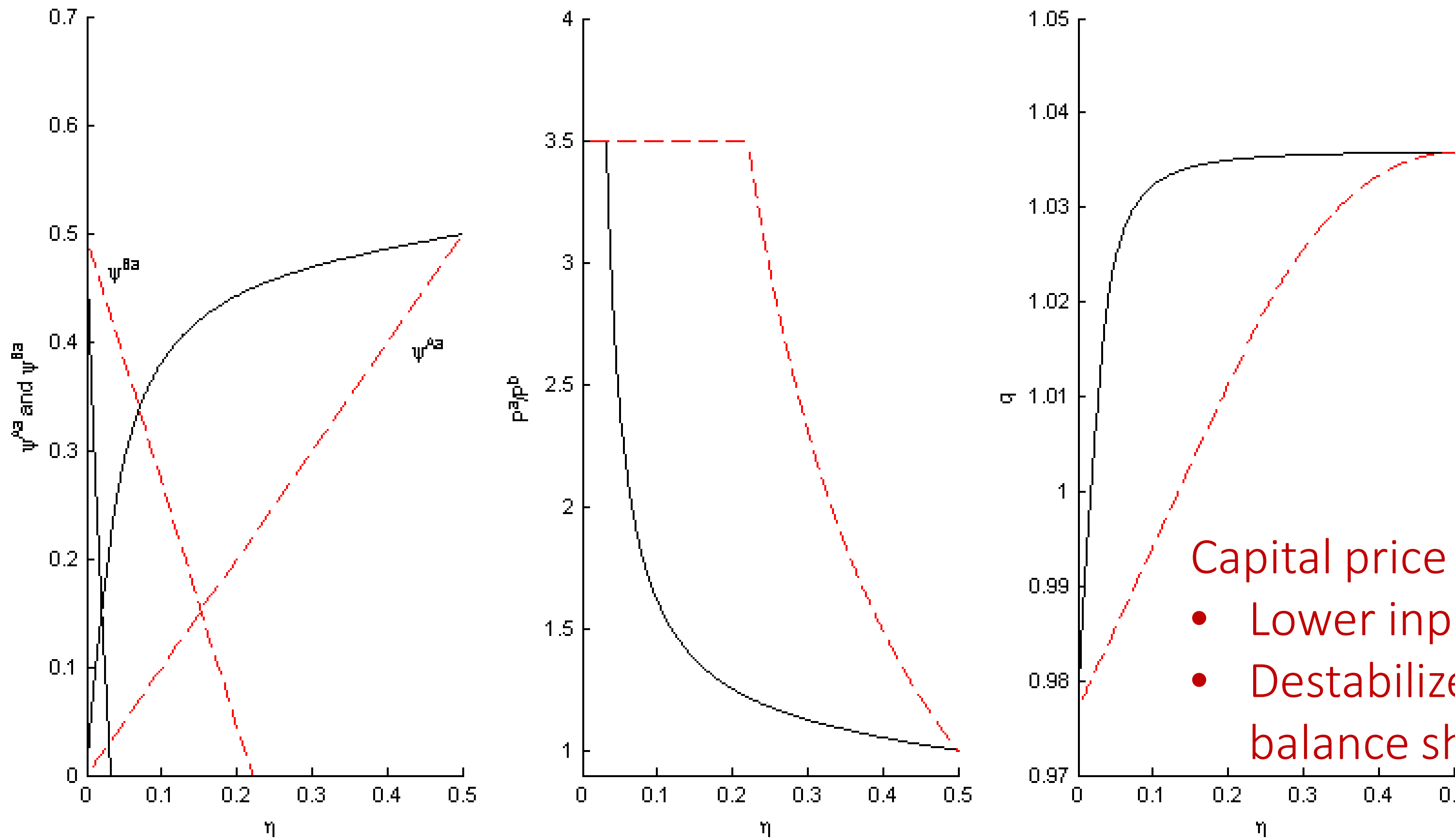
Market structures

	Trade		Finance	
Markets	Output y^a, y^b	Physical capital K	Debt	Equity
Complete Markets Full integration/First Best	X	X	X	X
Open credit account (equity home bias)	X	X	X	
Closed credit account	X	X		
	intratemporal		intertemporal	

Add taxes/capital controls

3. Credit account: open vs. closed

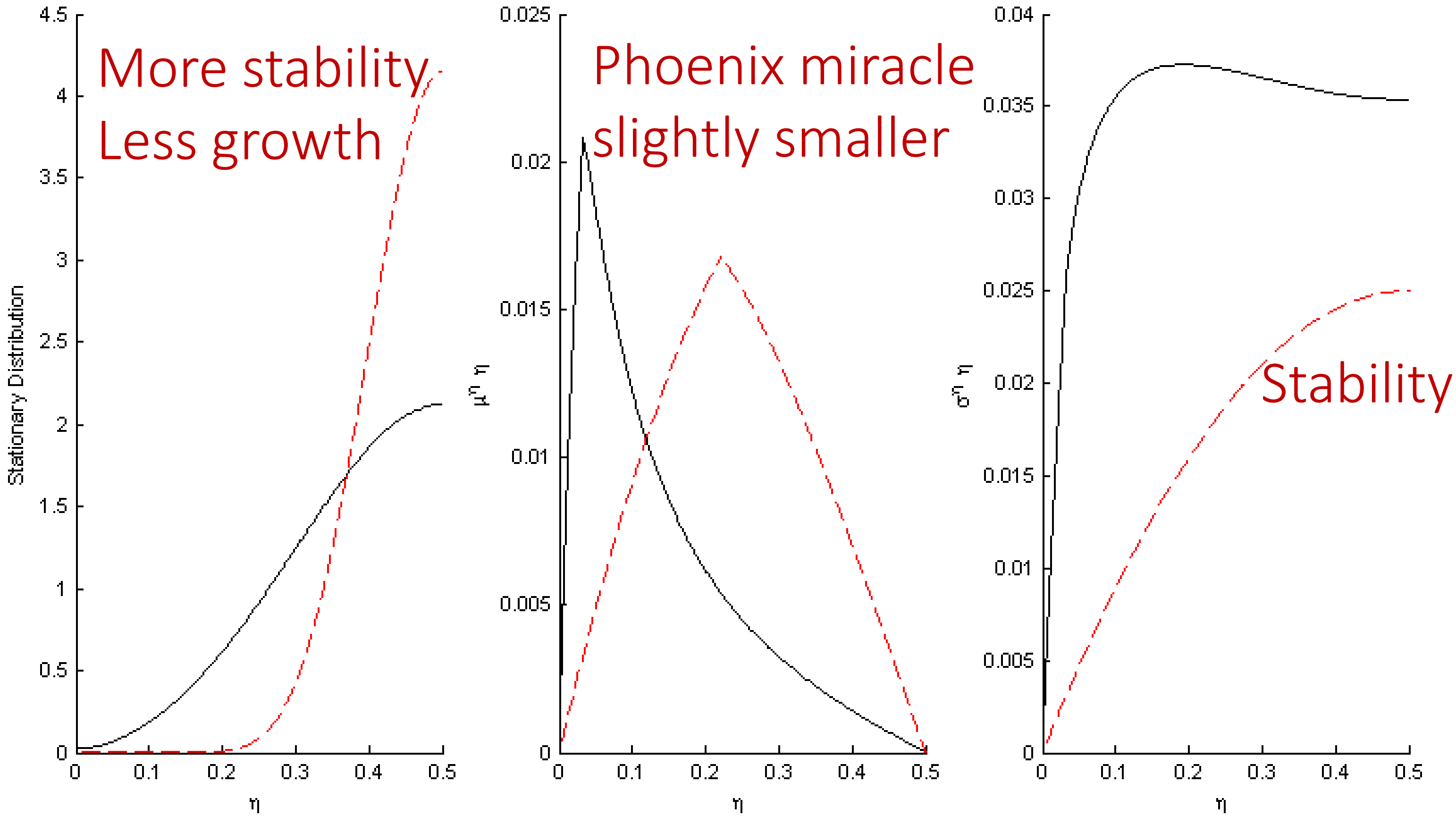
- $\rho = 5\%, \gamma = 1, \overline{a} = 14\%, \underline{a} = 4\%, \delta = 5\%, \kappa = 2, \sigma^A = \sigma^B = 10\%,$ $s = 1$



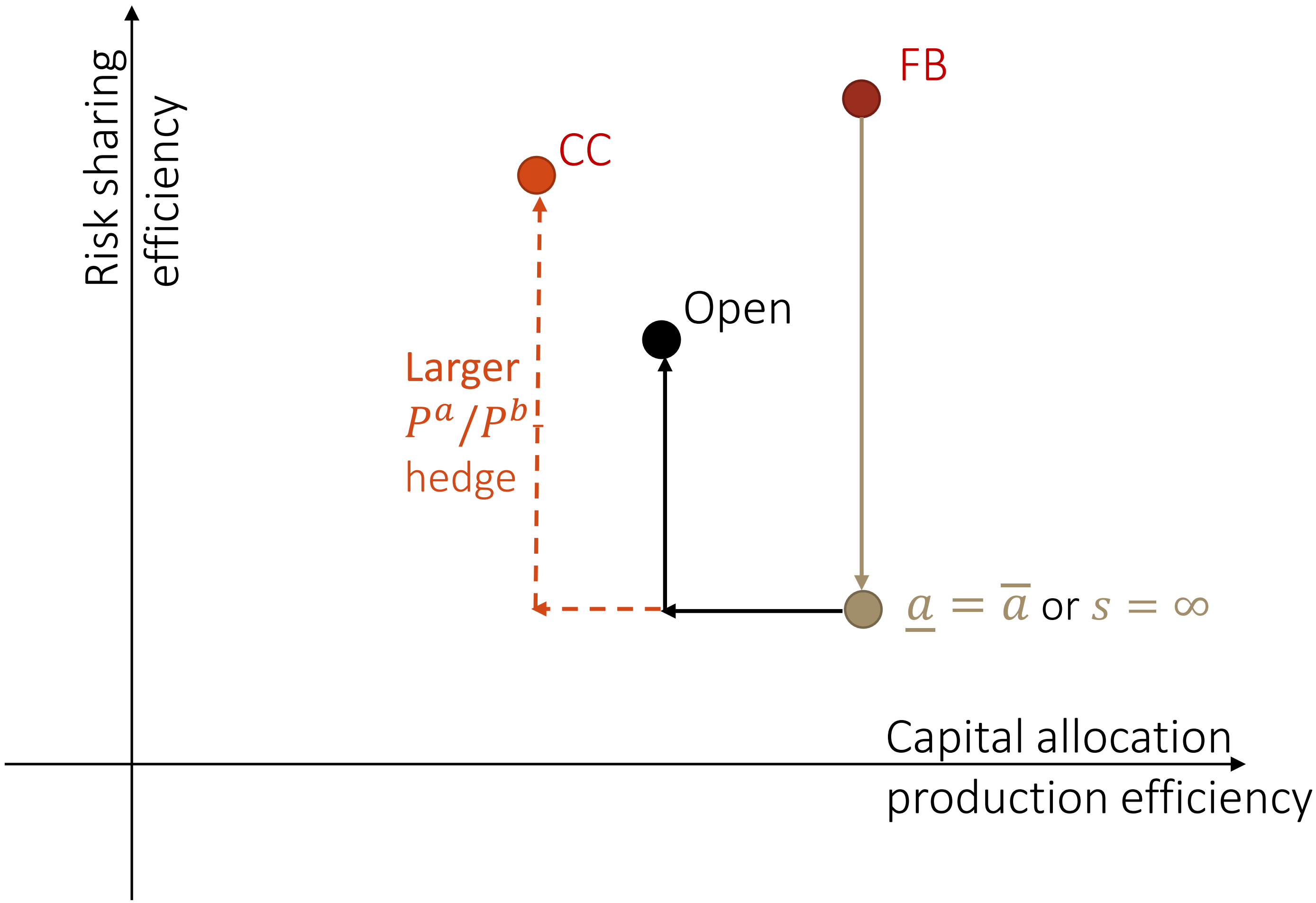
Capital price lower
- Lower input price
- Destabilizes balance sheet

3. Credit account: open vs. closed

- $\rho = 5\%, \overline{a} = 14\%, \underline{a} = 4\%, \delta = 5\%, \kappa = 2, \sigma^A = \sigma^B = 10\%,\ s = 1$



3. Efficiency trade-off



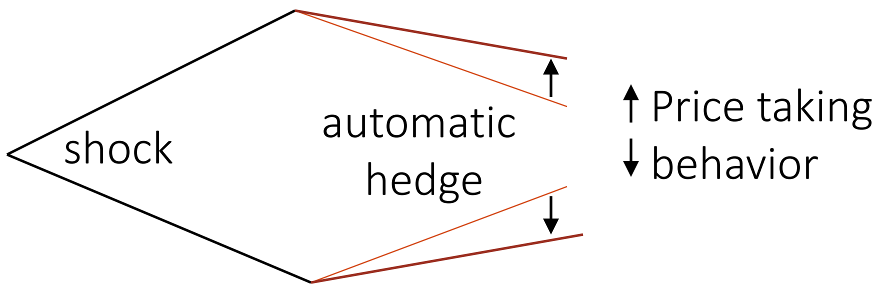
- Affect all subsequent dynamics

Overview

1. Complete markets ⇒ First best
2. Incomplete markets (equity home bias)
3. No equity, no debt: Closed capital account
4. Welfare analysis
   - Pecuniary externalities
   - Welfare calculations + Pareto improving redistributions

4. When are credit flows excessive?

- Constrained inefficiency (in incomplete market setting) due to pecuniary externalities
  - Price of capital: fire sale externality if leverage is high
  - Price of output good: "terms of trade hedge" restrained competition
    - Price taking behavior undermined this hedge



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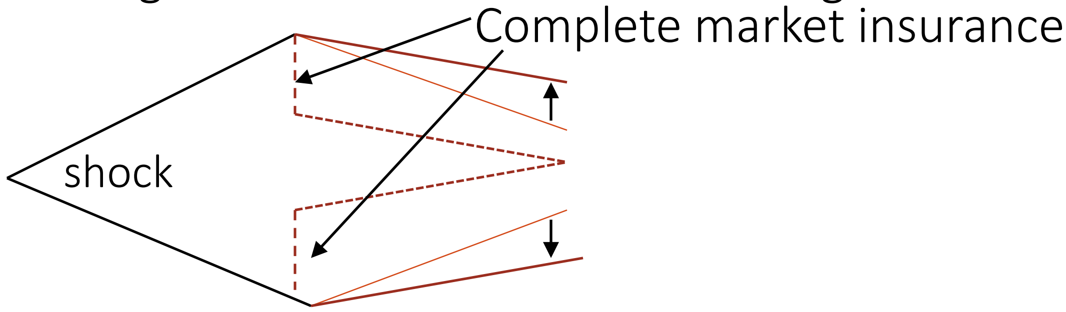

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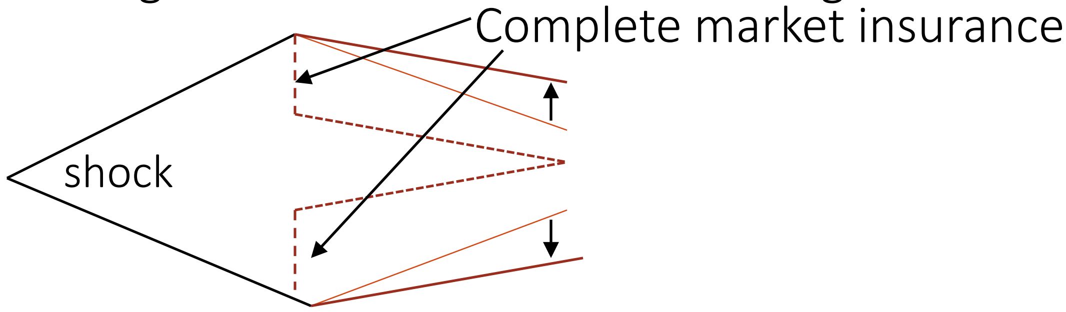
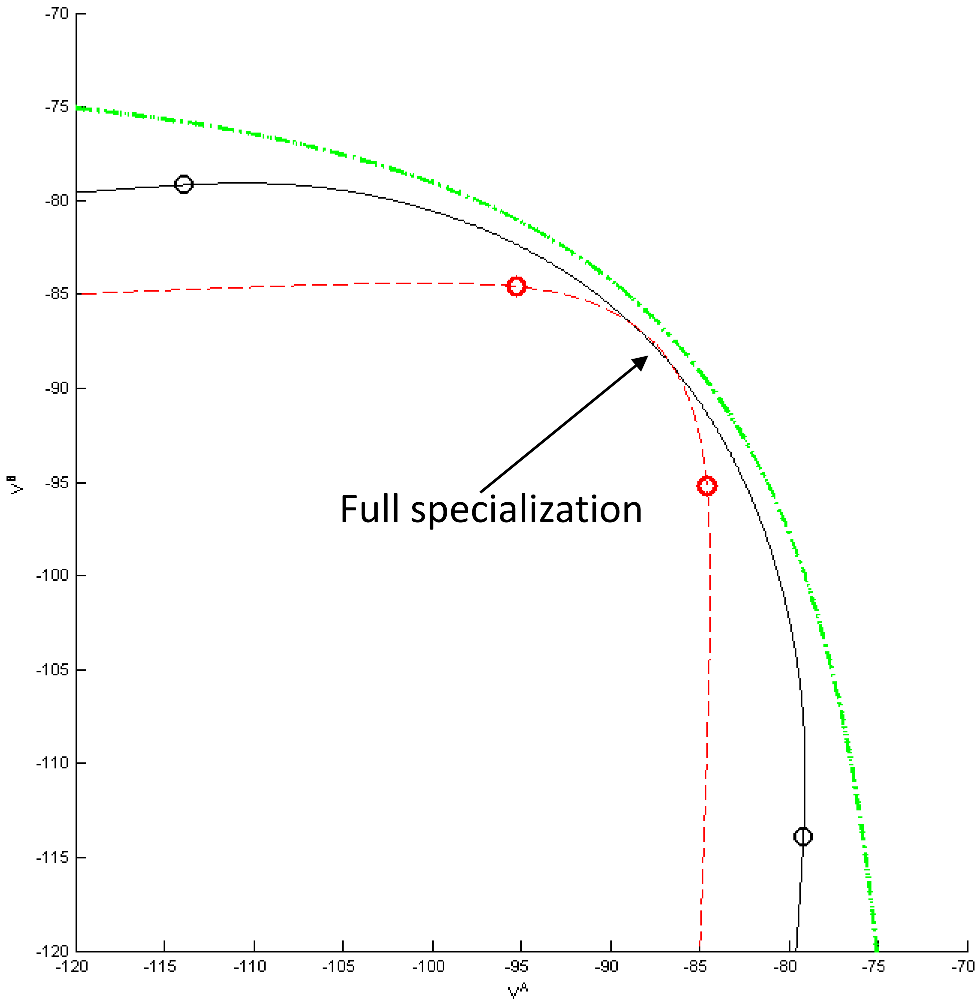


Price	Intention	Depends on
Capital price (input)	Buy cheaper but capital losses on existing k_t	Adjustment cost, $\Phi(\iota)$, κ
Output price	Sell output more expensive	Elasticity of substitution, s
Interest rate	Borrow cheaper	Intertemporal preference

4. Welfare comparison

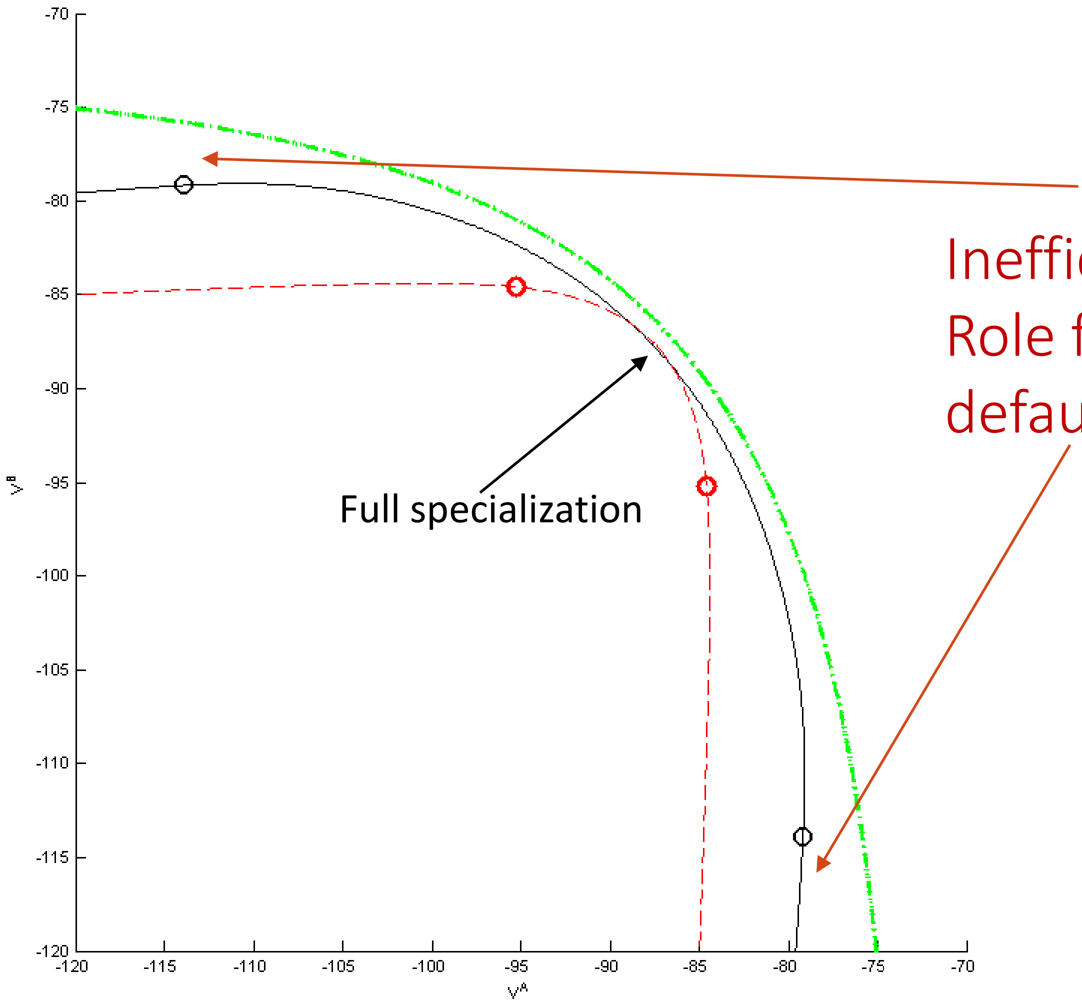
- $\rho = 5\%, \gamma = 1, \overline{a} = 14\%, \underline{a} = 4\%, \delta = 5\%, \kappa = 2, \sigma^A = \sigma^B = 10\%$



- No friction, first best
- No equity
- No equity, no debt

4. Welfare comparison

- $\rho = 5\%, \gamma = 1, \overline{a} = 14\%, \underline{a} = 4\%, \delta = 5\%, \kappa = 2, \sigma^A = \sigma^B = 10\%$



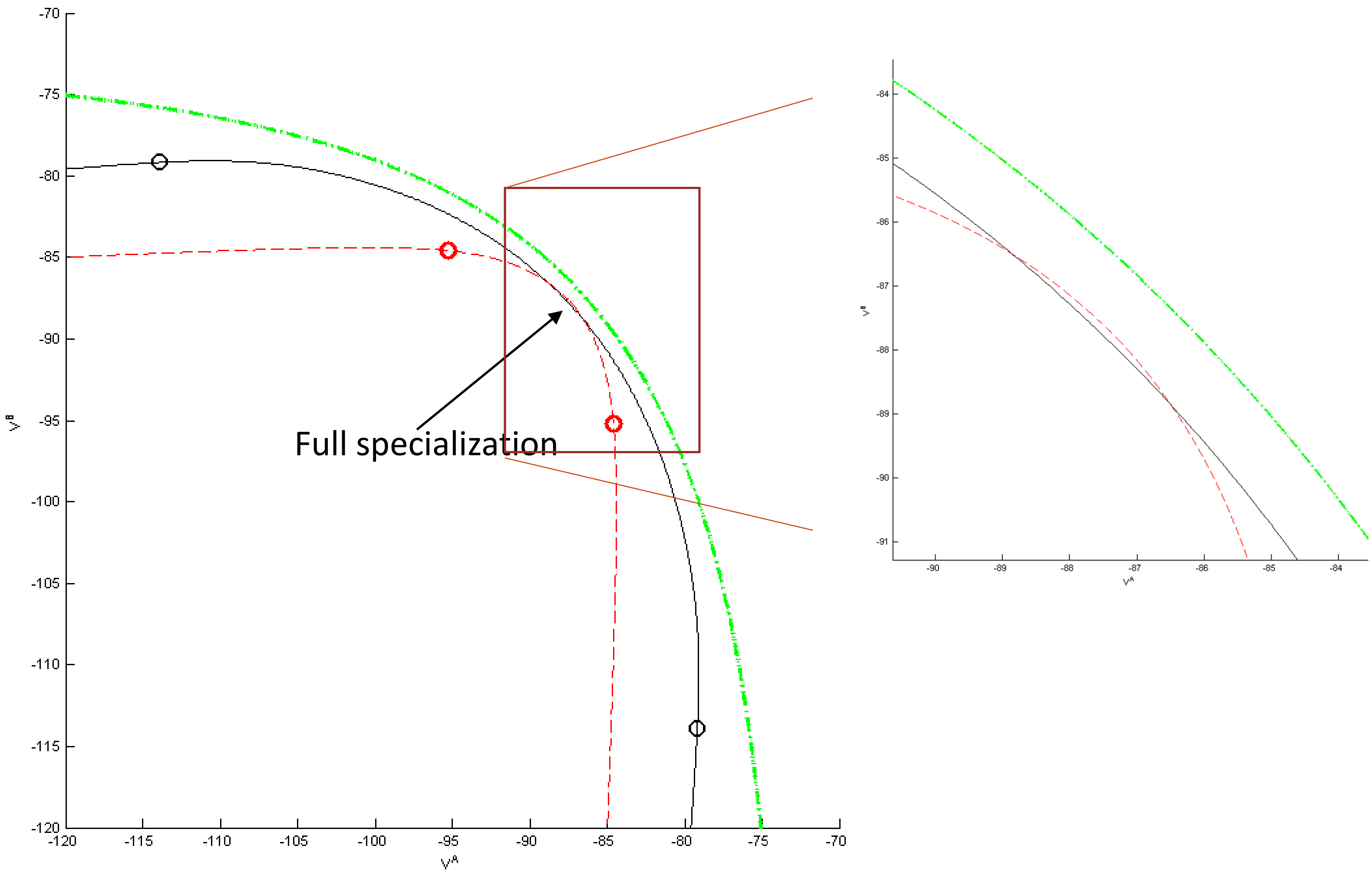
Inefficiency at the extremes:
Role for redistributive Policy
default/bail-out/debt-relief

Pareto improving

Intuition:
Other country's output price is high

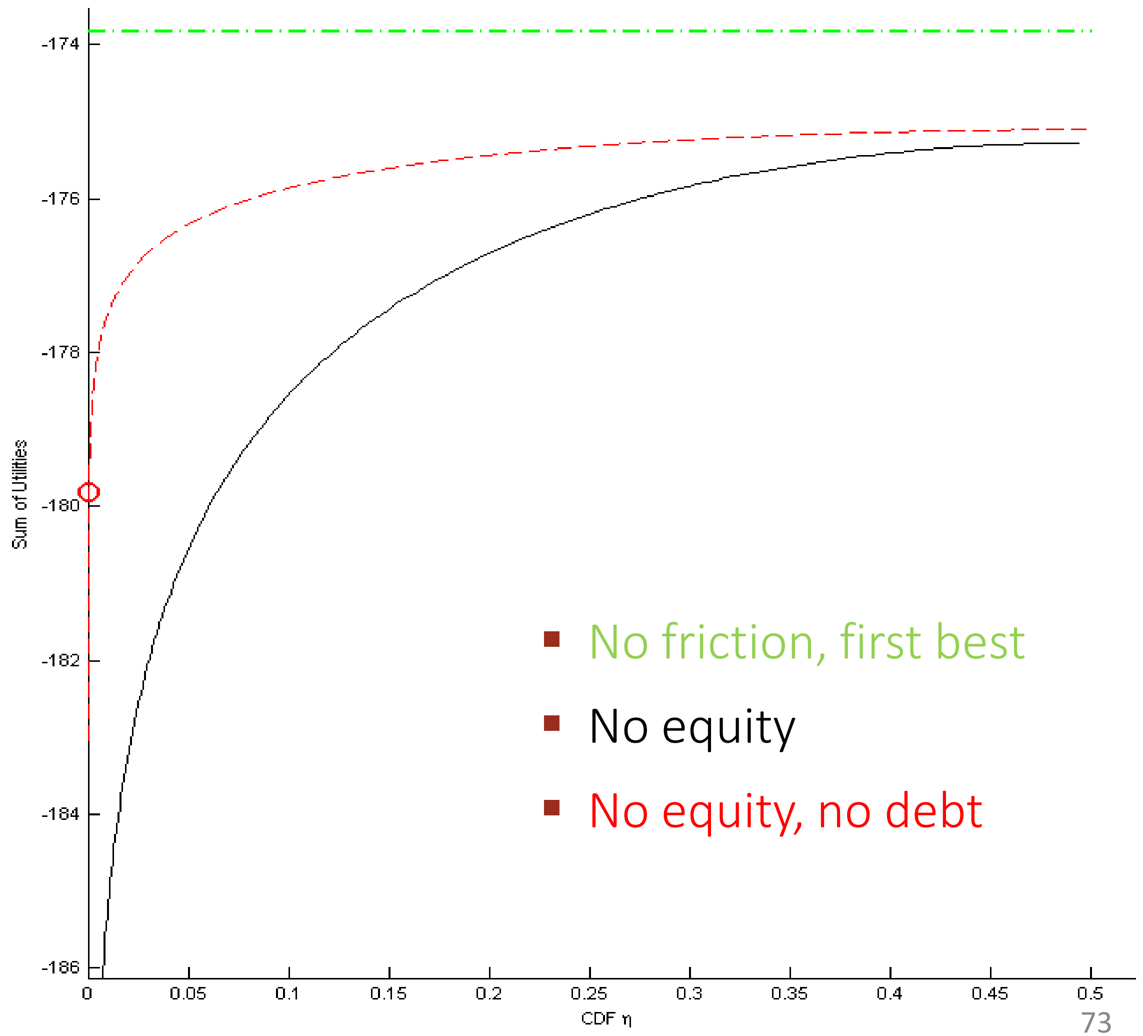
4. Welfare comparison

- $\rho = 5\%, \gamma = 1, \bar{a} = 14\%, \underline{a} = 4\%, \delta = 5\%, \kappa = 2, \sigma^A = \sigma^B = 10\%$



4. Welfare comparison

- Any monotone transformation of η would be equally good state variable
- Normalization:
  take CDF of η
  - Uniform stationary distribution!



- No friction, first best
- No equity
- No equity, no debt

Conclusion

- Symmetric setup (productivity, discount rate, …)
  - Derive $A(\psi)$
- Sudden stops
  - Amplification of fundamental shock
  - Runs due to sunspots – vulnerability region
- Phoenix miracle

- Tradeoff between capital allocation & risk sharing
  - "Terms of trade hedge"
- When are short-term credit flows excessive?
  - When can capital controls (financial liberalization) be welfare enhancing (reducing)?
  - Pecuniary externality
    - Price of physical capital   fire-sales externality – technological illiquidity
    - Price of output goods: "terms of trade hedge" externality
- Bailout/Restructuring
  Redistributive policy can be Pareto improving if one country is sufficiently balance sheet impaired
  - Reduces output good price

Next to do … Problem Set

- Solve model with CRRA utility functions numerically
  - Follow steps from previous lecture
- Allow for idiosyncratic risk in one country
- Plot fan charts and distribution impulse response functions
- Allow for anticipated jumps
  - Incorporate (compensated) jump process in probability space/proposed processes