

ECO 529: Lecture 8

Optimal policy in models of money and intermediation

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Goals

- Build some theoretical tools for finding and analyzing optimal policy
- One side: inefficiencies / tradeoffs
 - insurance vs. investment (one agent type)
 - allocation of assets / risk (across sectors / types)
- Other side: policy space
 - (1) controlling money growth rate
 - (2) risk redistribution
 - (3) macroprudential tools / wealth redistribution

Welfare with log utility

- Class of models: price capital q_t , value of money $p_t K_t$, two types of agents I and H, have wealth shares η_t and $1 - \eta_t$, idiosyncratic risk exposures

$$\tilde{\sigma}_t^I \quad \text{and} \quad \tilde{\sigma}_t^H$$

- Then the welfare of I is (similar formula for H)

$$E \left[\int_0^{\infty} e^{-\rho t} \log(c_t^I) dt \right] = E \left[\int_0^{\infty} e^{-\rho t} \log(\eta_t (a(\psi_t) - \iota_t) K_t \tilde{\eta}_t^I) dt \right],$$

$$\tilde{\eta}_0^I = 1, \quad \frac{d\tilde{\eta}_t^I}{\tilde{\eta}_t^I} = \tilde{\sigma}_t^I d\tilde{Z}_t$$

Welfare with log utility

- The welfare of I is

$$\begin{aligned}
 E \left[\int_0^{\infty} e^{-\rho t} \log(\eta_t (a(\psi_t) - \iota_t) K_t \tilde{\omega}_t) dt \right] &= E \left[\int_0^{\infty} e^{-\rho t} \log \eta_t dt \right] + \\
 &\quad \frac{\log \eta_0}{\rho} + E \left[\int_0^{\infty} e^{-\rho t} \left(\frac{\mu_t^\eta}{\rho} - \frac{|\sigma_t^\eta|^2}{2\rho} \right) dt \right] \\
 E \left[\int_0^{\infty} e^{-\rho t} \log(a(\psi_t) - \iota_t) dt \right] &+ E \left[\int_0^{\infty} e^{-\rho t} \log K_t dt \right] + E \left[\int_0^{\infty} e^{-\rho t} \log \tilde{\eta}_t dt \right] \\
 &\quad \frac{\log K_0}{\rho} + E \left[\int_0^{\infty} e^{-\rho t} \left(\frac{\Phi(\iota_t) - \delta}{\rho} - \frac{|\sigma_t^K|^2}{2\rho} \right) dt \right] \quad - E \left[\int_0^{\infty} e^{-\rho t} \frac{(\tilde{\sigma}_t^I)^2}{2\rho} dt \right]
 \end{aligned}$$

$$\tilde{\eta}_0^I = 1, \quad \frac{d\tilde{\eta}_t^I}{\tilde{\eta}_t^I} = \tilde{\sigma}_t^I d\tilde{Z}_t$$

Welfare

- We see that policy can affect welfare in several ways

$$E \left[\int_0^{\infty} e^{-\rho t} \log \eta_t dt \right] + E \left[\int_0^{\infty} e^{-\rho t} \log(a(\psi_t) - \iota_t) dt \right] +$$
$$E \left[\int_0^{\infty} e^{-\rho t} \left(\frac{\Phi(\iota_t) - \delta}{\rho} - \frac{|\sigma_t^K|^2}{2\rho} \right) dt \right] - E \left[\int_0^{\infty} e^{-\rho t} \frac{(\tilde{\sigma}_t^I)^2}{2\rho} dt \right]$$

- investment vs. consumption
- allocation of capital – idiosyncratic risk, total output
- η – the distribution of consumption and risk absorption capacity

One at a time: policy tools and equilibrium features

- Generally, idiosyncratic risk exposures $\tilde{\sigma}_t^I$ and $\tilde{\sigma}_t^H$ are stochastic (depend on η , risk absorption capacity, allocation)
- If intermediaries help reduce idiosyncratic risk, these may rise when η declines
- Let's see, how this matters with a simple model

Stochastic idiosyncratic risk

- One type of agents H, idiosyncratic risk of capital is stochastic (hence it is a state variable)

$$d\tilde{\sigma}_t = \tilde{\mu}(\tilde{\sigma}_t)dt + \tilde{\nu}(\tilde{\sigma}_t)dZ_t$$

e.g. as in Di Tella, CIR process

$$d\tilde{\sigma}_t = \lambda(\bar{\sigma} - \tilde{\sigma}_t)dt + \nu\sqrt{\tilde{\sigma}_t}dZ_t$$

- Global wealth as numeraire, agents' entire portfolio has return ρ (just the consumption rate)
- Money has return

$$\mu_t^\vartheta dt + \sigma_t^\vartheta dZ_t - \hat{\mu}_t^M dt$$

rate of money printing, which is distributed to capital

Remarks

- Printing money to distribute
 - 1) proportionately to money holdings
 - 2) proportionately to capital holdings
 - 3) proportionately to net worth
 - 4) per capita

Remarks

- Printing money to distribute
 - 1) proportionately to money holdings
 - this has no real effect, only nominal
 - 2) proportionately to capital holdings
 - money return goes down by $\hat{\mu}_t^M$
 - capital return goes up by $\vartheta \hat{\mu}_t^M / (1 - \vartheta)$
 - pushes people to hold more capital, invest more
 - 3) proportionately to net worth
 - with printing rate μ_t^M , $\hat{\mu}_t^M = (1 - \vartheta)\mu_t^M$ goes to capital (effect same as in 2) rest goes to money (same as 1)
 - 4) per capita
 - no real effect – people simply borrow against the transfers they expect to receive

Stochastic idiosyncratic risk

- Global wealth as numeraire, wealth has return ρ
Money has return

$$\mu_t^\vartheta dt + \sigma_t^\vartheta dZ_t - \hat{\mu}_t^M dt$$

- Money valuation equation

$$\rho - (\mu_t^\vartheta - \hat{\mu}_t^M) = \underbrace{(1 - \vartheta_t) \tilde{\sigma}_t}_{\text{idiosync. risk of wealth}} \underbrace{(1 - \vartheta_t) \tilde{\sigma}_t}_{\text{price of idiosync. risk}} \underbrace{-\sigma_t^\vartheta}_{\substack{\text{agg. risk} \\ \text{of wealth} \\ \text{rel. to money}}} \underbrace{0}_{\substack{\text{price of} \\ \text{agg. risk}}}$$

- Without policy, equation

$$\rho - \mu_t^\vartheta = (1 - \vartheta_t)^2 \tilde{\sigma}_t^2$$

has a unique solution in $\vartheta(\tilde{\sigma}_t) \in (0,1)$ (if idiosyncratic risk is sufficiently large)

Optimal Policy

- Market-clearing for output

$$a - \iota(q) = \rho \underbrace{\frac{q}{1 - \vartheta}}_{p+q}, \quad \text{if } \Phi(\iota) = \frac{\log(\kappa \iota + 1)}{\kappa}, \quad \iota(q) = \frac{q - 1}{\kappa}, \quad q = \frac{(a\kappa + 1)(1 - \vartheta)}{\rho\kappa + 1 - \vartheta}$$

- welfare is $\frac{\log K_0}{\rho} - \frac{\delta}{\rho^2} +$

$$\underbrace{E \left[\int_0^{\infty} e^{-\rho t} \log(a - \iota_t) dt \right]}_{E \left[\int_0^{\infty} e^{-\rho t} \log \left(\rho \frac{a\kappa + 1}{\rho\kappa + 1 - \vartheta_t} \right) dt \right]} + \underbrace{E \left[\int_0^{\infty} e^{-\rho t} \frac{\Phi(\iota_t)}{\rho} dt \right]}_{\frac{1}{\rho\kappa} E \left[\int_0^{\infty} e^{-\rho t} \log \left(\frac{(a\kappa + 1)(1 - \vartheta_t)}{\rho\kappa + 1 - \vartheta_t} \right) dt \right]} - E \left[\int_0^{\infty} e^{-\rho t} \frac{(1 - \vartheta_t)^2 \tilde{\sigma}_t^2}{2\rho} dt \right]$$

- let $\vartheta^*(\tilde{\sigma}^2)$ be the maximizer of (optimal baseline policy)

$$\frac{1}{\rho\kappa} \log(1 - \vartheta) - \frac{\rho\kappa + 1}{\rho\kappa} \log(\rho\kappa + 1 - \vartheta) - \frac{(1 - \vartheta)^2 \tilde{\sigma}^2}{2\rho}$$

Optimal policy

- If the planner could control ϑ_t directly, she would set $\vartheta_t = \vartheta^*(\tilde{\sigma}_t^2)$
- Controlling indirectly by choosing $\hat{\mu}_t^M$ the planner can achieve any function - including $\vartheta^*(\tilde{\sigma}_t^2)$ - by solving

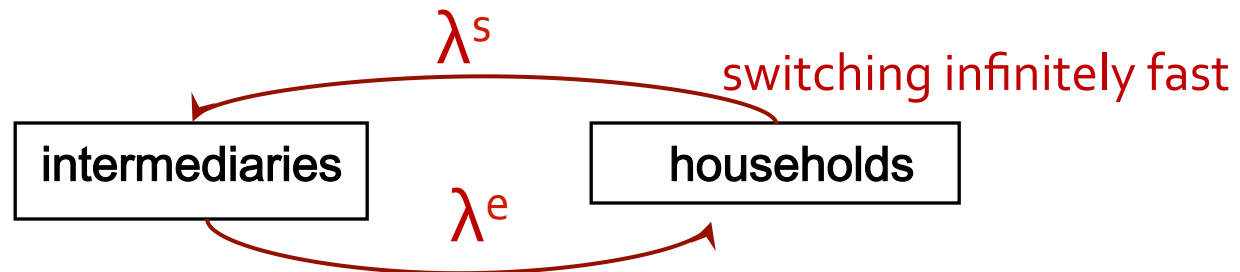
$$\rho - (\mu_t^\vartheta - \hat{\mu}_t^M) = (1 - \vartheta_t)^2 \tilde{\sigma}_t^2$$

for $\hat{\mu}_t^M$

- Optimal policy is easier to find than even the equilibrium outcome (differentiation vs. integration)
- Risk-free rate $\Phi(\iota_t) - \delta + \mu_t^\vartheta - \hat{\mu}_t^M = \rho - (1 - \vartheta_t)^2 \tilde{\sigma}_t^2 + \Phi(\iota_t) - \delta$ declines as $\tilde{\sigma}_t^2$ increases
- Nice relationship b/w baseline and dynamic model

- The relationship between idiosyncratic risk level $\tilde{\sigma}_t$ and optimal insurance $\vartheta^*(\tilde{\sigma}_t^2)$ is quite pervasive
- Let's consider another model, with heterogeneous agents but when monetary policy cannot change wealth distribution

Switching types



share of agents
also wealth
share

η

$1-\eta$

idiosyncratic
risk of capital

$\phi\tilde{\sigma}, \phi \in (0,1)$

$\tilde{\sigma}$

diversification

output per unit
of capital

the same, independently of the allocation

Policy maker can choose the money growth rate $\hat{\mu}_t^M$

Remarks

- Policy-maker cannot affect wealth shares (exogenously fixed by the switching process)
- Welfare weights on intermediaries and households, η and $1 - \eta$, follow from the setup
- The policy-maker may have tools to affect the allocation of capital (but will she choose to?)

Equilibrium capital allocation

- Fraction ψ of capital is held by the intermediaries
- Capital allocation must be such that

$$\underbrace{\phi\tilde{\sigma}}_{\text{idiosync. risk of I}} \underbrace{\frac{(1-\vartheta)\psi\phi\tilde{\sigma}}{\eta}}_{\text{I's price of idiosync. risk}} = \underbrace{\tilde{\sigma}}_{\text{idiosync. risk of H}} \underbrace{\frac{(1-\vartheta)(1-\psi)\tilde{\sigma}}{1-\eta}}_{\text{H's price of idiosync. risk}}$$

$$\Rightarrow \psi = \frac{\eta}{\phi^2(1-\eta) + \eta}$$

- Policy maker may try to affect ψ ...

Welfare

- Law of large numbers: switching risk does not matter. Everyone's wealth growth averages out to $\Phi(\iota_t) - \delta$ and idiosyncratic risk exposure, to

$$\eta(\tilde{\sigma}^I)^2 + (1-\eta)(\tilde{\sigma}^H)^2 = (1-\vartheta)^2 \underbrace{\tilde{\sigma}^2 \left(\frac{\psi^2 \phi^2}{\eta} + \frac{(1-\psi)^2}{1-\eta} \right)}_{(\tilde{\sigma}^A)^2}$$

$$\tilde{\sigma}^I = \frac{(1-\vartheta)\psi\phi\tilde{\sigma}}{\eta}, \quad \tilde{\sigma}^H = \frac{(1-\vartheta)(1-\psi)\tilde{\sigma}}{1-\eta}$$

- Welfare

$$\vartheta E \left[\int_0^{\infty} e^{-\rho t} \log \frac{\rho q}{1-\vartheta} dt \right] + E \left[\int_0^{\infty} e^{-\rho t} \frac{\Phi(\iota(q)) - \delta}{\rho} dt \right] - E \left[\int_0^{\infty} e^{-\rho t} \frac{(1-\vartheta)^2 (\tilde{\sigma}^A)^2}{2\rho} dt \right]$$

- Allocation determines $\tilde{\sigma}^A$. Then planner wants to choose $\vartheta = \vartheta^* ((\tilde{\sigma}^A)^2)$

Money valuation

- Numeraire: global wealth
- Global portfolio weights: money ϑ
capital held by I, $(1-\vartheta)\psi$,
capital held by H, $(1-\vartheta)(1-\psi)$
- Money valuation equation

$$\underbrace{\rho - (\mu_t^\vartheta - \hat{\mu}_t^M)}_{\text{difference in global wealth and money returns}} = \underbrace{(1-\vartheta)\psi\phi\tilde{\sigma} \overbrace{\tilde{\sigma}^I}^{\text{I price of risk}} + (1-\vartheta)(1-\psi)\tilde{\sigma} \overbrace{\tilde{\sigma}^H}^{\text{H price of risk}}}_{(1-\vartheta)^2(\tilde{\sigma}^A)^2 = \eta(\tilde{\sigma}^I)^2 + (1-\eta)(\tilde{\sigma}^H)^2}$$

Optimal allocation

- Average idiosyncratic risk of capital

$$(\tilde{\sigma}^A)^2 = \tilde{\sigma}^2 \left(\frac{\psi^2 \phi^2}{\eta} + \frac{(1-\psi)^2}{1-\eta} \right)$$

is minimized when

$$\frac{\psi \phi^2}{\eta} = \frac{1-\psi}{1-\eta} \Rightarrow \psi = \frac{\eta}{\phi^2(1-\eta) + \eta}$$

This is the equilibrium allocation already!!! Wow!

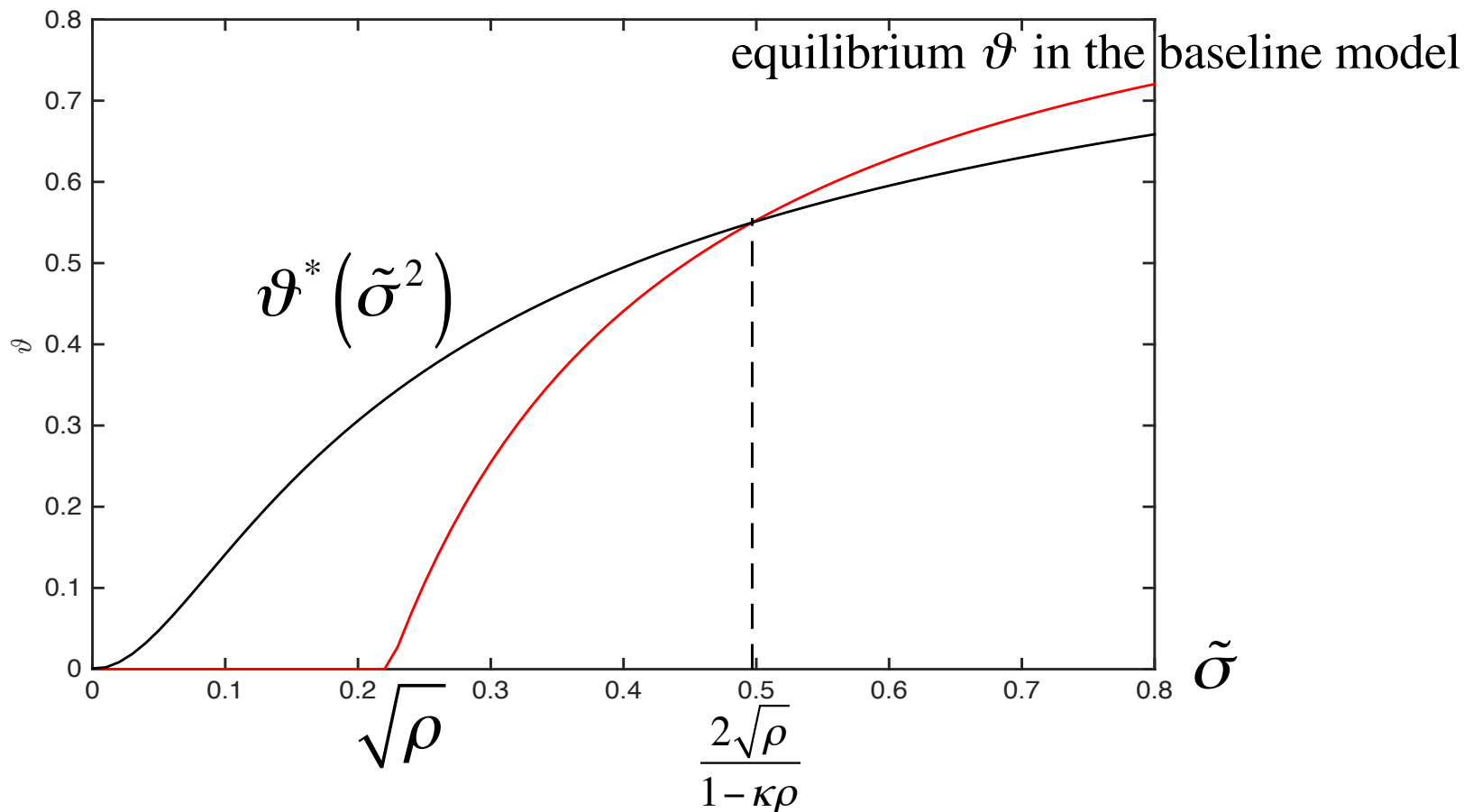
Remarks

- The trade-off between insurance and investment is the same even with heterogeneous agents when wealth shares correspond to welfare weights
- Equilibrium allocation is efficient as it minimizes the cost of risk exposure
- Policy space (1) money growth rate and (1) + (3) (also macroprudential tools) leads to the same outcome
- (2) risk redistribution is irrelevant here because no aggregate risk

A bit about the optimal policy function $\vartheta^*(\tilde{\sigma}^2)$

- it pops up everywhere...

$$\max_{\vartheta} \frac{\log(1-\vartheta)}{\rho\kappa} - \frac{\rho\kappa+1}{\rho\kappa} \log(\rho\kappa+1-\vartheta) - \frac{(1-\vartheta)^2 \tilde{\sigma}^2}{2\rho}$$



Endogenous law of motion of η

- Wealth distribution can change endogenously due to (1) risk exposure of intermediaries and households (2) risk premia and (3) consumption rates
- Consider the following relative of the last model

Fixed types (no switching)

	intermediaries	households	types fixed (no switching)
wealth shares	η	$1-\eta$	
welfare weights	λ	$1-\lambda$	
idiosyncratic risk of capital	$\phi\tilde{\sigma}, \phi \in (0,1)$	$\tilde{\sigma}$	
aggregate risk	σ	σ	
output per unit of capital	the same, independently of the allocation		

Two policy classes:

(1) policy maker can choose the money growth rate $\hat{\mu}_t^M$

(1) + (3) policy maker also choose allocation (macroprudential)
and transfer wealth between groups

Welfare of I and H

- Intermediaries (weight λ)

$$E \left[\int_0^{\infty} e^{-\rho t} \left(\log \eta_t + \log(a - \iota_t) + \frac{\Phi(\iota_t) - \delta}{\rho} - \frac{\sigma^2}{2\rho} - \frac{(1 - \vartheta)^2}{2\rho} \frac{\psi^2 \phi^2 \tilde{\sigma}^2}{\eta^2} \right) dt \right]$$

- households (weight $1 - \lambda$)

$$E \left[\int_0^{\infty} e^{-\rho t} \left(\log(1 - \eta_t) + \log(a - \iota_t) + \frac{\Phi(\iota_t) - \delta}{\rho} - \frac{\sigma^2}{2\rho} - \frac{(1 - \vartheta)^2}{2\rho} \frac{(1 - \psi)^2 \tilde{\sigma}^2}{(1 - \eta)^2} \right) dt \right]$$

Equilibrium

	intermediaries	households
wealth shares	η	$1-\eta$
idiosyncratic risk of capital	$\phi\tilde{\sigma}, \phi \in (0,1)$	$\tilde{\sigma}$

(as last model) intermediaries hold fraction of capital $\psi = \frac{\eta}{\phi^2(1-\eta) + \eta}$

$$\frac{d\eta}{\eta} = (1-\eta)((\tilde{\sigma}_t^I)^2 - (\tilde{\sigma}_t^H)^2)dt = (1-\eta)(1-\vartheta_t)^2 \tilde{\sigma}^2 \left(\frac{\psi^2 \phi^2}{\eta^2} - \frac{(1-\psi)^2}{(1-\eta)^2} \right) dt =$$

$$(1-\eta) \frac{(1-\vartheta_t)^2 \tilde{\sigma}^2 \phi^2 (1-\phi^2)}{(\phi^2(1-\eta) + \eta)^2} dt$$

In the long run, η converges to 1 and ϑ to $\left(1 - \frac{\sqrt{\rho}}{\phi\tilde{\sigma}} \right)$

Optimal policy, (1) + (3)

- Planner would like to maximize the disc. integral of

$$\lambda \log \eta_t + (1 - \lambda) \log(1 - \eta_t) + \log(a - u(\vartheta)) + \frac{\Phi(u(\vartheta)) - \delta}{\rho} - \frac{\sigma^2}{2\rho}$$

$$- \frac{(1 - \vartheta)^2 \tilde{\sigma}^2}{2\rho} \underbrace{\left(\lambda \frac{\psi^2 \phi^2}{\eta^2} + (1 - \lambda) \frac{(1 - \psi)^2}{(1 - \eta)^2} \right)}_{\frac{\lambda(1 - \lambda)\phi^2}{\lambda\phi^2(1 - \eta)^2 + (1 - \lambda)\eta^2}}$$

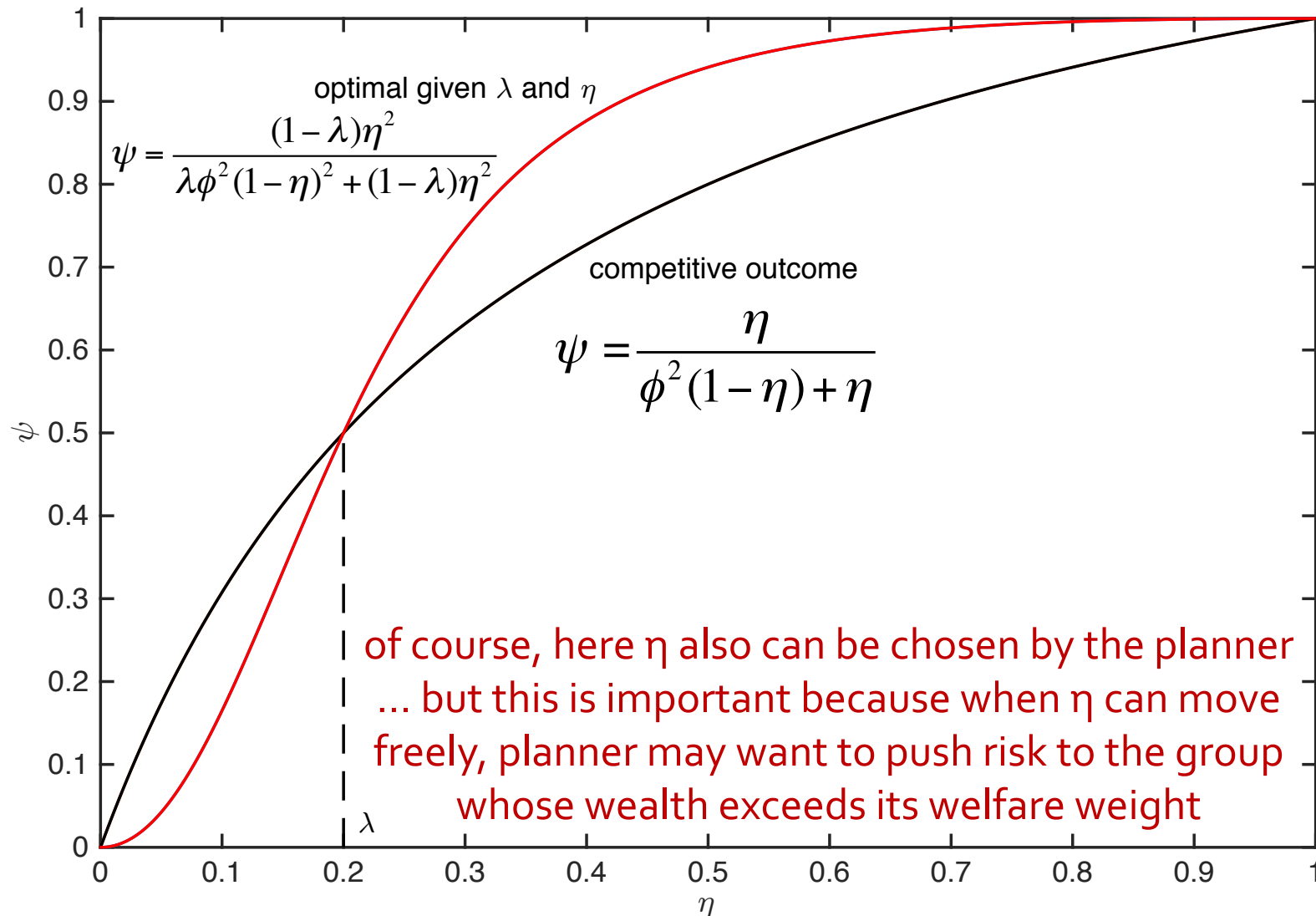
not the
competitive allocation
(unless $\eta = \lambda$)

given the optimal choice of $\psi = \frac{(1 - \lambda)\eta^2}{\lambda\phi^2(1 - \eta)^2 + (1 - \lambda)\eta^2}$

- given ψ and η , optimal to set ϑ to

$$\vartheta = \vartheta^* \underbrace{\left(\tilde{\sigma}^2 \frac{\lambda(1 - \lambda)\phi^2}{\lambda\phi^2(1 - \eta)^2 + (1 - \lambda)\eta^2} \right)}_{\text{welfare weighted average risk exposure}}$$

Competitive ψ vs. minimizing cost of risk



Optimal policy, (1) + (3)

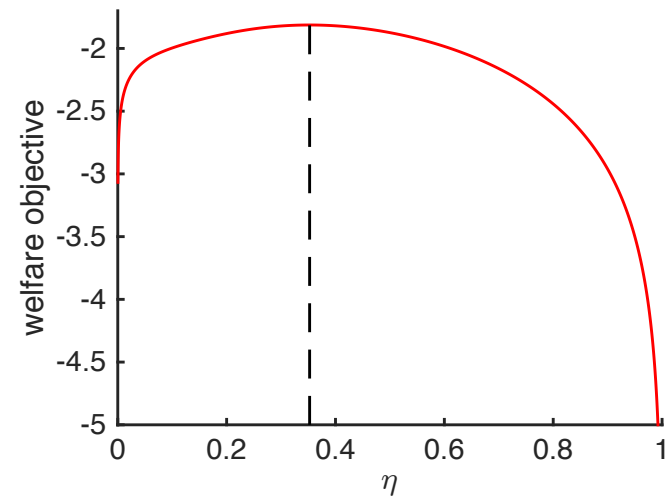
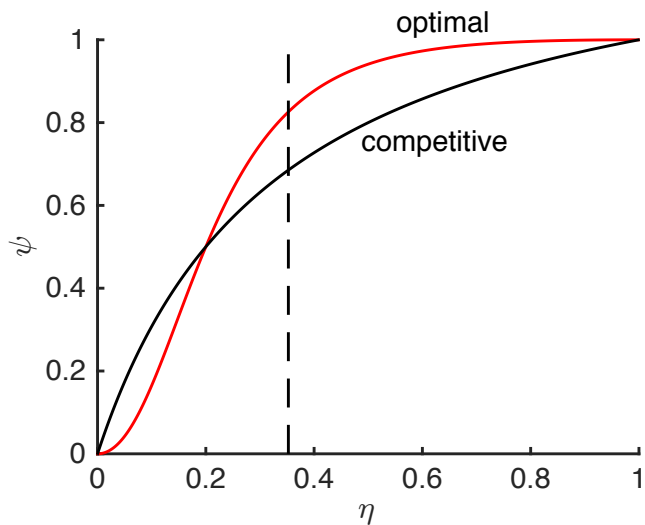
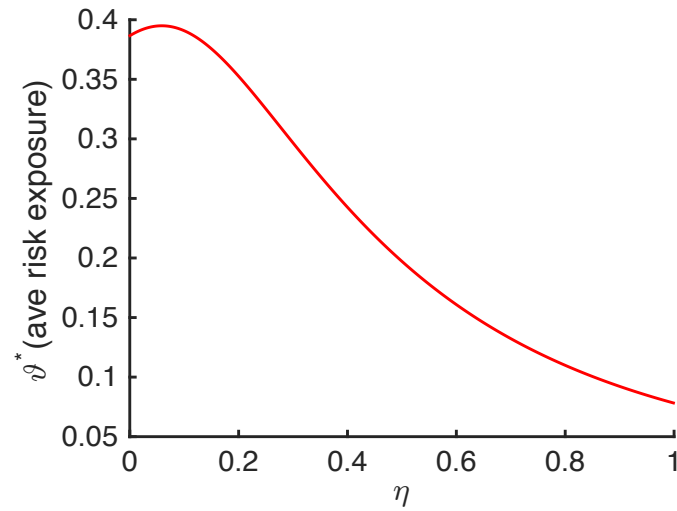
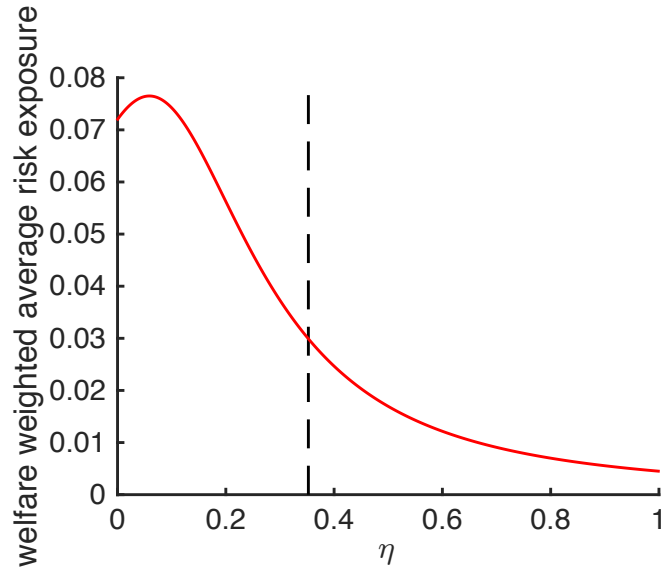
- Finally, optimal η (given ϑ) – let's look at terms containing η

$$\max_{\eta} \underbrace{\lambda \log \eta + (1 - \lambda) \log(1 - \eta)}_{\text{concave, max at } \eta = \lambda, \text{ goes to } -\infty \text{ at 0 and 1}} - \frac{(1 - \vartheta)^2 \tilde{\sigma}^2}{2\rho} \underbrace{\frac{\lambda(1 - \lambda)\phi^2}{\lambda\phi^2(1 - \eta)^2 + (1 - \lambda)\eta^2}}_{\text{concave (!) also, max at } \frac{\lambda\phi^2}{\lambda\phi^2 + 1 - \lambda} < \lambda}$$

- hence, it is optimal to set $\eta > \lambda$ (unfortunately I could not get a closed-form expression for the optimal η)
- push more risk to intermediaries than they'd take under competitive outcome
- relative to previous infinite switching model
 - it is optimal to give intermediaries more wealth, because they are more efficient at absorbing risk
 - overall risk is reduced and the value of money is lower (more intermediation)

Optimizing over η

$$\rho = 0.05, \kappa = 2, \tilde{\sigma} = 0.3, \phi = 0.5, \lambda = 0.2$$



Optimal policy, (1) only

- What about monetary policy alone?
- Planner cannot alter the comp. allocation, $\psi = \frac{\eta}{\phi^2(1-\eta) + \eta}$
- Welfare is the disc. integral of

$$\lambda \log \eta + (1 - \lambda) \log(1 - \eta) + \log(a - u(\vartheta)) + \frac{\Phi(u(\vartheta)) - \delta}{\rho} - \frac{\sigma^2}{2\rho}$$

$$- \frac{(1 - \vartheta)^2 \tilde{\sigma}^2}{2\rho} \underbrace{\left(\lambda \frac{\psi^2 \phi^2}{\eta^2} + (1 - \lambda) \frac{(1 - \psi)^2}{(1 - \eta)^2} \right)}_{\frac{\lambda \phi^2 + (1 - \lambda) \phi^4}{(\phi^2(1 - \eta) + \eta)^2}}$$

- s.t.

$$\frac{d\eta}{\eta} = (1 - \eta) \frac{(1 - \vartheta)^2 \tilde{\sigma}^2 \phi^2 (1 - \phi^2)}{(\phi^2(1 - \eta) + \eta)^2} dt$$

- the planner can select any path of ϑ by choosing $\hat{\mu}_t^M$

Optimal policy, (1) only

- It can be useful to decompose into initial wealth allocation + growth

$$\int_0^{\infty} e^{-\rho t} (\lambda \log \eta_t + (1 - \lambda) \log(1 - \eta_t)) dt =$$

$$\frac{\lambda \log \eta_0 + (1 - \lambda) \log(1 - \eta_0)}{\rho} + \int_0^{\infty} e^{-\rho t} \left(\lambda \frac{\mu_t^\eta}{\rho} + (1 - \lambda) \frac{\mu_t^{1-\eta}}{\rho} \right) dt$$

- in the long run, η goes to 1 under any policy. Then

$$\frac{d\eta}{\eta} = (1 - \eta) \frac{(1 - \vartheta)^2 \tilde{\sigma}^2 \phi^2 (1 - \phi^2)}{(\phi^2 (1 - \eta) + \eta)^2} dt \Rightarrow \mu^\eta \rightarrow 0, \mu^{1-\eta} \rightarrow -(1 - \vartheta)^2 \tilde{\sigma}^2 \phi^2 (1 - \phi^2)$$

- let's see if we can characterize opt. policy analytically in the long run. Our welfare objective is

$$-(1 - \lambda) \frac{(1 - \vartheta)^2 \tilde{\sigma}^2}{\rho} \phi^2 (1 - \phi^2) + \log \frac{q}{1 - \vartheta} + \frac{\Phi(\iota(q))}{\rho} - \frac{(1 - \vartheta)^2 \tilde{\sigma}^2}{2\rho} (\lambda \phi^2 + (1 - \lambda) \phi^4)$$

Optimal policy, (1) only

- In the long run, η goes to 1 and the planner maximizes

$$-(1-\lambda)\frac{(1-\vartheta)^2\tilde{\sigma}^2}{\rho}\phi^2(1-\phi^2) + \log\frac{q}{1-\vartheta} + \frac{\Phi(\iota(q))}{\rho} - \frac{(1-\vartheta)^2\tilde{\sigma}^2}{2\rho}(\lambda\phi^2 + (1-\lambda)\phi^4)$$

- it is optimal to set ϑ to

$$\vartheta^* \left(\underbrace{\tilde{\sigma}^2(\lambda\phi^2 + (1-\lambda)\phi^4)}_{\text{weighted average risk exposure}} + 2(1-\lambda)\tilde{\sigma}^2\phi^2(1-\phi^2) \right)$$

- planner cares about H, who become very poor in the long run. But planner has only monetary tools. Planner can help H by raising ϑ (which increases the return on money, slows down the rate at which H get poorer). Fiscal policy is much more efficient here. Using only monetary policy like hammering a nail with a screwdriver.

Optimal monetary policy

- Law of motion of η

$$\frac{d\eta}{\eta} = (1-\eta) \frac{(1-\vartheta)^2 \tilde{\sigma}^2 \phi^2 (1-\phi^2)}{(\phi^2(1-\eta) + \eta)^2} dt$$

Payoff flow

$$f(\eta, \vartheta) = \lambda \log \eta + (1-\lambda) \log(1-\eta) + \frac{\log(1-\vartheta)}{\rho\kappa} - \frac{\rho\kappa+1}{\rho\kappa} \log(\rho\kappa+1-\vartheta) - \frac{(1-\vartheta)^2 \tilde{\sigma}^2}{2\rho} \left(\lambda \frac{\psi^2 \phi^2}{\eta^2} + (1-\lambda) \frac{(1-\psi)^2}{(1-\eta)^2} \right), \quad \psi = \frac{\eta}{\phi^2(1-\eta) + \eta}$$

HJB equation

$$\rho V(\eta) = \max_{\vartheta} f(\eta, \vartheta) + V'(\eta) \mu^n \eta + \frac{1}{2} V''(\eta) (\sigma^n \eta)^2$$

Optimal ϑ

$$f(\eta, \vartheta) = \lambda \log \eta + (1 - \lambda) \log(1 - \eta) + \frac{\log(1 - \vartheta)}{\rho \kappa} - \frac{\rho \kappa + 1}{\rho \kappa} \log(\rho \kappa + 1 - \vartheta) - \frac{(1 - \vartheta)^2 \tilde{\sigma}^2}{2\rho} \left(\lambda \frac{\psi^2 \phi^2}{\eta^2} + (1 - \lambda) \frac{(1 - \psi)^2}{(1 - \eta)^2} \right), \quad \psi = \frac{\eta}{\phi^2(1 - \eta) + \eta}$$

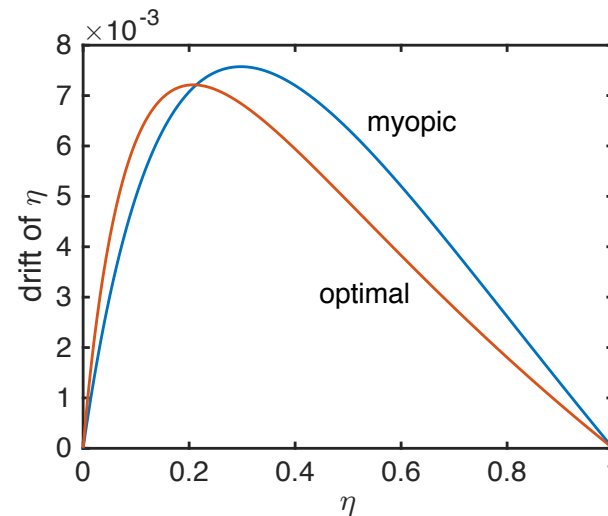
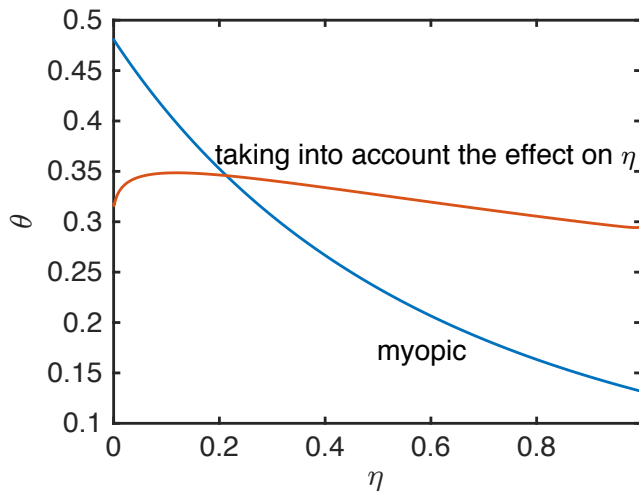
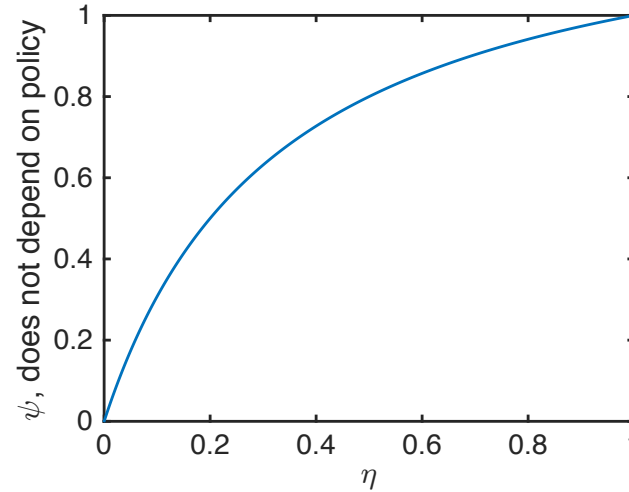
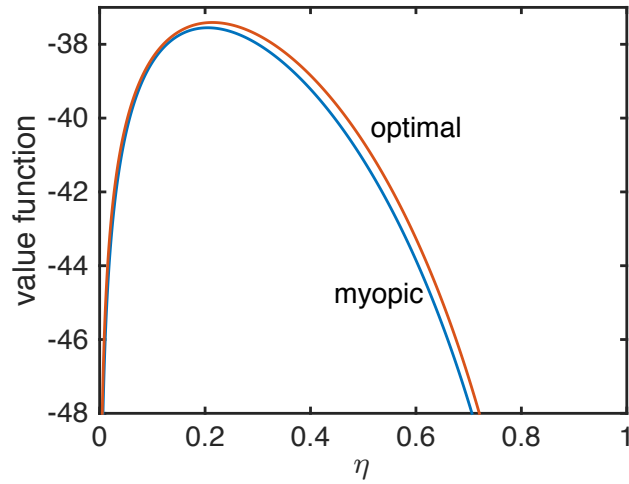
$$\mu^n \eta = (1 - \vartheta)^2 \frac{\eta(1 - \eta) \tilde{\sigma}^2 \phi^2 (1 - \phi^2)}{(\phi^2(1 - \eta) + \eta)^2} \quad \max_{\vartheta} f(\eta, \vartheta) + V'(\eta) \mu^n \eta + \frac{1}{2} V''(\eta) (\sigma^n \eta)^2$$

- ϑ does not affect the allocation but it affects the drift of η . Hence optimal to choose

$$\vartheta^* \left(\tilde{\sigma}^2 \left(\lambda \frac{\psi^2 \phi^2}{\eta^2} + (1 - \lambda) \frac{(1 - \psi)^2}{(1 - \eta)^2} \right) - 2\rho V'(\eta) \frac{\eta(1 - \eta) \tilde{\sigma}^2 \phi^2 (1 - \phi^2)}{(\phi^2(1 - \eta) + \eta)^2} \right)$$

Example: using ϑ to push η

$$\rho = 0.05, \kappa = 2, \tilde{\sigma} = 0.3, \phi = 0.5, \lambda = 0.2$$





Risk of money: stochastic money growth

- Money return (global wealth as numeraire) in the absence of policy

$$dr_t^M = \mu_t^\vartheta dt + \sigma_t^\vartheta dZ_t$$

- (Stochastic) money growth: monetary authority prints money to transfer to capital. Money return becomes

$$dr_t^M = (\mu_t^\vartheta - \hat{\mu}_t^M)dt + (\sigma_t^\vartheta - \hat{\sigma}_t^M)dZ_t$$

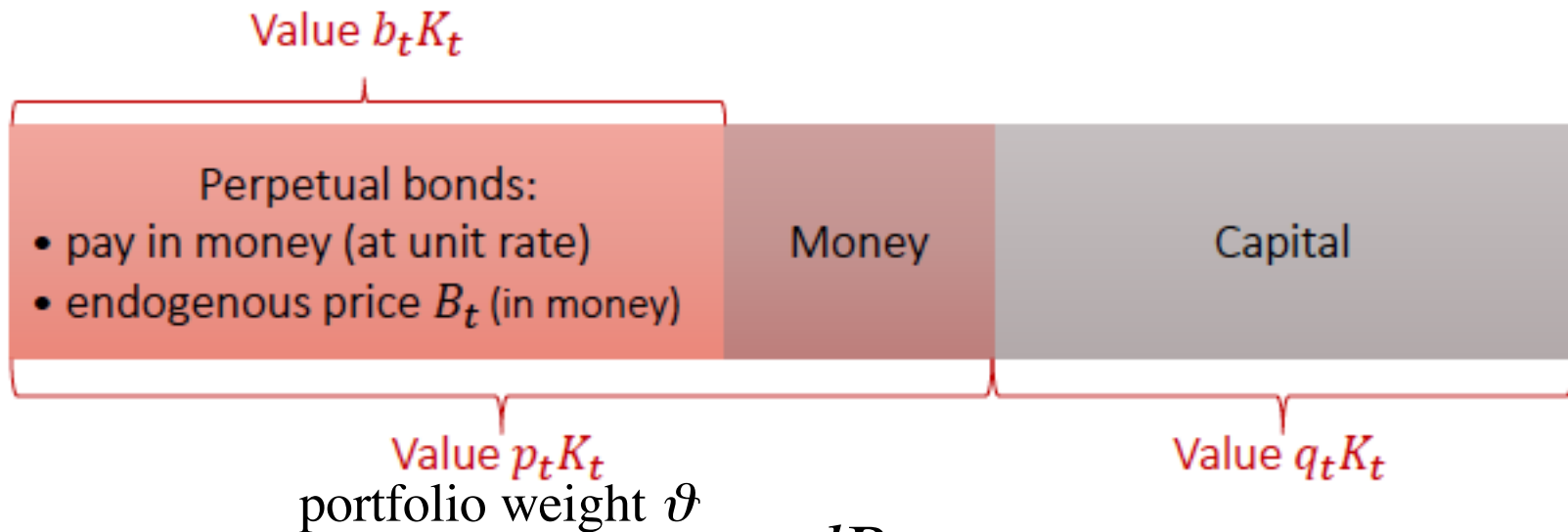
- Capital must have return

$$\frac{\rho}{1 - \vartheta} dt - \frac{\vartheta}{1 - \vartheta} dr_t^M$$

- Notice that a portfolio of $1 - \vartheta$ capital and ϑ money (i.e. world wealth) has return ρ !

Monetary policy and risk transfer

- Money and bonds



- Value of bonds in money $\frac{dB_t}{B_t} = \mu_t^B dt + \sigma_t^B dZ_t$
- With bonds, the risk of money is not $(\sigma_t^\vartheta - \hat{\sigma}_t^M) dZ_t$

but $(\sigma_t^\vartheta - \hat{\sigma}_t^M - \frac{b_t}{p_t} \sigma_t^B) dZ_t$

Monetary policy and risk transfer

- Value of money and bonds: $p_t K_t$
- Value of bonds: $b_t K_t$

- Value of bonds in money $\frac{dB_t}{B_t} = \mu_t^B dt + \sigma_t^B dZ_t$

- Portfolio of money and bonds (value $p_t K_t$) has risk

$$\begin{aligned}
 & (\sigma_t^\vartheta - \hat{\sigma}_t^M) dZ_t = \\
 & \underbrace{\frac{p_t - b_t}{p_t}}_{\text{weight on money}} (\sigma_t^\vartheta - \hat{\sigma}_t^M - \frac{b_t}{p_t} \sigma_t^B) dZ_t + \underbrace{\frac{b_t}{p_t}}_{\text{weight on bonds}} (\sigma_t^\vartheta - \hat{\sigma}_t^M - \frac{b_t}{p_t} \sigma_t^B + \underbrace{\sigma_t^B}_{\text{risk of bonds relative to money}}) dZ_t \\
 & = \underbrace{(\sigma_t^\vartheta - \hat{\sigma}_t^M - \frac{b_t}{p_t} \sigma_t^B)}_{\text{risk of money}} dZ_t + \underbrace{\frac{b_t}{p_t}}_{\text{weight on bonds}} \underbrace{\sigma_t^B}_{\text{risk of bonds relative to money}} dZ_t
 \end{aligned}$$

Monetary policy and risk transfer

- If $\sigma^B < 0$ (bonds appreciate when Z goes down, because interest rate is cut) and $\sigma^\eta > 0$ (intermediaries suffer losses when Z goes down), then intermediaries can hold bonds and use them as a hedge
- The effect of this hedge on σ^η is

$$\frac{\vartheta_t}{\eta_t} \frac{b_t}{p_t} \sigma_t^B$$

- Equivalently monetary authority can offer to sell a derivative (money-bond swap) backed by money
- Money risk increases by $\sigma^+ = -b/p \sigma^B > 0$, while the direct effect of the derivative on the risk of η is $-\vartheta/\eta \sigma^+$.
- Monetary authority can also sell opposite derivatives, which households would buy. Money risk decreases by $\sigma^- > 0$; the direct effect on the risk of $1 - \eta$ is $\vartheta/(1-\eta) \sigma^-$.

Remarks

- This captures the general idea that monetary policy can transfer risk
- Important that bonds cannot be shorted (or else markets become complete wrt aggregate risk)
- If markets are complete wrt aggregate risk, then stochastic money growth or risk transfer have no bite (fully undone by private markets)
- If limited supply of 2 types of derivatives are issued, one pays when dZ goes up and another when dZ goes down, H buy the former and I buy the latter assuming $\sigma^\eta > 0$ (and so $\sigma^{1-\eta} < 0$)... until the gap between σ^η and $\sigma^{1-\eta}$ closes (both become 0)
- If backed by money, these derivatives affect the risk of money (and there is also an effect on the agents' portfolios). This risk can be transferred to capital (through stochastic money growth)

To sum up...

- Denote the risk of money by σ^M (global wealth as numeraire).
- I sell off risk $\vartheta\sigma^+$ through derivatives, H buy risk $\vartheta\sigma^-$.
- Risk of the global portfolio is

$$\vartheta\sigma_t^M - \vartheta\sigma_t^+ + \vartheta\sigma_t^- + (1 - \vartheta)\sigma_t^K = 0,$$

where σ^K is the risk of capital. Hence,

$$\sigma_t^K = \frac{\vartheta}{1 - \vartheta} (\sigma_t^+ - \sigma_t^- - \sigma_t^M).$$

Let's come back to the last model

	intermediaries	households	types fixed (no switching)
wealth shares	η	$1-\eta$	
welfare weights	λ	$1-\lambda$	
idiosyncratic risk of capital	$\phi\tilde{\sigma}, \phi \in (0,1)$	$\tilde{\sigma}$	
aggregate risk	σ	σ	
output per unit of capital	the same, independently of the allocation		

Two policy classes:

- (a) policy maker chooses stochastic money growth $\hat{\mu}_t^M dt + \hat{\sigma}_t^M dZ_t$
- (b) policy maker can also provide hedges

Capital allocation equation

- Can choose any risk of money σ^M and any ϑ
- Without hedges/derivatives, the risk of capital is $-\frac{\vartheta}{1-\vartheta}\sigma_t^M$

- I's portfolio weight on capital is $\frac{(1-\vartheta)\psi}{\eta}$,

so

$$\sigma^\eta = \sigma_t^M + \frac{(1-\vartheta)\psi}{\eta} \underbrace{\left(-\frac{\vartheta}{1-\vartheta}\sigma_t^M - \sigma_t^M \right)}_{-\frac{\sigma_t^M}{1-\vartheta}, \text{ risk of capital relative to money}} = -\frac{\psi-\eta}{\eta}\sigma_t^M$$

- Capital allocation equation

$$\frac{\sigma_t^M}{1-\vartheta} \frac{\psi-\eta}{\eta} \sigma_t^M + \frac{(1-\vartheta)\psi\phi^2\tilde{\sigma}^2}{\eta} = -\frac{\sigma_t^M}{1-\vartheta} \frac{\psi-\eta}{1-\eta} \sigma_t^M + \frac{(1-\vartheta)(1-\psi)\tilde{\sigma}^2}{1-\eta} \Rightarrow$$

$$\frac{\psi-\eta}{\eta(1-\eta)} (\sigma_t^M)^2 + \frac{(1-\vartheta)^2\psi\phi^2\tilde{\sigma}^2}{\eta} = \frac{(1-\vartheta)^2(1-\psi)\tilde{\sigma}^2}{1-\eta}$$

- Higher σ^M pushes ψ from $\psi = \frac{\eta}{\phi^2(1-\eta) + \eta}$ to η

Driving η with monetary policy

- Recall that in general, the drift of η can be expressed in terms of risks

$$d\eta = \eta\sigma^\eta dZ + \frac{1-2\eta}{\eta(1-\eta)} (\eta\sigma^\eta)^2 dt - \eta\sigma^\eta\sigma^M dt + \eta(1-\eta) \underbrace{((\tilde{\sigma}_t^I)^2 - (\tilde{\sigma}_t^H)^2)}_{\tilde{\sigma}^2 \left(\frac{\psi^2\phi^2}{\eta^2} - \frac{(1-\psi)^2}{(1-\eta)^2} \right)} dt$$

where (from the last slide) $\sigma^\eta = -\frac{\psi-\eta}{\eta}\sigma_t^M$

and ψ and σ^M are related through

$$\frac{\psi-\eta}{\eta(1-\eta)} (\sigma_t^M)^2 + \frac{(1-\vartheta)^2\psi\phi^2\tilde{\sigma}^2}{\eta} = \frac{(1-\vartheta)^2(1-\psi)\tilde{\sigma}^2}{1-\eta}$$

Welfare

$$f(\eta, \vartheta, \psi) = \lambda \log \eta_t + (1 - \lambda) \log(1 - \eta_t) + \log(a - \iota_t) + \frac{\Phi(\iota_t) - \delta}{\rho} - \frac{\sigma^2}{2\rho}$$

$$- \frac{(1 - \vartheta)^2 \tilde{\sigma}^2}{2\rho} \left(\lambda \frac{\psi^2 \phi^2}{\eta^2} + (1 - \lambda) \frac{(1 - \psi)^2}{(1 - \eta)^2} \right)$$

minimum $\frac{\lambda(1-\lambda)\phi^2}{\lambda\phi^2(1-\eta)^2 + (1-\lambda)\eta^2}$, when $\psi = \frac{(1-\lambda)\eta^2}{\lambda\phi^2(1-\eta)^2 + (1-\lambda)\eta^2}$

higher than
equilibrium

$$\frac{\eta}{\phi^2(1-\eta) + \eta}$$

iff $\eta > \lambda$

drift of η $\left(\frac{1 - 2\eta}{\eta(1 - \eta)} (\psi - \eta)^2 + (\psi - \eta) \right) (\sigma^M)^2 +$

$$(1 - \vartheta)^2 \tilde{\sigma}^2 \eta(1 - \eta) \left(\frac{\psi^2 \phi^2}{\eta^2} - \frac{(1 - \psi)^2}{(1 - \eta)^2} \right)$$

Law of motion of η

$$(\eta\sigma^n)^2 = (\psi - \eta)^2 (\sigma_t^M)^2 = (1 - \vartheta)^2 \tilde{\sigma}^2 (\psi - \eta) (\eta - \psi(\eta + \phi^2(1 - \eta)))$$

$$\mu^n \eta =$$

$$\left(\frac{1 - 2\eta}{\eta(1 - \eta)} (\psi - \eta)^2 + (\psi - \eta) \right) (\sigma^M)^2 + (1 - \vartheta)^2 \tilde{\sigma}^2 \eta(1 - \eta) \left(\frac{\psi^2 \phi^2}{\eta^2} - \frac{(1 - \psi)^2}{(1 - \eta)^2} \right)$$

$$= (1 - \vartheta)^2 \tilde{\sigma}^2 (\psi - \eta) (\psi \phi^2 + 1 - \psi)$$

these and welfare “payoff flow” are functions of ϑ and

$$\psi \in \left[\eta, \frac{\eta}{\phi^2(1 - \eta) + \eta} \right].$$

Nice!

$$f(\eta, \vartheta, \psi) = \lambda \log \eta + (1 - \lambda) \log(1 - \eta) + \frac{\log(1 - \vartheta)}{\rho \kappa} - \frac{\rho \kappa + 1}{\rho \kappa} \log(\rho \kappa + 1 - \vartheta)$$

$$- \frac{(1 - \vartheta)^2 \tilde{\sigma}^2}{2\rho} \left(\lambda \frac{\psi^2 \phi^2}{\eta^2} + (1 - \lambda) \frac{(1 - \psi)^2}{(1 - \eta)^2} \right)$$

HJB equation and the optimal ϑ

$$(\eta\sigma^n)^2 = (1 - \vartheta)^2 \tilde{\sigma}^2 (\psi - \eta) (\eta - \psi(\eta + \phi^2(1 - \eta)))$$

$$\mu^n \eta = (1 - \vartheta)^2 \tilde{\sigma}^2 (\psi - \eta) (\psi\phi^2 + 1 - \psi)$$

$$f(\eta, \vartheta, \psi) = \lambda \log \eta + (1 - \lambda) \log(1 - \eta) + \frac{\log(1 - \vartheta)}{\rho\kappa} - \frac{\rho\kappa + 1}{\rho\kappa} \log(\rho\kappa + 1 - \vartheta)$$

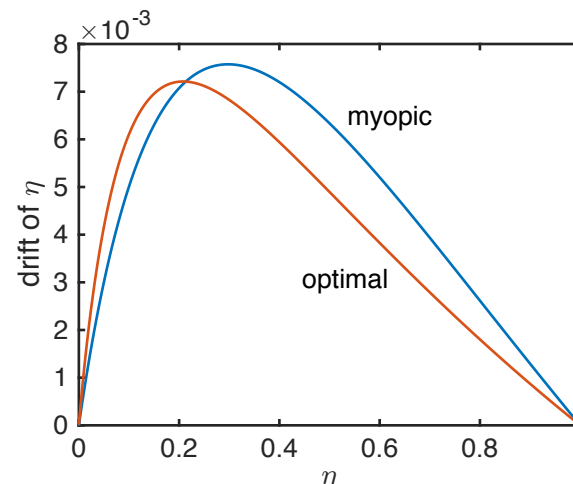
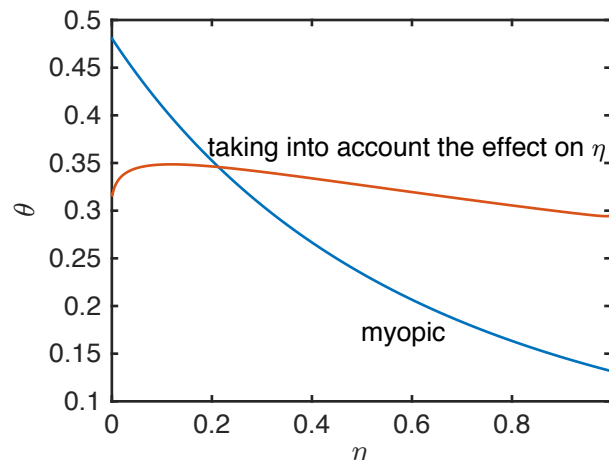
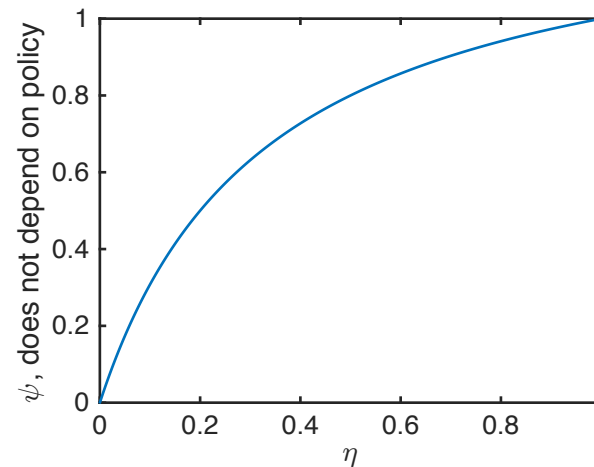
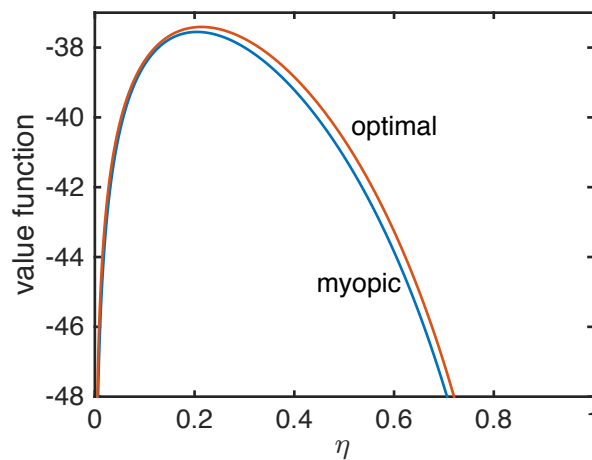
$$- \frac{(1 - \vartheta)^2 \tilde{\sigma}^2}{2\rho} \left(\lambda \frac{\psi^2 \phi^2}{\eta^2} + (1 - \lambda) \frac{(1 - \psi)^2}{(1 - \eta)^2} \right)$$

$$\max_{\vartheta, \psi} f(\eta, \vartheta) + V'(\eta) \mu^n \eta + \frac{1}{2} V''(\eta) (\sigma^n \eta)^2$$

$$\vartheta^* \left(\begin{array}{c} \tilde{\sigma}^2 \left(\lambda \frac{\psi^2 \phi^2}{\eta^2} + (1 - \lambda) \frac{(1 - \psi)^2}{(1 - \eta)^2} \right) \\ -2\rho V'(\eta) \tilde{\sigma}^2 (\psi - \eta) (\psi\phi^2 + 1 - \psi) - \rho V''(\eta) \tilde{\sigma}^2 (\psi - \eta) (\eta - \psi(\eta + \phi^2(1 - \eta))) \end{array} \right)$$

Optimal Policy

- It turns out that (perhaps not too surprisingly) while the policy maker could affect the allocation by making money risky, she would not want to do it. The optimal policy is the same as before



Including aggregate risk

	intermediaries	households	types fixed (no switching)
wealth shares	η	$1-\eta$	
welfare weights	λ	$1-\lambda$	
idiosyncratic risk of capital	$\phi\tilde{\sigma}, \phi \in (0,1)$	$\tilde{\sigma}$	
aggregate risk: fraction of capital	fraction $\bar{\psi}$ of capital has aggregate risk σ , the rest risk 0		
	$\psi \leq \bar{\psi}$ only capital which has aggregate risk		
output per unit of capital	the same, independently of the allocation		

Calculating risk

- Can choose any risk of money σ^M
- Without hedges/derivatives, the risk of capital is

$$\underbrace{(1 - \bar{\psi})\sigma}_\text{fundamental risk of I capital rel. global wealth} - \frac{\vartheta}{1 - \vartheta} \sigma_t^M \quad \text{and} \quad -\bar{\psi}\sigma - \frac{\vartheta}{1 - \vartheta} \sigma_t^M$$

fundamental risk of I capital
rel. global wealth

- I's portfolio weight on capital is $\frac{(1 - \vartheta)\bar{\psi}}{\eta}$,

so

$$\sigma^\eta = \sigma_t^M + \frac{(1 - \vartheta)\psi}{\eta} \left((1 - \bar{\psi})\sigma - \frac{\vartheta}{1 - \vartheta} \sigma_t^M - \sigma_t^M \right) =$$

$$(1 - \vartheta) \frac{\psi(1 - \bar{\psi})}{\eta} \sigma - \frac{\psi - \eta}{\eta} \sigma_t^M$$

- With hedges, keeping σ^M fixed, can get any σ^η between 0 and this level

Capital allocation equation

- Capital allocation equation, equality if $\psi < \bar{\psi}$

$$\left((1 - \bar{\psi})\sigma - \frac{\sigma_t^M}{1 - \vartheta} \right) \sigma_t^\eta + \frac{(1 - \vartheta)\psi\phi^2\tilde{\sigma}^2}{\eta} \leq \left((1 - \bar{\psi})\sigma - \frac{\sigma_t^M}{1 - \vartheta} \right) \frac{-\eta\sigma_t^\eta}{1 - \eta} + \frac{(1 - \vartheta)(1 - \psi)\tilde{\sigma}^2}{1 - \eta}$$

where $\sigma^\eta = (1 - \vartheta) \frac{\psi(1 - \bar{\psi})}{\eta} \sigma - \frac{\psi - \eta}{\eta} \sigma_t^M$

- The allocation equation simplifies to

$$\left((1 - \bar{\psi})\sigma - \frac{\sigma_t^M}{1 - \vartheta} \right) \frac{\sigma_t^\eta}{1 - \eta} + (1 - \vartheta)\tilde{\sigma}^2 \left(\frac{\psi\phi^2}{\eta} - \frac{1 - \psi}{1 - \eta} \right) = 0$$

or

$$\frac{\psi(1 - \bar{\psi})\sigma - (\psi - \eta) \frac{\sigma_t^M}{1 - \vartheta}}{\eta(1 - \eta)} \left((1 - \bar{\psi})\sigma - \frac{\sigma_t^M}{1 - \vartheta} \right) + \tilde{\sigma}^2 \left(\frac{\psi\phi^2}{\eta} - \frac{1 - \psi}{1 - \eta} \right) = 0$$

Given $\sigma^M/(1 - \vartheta)$, this gives us ψ (independently of ϑ)

Driving η with monetary policy

- We have $\eta\sigma^\eta = (1 - \vartheta) \left(\psi(1 - \bar{\psi})\sigma - (\psi - \eta) \frac{\sigma^M}{1 - \vartheta} \right)$

$$\eta\mu^\eta = (1 - \vartheta)^2 \left(\psi(1 - \bar{\psi})\sigma - (\psi - \eta) \frac{\sigma^M}{1 - \vartheta} \right) \left(\frac{\psi - 2\eta\psi + \eta^2}{\eta(1 - \eta)} \left((1 - \bar{\psi})\sigma - \frac{\sigma^M}{1 - \vartheta} \right) - \frac{\eta}{1 - \eta} (1 - \bar{\psi})\sigma \right)$$

$$+ \eta(1 - \eta)(1 - \vartheta)^2 \tilde{\sigma}^2 \left(\frac{\psi^2 \phi^2}{\eta^2} - \frac{(1 - \psi)^2}{(1 - \eta)^2} \right)$$

where

$$\frac{\psi(1 - \bar{\psi})\sigma - (\psi - \eta) \frac{\sigma_t^M}{1 - \vartheta}}{\eta(1 - \eta)} \left((1 - \bar{\psi})\sigma - \frac{\sigma_t^M}{1 - \vartheta} \right) + \tilde{\sigma}^2 \left(\frac{\psi\phi^2}{\eta} - \frac{1 - \psi}{1 - \eta} \right) \leq 0$$

with equality if $\psi < \bar{\psi}$

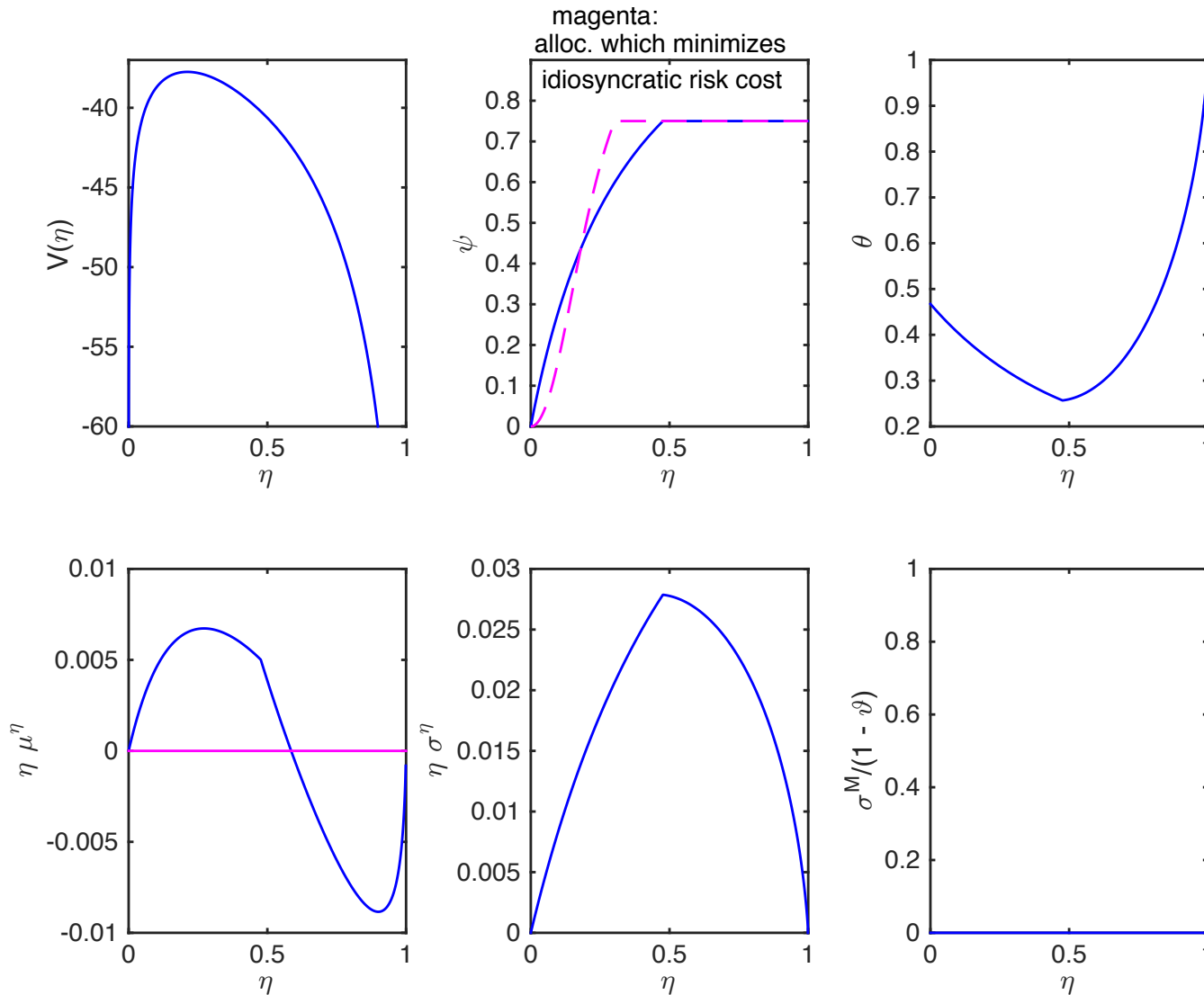
HJB equation

$$\rho V(\eta) = \max_{\vartheta, \sigma^{M/(1-\vartheta)}} f(\eta, \vartheta) + V'(\eta)\mu^n\eta + \frac{V''(\eta)}{2}(\sigma^n\eta)^2$$

$$\begin{aligned} \rho V(\eta) = & \max_{\vartheta, \sigma^{M/(1-\vartheta)}} \lambda \log \eta + (1-\lambda) \log(1-\eta) + \frac{\log(1-\vartheta)}{\rho\kappa} - \frac{\rho\kappa+1}{\rho\kappa} \log(\rho\kappa+1-\vartheta) \\ & - \frac{(1-\vartheta)^2 \tilde{\sigma}^2}{2\rho} \left(\lambda \frac{\psi^2 \phi^2}{\eta^2} + (1-\lambda) \frac{(1-\psi)^2}{(1-\eta)^2} \right) + (1-\vartheta)^2 \frac{V''(\eta)}{2} \left(\psi(1-\bar{\psi})\sigma - (\psi-\eta) \frac{\sigma^M}{1-\vartheta} \right)^2 \\ & + (1-\vartheta)^2 V'(\eta) \left(\eta(1-\eta) \tilde{\sigma}^2 \left(\frac{\psi^2 \phi^2}{\eta^2} - \frac{(1-\psi)^2}{(1-\eta)^2} \right) \right. \\ & \left. + \left(\psi(1-\bar{\psi})\sigma - (\psi-\eta) \frac{\sigma^M}{1-\vartheta} \right) \left(\frac{\psi - 2\eta\psi + \eta^2}{\eta(1-\eta)} \left((1-\bar{\psi})\sigma - \frac{\sigma^M}{1-\vartheta} \right) - \frac{\eta}{1-\eta} (1-\bar{\psi})\sigma \right) \right) \end{aligned}$$

Example: myopic choice of ϑ

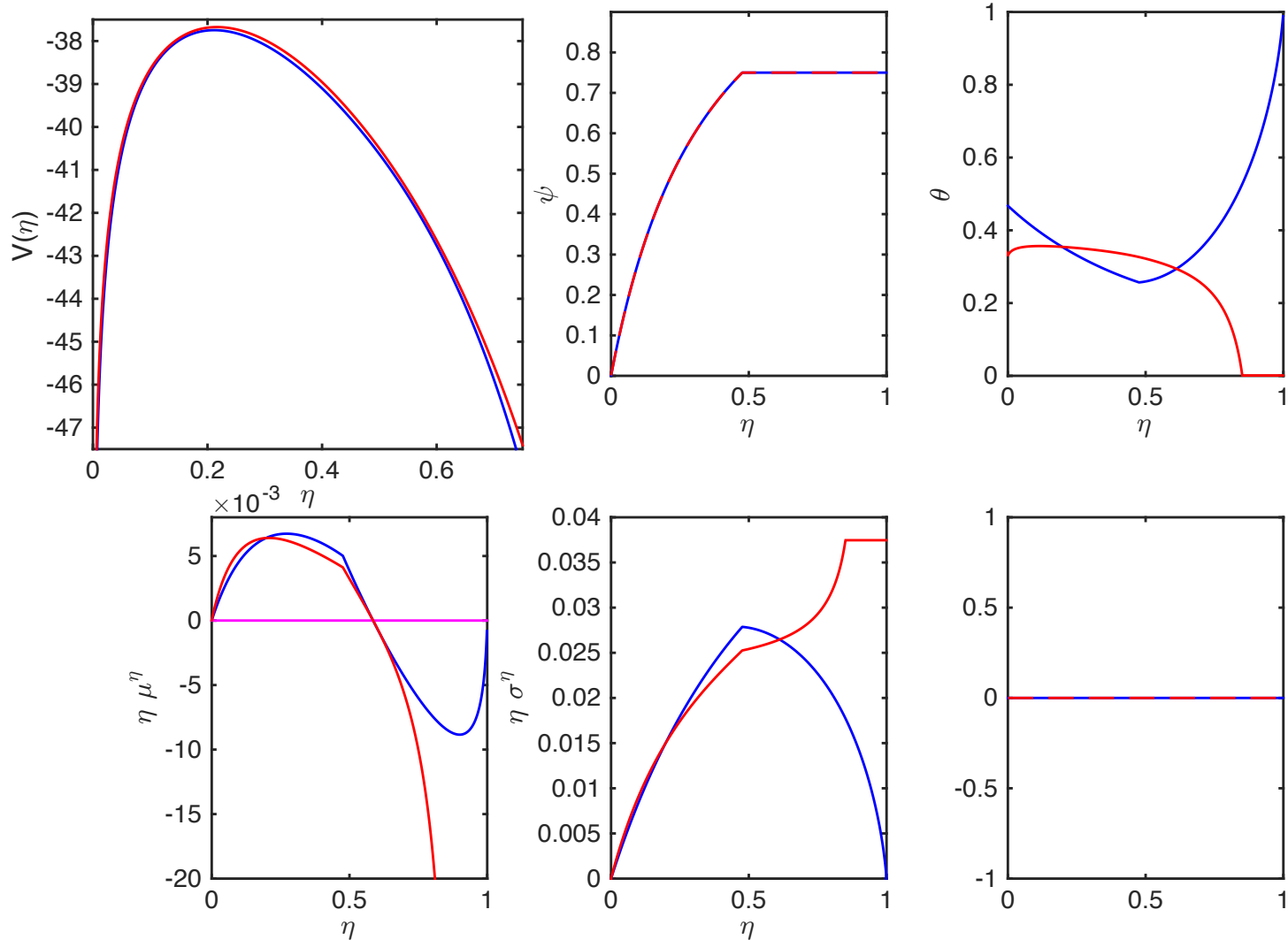
$$\rho = 0.05, \sigma = 0.2, \tilde{\sigma} = 0.3, \phi = 0.5, \kappa = 2, \bar{\psi} = 0.75, \lambda = 0.2$$



planner pushes ϑ very high to protect H from idiosyncr. risk

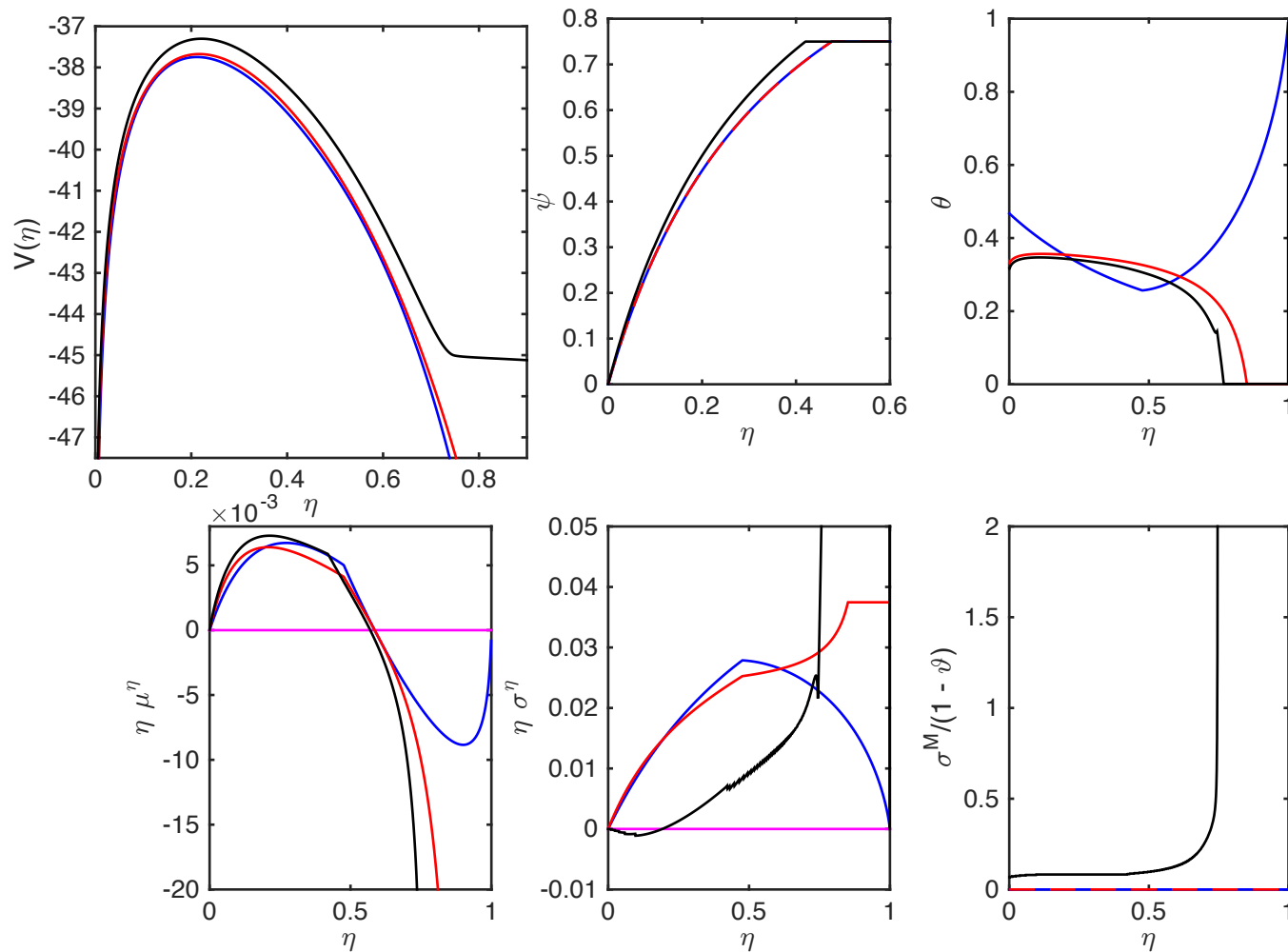
Example: forward-looking choice of ϑ

$$\rho = 0.05, \sigma = 0.2, \tilde{\sigma} = 0.3, \phi = 0.5, \kappa = 2, \bar{\psi} = 0.75, \lambda = 0.2$$

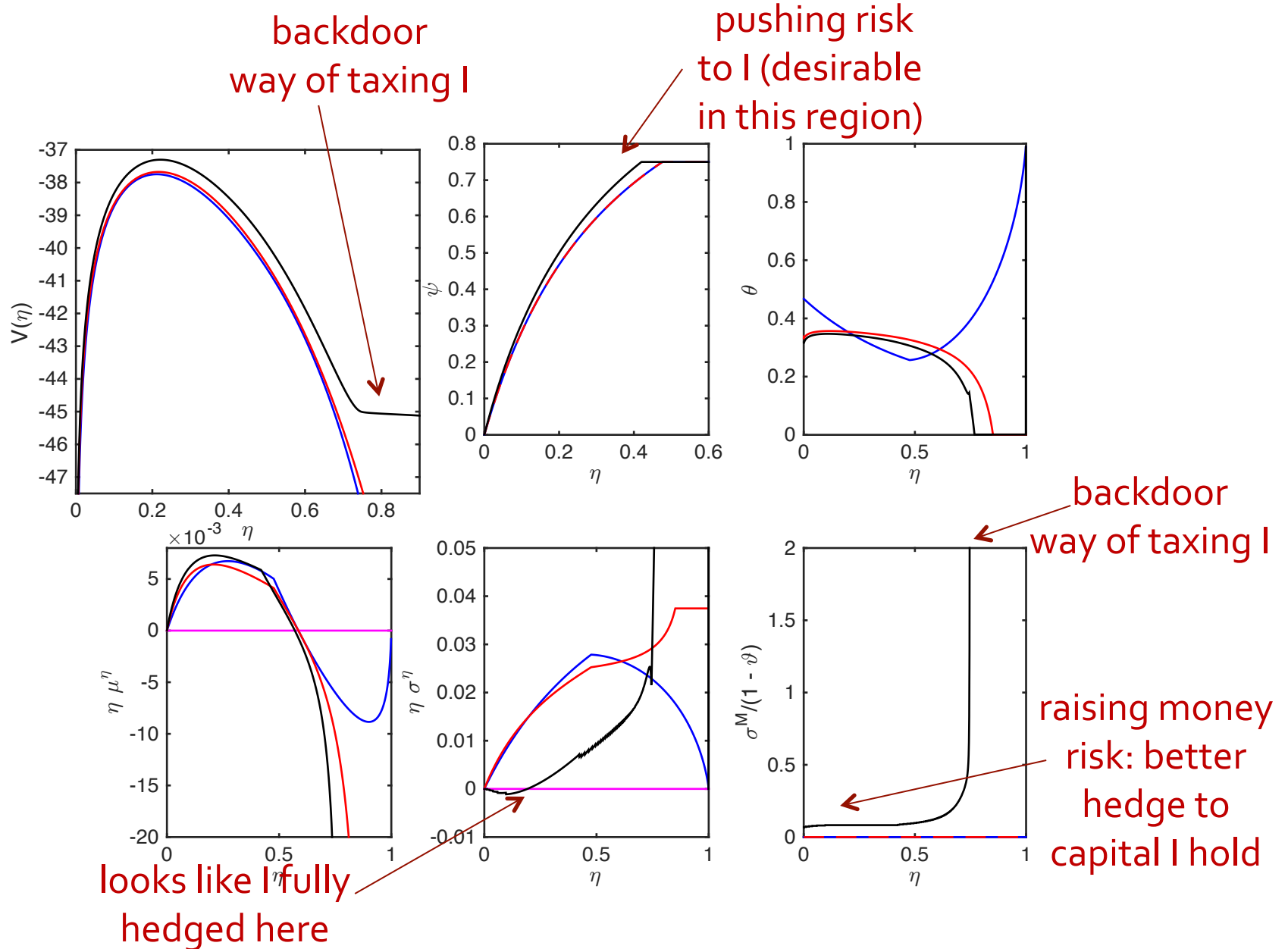


Example: optimal ϑ and σ^M

$\rho = 0.05$, $\sigma = 0.2$, $\tilde{\sigma} = 0.3$, $\phi = 0.5$, $\kappa = 2$, $\bar{\psi} = 0.75$, $\lambda = 0.2$



Example: optimal ϑ and σ^M



THANK YOU

