Macro, Money and Finance

Lecture 02: A Simple Macro-Finance Model

Markus Brunnermeier, Lars Hansen, Yuliy Sannikov

Princeton, Chicago, NYU, UPenn, Northwerstern, EPFL, Stanford, Chicago Fed

Spring 2019

Course on Continuous-time Macro

1. Introduction: Liquidity, Run-up, Crisis-Amplification, Recovery

Real Macro-Finance Models with Heterogeneous Agents

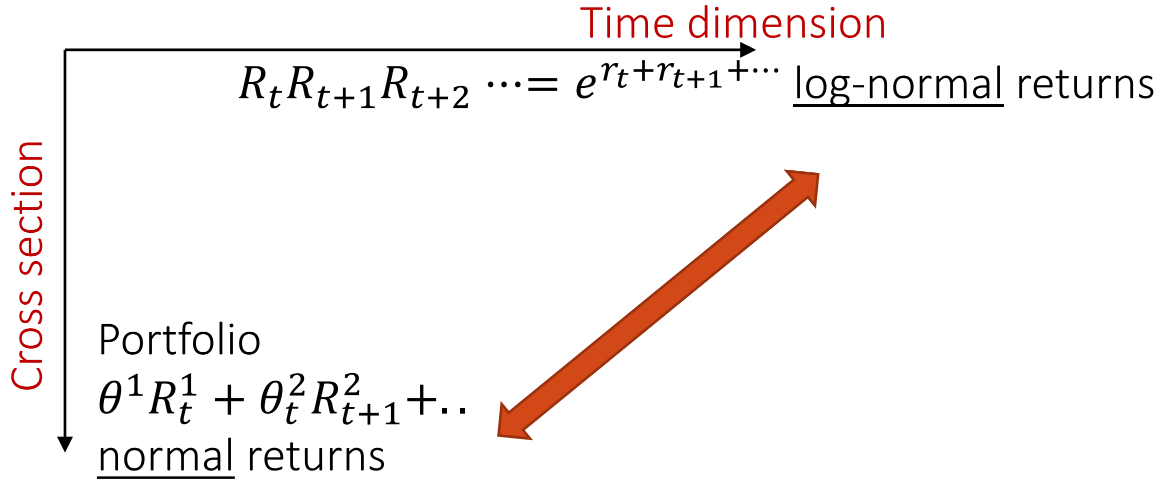
2. A Simple Model
3. General Solution Technique for Real Models
4. International Macro-Finance Model with Sudden Stops/Runs

Money Models

5. A Simple Money Model
6. General Solution Technique for Money Models
7. The I Theory of Money
8. Welfare Analysis & Optimal Policy
    - Monetary and Macroprudential Policy
9. International Financial Architecture*
10. Robust Computational Methods – Comparing Nonlinear Models
11. Calibration and Empirical Implications

Why Continuous Time Modeling?

- Discrete time consumption
  - IES/RA within period $= \infty$, but across periods $= 1/\gamma$
- Discrete time: Portfolio choice



Time dimension

$R_t R_{t+1} R_{t+2} \cdots = e^{r_t + r_{t+1} + \cdots}$ log-normal returns

Cross section

Portfolio

$\theta^1 R_t^1 + \theta_t^2 R_{t+1}^2 + ..$

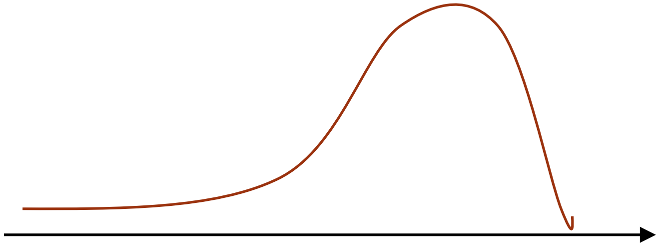
normal returns

  - Linearize: kills σ-term and all assets are equivalent
  - 2nd order approximation: kills time-varying σ
  - Log-linearize a la Campbell-Shiller
- As $\Delta t \to 0$ (net)returns converge to normal distribution
  - Constantly adjust the approximation point
  - Continuous compounding

Why Continuous Time Modeling?

- Ito processes… fully characterized by drift and volatility
  - Geometric Ito Process $dX_t = \mu_t^X X_t dt + \sigma_t^X X_t dZ_t$

- Characterization for full volatility dynamics on Prob.-space
  - Discrete time: Probability-loading on states
  - Cts. time: Loading on a Brownian Motion dZ_t (captured by σ)
- Normal distribution for dt, yet with skewness for $\Delta t > 0$
  - If σ_t is time-varying



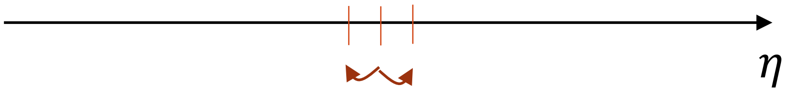
- How restrictive?

Why Continuous Time Modeling with Ito?

- Continuous path
  - Information arrives continuously "smoothly" – not in lumps
  - Implicit assumption: can react continuously to continuous info flow
  - Never jumps over a specific point, e.g. insolvency point
  - Simplifies numerical analysis:
    - Only need change from grid-point to grid-point (since one never jumps beyond the next grid-points)
  - No default risk
    - Can continuously delever as wealth declines
      - Might embolden investors ex-ante
  - Collateral constraint
    - Discrete time: $b_t R_{t,t+1} \leq \min\{q_{t+1}\} k_t$
    - Cts. time: $b_t \leq (p_t + \underbrace{dp_t}_{\to 0}) k_t$
      - For short-term debt – not for long-term debt … or if there are jumps
- Levy processes… with jumps

Why Continuous Time Modeling with Ito?

- $E[dV(\eta)] = V'(\eta)\mu^{\eta}\eta dt + \frac{1}{2}V''(\eta)(\sigma^{\eta})^2\eta^2 dt$



Just need the two
neighboring grid points
(not whole function)

- More analytical steps
  - Return equations
    - Next instant returns are essentially normal
      (easy to take expectations)
  - Explicit net worth and state variable dynamics
    - Continuous: only slope of price function determines amplification
    - Discrete: need whole price function (as jump size can vary)

Basics of Ito Calculus

- Geometric Ito Process $dX_t = \mu_t^X X_t dt + \sigma_t^X X_t dZ_t$
  - Stock goes up 32% or down 32% over a year.
256 trading days $\frac{32\%}{\sqrt{256}} = 2\%$
- Ito's Lemma:

$$df(X_t) = f'(X_t)\mu_t^X X_t dt + \frac{1}{2} f''(X_t)(\sigma_t^X X_t)^2 dt + f'(X_t)\sigma_t^X X_t dZ_t$$

  - $u(c) = \frac{c^{1-\gamma}-1}{1-\gamma}, u'(c) = c^{-\gamma}$ volatility of process $\frac{dc_t^{-\gamma}}{c_t^{-\gamma}}$ is $-\gamma\sigma_t^c$
- Ito product rule: stock price * exchange rate

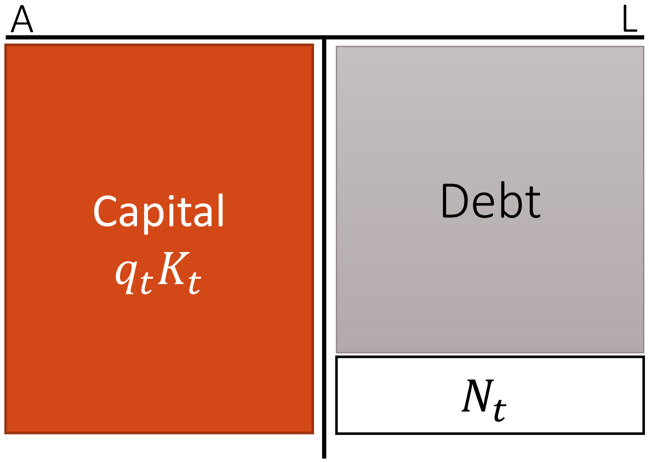
$$\frac{d(X_t Y_t)}{X_t Y_t} = (\mu_t^X + \mu_t^Y + \sigma_t^X \sigma_t^Y)dt + (\sigma_t^X + \sigma_t^Y)dZ_t$$

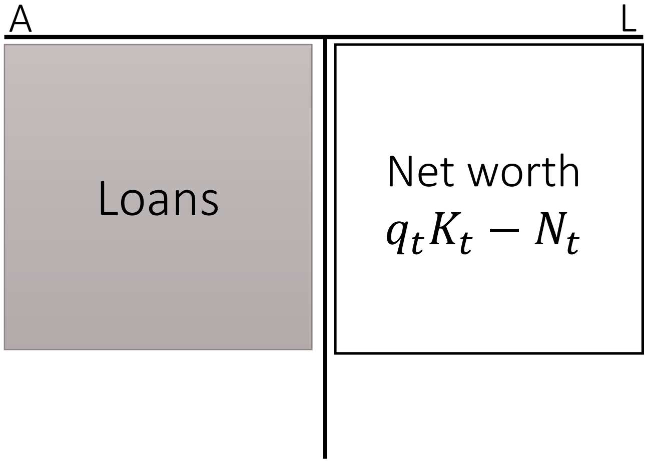
- Ito ratio rule:

$$\frac{d(X_t/Y_t)}{X_t/Y_t} = \mu_t^X - \mu_t^Y + \sigma_t^Y(\sigma_t^Y - \sigma_t^X)dt + (\sigma_t^X - \sigma_t^Y)dZ$$

Simple Two Sector Model: Basak Cuoco (1998)

- Expert sector Household sector





See Handbook of Macroeconomics 2017, Chapter 18

Two Sector Model Setup

Expert sector

- Output: $y_t = ak_t$

Household sector

Two Sector Model Setup

Expert sector

- Output: $y_t = ak_t$
- Consumption rate: c_t
- Investment rate: ι_t

$$\frac{dk_t^{\tilde{\imath}}}{k_t^{\tilde{\imath}}} = (\Phi(\iota_t) - \delta)dt + \sigma dZ_t$$

Household sector

- Consumption rate: $\underline{c}_t$

agent $\tilde{\imath}$ of type i (expert, HH)

Two Sector Model Setup

Expert sector

- Output: $y_t = ak_t$
- Consumption rate: c_t
- Investment rate: ι_t

$$\frac{dk_t^{\tilde{\imath}}}{k_t^{\tilde{\imath}}}=(\Phi(\iota_t)-\delta)dt+\sigma dZ_t$$

- $E_0[\int_0^\infty e^{-\rho t}\frac{c_t^{1-\gamma}}{1-\gamma}dt]$

Household sector

- Consumption rate: $\underline{c}_t$

- $E_0[\int_0^\infty e^{-\rho t}\frac{\underline{c}_t^{1-\gamma}}{1-\gamma}dt]$

Log-utility in Basak Cuoco 1998

Two Sector Model Setup

Expert sector

- Output: $y_t = ak_t$
- Consumption rate: c_t
- Investment rate: ι_t

$$\frac{dk_t^{\tilde{i}}}{k_t^{\tilde{i}}} = (\Phi(\iota_t) - \delta)dt + \sigma dZ_t$$

- $E_0[\int_0^\infty e^{-\rho t} \frac{c_t^{1-\gamma}}{1-\gamma} dt]$

Household sector

- Consumption rate: $\underline{c}_t$

- $E_0[\int_0^\infty e^{-\rho t} \frac{\underline{c}_t^{1-\gamma}}{1-\gamma} dt]$

Friction: Can only issue

- Risk-free debt

Solving MacroModels Step-by-Step

0. Postulate aggregates, price processes & obtain return processes
1. For given SDF processes *static*
   - a. Real investment ι, (portfolio $\boldsymbol{\theta}$, & consumption choice of *each agent*)
     - *Toolbox 1:* Martingale Approach
   - b. Asset/Risk Allocation *across types/sectors* & asset market clearing
     - *Toolbox 2:* "price-taking social planner approach" – Fisher separation theorem
2. Value functions *backward equation*
   - a. Value fcn. as fcn. of individual investment opportunities ω
     - *Special cases*
   - b. De-scaled value fcn. as function of state variables η
     - *Digression:* HJB-approach (instead of martingale approach & envelop condition)
   - c. Derive ς price of risk, C/N-ratio from value fcn. envelop condition
3. Evolution of state variable η *forward equation*
   - *Toolbox 3:* Change in numeraire to total wealth (including SDF)
   - ("Money evaluation equation" μ^{ϑ})
4. Value function iteration & goods market clearing
   - a. PDE of de-scaled value fcn.
   - b. Value function iteration by solving PDE

0. Postulate Aggregates and processes

- Individual capital evolution

$$\frac{dk_t^{\tilde{\imath}}}{k_t^{i}} = \left(\Phi\left(\iota_t^{\tilde{\imath}}\right) - \delta\right)dt + \sigma dZ_t + d\Delta_t^{k,\tilde{\imath}}$$

  - Where $\Delta_t^{k,\tilde{\imath}}$ is the individual cumulative capital purchase process
- Capital aggregation: $K \equiv \int k_t^{\tilde{\imath}}\, d\tilde{\imath}$

$$\frac{dK_t}{K_t} = \int \left(\Phi\left(\iota_t^{\tilde{\imath}}\right) - \delta\right)d\tilde{\imath}dt + \sigma dZ_t$$

$\Delta_t^{k,\tilde{\imath}}$ add up to zero

0. Postulate Aggregates and processes

- Individual capital evolution

$$\frac{dk_t^{\tilde{\imath}}}{k_t^{i}} = \left(\Phi\left(\iota_t^{\tilde{\imath}}\right) - \delta\right)dt + \sigma dZ_t + d\Delta_t^{k,\tilde{\imath}}$$

  - Where $\Delta_t^{k,\tilde{\imath}}$ is the individual cumulative capital purchase process
- Capital aggregation: $K \equiv \int k_t^{\tilde{\imath}}\, d\tilde{\imath}$

$$\frac{dK_t}{K_t} = \int \left(\Phi\left(\iota_t^{\tilde{\imath}}\right) - \delta\right)d\tilde{\imath}dt + \sigma dZ_t$$

$\Delta_t^{k,\tilde{\imath}}$ add up to zero

- Networth aggregation:
  - Within sector: $N_t \equiv \int n_t^{\tilde{\imath}} d\tilde{\imath}, \quad \underline{N}_t \equiv \int \underline{n}_t^{\tilde{\imath}} d\tilde{\imath}$
  - Wealth share: $\eta_t \equiv N_t/(N_t + \underline{N}_t)$

0. Postulate Aggregates and processes

- Individual capital evolution

$$\frac{dk_t^{\tilde{\imath}}}{k_t^{i}} = \left(\Phi\left(\iota_t^{\tilde{\imath}}\right) - \delta\right)dt + \sigma dZ_t + d\Delta_t^{k,\tilde{\imath}}$$

  - Where $\Delta_t^{k,\tilde{\imath}}$ is the individual cumulative capital purchase process
- Capital aggregation: $K \equiv \int k_t^{\tilde{\imath}}\, d\tilde{\imath}$

$$\frac{dK_t}{K_t} = \int \left(\Phi\left(\iota_t^{\tilde{\imath}}\right) - \delta\right)d\tilde{\imath}dt + \sigma dZ_t$$

$\Delta_t^{k,\tilde{\imath}}$ add up to zero

- Networth aggregation:
  - Within sector: $N_t \equiv \int n_t^{\tilde{\imath}} d\tilde{\imath}, \quad \underline{N}_t \equiv \int \underline{n}_t^{\tilde{\imath}} d\tilde{\imath}$
  - Wealth share: $\eta_t \equiv N_t/(N_t + \underline{N}_t)$
- Value of capital stock: $q_t K_t$

Same Brownian

Postulate $dq_t/q_t = \mu_t^q dt + \sigma_t^q dZ_t$

0. Postulate Aggregates and processes

- Individual capital evolution

$$\frac{dk_t^{\tilde{i}}}{k_t^{i}} = \left(\Phi\left(\iota_t^{\tilde{i}}\right) - \delta\right)dt + \sigma dZ_t + d\Delta_t^{k,\tilde{i}}$$

  - Where $\Delta_t^{k,\tilde{i}}$ is the individual cumulative capital purchase process

$\Delta_t^{k,\tilde{i}}$ add up to zero

- Capital aggregation: $K \equiv \int k_t^{\tilde{i}}\, d\tilde{i}$

$$\frac{dK_t}{K_t} = \int \left(\Phi\left(\iota_t^{\tilde{i}}\right) - \delta\right)d\tilde{i}dt + \sigma dZ_t$$

- Networth aggregation:
  - Within sector: $N_t \equiv \int n_t^{\tilde{i}} d\tilde{i}, \quad \underline{N}_t \equiv \int \underline{n}_t^{\tilde{i}} d\tilde{i}$
  - Wealth share: $\eta_t \equiv N_t/(N_t + \underline{N}_t)$
- Value of capital stock: $q_t K_t$

Same Brownian

Postulate $dq_t/q_t = \mu_t^q dt + \sigma_t^q dZ_t$

- Postulate SDF-process: $\frac{d\xi_t}{\xi_t} = -r_t dt - \varsigma_t dZ_t, \frac{d\underline{\xi}_t}{\underline{\xi}_t} = -r_t dt - \underline{\varsigma}_t dZ_t$

Price of risk

Price of risk

Aside: Basics of Ito Calculus

- Geometric Ito Process $dX_t = \mu_t^X X_t dt + \sigma_t^X X_t dZ_t$

- Ito's Lemma:

$$df(X_t)$$

$$= f'(X_t)\mu_t^X X_t dt + \frac{1}{2} f''(X_t)(\sigma_t^X X_t)^2 dt + f'(X_t)\sigma_t^X X_t dZ_t$$

  - $u(c) = \frac{c^{1-\gamma}-1}{1-\gamma}$, $u'(c) = c^{-\gamma}$ volatility of process $\frac{dc_t^{-\gamma}}{c_t^{-\gamma}}$ is $-\gamma\sigma_t^c$

- Ito product rule: stock price * exchange rate

$$\frac{d(X_t Y_t)}{X_t Y_t} = (\mu_t^X + \mu_t^Y + \sigma_t^X \sigma_t^Y)dt + (\sigma_t^X + \sigma_t^Y)dZ_t$$

0. Postulate Aggregates and Processes

- … from price processes to return processes (using Ito)
  - Use Ito product rule to obtain capital gain rate (in absence of purchases/sales)
    - Define $\check{k}_t^{\tilde{\imath}}$: $\frac{d\check{k}_t^{\tilde{\imath}}}{\check{k}_t^{\tilde{\imath}}} = \left(\Phi\left(\iota_t^{\tilde{\imath}}\right) - \delta\right)dt + \sigma dZ_t \cancel{+ d\Delta_t^{k,\tilde{\imath}}}$ without purchases/sales

$$dr_t^K\left(\iota_t^{\tilde{\imath}}\right) = \left(\overbrace{\frac{a - \iota_t^i}{q}}^{\text{Dividend yield}} + \overbrace{\Phi\left(\iota_t^i\right) - \delta + \mu_t^q + \sigma\sigma_t^q}^{\text{E[Capital gain rate]}=\frac{d(q_t\check{k}_t)}{(q_t\check{k}_t)}}\right)dt + \left(\sigma + \sigma_t^q\right)dZ_t$$

- Postulate SDF-process: (Example: $\xi_t = e^{-\rho t}u'(c_t)$)

$$\frac{d\xi_t}{\xi_t} = -r_t dt - \varsigma_t dZ_t \qquad \frac{d\underline{\xi}_t}{\underline{\xi}_t} = -r_t dt - \underline{\varsigma}_t dZ_t$$

Price of risk

Recall discrete time $e^{-r^F} = E[SDF]$

0. Postulate Aggregates and Processes

- … from price processes to return processes (using Ito)
  - Use Ito product rule to obtain capital gain rate (in absence of purchases/sales)
    - Define $\check{k}_t^{\tilde{i}}$: $\frac{d\check{k}_t^{\tilde{i}}}{\check{k}_t^{i}} = \left(\Phi\left(\iota_t^{\tilde{i}}\right) - \delta\right)dt + \sigma dZ_t \cancel{+ d\Delta_t^{k,\tilde{i}}}$ without purchases/sales

$$dr_t^K\left(\iota_t^{\tilde{i}}\right) = \left(\overbrace{\frac{a - \iota_t^i}{q}}^{\text{Dividend yield}} + \overbrace{\Phi\left(\iota_t^i\right) - \delta + \mu_t^q + \sigma\sigma_t^q}^{\text{E[Capital gain rate]}= \frac{d(q_t\check{k}_t)}{(q_t\check{k}_t)}}\right)dt + \left(\sigma + \sigma_t^q\right)dZ_t$$

- Postulate SDF-process: (Example: $\xi_t = e^{-\rho t}u'(c_t)$)

$$\frac{d\xi_t}{\xi_t} = -r_t dt - \varsigma_t dZ_t \qquad \frac{d\underline{\xi}_t}{\underline{\xi}_t} = -r_t dt - \underline{\varsigma}_t dZ_t$$

Price of risk

Recall discrete time $e^{-r^F} = E[SDF]$

Poll 20:

Why is r_t for HH not underlined?

a) Typo

b) Debt can be traded

Solving MacroModels Step-by-Step

0. Postulate aggregates, price processes & obtain return processes
1. For given SDF processes *static*
   a. Real investment ι, (portfolio $\boldsymbol{\theta}$, & consumption choice of *each agent*)
      - *Toolbox 1:* Martingale Approach

   b. Asset/Risk Allocation *across types/sectors* & asset market clearing
      - *Toolbox 2:* "price-taking social planner approach" – Fisher separation theorem
2. Value functions *backward equation*
   a. Value fcn. as fcn. of individual investment opportunities ω
      - *Special cases*

   b. De-scaled value fcn. as function of state variables η
      - *Digression:* HJB-approach (instead of martingale approach & envelop condition)

   c. Derive ς price of risk, C/N-ratio from value fcn. envelop condition
3. Evolution of state variable η *forward equation*
   - *Toolbox 3:* Change in numeraire to total wealth (including SDF)
   - ("Money evaluation equation" μ^{ϑ})
4. Value function iteration & goods market clearing
   a. PDE of de-scaled value fcn.
   b. Replace endogenous μs, σs with $f, f', f''(\eta)$
   c. Value function iteration by solving PDE

1. Individual Agent Choice of ι, θ, c

- Choice of ι is static problem (and separable) for each t
- $\max_{\iota^i} dr^K(\iota^i)$

$$= \max_{\iota^i}\left(\frac{a-\iota^i}{q}+\Phi(\iota^i)-\delta+\mu^q+\sigma\sigma^q\right)$$

- FOC: $\frac{1}{q}=\Phi'(\iota^i)$ Tobin's q
  - All agents $\iota^i=\iota$ $\Rightarrow \frac{dK_t}{K_t}=(\Phi(\iota)-\delta)\,dt+\sigma dZ_t$
  - Special functional form:
    - $\Phi(\iota)=\frac{1}{\kappa}\log(\kappa\iota+1)\Rightarrow \kappa\iota=q-1$

1a. Martingale Approach – Discrete Time

$$\max_{\{c,\boldsymbol{\theta}\}} E_t\left[\sum_{\tau=t}^{T}\frac{1}{(1+\rho)^{\tau-t}}u(c_\tau)\right]$$

$$\text{s.t. } \boldsymbol{\theta}_t\boldsymbol{p}_t = \boldsymbol{\theta}_{t-1}(\boldsymbol{p}_t+\boldsymbol{d}_t)-c_t \text{ for all } t$$

- FOC w.r.t. θ_t: (deviate from optimal at t and $t+1$)

$$\xi_t p_t = E_t[\xi_{t+1}(p_{t+1}+d_{t+1})]$$

  - where $\xi_t = \frac{1}{(1+\rho)^t}\frac{u'(c_t)}{u'(c_0)}$ is the (multi-period) stochastic discount factor (SDF)
  - If projected on asset span, then pricing kernel ξ_t^*
  - Note: $\mathrm{MRS}_{\mathrm{t},\tau} = \xi_{t+\tau}/\xi_t$
- Consider portfolio, where one reinvests dividend d
  - Portfolio is a self-financing trading strategy, A, with price, p_t^A

$$\xi_t p_t^A = E_t[\xi_{t+1}p_{t+1}^A]$$

- Stochastic process, $\xi_t p_t^A$, is a martingale

1a. Martingale Approach – Cts. Time

$$\max_{\{\iota_t, \boldsymbol{\theta}_t, c_t\}_{t=0}^{\infty}} E\left[\int_0^{\infty} e^{-\rho} u(c_t) dt\right]$$

s.t. $\frac{dn_t}{n_t} = -\frac{c_t}{n_t} dt + \sum_j \theta_t^j dr_t^j$ + labor income/endow/taxes

n_0 given

- Portfolio Choice: Martingale Approach
  - Let x_t^A be the value of a "self-financing trading strategy"(reinvest dividends)
- Theorem: $\xi_t x_t^A$ follows a Martingale, i.e. drift $= 0$.
  - Let $\frac{dx_t^A}{x_t^A} = \mu_t^A dt + \sigma_t^A dZ_t,$
  - Recall $\frac{d\xi_t^i}{\xi_t^i} = -r_t dt - \varsigma_t^i dZ_t$
  - By Ito product rule

$$\frac{d(\xi_t^i x_t^A)}{\xi_t^i x_t^A} = \Big(\underbrace{-r_t + \mu_t^A - \varsigma_t^i \sigma_t^A}_{=0}\Big) dt + \text{volatility terms}$$

  - Expected return: $\boxed{\mu_t^A = r_t + \varsigma_t^i \sigma_t^A}$
    - For risk-free asset, i.e. $\sigma_t^A = 0$: $r_t^F = r_t$
    - Excess expected return to risky asset B: $\mu_t^A - \mu_t^B = \varsigma_t^i (\sigma_t^A - \sigma_t^B)$

1a. Martingale Approach – Cts. Time

- Proof 1: Stochastic Maximum Principle (see Handbook chapter)
- Proof 2: Intuition (calculus of variation)

  remove from optimum Δ at t_1 and add back at t_2

  $$V(n,\omega,t)=\max_{\{\iota_s,\boldsymbol{\theta}_s,c_s\}_{s=t}^{\infty}} E_t\left[\int_0^{\infty} e^{-\rho(s-t)}u(c_s)ds|\omega_t=\omega\right]$$

  - s.t. $n_t=n$

  $$e^{-\rho t_1}\frac{\partial V}{\partial n}(n_{t_1}^{*},x_{t_1},t_1)x_{t_1}^{A}=E_{t_1}\left[e^{-\rho t_2}\frac{\partial V}{\partial n}(n_{t_2}^{*},x_{t_2},t_2)x_{t_2}^{A}\right]$$

  - See Merkel Handout

1a. Optimal Portfolio Choice

- Using $\mu_t^A - r_t = \varsigma_t^i \sigma_t^A$ for capital return (instead of generic asset A)

$$\frac{a-\iota_t}{q_t} + \Phi(\iota_t) - \delta + \mu_t^q + \sigma\sigma_t^q - r_t = \varsigma_t(\sigma + \sigma_t^q)$$

- Recall
  - θ_t portfolio share in risk-free debt (short position)
  - $(1-\theta_t)$ portfolio share in (physical) capital k_t

Solving MacroModels Step-by-Step

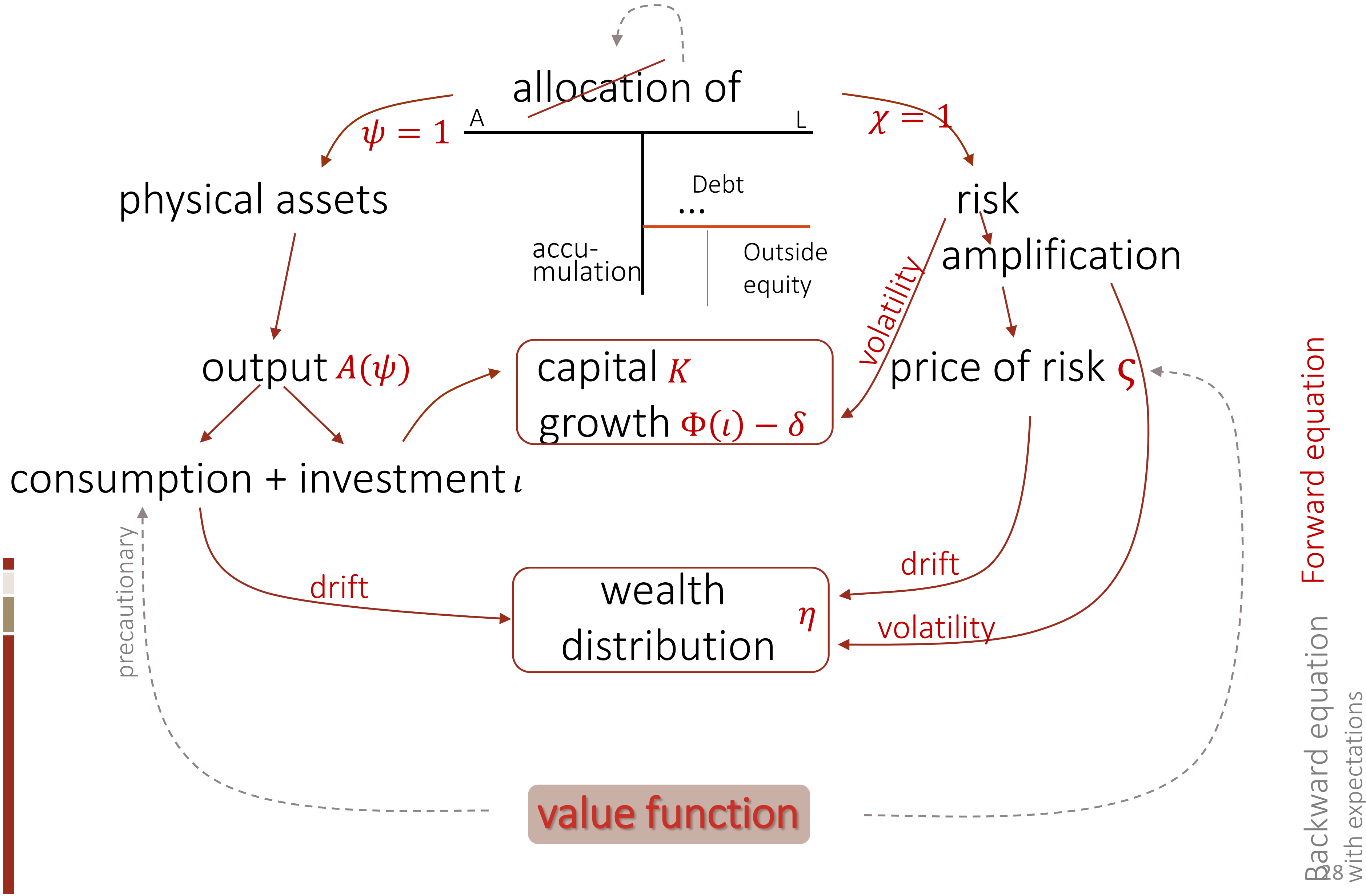
0. Postulate aggregates, price processes & obtain return processes
1. For given SDF processes *static*
   a. Real investment ι, (portfolio $\boldsymbol{\theta}$, & consumption choice of *each agent*)
      - *Toolbox 1:* Martingale Approach

   b. Asset/Risk Allocation *across types/sectors* & asset market clearing
      - *Toolbox 2:* "price-taking social planner approach" – Fisher separation theorem
2. Value functions *backward equation*
   a. Value fcn. as fcn. of individual investment opportunities ω
      - *Special cases*

   b. De-scaled value fcn. as function of state variables η
      - *Digression:* HJB-approach (instead of martingale approach & envelop condition)

   c. Derive ς price of risk, C/N-ratio from value fcn. envelop condition
3. Evolution of state variable η *forward equation*
   - *Toolbox 3:* Change in numeraire to total wealth (including SDF)
   - ("Money evaluation equation" μ^{ϑ})
4. Value function iteration & goods market clearing
   a. PDE of de-scaled value fcn.
   b. Value function iteration by solving PDE

The Big Picture



~~allocation of~~

A

L

$\psi = 1$

$\chi = 1$

Debt

…

accu-
mulation

Outside
equity

physical assets

risk

amplification

volatility

output $A(\psi)$

capital K
growth $\Phi(\iota) - \delta$

price of risk ς

consumption + investment ι

precautionary

drift

wealth
distribution η

drift

volatility

value function

Forward equation

Backward equation
with expectations

2a. CRRA Value Function: relate to ω

Applies separately for each type of agent

- ω_t Investment opportunity/ "networth multiplier"
- CRRA/power utility $u(c) = \frac{c^{1-\gamma}-1}{1-\gamma}$

  $\Rightarrow$ increase networth by factor, optimal consumption for all future states increases by same factor

  $\Rightarrow \left(\frac{c}{n}\right)$-ratio is invariant in n
- $\Rightarrow$ value function can be written as $\frac{u(\omega_t n_t)}{\rho}$, that is

$$= \frac{1}{\rho} \frac{(\omega_t n_t)^{1-\gamma}-1}{1-\gamma} = \frac{1}{\rho} \frac{\omega_t^{1-\gamma} n_t^{1-\gamma}-1}{1-\gamma}$$

- $\frac{\partial V}{\partial n} = u'(c)$ by optimal consumption (if no corner solution)

$$\frac{\omega_t^{1-\gamma} n_t^{-\gamma}}{\rho} = c_t^{-\gamma} \Leftrightarrow \boxed{\frac{c_t}{n_t} = \rho^{1/\gamma} \omega_t^{1-1/\gamma}}$$

2a. CRRA Value Function: Special Cases

$$\boxed{\frac{c_t}{n_t} = \rho^{1/\gamma}\omega_t^{1-1/\gamma}}$$

- For log utility $\gamma = 1$: $\boxed{\frac{c_t}{n_t} = \rho}$

  $\xi_t = e^{-\rho t}/c_t = e^{-\rho t}/(\rho n_t)$ for any $\omega_t \Rightarrow \boxed{\sigma_t^n = \sigma_t^c = \varsigma_t}$

  - Expected excess return: $\mu_t^A - r_t^F = \sigma_t^n \sigma_t^A$
  - Recall $\frac{dn_t}{n_t} = -\frac{c_t}{n_t}dt + (1-\theta)dr_t^K + \theta dr_t$

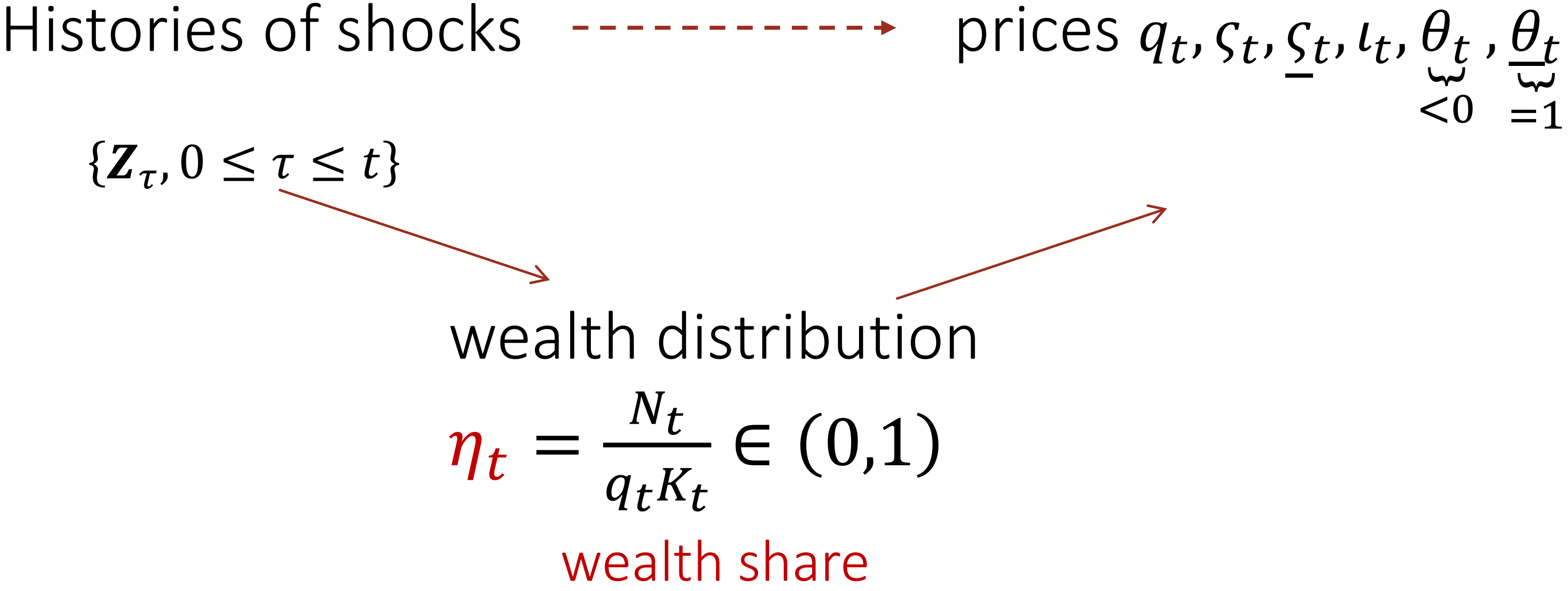
- For both types: experts and HHs,
  - $\frac{\underline{c}_t}{\underline{n}_t} = \rho$ and $\underline{\sigma}_t^n = \underline{\sigma}_t^c = \underline{\varsigma}_t$

Solving MacroModels Step-by-Step

0. Postulate aggregates, price processes & obtain return processes
1. For given SDF processes *static*
    a. Real investment ι, (portfolio $\boldsymbol{\theta}$, & consumption choice of *each agent*)
        - *Toolbox 1:* Martingale Approach
    b. Asset/Risk Allocation *across types/sectors* & asset market clearing
        - *Toolbox 2:* "price-taking social planner approach" – Fisher separation theorem
2. Value functions *backward equation*
    a. Value fcn. as fcn. of individual investment opportunities ω
        - *Special cases*
    b. De-scaled value fcn. as function of state variables η
        - *Digression:* HJB-approach (instead of martingale approach & envelop condition)
    c. Derive ς price of risk, C/N-ratio from value fcn. envelop condition
3. Evolution of state variable η *forward equation*
    - *Toolbox 3:* Change in numeraire to total wealth (including SDF)
    - ("Money evaluation equation" μ^{ϑ})
4. Value function iteration & goods market clearing
    a. PDE of de-scaled value fcn.
    b. Value function iteration by solving PDE

3. GE: Markov States and Equilibria

- Equilibrium is a map



Histories of shocks --------→ prices $q_t, \varsigma_t, \underline{\varsigma}_t, \iota_t, \underbrace{\theta_t}_{<0}, \underbrace{\underline{\theta}_t}_{=1}$

$\{\boldsymbol{Z}_\tau, 0 \leq \tau \leq t\}$

wealth distribution

$$\eta_t = \frac{N_t}{q_t K_t} \in (0,1)$$

wealth share

Aside: Basics of Ito Calculus

- Geometric Ito Process $dX_t = \mu_t^X X_t dt + \sigma_t^X X_t dZ_t$
- Ito's Lemma:

$$df(X_t) = f'(X_t)\mu_t^X X_t dt + \frac{1}{2} f''(X_t)(\sigma_t^X X_t)^2 dt + f'(X_t)\sigma_t^X X_t dZ_t$$

- Ito product rule:

$$\frac{d(X_t Y_t)}{X_t Y_t} = (\mu_t^X + \mu_t^Y + \sigma_t^X \sigma_t^Y)dt + (\sigma_t^X + \sigma_t^Y)dZ_t$$

- Ito ratio rule:

$$\frac{d(X_t/Y_t)}{X_t/Y_t} = \mu_t^X - \mu_t^Y + \sigma_t^Y(\sigma_t^Y - \sigma_t^X)dt + (\sigma_t^X - \sigma_t^Y)dZ_t$$

3. Law of Motion of Wealth Share η_t

- **Method 1:** Using Ito's quotation rule $\eta_t = N_t/(q_t K_t)$
- $\frac{dn_t}{n_t} = -\frac{c_t}{n_t}dt + r_t dt + (1-\theta_t)[dr_t^K - r_t dt]$

$$\frac{dN_t}{N_t} = -\rho dt + r_t dt + (1-\theta_t)\left[\underbrace{\left(\frac{a-\iota_t}{q_t} + \Phi(\iota_t) - \delta + \mu_t^q + \sigma\sigma_t^q - r_t\right)}_{=\varsigma_t(\sigma+\sigma^q)} dt + (\sigma+\sigma_t^q)dZ_t\right]$$

$$\frac{dq_t K_t}{q_t K_t} = \underbrace{(\mu_t^q + \Phi(\iota_t) - \delta + \sigma\sigma_t^q)}_{=r_t - \frac{a-\iota_t}{q_t} + \varsigma_t(\sigma+\sigma^q)} dt + (\sigma+\sigma_t^q)dZ_t$$

Using portfolio choice equation

- Note that $\varsigma_t = \sigma_t^n = (1-\theta_t)(\sigma+\sigma_t^q)$
- Ito ratio rule:

$$\frac{d(X_t/Y_t)}{X_t/Y_t} = \mu_t^X - \mu_t^Y + \sigma_t^Y(\sigma_t^Y - \sigma_t^X)dt + (\sigma_t^X - \sigma_t^Y)dZ_t$$

- $\frac{d\eta_t}{\eta_t} = \left(\frac{a-\iota_t}{q_t} - \rho + \theta_t^2(\sigma+\sigma_t^q)^2\right)dt - \underbrace{\theta_t}_{<0}(\sigma+\sigma_t^q)dZ_t$
- **Method 2:** Change of numeraire + Martingale (Lecture 03)

Solving MacroModels Step-by-Step

0. Postulate aggregates, price processes & obtain return processes
1. For given SDF processes *static*
   a. Real investment ι, (portfolio $\boldsymbol{\theta}$, & consumption choice of *each agent*)
      - *Toolbox 1:* Martingale Approach
   b. Asset/Risk Allocation *across types/sectors* & asset market clearing
      - *Toolbox 2:* "price-taking social planner approach" – Fisher separation theorem
2. Value functions *backward equation*
   a. Value fcn. as fcn. of individual investment opportunities ω
      - *Special cases*
   b. De-scaled value fcn. as function of state variables η
      - *Digression:* HJB-approach (instead of martingale approach & envelop condition)
   c. Derive ς price of risk, C/N-ratio from value fcn. envelop condition
3. Evolution of state variable η *forward equation*
   - *Toolbox 3:* Change in numeraire to total wealth (including SDF)
   - ("Money evaluation equation" μ^{ϑ})
4. Value function iteration & goods market clearing
   a. PDE of de-scaled value fcn.
   b. Value function iteration by solving PDE

4a. Market Clearing

- Output good market

$$C_t = (a - \iota)K_t$$
$$\rho q_t K_t = \big(a - \iota(q_t)\big)K_t$$
$$\rho q_t = (a - \iota(q_t)) \qquad \Rightarrow q_t = q \ \forall t$$

- Capital market

$$1 - \theta_t = \underbrace{\frac{q_t K_t}{N_t}}_{=1/\eta_t}$$

- Money market (by Walras Law)

4b. Model Solution

- Using $\rho q_t = (a - \iota(q_t))$, $\kappa\iota_t = q_t - 1$, $\Phi(\iota) = \frac{1}{\kappa}\log(\kappa\iota + 1)$

$$q = \frac{1+\kappa a}{1+\kappa\rho}$$

- Using portfolio choice, goods & capital market clearing

$$r_t = \frac{a-\iota_t}{q_t} + \Phi(\iota_t) - \delta + \cancel{\mu_t^q} + \sigma\cancel{\sigma_t^q} - \varsigma_t(\sigma + \cancel{\sigma_t^q})$$

$$= \rho + \Phi(\iota_t) - \delta - (1-\theta_t)\sigma^2$$

$$= \rho + \Phi(\iota_t) - \delta - \frac{\sigma^2}{\eta_t}$$

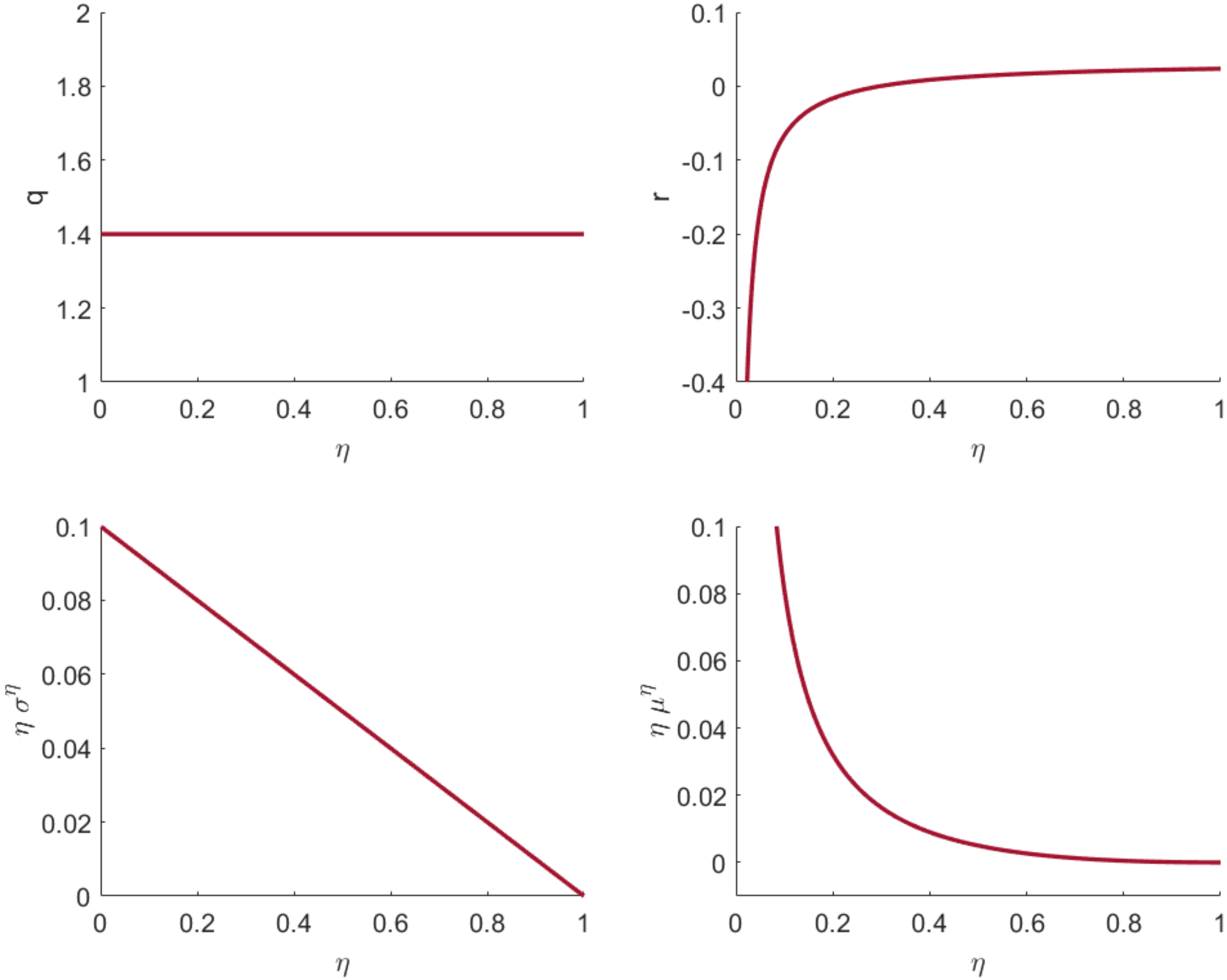
from capital market clearing

$$r_t = \rho + \frac{1}{\kappa}\log\left(\frac{1+\kappa a}{1+\kappa\rho}\right) - \delta - \frac{\sigma^2}{\eta_t}$$

- Goods & capital market clearing and η-evolution

$$\frac{d\eta_t}{\eta_t} = \frac{(1-\eta_t)^2}{\eta_t^2}\sigma^2 dt + \frac{1-\eta_t}{\eta_t}\sigma dZ_t$$

Numerical Example



$$a = .11, \rho = 5\%, \sigma = .1, \Phi(\iota) = \frac{\log(\kappa\iota + 1)}{\kappa}, \kappa = 10$$

Observation of Basak-Cuoco Model

- η_t fluctuates with macro shocks, since experts are levered
- Price of risk, i.e. Sharpe ratio, is

$$\frac{\sigma}{\eta_t} = \frac{\rho + \Phi(\iota) - \delta - r_t}{\sigma}$$

  - Goes to ∞ as η_t goes to zero
  - Achieved via risk-free rate

$$r_t = \rho + \Phi(\iota) - \delta - \sigma^2/\eta_t \to -\infty$$

  - Rather than depressing price of risky asset, $q_t = q \ \forall t$
- No endogenous risk $\sigma^q = 0$
  - No amplification
  - No volatility effects
- $\mu_t^\eta = \frac{(1-\eta_t)^2}{\eta_t^2}\sigma^2 > 0 \Rightarrow$ in the long run HH vanish
  - Way out:
    - Different discount rates ρ (KM)
    - Switching types (BGG)

Desired Model Properties

- Normal regime: stable around steady state
  - Experts are adequately capitalized
  - Experts can absorb macro shock
- Endogenous risk
  - Fire-sales, liquidity spirals, fat tails
  - Spillovers across assets and agents
  - Market and funding liquidity connection
  - SDF vs. cash-flow news
- Volatility paradox
- Financial innovation   less stable economy
- (“Net worth trap” double-humped stationary distribution)