Macro, Money and Finance

Lecture 06: Money versus Debt

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Towards the I Theory of Money

- One sector model with idio risk - “The I Theory without I” (steady state focus)
  - Store of value
    - Insurance role of money *within sector*
  - Money as bubble or not
  - Fiscal Theory of the Price Level
  - Medium of Exchange Role ⇒ SDF-Liquidity multiplier ⇒ Money bubble
- 2 sector/type model with money and idio risk
  - Generic Solution procedure (compared to lecture 03)
  - Real debt vs. Money
    - Implicit insurance role of money *across sectors*
  - The curse of insurance
    - Reduces insurance premia and net worth gains
- I Theory with Intermediary sector
  - Intermediaries as diversifiers
- Welfare analysis
- Optimal Monetary Policy and Macroprudential Policy

Last week

Today

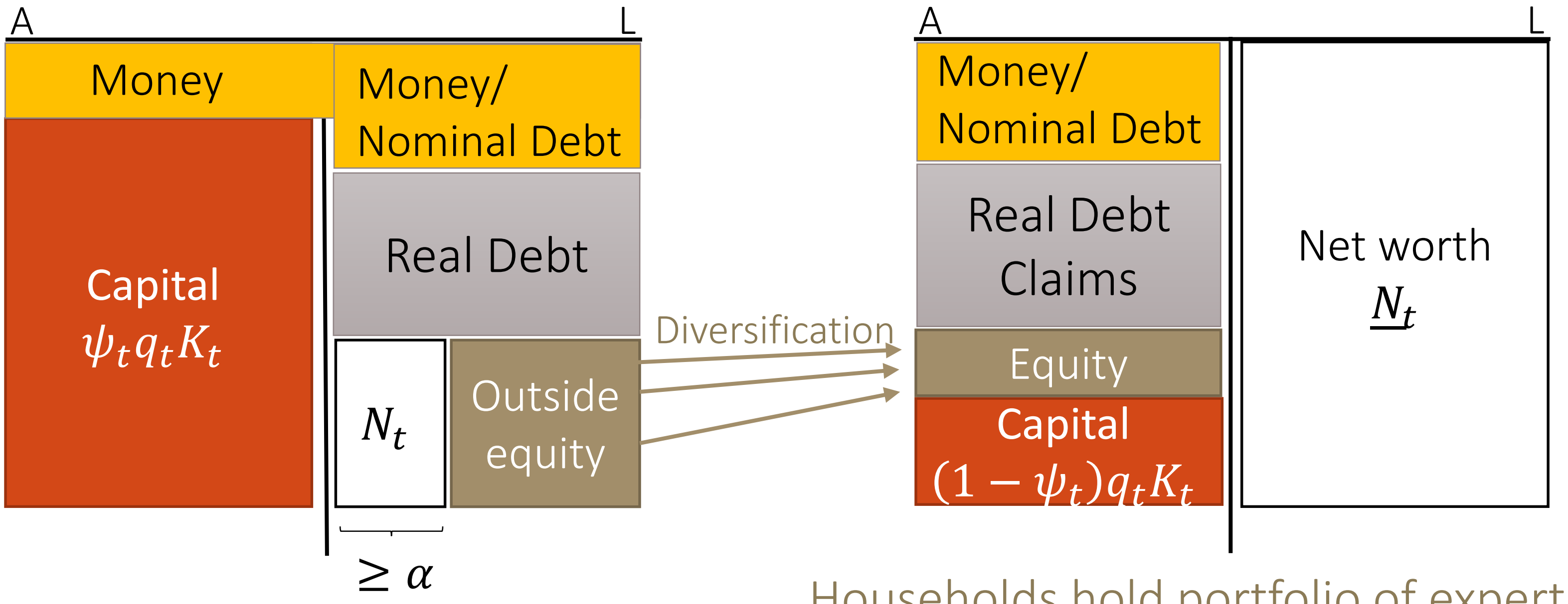
Next lectures

Two Sector Model w/ Outside Equity & Money

Outside money

- Expert sector

Household sector



A

L

Money

Money/ Nominal Debt

Real Debt

Capital $\psi_t q_t K_t$

N_t

Outside equity

$\geq \alpha$

Diversification

A

L

Money/ Nominal Debt

Real Debt Claims

Equity

Capital $(1-\psi_t) q_t K_t$

Net worth $\underline{N_t}$

Households hold portfolio of experts' outside equity and diversify idio risk away

- Experts must hold fraction $\chi_t \geq \alpha\psi_t$ (skin in the game constraint)

Expanded on Handbook of Macroeconomics 2017, Chapter 18

- Includes now money and idiosyncratic risk

Two Sector Model Setup

Expert sector

- Output: $y_t = ak_t$
- Consumption rate: c_t
- Investment rate: ι_t

$$\frac{dk_t^{\tilde{i}}}{k_t^{\tilde{i}}} = (\Phi(\iota_t)-\delta)dt+\sigma dZ_t+\tilde{\sigma} d\tilde{Z}_t^{\tilde{i}}+d\Delta_t^{k,\tilde{i}}$$

- $E_0[\int_0^\infty e^{-\rho t}\frac{c_t^{1-\gamma}}{1-\gamma}dt]$

$a \geq \underline{a}$

$\delta \leq \underline{\delta}$

$\tilde{\sigma} \leq \underline{\tilde{\sigma}}$

$\rho \geq \underline{\rho}$

Household sector

- Output: $\underline{y}_t = \underline{a}\underline{k}_t$
- Consumption rate: $\underline{c}_t$
- Investment rate: ι_t

$$\frac{d\underline{k}_t^{\tilde{i}}}{\underline{k}_t^{\tilde{i}}} = (\Phi(\underline{\iota}_t)-\underline{\delta})dt+\sigma dZ_t+\underline{\tilde{\sigma}} d\tilde{Z}_t^{\tilde{i}}+d\underline{\Delta}_t^{k,\tilde{i}}$$

- $E_0[\int_0^\infty e^{-\underline{\rho} t}\frac{\underline{c}_t^{1-\gamma}}{1-\gamma}dt]$

Friction: Can only issue

- Risk-free debt
- Equity, but most hold $\chi_t \geq \underline{\alpha}$

Solving MacroModels Step-by-Step

0. Postulate aggregates, price processes & obtain return processes
1. For given SDF processes *static*
   a. Real investment ι, (portfolio $\boldsymbol{\theta}$, & consumption choice of *each agent*)
      - *Toolbox 1:* Martingale Approach
   b. Asset/Risk Allocation *across types/sectors* & asset market clearing
      - *Toolbox 2:* "price-taking social planner approach" – Fisher separation theorem
2. Value functions *backward equation*
   a. Value fcn. as fcn. of individual investment opportunities ω
      - *Special cases*
   b. De-scaled value fcn. as function of state variables η
      - *Digression:* HJB-approach (instead of martingale approach & envelop condition)
   c. Derive ς price of risk, C/N-ratio from value fcn. envelop condition
3. Evolution of state variable η *forward equation*
   - *Toolbox 3:* Change in numeraire to total wealth (including SDF)
   - "Money evaluation equation" μ^{ϑ}
4. Value function iteration & goods market clearing
   a. PDE of de-scaled value fcn.
   b. Value function iteration by solving PDE

0. Postulate Aggregates

- Individual capital evolution:

$$\frac{dk_t^{i,\tilde{\imath}}}{k_t^{i,\tilde{\imath}}} = \left(\Phi\left(\iota^{i,\tilde{\imath}}\right) - \delta\right)dt + \sigma dZ_t + \tilde{\sigma}^i d\tilde{Z}_t^{i,\tilde{\imath}} + d\Delta_t^{k,i,\tilde{\imath}}$$

  - Where $\Delta_t^{k,\tilde{\imath},i}$ is the individual cumulative capital purchase process
- Capital aggregation:
  - Within sector i: $K_t^i \equiv \int k_t^{i,\tilde{\imath}} d\tilde{\imath}$
  - Across sectors: $K_t \equiv \sum_i K_t^i$
  - Capital share: $\psi_t^i \equiv K_t^i / K_t$

$$\frac{dK_t}{K_t} = \int \left(\Phi\left(\iota^i\right) - \delta\right) di\, dt + \sigma dZ_t$$

- Networth aggregation:
  - Within sector i: $N_t^i \equiv \int n_t^{i,\tilde{\imath}} d\tilde{\imath}$
  - Across sectors: $\overline{N}_t \equiv \sum_i N_t^i$
  - Wealth share: $\eta_t^i \equiv N_t^i / \overline{N}_t$
- Value of capital: $q_t K_t$
- Value of money: $p_t K_t$

0. Postulate Processes

- Value of capital: $q_t K_t$
- Value of money: $p_t K_t$
- Postulate

$$dq_t/q_t = \mu_t^q dt + \sigma_t^q dZ_t$$
$$dp_t/p_t = \mu_t^p dt + \sigma_t^p dZ_t$$

$$d\xi_t^i/\xi_t^i = \underbrace{\mu_t^\xi}_{\equiv - r_t} dt + \underbrace{\sigma_t^{\xi^i}}_{\equiv - \varsigma_t^i} dZ_t + \underbrace{\tilde{\sigma}^{\xi^i}}_{\equiv - \tilde{\varsigma}_t^i} d\tilde{Z}_t^{\tilde{i}}$$

- Derive return processes

$$dr_t^{K,i,\tilde{i}} = \left(\frac{a^i - \iota_t^{\tilde{i}}}{q_t} + \Phi(\iota_t^{\tilde{i}}) - \delta + \mu_t^q + \sigma\sigma_t^q\right) dt + (\sigma + \sigma_t^q) dZ_t + \tilde{\sigma}^i d\tilde{Z}_t^{\tilde{i}}$$

$$dr_t^M = (\Phi(\iota_t) - \delta + \mu_t^p + \sigma\sigma_t^p - \mu^M) dt + (\sigma + \sigma_t^p) dZ_t$$

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1a. Agent Choice of ι, θ, c

- Portfolio Choice: Martingale Approach
  - Let x_t^A be the value of a "self-financing trading strategy" (reinvest dividends)
  - Theorem: $\xi_t x_t^A$ follows a Martingale, i.e. drift $= 0$.
  - Let $\frac{dx_t^A}{x_t^A} = \mu_t^A dt + \sigma_t^A dZ_t + \tilde{\boldsymbol{\sigma}}_t^A d\tilde{\mathbf{Z}}_t,$
  - Recall $\frac{d\xi_t^i}{\xi_t^i} = -r_t dt - \varsigma_t^i dZ_t - \tilde{\boldsymbol{\varsigma}}_t^i d\tilde{\mathbf{Z}}_t^i$
  - By Ito product rule

$$\frac{d(\xi_t^i x_t^A)}{\xi_t^i x_t^A} = \big(\underbrace{-r_t + \mu_t^A - \varsigma_t^i \sigma_t^A - \tilde{\boldsymbol{\varsigma}}_t^i \tilde{\boldsymbol{\sigma}}_t^A}_{=0}\big)\, dt + \text{volatility terms}$$

  - Expected return: $\boxed{\mu_t^A = r_t + \varsigma_t^i \sigma_t^A + \tilde{\boldsymbol{\varsigma}}_t^i \tilde{\boldsymbol{\sigma}}_t^A}$
    - For risk-free asset, i.e. $\sigma_t^A = \tilde{\boldsymbol{\sigma}}_t^A = 0$:

$$r_t^f = r_t$$

    - Excess expected return to risky asset B:

$$\mu_t^A - \mu_t^B = \varsigma_t^i(\sigma_t^A - \sigma_t^B) + \tilde{\boldsymbol{\varsigma}}_t^i(\tilde{\boldsymbol{\sigma}}_t^A - \tilde{\boldsymbol{\sigma}}_t^B)$$

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1b. Asset/Risk Allocation across Types

- Price-Taking Planner's Theorem:
  A social planner that takes prices as given chooses an physical asset and money allocation, $\boldsymbol{\psi}_t$, and risk allocation $\boldsymbol{\chi}_t, \widetilde{\boldsymbol{\chi}}_t$, that coincides with the choices implied by all individuals' portfolio choices.

$$\begin{aligned}
\boldsymbol{\varsigma}_t &= (\varsigma_t^1, \dots, \varsigma_t^I) \\
\boldsymbol{\chi}_t &= (\chi_t^1, \dots, \chi_t^I) \\
\boldsymbol{\sigma}(\boldsymbol{\psi}_t, \boldsymbol{\chi}_t) &= \left(\boldsymbol{\chi}_t^1 \sigma^{\bar{N}}(\psi_t), \dots, \boldsymbol{\chi}_t^I \sigma^{\bar{N}}(\psi_t)\right) \\
\widetilde{\boldsymbol{\sigma}}(\boldsymbol{\psi}_t, \boldsymbol{\chi}_t) &= \left(\tilde{\sigma}^{n^1}(\boldsymbol{\psi}_t, \widetilde{\boldsymbol{\chi}}_t), \dots, \tilde{\sigma}^{n^I}(\boldsymbol{\psi}_t, \widetilde{\boldsymbol{\chi}}_t)\right)
\end{aligned}$$

Return on total wealth (including money)

- Planner's problem

$$\max_{\{\boldsymbol{\psi}_t, \boldsymbol{\chi}_t, \widetilde{\boldsymbol{\chi}}_t\}} E_t[dr_t^{\bar{N}}(\psi_t)] - \boldsymbol{\varsigma}_t \sigma(\boldsymbol{\psi}_t, \boldsymbol{\chi}_t) - \tilde{\boldsymbol{\varsigma}}_t \tilde{\sigma}(\boldsymbol{\psi}_t, \widetilde{\boldsymbol{\chi}}_t)$$

$= dr^F$ in equilibrium

subject to friction: $F(\boldsymbol{\psi}_t, \boldsymbol{\chi}_t, \widetilde{\boldsymbol{\chi}}_t) \leq 0$

  - Example:
    1. $\chi_t = \psi_t$ (if one holds capital, one has to hold risk)
    2. $\chi_t \geq \alpha \psi_t$ (skin in the game constraint, outside equity up to a limit)

Note: By holding a portfolio of various experts' outside equity HH can diversify idio risk away

1b. Allocation of Capital, ψ, and Risk, χ

If you shift one capital unit from HH to experts
- Dividend yield rises by LHS,
- Change the aggregate required risk premium (alpha fraction due to skin of the game constraint)
- HH reduce their risk by one unit, sell back $(1-\alpha)$ and diversified away

Note that HH, which hold a portfolio of different experts' outside equity can diversify idiosyncratic risk away

Cases	$\chi_t \geq \alpha\psi_t$	$\psi_t \leq 1$	$\frac{(a-\underline{a})}{q_t} \geq \alpha\left(\varsigma_t - \underline{\varsigma}_t\right)\left(\sigma + \sigma_t^q\right) + \left(\alpha\tilde{\varsigma}_t - \underline{\tilde{\varsigma}}_t\right)\tilde{\sigma}$	$\varsigma_t\left(\sigma + \sigma_t^q\right) + \tilde{\varsigma}_t\tilde{\sigma} > \underline{\varsigma}_t\left(\sigma + \sigma_t^q\right)$
1a	=	<	=	>
1b	=	=	>	>
2a	>	=	>	=
Impossible				

1b. Allocation of Capital, ψ, and Risk, χ

Cases	$\chi_t \geq \alpha\psi_t$	$\psi_t \leq 1$	$\frac{(a - \underline{a})}{q_t} \geq \alpha\left(\varsigma_t - \underline{\varsigma}_t\right)(\sigma + \sigma_t^q) + \left(\alpha\tilde{\varsigma}_t - \underline{\tilde{\varsigma}}_t\right)\tilde{\sigma}$	$\varsigma_t(\sigma + \sigma_t^q) + \tilde{\varsigma}_t\tilde{\sigma} > \underline{\varsigma}_t(\sigma + \sigma_t^q)$
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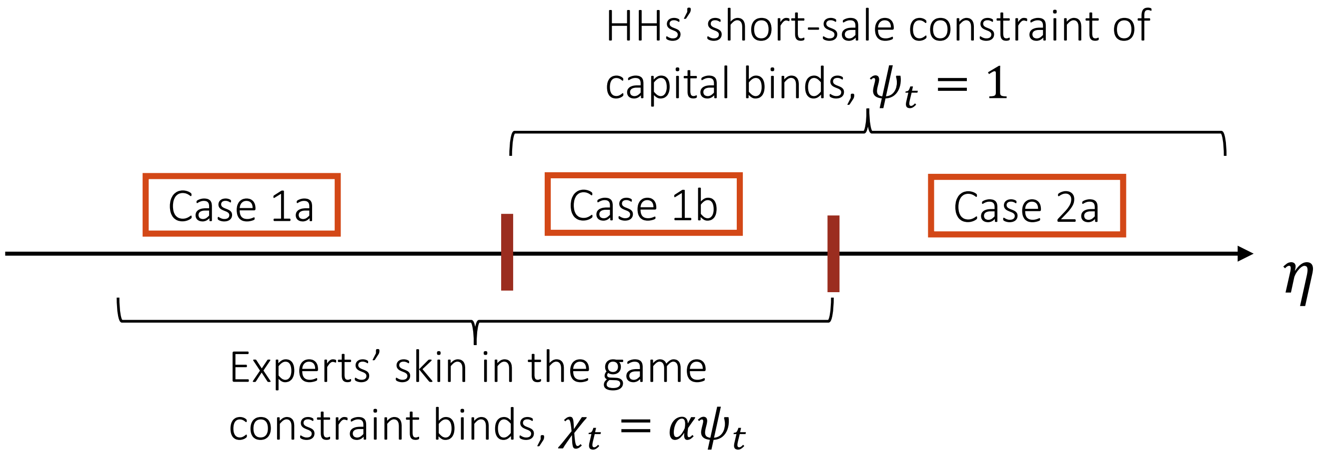
Impossible				



HHs' short-sale constraint of capital binds, $\psi_t = 1$

Experts' skin in the game constraint binds, $\chi_t = \alpha\psi_t$

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2b. CRRA Value Fcn: Isolating Idio. Risk

- Rephrase the conjecture value function as

$$V_t = \frac{1}{\rho}\frac{(\omega_t n_t)^{1-\gamma}}{1-\gamma} = \underbrace{\frac{1}{\rho(1-\gamma)}\left(\omega_t \frac{N_t}{K_t}\right)^{1-\gamma}}_{=:v_t} \underbrace{\left(\frac{n_t}{N_t}\right)^{1-\gamma}}_{=:\left(\tilde{\eta}_t^{\tilde{\imath}}\right)^{1-\gamma}} K_t^{1-\gamma}$$

- v_t depends only on aggregate state η_t
- Ito's quotation rule

$$\frac{d\tilde{\eta}_t^{\tilde{\imath}}}{\tilde{\eta}_t^{\tilde{\imath}}} = \frac{d(n_t/N_t)}{n_t/N_t} = (\mu_t^n - \mu_t^N + (\sigma_t^N)^2 - \sigma^N\sigma^n)dt + (\sigma_t^n - \sigma_t^N)dZ_t + \tilde{\sigma}^n d\tilde{Z}_t^{\tilde{\imath}} = \tilde{\sigma}^n d\tilde{Z}_t^{\tilde{\imath}}$$

- Ito's Lemma

$$\frac{d\left(\tilde{\eta}_t^{\tilde{\imath}}\right)^{1-\gamma}}{\left(\tilde{\eta}_t^{\tilde{\imath}}\right)^{1-\gamma}} = -\frac{1}{2}\gamma(1-\gamma)(\tilde{\sigma}^n)^2 dt + (1-\gamma)\tilde{\sigma}^n d\tilde{Z}_t^{\tilde{\imath}}$$

2b. CRRA Value Function

$$\frac{dV_t}{V_t} = \frac{d\left(v_t\left(\tilde{\eta}_t^{\tilde{\imath}}\right)^{1-\gamma} K_t^{1-\gamma}\right)}{v_t\left(\tilde{\eta}_t^{\tilde{\imath}}\right)^{1-\gamma} K_t^{1-\gamma}}$$

- By Ito's product rule

$$= \left(\mu_t^v + (1-\gamma)(\Phi(\iota) - \delta) - \frac{1}{2}\gamma(1-\gamma)(\sigma^2 + (\tilde{\sigma}^n)^2) + (1-\gamma)\sigma\sigma_t^v\right) dt$$
$$+\, volatility\ terms$$

- Recall by consumption optimality

$\frac{dV_t}{V_t} - \rho dt + \frac{c_t}{n_t} dt$ follows a martingale

- Hence, drift above $= \rho - \frac{c_t}{n_t}$

Still have to solve for μ_t^v, σ_t^v

Poll 16: Why martingale?

a) Because we can "price" networth with SDF

b) because ρ and c_t/n_t cancel out

2b. CRRA Value Fcn BSDE

- Only conceptual interim solution
    - We will transform it into a PDE in Step 4 below
- From last slide

$$\underbrace{\mu_t^v + (1-\gamma)(\Phi(\iota)-\delta) - \frac{1}{2}\gamma(1-\gamma)\left(\sigma^2 + (\tilde{\sigma}^n)^2\right) + (1-\gamma)\sigma\sigma_t^v}_{=:\mu_t^V} = \rho - \frac{c_t}{n_t}$$

- Can solve for μ_t^v, then v_t must follow

$$\frac{dv_t}{v_t} = f(\eta_t, v_t, \sigma_t^v)dt + \sigma_t^v dZ_t$$

with

$$f(\eta_t, v_t, \sigma_t^v) = \rho - \frac{c_t}{n_t} - (1-\gamma)(\Phi(\iota)-\delta) + \frac{1}{2}\gamma(1-\gamma)\left(\sigma^2 + (\tilde{\sigma}^n)^2\right) - (1-\gamma)\sigma\sigma_t^v$$

- Together with terminal condition v_T (possibly a constant for 1000 periods ahead), this is a backward stochastic differential equation (BSDE)
- A solution consists of processes v and σ^v
- Can use numerical BSDE solution methods (as random objects, so only get simulated paths)
- To solve this via a PDE we also need to get state evolution

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2c. Get ςs from Value Function Envelop

- Experts value function

HH's value function

$$v_t \frac{K_t^{1-\gamma}}{1-\gamma}\left(\tilde{\eta}_t^{\tilde{i}}\right)^{1-\gamma}$$

$$\underline{v}_t \frac{K_t^{1-\underline{\gamma}}}{1-\underline{\gamma}}\left(\underline{\tilde{\eta}}_t^{\tilde{i}}\right)^{1-\gamma}$$

- To obtain $\frac{\partial V_t(n)}{\partial n_t}$ use $K_t = \frac{N_t}{\eta_t(q_t+p_t)} = \frac{1}{\tilde{\eta}_t^{\tilde{i}}} \frac{n_t}{\eta_t(q_t+p_t)}$

$$V_t(n) = v_t \frac{n_t^{1-\gamma}/(\eta_t(q_t+p_t))^{1-\gamma}}{1-\gamma}$$

…

- Envelop condition $\frac{\partial V_t(n)}{\partial n_t} = \frac{\partial u(c_t)}{\partial c_t}$

$$v_t \frac{n_t^{-\gamma}}{(\eta_t(q_t+p_t))^{1-\gamma}} = c_t^{-\gamma}$$

…

- Using $K_t = \frac{1}{\tilde{\eta}_t^{\tilde{i}}} \frac{n_t}{\eta_t(q_t+p_t)}$, $C_t = \frac{1}{\tilde{\eta}_t^{\tilde{i}}} c_t$

$$\frac{v_t}{\eta_t(q_t+p_t)} K_t^{-\gamma} = C_t^{-\gamma}$$

…

$$\sigma_t^{v} - \sigma_t^{\eta} - \sigma_t^{q+p} - \gamma\sigma = -\gamma\sigma_t^{c},$$

$$= -\varsigma_t$$

$$\sigma_t^{\underline{v}} - \sigma_t^{\underline{\eta}} - \sigma_t^{q+p} - \underline{\gamma}\sigma = -\underline{\gamma}\sigma_t^{\underline{c}}$$

$$= -\underline{\varsigma}_t$$

2c. Get ςs from Value Function Envelop

- Experts value function

$$v_t \frac{K_t^{1-\gamma}}{1-\gamma}\left(\tilde{\eta}_t^{\tilde{\iota}}\right)^{1-\gamma}$$

HH's value function

$$\underline{v}_t \frac{K_t^{1-\underline{\gamma}}}{1-\underline{\gamma}}\left(\underline{\tilde{\eta}}_t^{\tilde{\iota}}\right)^{1-\gamma}$$

- To obtain $\frac{\partial V_t(n)}{\partial n_t}$ use $K_t = \frac{N_t}{\eta_t(q_t+p_t)} = \frac{1}{\tilde{\eta}_t^{\tilde{\iota}}} \frac{n_t}{\eta_t(q_t+p_t)}$

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…

- Using $K_t = \frac{1}{\tilde{\eta}_t^{\tilde{\iota}}} \frac{n_t}{\eta_t(q_t+p_t)}$, $C_t = \frac{1}{\tilde{\eta}_t^{\tilde{\iota}}} c_t$

$$\frac{v_t}{\eta_t(q_t+p_t)} K_t^{-\gamma} = C_t^{-\gamma}$$

By Ito's Lemma

$$(q_t+p_t)\sigma_t^{q+p} = q_t\sigma_t^q + p_t\sigma_t^p$$

…

$$\sigma_t^v - \sigma_t^\eta - \sigma_t^{q+p} - \gamma\sigma = -\gamma\sigma_t^c,$$

$$= -\varsigma_t$$

$$\sigma_t^{\underline{v}} - \sigma_t^{\underline{\eta}} - \sigma_t^{q+p} - \underline{\gamma}\sigma = -\underline{\gamma}\sigma_t^{\underline{c}}$$

$$= -\underline{\varsigma}_t$$

2c. Get ςs from Value Function Envelop

- Experts risk-premia

$$v_t \frac{K_t^{1-\gamma}}{1-\gamma}\left(\tilde{\eta}_t^{\tilde{\imath}}\right)^{1-\gamma}$$

$$\varsigma_t = \gamma\sigma_t^c =$$

$$-\sigma_t^v + \sigma_t^\eta + \sigma_t^{q+p} + \gamma\sigma$$

HH's risk premia

$$\underline{v}_t \frac{K_t^{1-\gamma}}{1-\gamma}\left(\underline{\tilde{\eta}}_t^{\tilde{\imath}}\right)^{1-\gamma}$$

$$\underline{\varsigma}_t = \underline{\gamma}\sigma_t^{\underline{c}} =$$

$$-\sigma_t^{\underline{v}} - \frac{\eta_t\sigma_t^\eta}{1-\eta_t} + \sigma_t^{q+p} + \underline{\gamma}\sigma$$

- For $\tilde{\varsigma}_t$

  note that from

$$v_t \frac{n_t^{-\gamma}}{(\eta_t(q_t+p_t))^{1-\gamma}} = c_t^{-\gamma} \text{ follows } \tilde{\sigma}_t^n = \tilde{\sigma}_t^c$$

Hence, $\tilde{\varsigma}_t = \gamma\tilde{\sigma}_t^n$

$$\underline{\tilde{\varsigma}}_t = \underline{\gamma}\tilde{\sigma}_t^{\underline{n}}$$

2c. Get $\frac{C_t}{N_t}, \frac{\underline{C}_t}{\underline{N}_t}$ from Value Function Envelop

- Experts
- Recall $v_t \frac{n_t^{-\gamma}}{(\eta_t(q_t+p_t))^{1-\gamma}} = c_t^{-\gamma}$

$$\frac{c_t}{n_t} = \frac{(\eta_t(q_t+p_t))^{1/\gamma-1}}{v_t^{1/\gamma}}$$

$$\frac{C_t}{N_t} = \frac{(\eta_t(q_t+p_t))^{1/\gamma-1}}{v_t^{1/\gamma}}$$

Households

$$\frac{\underline{C}_t}{\underline{N}_t} = \frac{((1-\eta_t)(q_t+p_t))^{1/\underline{\gamma}-1}}{\underline{v}_t^{1/\underline{\gamma}}}$$

$$\frac{C_t+\underline{C}_t}{N_t+\underline{N}_t} = \eta_t \frac{C_t}{N_t} + (1-\eta_t)\frac{\underline{C}_t}{\underline{N}_t}$$

Plug in from above

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3. Evolution of state variable η *forward equation*
   - *Toolbox 3:* Change in numeraire to total wealth (including SDF)
   - "Money evaluation equation" μ^{ϑ}
4. Value function iteration & goods market clearing
   a. PDE of de-scaled value fcn.
   b. Value function iteration by solving PDE

3. μ^{η} Drift of Wealth Share: Two Types

- Asset pricing formula (relative to benchmark asset)

$$\mu_t^{\eta} + \frac{C_t}{N_t} - \vartheta_t \mu^M - r_t^M = (\varsigma - \sigma^N)(\sigma^{\eta} - \sigma^M) + \tilde{\varsigma}\tilde{\sigma}_t^n$$

Seignorage due to money supply growth leads to **transfers**

- Add up across types (weighted),

(capital letters without superscripts are aggregates for total economy)

$$\underbrace{(\eta_t \mu_t^{\eta} + (1-\eta_t)\mu_t^{\underline{\eta}})}_{=0} + \frac{\bar{C}_t}{\bar{N}_t} - \vartheta_t \mu^M - r_t^M =$$

$$\eta_t(\varsigma_t - \sigma_t^{\bar{N}})(\sigma_t^{\eta} - \sigma_t^M) + (1-\eta_t)\left(\underline{\varsigma}_t - \sigma_t^{\bar{N}}\right)\left(\sigma_t^{\underline{\eta}} - \sigma_t^M\right) + \eta_t \tilde{\varsigma}\tilde{\sigma}_t^n + (1-\eta_t)\underline{\tilde{\varsigma}}_t \underline{\tilde{\sigma}}_t^n$$

- Subtract from each other yields wealth share drift

$$\mu_t^{\eta} = (1-\eta_t)(\varsigma_t - \sigma_t^{\bar{N}})(\sigma_t^{\eta} - \sigma_t^M) - (1-\eta_t)\left(\underline{\varsigma}_t - \sigma_t^{\bar{N}}\right)\left(\sigma_t^{\underline{\eta}} - \sigma_t^M\right)$$

$$+ (1-\eta_t)\tilde{\varsigma}_t\tilde{\sigma}_t^n - (1-\eta_t)\underline{\tilde{\varsigma}}_t\underline{\tilde{\sigma}}_t^n - \left(\frac{C_t}{N_t} - \frac{C_t + \underline{C}_t}{(q_t + p_t)K_t}\right)$$

3. σ^η Volatility of Wealth Share

- Since $\eta_t^i = N_t^i / \bar{N}_t$,

$$\sigma_t^{\eta^i} = \sigma_t^{N^i} - \sigma_t^{\bar{N}} = \sigma_t^{N^i} - \sum_{i'} \eta_t^{i'} \sigma_t^{N^{i'}}$$

$$= \left(1 - \eta_t^i\right)\sigma_t^{N^i} - \sum_{i^- \neq i} \eta_t^{i^-} \sigma_t^{N^{i^-}}$$

- Note for 2 types example

Change in notation in 2 type setting
Type-networth is $n = N^i$

$$\sigma_t^\eta = (1 - \eta_t)\left(\sigma_t^n - \sigma_t^{\underline{n}}\right)$$

$$\sigma_t^n = \left(\sigma + \sigma_t^p\right) + \underbrace{\frac{\chi_t}{\eta_t}(1 - \vartheta)}_{=\theta^k + \theta^{oe}} \left(\sigma_t^q - \sigma_t^p\right), \quad \sigma_t^{\underline{n}} = \left(\sigma + \sigma_t^p\right) + \frac{1 - \chi_t}{1 - \eta_t}(1 - \vartheta)\left(\sigma_t^q - \sigma_t^p\right)$$

  - Hence,

$$\sigma_t^\eta = \frac{\chi_t - \eta_t}{\eta_t} \underbrace{(1 - \vartheta)\left(\sigma_t^q - \sigma_t^p\right)}_{= -\sigma_t^\vartheta}$$

Apply Ito's Lemma on ϑ

- Note also, $\eta_t \sigma_t^\eta + (1 - \eta_t)\sigma_t^{\underline{\eta}} = 0 \Rightarrow \sigma_t^{\underline{\eta}} = -\frac{\eta_t}{1 - \eta_t}\sigma_t^\eta$

Solving MacroModels Step-by-Step

0. Postulate aggregates, price processes & obtain return processes
1. For given SDF processes *static*
   a. Real investment ι, (portfolio $\boldsymbol{\theta}$, & consumption choice of *each agent*)
      - *Toolbox 1:* Martingale Approach

   b. Asset/Risk Allocation *across types/sectors* & asset market clearing
      - *Toolbox 2:* "price-taking social planner approach" – Fisher separation theorem
2. Value functions *backward equation*
   a. Value fcn. as fcn. of individual investment opportunities ω
      - *Special cases*

   b. De-scaled value fcn. as function of state variables η
      - *Digression:* HJB-approach (instead of martingale approach & envelop condition)

   c. Derive ς price of risk, C/N-ratio from value fcn. envelop condition
3. Evolution of state variable η *forward equation*
   - *Toolbox 3:* Change in numeraire to total wealth (including SDF)
   - "Money evaluation equation" μ^{ϑ}
4. Value function iteration & goods market clearing
   a. PDE of de-scaled value fcn.
   b. Value function iteration by solving PDE

3. “Money evaluation equation” μ^{ϑ}

- Recall $\frac{\bar{C}_t}{\bar{\bar{N}}_t} - r_t^M - \vartheta_t \mu^M =$

$$\eta_t\left(\varsigma_t - \sigma_t^{\bar{N}}\right)\left(\sigma_t^{\eta} - \sigma_t^{M}\right) + (1-\eta_t)\left(\underline{\varsigma}_t - \sigma_t^{\bar{N}}\right)\left(\sigma_t^{\underline{\eta}} - \sigma_t^{M}\right) + \eta_t \tilde{\varsigma} \tilde{\sigma}_t^{n} + (1-\eta_t)\underline{\tilde{\varsigma}}_t \underline{\tilde{\sigma}}_t^{\underline{n}}$$

- If benchmark asset is money
  - Replace $r_t^M = \mu_t^{\vartheta} - \mu^M$ and $\sigma_t^M = \sigma_t^{\vartheta}$ (in the total wealth numeraire)

$$-\mu_t^{\vartheta} = -(1-\vartheta)\mu^M - \frac{\bar{C}_t}{\bar{\bar{N}}_t} + \eta_t\left(\varsigma_t - \sigma_t^{\bar{N}}\right)\left(\sigma_t^{\eta} - \sigma_t^{\vartheta}\right) + (1-\eta_t)\left(\underline{\varsigma}_t - \sigma_t^{\bar{N}}\right)\left(\sigma_t^{\underline{\eta}} - \sigma_t^{\vartheta}\right) + \eta_t \tilde{\varsigma} \tilde{\sigma}_t^{n} + (1-\eta_t)\underline{\tilde{\varsigma}}_t \underline{\tilde{\sigma}}_t^{\underline{n}}$$

  - Why is return on money in new numeriare $\frac{d\vartheta_t}{\vartheta_t} - \mu^M$?
    - $\vartheta_t = p_t K_t / \bar{N}_t$ is the value of money stock in total networth units

Solving MacroModels Step-by-Step

0. Postulate aggregates, price processes & obtain return processes
1. For given SDF processes *static*
    - a. Real investment ι, (portfolio $\boldsymbol{\theta}$, & consumption choice of *each agent*)
        - *Toolbox 1:* Martingale Approach
    - b. Asset/Risk Allocation *across types/sectors* & asset market clearing
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    - a. Value fcn. as fcn. of individual investment opportunities ω
        - *Special cases*
    - b. De-scaled value fcn. as function of state variables η
        - *Digression:* HJB-approach (instead of martingale approach & envelop condition)
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    - b. Value function iteration by solving PDE

4. PDE Value Function Iteration

Short-hand notation:
$\partial_x f$ for $\partial f / \partial x$

- Postulate $v_t = v(\eta_t, t)$
- By Ito's Lemma
- $\frac{dv_t}{v_t} = \underbrace{\frac{\partial_t v_t + \partial_\eta v_t \eta \mu_t^\eta + \frac{1}{2}\partial_{\eta\eta} v_t (\eta_t \sigma_t^\eta)^2}{v_t}}_{\mu_t^v} dt + \underbrace{\frac{\partial_\eta v_t \eta \sigma_t^\eta}{v_t}}_{\sigma_t^v} dZ_t$
- That is,
- $\mu_t^v v_t = \partial_t v_t + \partial_\eta v_t \eta \mu_t^\eta + \frac{1}{2}\partial_{\eta\eta} v_t (\eta_t \sigma_t^\eta)^2$
- $\sigma_t^v v_t = \partial_\eta v_t \eta \sigma_t^\eta$
- Plugging in previous slides drift equation ⇒"growth equation"

$$\boxed{\begin{aligned} &\partial_t v_t + (\eta \mu_t^\eta + (1-\gamma)\sigma \eta_t \sigma_t^\eta v_t)\partial_\eta v_t + \frac{1}{2}\partial_{\eta\eta} v_t (\eta_t \sigma_t^\eta)^2 = \\ &= \left(\rho - (1-\gamma)(\Phi(\iota) - \delta) + \frac{1}{2}\gamma(1-\gamma)(\sigma^2 + (\tilde{\sigma}^n)^2)\right) v_t - \frac{c_t}{n_t} v_t \end{aligned}}$$

4a. Algorithm

- *Dynamic steps* involves now iterating $v(\eta), \underline{v}(\eta)$, and $\vartheta(\eta)$

- *Static step* only involve planner's conditions (which implicitly includes asset market clearing),
  - Solve everything in terms of $\vartheta(\eta)$
    - $q(\eta)$ and $p(\eta)$ can be easily derived since we have it as a function of ϑ and ψ in closed form
    - $q_t = (1 - \vartheta_t) \frac{1+\kappa A(\psi_t)}{1-\vartheta_t+\kappa\bar{\zeta}_t}$, where $\bar{\zeta}_t := \bar{C}_t / \bar{N}_t$
    - $p_t = \vartheta_t \frac{1+\kappa A(\psi_t)}{1-\vartheta_t+\kappa\bar{\zeta}_t}$

- Remark:
  One can obtain the moneyless equilibrium with $\vartheta(\eta) = 0$ by setting $\sigma^p = -\sigma$ (in models with real risk-free debt)
  - Why? recall $dr^M = [(\Phi(\iota) - \delta) - \mu^M]dt + (\sigma + \sigma^p)dZ$
    - We never used the drift to solve the model.
    - To make money, risk-free asset we have to set $\sigma^p = -\sigma$

Roadmap

- Changes in solution procedure in a setting with idiosyncratic risk and money
  - Compare to lecture 03 without idiosyncratic risk and money

- Simple two sector model
  1. Real Debt

  2. Money/Nominal Debt

Two Sector Model Setup

Expert sector

- Output: $y_t = ak_t$
- Consumption rate: c_t
- Investment rate: ι_t

$$\frac{dk_t^{\tilde{\imath}}}{k_t^{\tilde{\imath}}} = (\Phi(\iota_t) - \delta)dt + \sigma dZ_t + \tilde{\sigma} d\tilde{Z}_t^{\tilde{\imath}} + d\Delta_t^{k,\tilde{\imath}}$$

- $E_0[\int_0^\infty e^{-\rho t} \log c_t \, dt]$

Friction: Can only issue

- Risk-free debt

$$a = \underline{a}$$

$$\delta = \underline{\delta}$$

$$\boxed{\tilde{\sigma} \leq \underline{\tilde{\sigma}}}$$

$$\rho = \underline{\rho}$$

Household sector

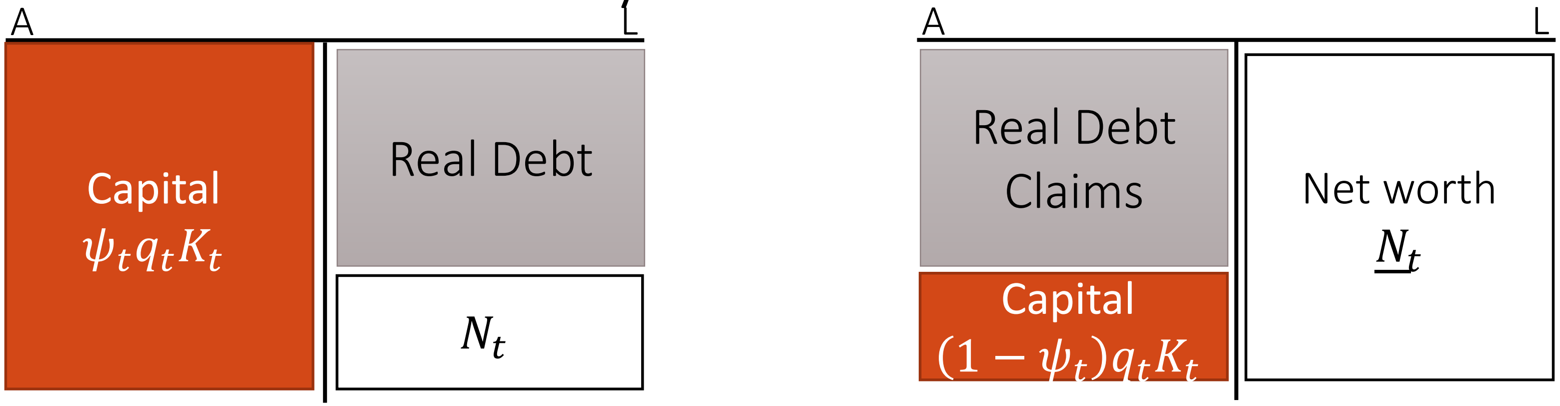
- Output: $\underline{y}_t = \underline{a}\underline{k}_t$
- Consumption rate: $\underline{c}_t$
- Investment rate: $\underline{\iota}_t$

$$\frac{d\underline{k}_t^{\tilde{\imath}}}{\underline{k}_t^{\tilde{\imath}}} = (\Phi(\underline{\iota}_t) - \underline{\delta})dt + \sigma dZ_t + \underline{\tilde{\sigma}} d\tilde{Z}_t^{\tilde{\imath}} + d\underline{\Delta}_t^{k,\tilde{\imath}}$$

- $E_0[\int_0^\infty e^{-\underline{\rho} t} \log \underline{c}_t \, dt]$

Two Sector Model with & without Money

- Idio risk without money and real debt

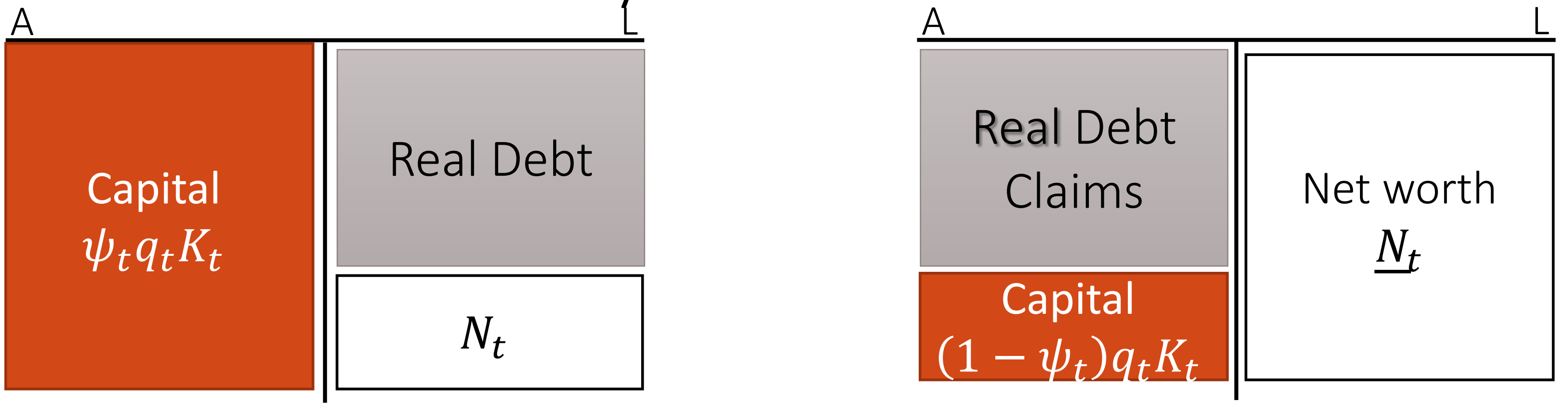


Poll 33: Increasing experts idiosyncratic risk $\tilde{\sigma}$

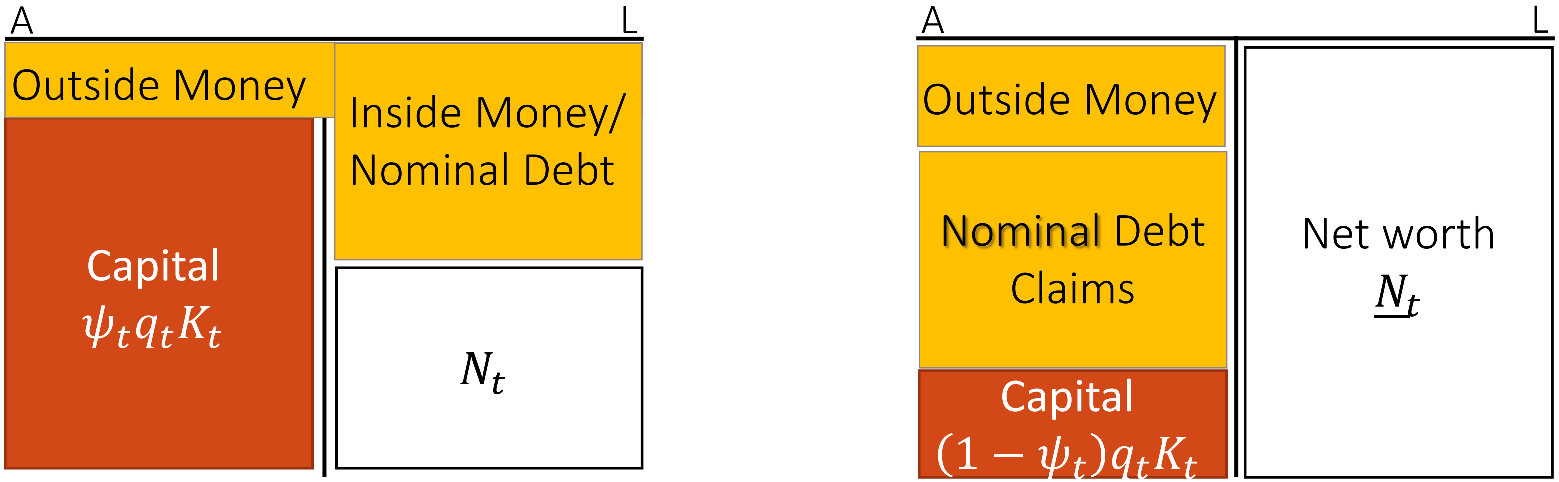
- *a) Lowers experts wealth share drift μ^η*
- *b) Increases experts wealth share drift μ^η, as they earn some extra risk premium*
- *c) Hurts the households, as it depresses r_t^f*

Two Sector Model with & without Money

- Idio risk without money and real debt



- Idio risk with money (and nominal short-term debt)



  - Value of money covaries with K-shocks ⇒ implicit insurance

Solution Procedure for Both Settings

- Goods market clearing

$$\rho(p_t + q_t) = a - \iota_t$$ divide by q and use $q = 1 + \kappa\iota$

$$\rho \frac{1}{1 - \vartheta_t} = \frac{a - i}{1 + \kappa\iota}$$

  - $\iota_t = \frac{(1-\vartheta_t)a-\rho}{1-\vartheta_t+\kappa\rho}$
  - $q_t = 1 + \kappa\iota_t = (1 - \vartheta_t)\frac{1+\kappa a}{1-\vartheta_t+\kappa\rho}$
  - $p_t = q_t\frac{\vartheta_t}{1-\vartheta_t} = \vartheta_t\frac{1+\kappa a}{1-\vartheta_t+\kappa\rho}$

Solution Procedure for Both Settings

- Goods market clearing

$$\rho(p_t + q_t) = a - \iota_t$$ divide by q and use $q = 1 + \kappa\iota$

$$\rho\frac{1}{1-\vartheta_t} = \frac{a-i}{1+\kappa\iota}$$

  - $\iota_t = \frac{(1-\vartheta_t)a-\rho}{1-\vartheta_t+\kappa\rho}$
  - $q_t = 1 + \kappa\iota_t = (1-\vartheta_t)\frac{1+\kappa a}{1-\vartheta_t+\kappa\rho}$
  - $p_t = q_t\frac{\vartheta_t}{1-\vartheta_t} = \vartheta_t\frac{1+\kappa a}{1-\vartheta_t+\kappa\rho}$

Poll 36:

How would equations change if $a \neq \underline{a}$

a) Replace a with $A(\psi_t)$

b) Nothing

c) Whole approach has to be different.

Solution Procedure for Both Settings

- Goods market clearing

$$\rho(p_t + q_t) = a - \iota_t$$ divide by q and use $q = 1 + \kappa\iota$

$$\rho \frac{1}{1-\vartheta_t} = \frac{a-i}{1+\kappa\iota}$$

  - $\iota_t = \frac{(1-\vartheta_t)a-\rho}{1-\vartheta_t+\kappa\rho}$
  - $q_t = 1 + \kappa\iota_t = (1-\vartheta_t)\frac{1+\kappa a}{1-\vartheta_t+\kappa\rho}$
  - $p_t = q_t \frac{\vartheta_t}{1-\vartheta_t} = \vartheta_t \frac{1+\kappa a}{1-\vartheta_t+\kappa\rho}$

- Capital market clearing (defines ψ_t for planner)

$$(1-\theta_t) = \frac{\psi_t}{\eta_t}(1-\vartheta_t) \qquad \left(1-\underline{\theta}_t\right) = \frac{1-\psi_t}{1-\eta_t}(1-\vartheta_t)$$

- Money market clearing by Walras law

Solution Procedure for Both Settings

- Price-taking Social Planner Problem
- $$\begin{aligned}\max_{\psi_t} \vartheta_t E[r_t^M] + (1-\vartheta_t) E[r_t^K] \\ -(\varsigma_t \psi_t + \underline{\varsigma}_t (1-\psi_t))(\sigma + \sigma_t^{p+q}) \\ -(\tilde{\varsigma}_t \psi_t \tilde{\sigma} + \underline{\tilde{\varsigma}} (1-\psi_t) \underline{\tilde{\sigma}})\end{aligned}$$

Poll 38: Does $E_t[r_t^K]$ depend on ψ?

a) Yes
b) No

Solution Procedure for Both Settings

- Price-taking Social Planner Problem
- $\max_{\psi_t} \vartheta_t E[r_t^M] + (1-\vartheta_t)E[r_t^K]$

$$-(\varsigma_t\psi_t + \underline{\varsigma}_t(1-\psi_t))(\sigma + \sigma_t^{p+q})$$

$$-(\tilde{\varsigma}_t\psi_t\tilde{\sigma} + \underline{\tilde{\varsigma}}(1-\psi_t)\underline{\tilde{\sigma}})$$

- FOC: $\varsigma_t\sigma + \tilde{\varsigma}_t\tilde{\sigma} = \underline{\varsigma}_t\sigma + \underline{\tilde{\varsigma}}\underline{\tilde{\sigma}}$ (prices of risks adjust for interior solution)

1. Real Debt Setting: q, ς, Planner's prob.

- Set $\vartheta_t = 0, \Rightarrow p = 0$
- $q = \frac{1+\kappa a}{1+\kappa\rho} \;\; \forall t \;\Rightarrow \sigma^q = \sigma^{p+q} = 0$ (as in Basak Cuoco)
- Prices of Risk

$$\varsigma_t = \sigma_t^n = (1-\theta_t)\sigma = \frac{\psi_t}{\eta_t}\sigma, \;\; \underline{\varsigma}_t = \sigma_t^{\underline{n}} = \left(1-\underline{\theta}_t\right)\sigma = \frac{1-\psi_t}{1-\eta_t}\sigma$$

$$\tilde{\varsigma}_t = \tilde{\sigma}_t^n = (1-\theta_t)\tilde{\sigma} = \frac{\psi_t}{\eta_t}\tilde{\sigma}, \;\; \underline{\tilde{\varsigma}}_t = \tilde{\sigma}_t^{\underline{n}} = \left(1-\underline{\theta}_t\right)\underline{\tilde{\sigma}} = \frac{1-\psi_t}{1-\eta_t}\underline{\tilde{\sigma}}$$

- Plug in planners FOC: $\varsigma_t\sigma^2 + \tilde{\varsigma}_t\tilde{\sigma} = \underline{\varsigma}_t\sigma + \underline{\tilde{\varsigma}}\underline{\tilde{\sigma}}$

$$\psi_t = \frac{\eta_t(\sigma^2 + \underline{\tilde{\sigma}}^2)}{\sigma^2 + (1-\eta_t)\tilde{\sigma}^2 + \eta_t\underline{\tilde{\sigma}}^2}$$

- $\mu_t^\eta = \cdots, \sigma_t^\eta = \cdots$

1. Real Debt Setting: η-Evolution

- $\sigma_t^\eta = (1-\eta_t)\left(\sigma_t^N - \sigma_t^{\bar{N}}\right) = (1-\eta_t)\left(\frac{\psi_t}{\eta_t} - \frac{1-\psi_t}{1-\eta_t}\right)\sigma =$

  $= \frac{\psi_t - \eta_t}{\eta_t}\sigma$
- $\mu_t^\eta = (1-\eta_t)(\varsigma_t - \sigma_t^{\bar{N}})(\sigma_t^\eta - \sigma_t^M) - (1-\eta_t)\left(\underline{\varsigma}_t - \sigma_t^{\bar{N}}\right)\left(\sigma_t^{\underline{\eta}} - \sigma_t^M\right)$

  $+(1-\eta_t)\tilde{\varsigma}_t\tilde{\sigma}_t^n - (1-\eta_t)\underline{\tilde{\varsigma}}_t\underline{\tilde{\sigma}}_t^{\underline{n}} - \left(\frac{C_t}{N_t} - \frac{C_t + \underline{C}_t}{(q_t + p_t)K_t}\right)$

- Benchmark asset is risk-free asset in $\bar{N}$-numeraire
  - $\sigma_t^M = -\sigma$ because $\sigma_t^{\bar{N}} = \sigma$ (since $\sigma^q = 0$), $\frac{C}{N} = \rho$
  - $\mu_t^\eta = (1-\eta_t)(\varsigma_t - \sigma)(\sigma_t^\eta + \sigma) - (1-\eta_t)\left(\underline{\varsigma}_t - \sigma\right)\left(\sigma_t^{\underline{\eta}} + \sigma\right)$

    $+(1-\eta_t)\tilde{\varsigma}_t\tilde{\sigma}_t^n - (1-\eta_t)\underline{\tilde{\varsigma}}_t\underline{\tilde{\sigma}}_t^{\underline{n}}$
  - $\mu_t^\eta = \frac{\psi_t - \eta_t}{\eta_t}\frac{\psi_t - 2\eta_t\psi_t + \eta_t^2}{\eta_t(1-\eta_t)}\sigma^2 + (1-\eta_t)\left[\left(\frac{\psi_t}{\eta_t}\right)^2\tilde{\sigma}^2 - \left(\frac{1-\psi_t}{1-\eta_t}\right)^2\underline{\tilde{\sigma}}^2\right]$

1. Real Debt Setting: risk free rate

- $\frac{a-\iota}{q} + \Phi(\iota) - \delta = r_t^f + \varsigma_t \sigma + \tilde{\varsigma}_t \tilde{\sigma}$
- $r_t^f = \rho + (\Phi(\iota) - \delta) - \frac{\psi_t}{\eta_t}(\sigma^2 + \tilde{\sigma}^2)$, where $\frac{\psi_t}{\eta_t} = \frac{(\sigma^2 + \underline{\tilde{\sigma}}^2)}{\sigma^2 + (1-\eta_t)\tilde{\sigma}^2 + \eta_t \underline{\tilde{\sigma}}^2}$

$$r_t^f = \rho + (\Phi(\iota) - \delta) - \frac{(\sigma^2 + \tilde{\sigma}^2)\left(\sigma^2 + \underline{\tilde{\sigma}}^2\right)}{\sigma^2 + (1-\eta_t)\tilde{\sigma}^2 + \eta_t \underline{\tilde{\sigma}}^2}$$

- Proposition: r_t^f is decreasing in $\tilde{\sigma}^2$
    - HH suffer from experts' idiosyncratic risk exposure via a lower r_t^f
    - Experts have more idio risk, but benefit from lower r_t^f (since they have to earn risk premium for idio risk)

- Difference to
    - Basak-Cuoco: limited participation $\psi = 1$, HH fully at mercy of experts' ability to hedge idio risk
    - Here: HH participate in capital holding

2. Money/Nominal Debt Setting: ςs

- Experts' price of risk

$$\varsigma_t = \sigma_t^n = \sigma + \sigma_t^p + (1 - \theta_t)\left(\sigma_t^q - \sigma_t^p\right)$$
$$= \sigma + \sigma_t^p + \frac{\psi_t}{\eta_t}(1 - \vartheta_t)\left(\sigma_t^q - \sigma_t^p\right)$$

$$\tilde{\varsigma}_t = \sigma_t^{\tilde{n}} = (1 - \theta_t)\tilde{\sigma} = \frac{\psi_t}{\eta_t}(1 - \vartheta_t)\tilde{\sigma}$$

- Households' price of risk

$$\underline{\varsigma}_t = \sigma_t^{\underline{n}} = \sigma + \sigma^p + \left(1 - \underline{\theta}_t\right)\left(\sigma_t^q - \sigma^p\right)$$
$$= \sigma + \sigma^p + \frac{1 - \psi_t}{1 - \eta_t}(1 - \vartheta_t)\left(\sigma_t^q - \sigma^p\right)$$

$$\underline{\tilde{\varsigma}}_t = \tilde{\sigma}_t^{\underline{n}} = \left(1 - \underline{\theta}_t\right)\underline{\tilde{\sigma}} = \frac{1-\psi_t}{1-\eta_t}(1 - \vartheta_t)\underline{\tilde{\sigma}}$$

2. Money Setting: Planner's Problem

- Conjecture: $\sigma_t^q = \sigma_t^p = 0 \; \forall t$

$$\Rightarrow \varsigma = \sigma = \underline{\varsigma} = \sigma$$

- Proposition: Aggregate risk is perfectly shared!
  - Via inflation risk
  - Stable inflation (targeting) would ruin risk-sharing
    - Example: Brexit uncertainty. Use inflation reaction to share risks within UK

- Planner's FOC: $\tilde{\varsigma}\tilde{\sigma} = \underline{\tilde{\varsigma}}\underline{\tilde{\sigma}}$
  - $\frac{\psi_t}{\eta_t}\cancel{(1-\vartheta_t)}\tilde{\sigma}^2 = \frac{1-\psi_t}{1-\eta_t}\cancel{(1-\vartheta_t)}\underline{\tilde{\sigma}}^2$
  - $\psi(\eta,\vartheta)$ does not depend on ϑ

$$\psi(\eta) = \frac{\eta\underline{\tilde{\sigma}}^2}{(1-\eta)\tilde{\sigma}^2 + \eta\underline{\tilde{\sigma}}^2}$$

2. Money Setting: η-Evolution

- $\sigma_t^\eta = \underbrace{(1-\eta_t)\left(\frac{\psi_t}{\eta_t} - \frac{1-\psi_t}{1-\eta_t}\right)}_{\frac{\psi_t-\eta_t}{\eta_t}}(1-\vartheta_t)\left(\sigma_t^q - \sigma^p\right)$
  - If $\sigma^q = \sigma^p = 0$, then $\sigma_t^\eta = 0\ \forall t$, if $\sigma_t^\eta = 0$, then $\sigma^q = \sigma^p = 0$

By Ito's lemma on $q(\eta)$ and $p(\eta)$

- $\mu_t^\eta = (1-\eta_t)\left(\varsigma_t - \sigma_t^{\bar{N}}\right)\left(\sigma_t^\eta - \sigma_t^M\right) - (1-\eta_t)\left(\underline{\varsigma}_t - \sigma_t^{\bar{N}}\right)\left(\underline{\sigma}_t^\eta - \sigma_t^M\right)$
  $+(1-\eta_t)\tilde{\varsigma}_t\tilde{\sigma}_t^n - (1-\eta_t)\underline{\tilde{\varsigma}}_t\underline{\tilde{\sigma}}_t^n - \left(\frac{C_t}{N_t} - \frac{C_t+\underline{C}_t}{(q_t+p_t)K_t}\right)$

- Benchmark asset is risk-free asset in $\bar{N}$-numeraire
  - $\sigma_t^M = 0$ and $\sigma_t^{\bar{N}} = \sigma, \frac{C}{N} = \rho$
  - $\mu_t^\eta/(1-\eta_t) = (\varsigma_t - \sigma)\sigma_t^\eta - \left(\underline{\varsigma}_t - \sigma\right)\underline{\sigma}_t^\eta + \tilde{\varsigma}_t\frac{\psi_t}{\eta_t}(1-\vartheta_t)\tilde{\sigma} - \underline{\tilde{\varsigma}}_t\frac{1-\psi_t}{1-\eta_t}(1-\vartheta_t)\underline{\tilde{\sigma}}$
  - $\mu_t^\eta = (1-\eta_t)(1-\vartheta_t)^2\left[\left(\frac{\psi_t}{\eta_t}\right)^2\tilde{\sigma}^2 - \left(\frac{1-\psi_t}{1-\eta_t}\right)^2\underline{\tilde{\sigma}}^2\right]$

2. Money Setting: Money Evaluation

- Recall $-\mu_t^{\vartheta} = -(1-\vartheta)\mu^M - \frac{\bar{C}_t}{\bar{N}_t} + \eta_t(\varsigma_t - \sigma_t^{\bar{N}})(\sigma_t^{\eta} - \sigma_t^{\vartheta})$

$$+(1-\eta_t)\left(\underline{\varsigma}_t - \sigma_t^{\bar{N}}\right)\left(\sigma_t^{\underline{\eta}} - \sigma_t^{\vartheta}\right) + \eta_t\tilde{\varsigma}\tilde{\sigma}_t^{n} + (1-\eta_t)\underline{\tilde{\varsigma}}_t\tilde{\sigma}_t^{\underline{n}}$$

- Plug in $\mu^M = 0, \frac{\bar{C}_t}{\bar{N}_t} = \rho, \varsigma_t = \underline{\varsigma}_t = \sigma, \sigma_t^{\bar{N}} = \sigma$

$$\tilde{\varsigma}\tilde{\sigma}_t^{n} = (1-\vartheta_t)^2\left(\frac{\psi_t}{\eta_t}\right)^2\tilde{\sigma}^2, \qquad \underline{\tilde{\varsigma}}_t = \tilde{\sigma}_t^{\underline{n}} = (1-\vartheta_t)^2\left(\frac{1-\psi_t}{1-\eta_t}\right)^2\underline{\tilde{\sigma}}^2$$

- $-\mu_t^{\vartheta} = -\rho + (1-\vartheta_t)^2\left[\eta_t\left(\frac{\psi_t}{\eta_t}\right)^2\tilde{\sigma}^2 + (1-\eta_t)\left(\frac{1-\psi_t}{1-\eta_t}\right)^2\underline{\tilde{\sigma}}^2\right]$
  - where $\psi_t = \frac{\eta_t\underline{\tilde{\sigma}}^2}{(1-\eta_t)\tilde{\sigma}^2 + \eta_t\underline{\tilde{\sigma}}^2}$

2. Money Setting: Adding Real Debt

- Adding Real Debt does not alter the equilibrium, since
  - Markets are complete w.r.t. to aggregate risk (perfect aggregate risk sharing)
  - Markets are incomplete w.r.t. to idiosyncratic risk only

- Note: Result relies on absence of price stickiness

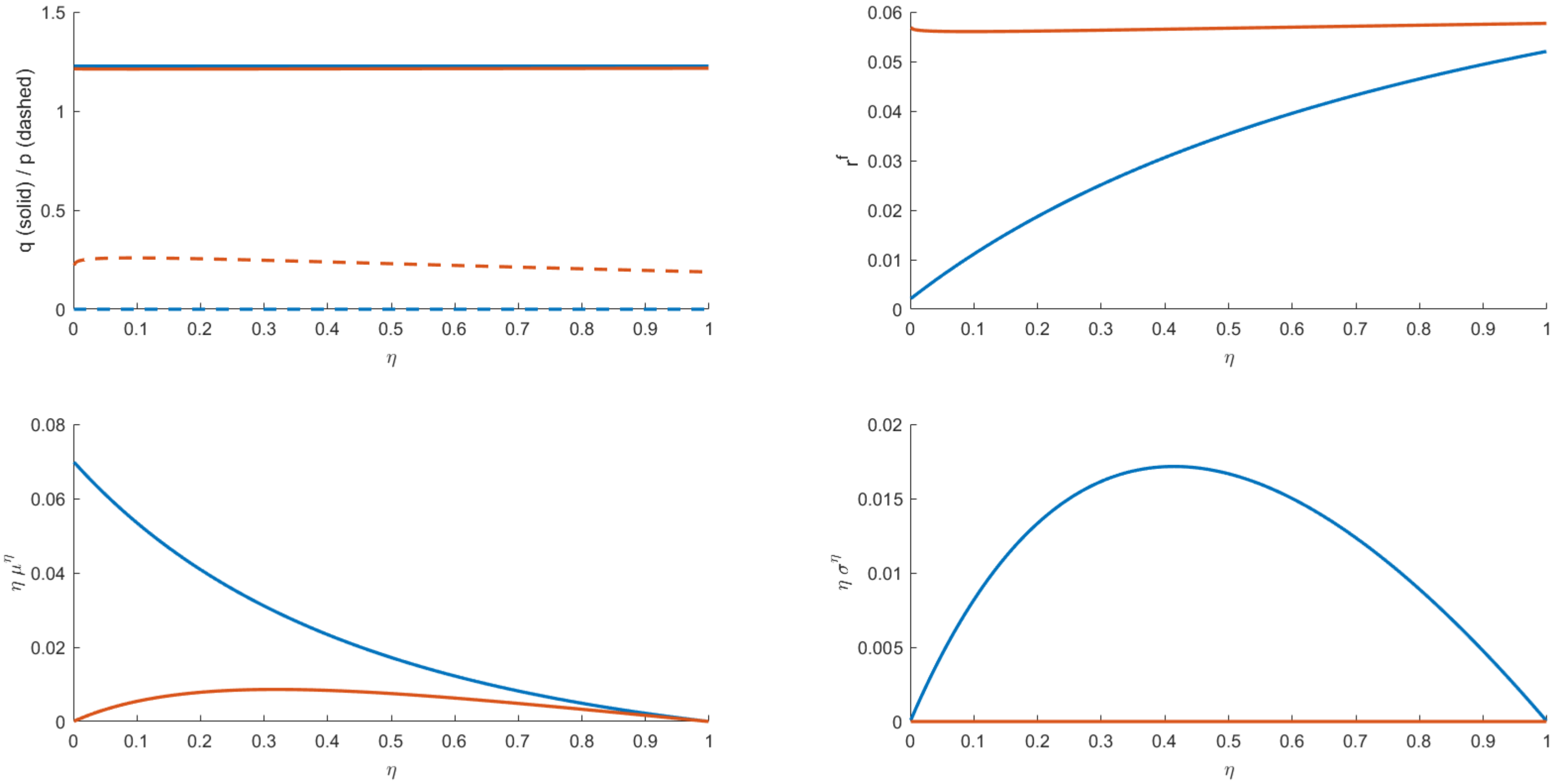
- *Both Settings: Real Debt and Money/Nominal Debt converge in the long-run to the "I Theory without I" steady state model of Lecture 05.*

Example: Real vs. Nominal Debt/Money

- $a = .15, \rho = .03, \sigma = .1, \kappa = 2, \delta = .03, \tilde{\sigma} = .2, \underline{\tilde{\sigma}} = .3$

Blue: real debt model

Red: nominal model



Towards the I Theory of Money

- One sector model with idio risk - “The I Theory without I” (steady state focus)
  - Store of value
    - Insurance role of money *within sector*
  - Money as bubble or not
  - Fiscal Theory of the Price Level
  - Medium of Exchange Role ⇒ SDF-Liquidity multiplier ⇒ Money bubble
- 2 sector/type model with money and idio risk
  - Generic Solution procedure (compared to lecture 03)
  - Real debt vs. Money
    - Implicit insurance role of money *across sectors*
  - The curse of insurance
    - Reduces insurance premia and net worth gains
- I Theory with Intermediary sector
  - Intermediaries as diversifiers
- Welfare analysis
- Optimal Monetary Policy and Macroprudential Policy

Last week

Today

Next lectures