The I Theory of Money*

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Abstract

A theory of money needs a proper place for financial intermediaries. Intermediaries diversify risks and create inside money. In downturns, micro-prudent intermediaries shrink their lending activity, fire-sell assets and supply less inside money, exactly when money demand rises. The resulting Fisher disinflation hurts intermediaries and other borrowers. Shocks are amplified, volatility spikes and risk premia rise. Monetary policy is redistributive. Accommodative monetary policy that boosts assets held by balance sheet-impaired sectors, recapitalizes them and mitigates the adverse liquidity and disinflationary spirals. Since monetary policy cannot provide insurance and control risk-taking separately, adding macroprudential policy that limits leverage attains higher welfare.

Keywords: Monetary Economics, (Inside) Money, Endogenous Risk Dynamics, Volatility Paradox, Paradox of Prudence, Financial Frictions.

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1 Introduction

A theory of money needs a proper place for financial intermediaries. Financial institutions are able to create money – when they extend loans to businesses and home buyers, they credit the borrowers with deposits and so create inside money. Money creation by financial intermediaries depends crucially on the health of the banking system and the presence of profitable investment opportunities. This paper proposes a theory of money and provides a framework for analyzing the interaction between price stability and financial stability. It therefore provides a unified way of thinking about monetary and macroprudential policy.

We model money supply and demand, and the role of financial intermediaries as follows. Households manage productive projects that use capital and expose them to idiosyncratic risk. They hold money for self insurance against this risk. This creates money demand - as in Samuelson (1958) and Bewley (1980) money has value in equilibrium even though it never pays dividends - in other words money is a bubble. Money supply consists of outside money and inside money created by intermediaries. Intermediaries take stakes in the households' risky projects, absorbing and diversifying some of households' risk. They are active in maturity and liquidity *transformation*, as they issue liquid, at notice redeemable, (inside) money and invest in illiquid long-term assets. The mismatch between assets and liabilities exposes intermediaries to risk. When intermediaries suffer losses, they shrink their balance sheets, creating less inside money and financing fewer household projects. In this case money supply shrinks and money demand rises. Together, both effects lead to increase in the value of outside money, i.e. disinflation à la Fisher (1933) occurs.

The relationship between the value of money and the state of the financial system can be understood through two polar cases. In one polar case intermediaries are undercapitalized and cannot perform their functions. Without inside money, money supply is scarce and the value of money is high. Households have a desire to hold money which, unlike the households' risky projects, is subject only to aggregate, not idiosyncratic, risk. In the opposite polar case, intermediaries are well capitalized and so well equipped to mitigate financial frictions. They are able to exploit the diversification benefits by investing across many different projects. Intermediaries also create inside money and hence the money multiplier is high. At the same time, since households can offload some of their idiosyncratic risks to the intermediary sector, their demand for money is low. Hence, the value of money is low in this polar case.

An adverse shock to end borrowers not only hurts the intermediaries directly, but also moves the economy closer to the first polar regime with high value of money. Shocks are amplified by spirals on both sides of intermediaries' balance sheets. On the asset side, intermediaries are exposed to productivity shocks of their end-borrowers. End-borrowers' fire sales depress the price of physical capital and liquidity spirals further erode intermediaries' net worth (as shown in Brunnermeier and Sannikov (2014)). On the liabilities side, intermediaries are hurt by the Fisher disinflation. As lending and inside money creation shrink, money demand rises and the real value of nominal liabilities expands. The "Paradox of Prudence" arises when intermediaries shrink their balance sheet and households tilt their portfolio away from real investment towards the safe asset, money. Scaling back risky asset holding is micro-prudent, but makes the economy more risky, i.e. it is macro-imprudent. Our Paradox of Prudence is in the risk space what Keynes' Paradox of Thrift is for the consumption-savings decision. The Paradox of thrift describes how each person's attempt to save more paradoxically lowers overall aggregate savings. In our model attempts to reduce individual risks increases endogenous risks as the economy's capacity to diversify idiosyncratic risk moves around endogenously.

Monetary policy can work against the adverse feedback loops that precipitate crises, by affecting the prices of assets held by constrained agents and redistributing wealth. That is, monetary policy works through wealth/income effects, unlike conventional New Keynesian models in which monetary policy gains traction by changing intertemporal incentives – a substitution effect. Specifically, in our model, monetary policy softens the blow of negative shocks and helps the intermediary sector to maintain the capacity to diversify idiosyncratic risk. Thus, it reduces endogenous (self-generated) risk and overall risk premia. Monetary policy is redistributive, but it is not a zero-sum game – redistribution can actually improve welfare. Unexpected monetary policy redistributes *wealth*, but anticipated loosening redistributes *risk* by affecting prices and returns on assets in different states. Thus, monetary policy can provide insurance.

Simple interest rate cuts in downturns improve economic outcomes only if they boost prices of assets, such as long-term government bonds, that are held by constrained sectors. Wealth redistribution towards the constrained sector leads to a rise in economic activity and an increase in the price of physical capital. As the constrained intermediary sector recovers, it creates more (inside) money and reverses the disinflationary pressure. The appreciation of long-term bonds also mitigates money demand, since long-term bonds are also safe assets and hence can be used as a store of value as well. As banks are recapitalized, they are able to take on more idiosyncratic household risks, so economy-wide diversification of risk improves and the overall economy becomes, somewhat paradoxically, safer. Importantly, monetary policy also affects risk premia. As interest rate moves affect the equilibrium allocations, they also affect the long-term *real* interest rate as documented by Hanson and Stein (2014) and term premia and credit spread as documented by Gertler and Karadi (2014). From an ex-ante perspective long-term bonds are a good hedging instrument for intermediaries if the central bank follows an appropriate monetary policy *rule*. After an adverse shock, central banks cut short-term interest rates and bonds appreciate.

Like any insurance, "stealth recapitalization" of the financial system through monetary policy can potentially create a moral hazard problem. However, moral hazard from monetary policy is less severe than that associated with explicit bailouts of failing institutions. The reason is that monetary policy is a crude redistributive tool that helps the strong institutions more than the weak. The cautious institutions that bought long-term bonds as a hedge against downturns benefit more from interest rate cuts than the opportunistic institutions that increased leverage to take on more risk. In contrast, ex-post bailouts of the weakest institutions create strong risk-taking incentives ex-ante.

While monetary policy improves welfare, the right amount of risk redistribution is not always clearcut. It comes with side effects since several quantities adjust endogenously. Monetary policy cannot control risk separately from risk-taking and risk premia. Policy that only partially completes the markets need not be welfare-improving, as originally shown in the famous Hart (1975) example. Monetary policy is just one tool, which cannot perfectly control the many quantities determined by a system of equilibrium equations, i.e. moral hazard exists.

We show that combining monetary policy with macroprudential policy measures that control individual households' undiversifiable risk-taking significantly increases welfare. There are various reasons for this - even without intermediaries household portfolio decisions create pecuniary externalities, as they affect the price of capital, and thus idiosyncratic risk exposure of others, as well as the rate of economic growth. With intermediaries, macroprudential policies in addition affect risk premia, and thus earnings and the law of motion of the wealth distribution.

Related Literature. Our approach differs in important ways from both the prominent New Keynesian approach but also from the monetarist approach. The New Keynesian approach emphasizes the interest rate channel. It stresses the role of money as unit of account and price and wage rigidities as the key frictions. Price stickiness implies that a lowering of the nominal interest rate also reduces the real interest rate. Households bring consumption forward and investment projects become more profitable. Within the class of New Keynesian models, Christiano, Moto and Rostagno (2003) is closest to our analysis as it studies the disinflationary spiral during the Great Depression. More recently, Cúrdia and Woodford (2010) introduced financial frictions in the new Keynesian framework.

In contrast, our I Theory stresses the role of money as a store of value and the redistributive channel of monetary policy. The key frictions are financial. Prices are fully flexible. The transmission of monetary policy works primarily through capital gains, as in the asset pricing channel promoted by Tobin (1969) and Brunner and Meltzer (1972). As assets are held asymmetrically in our setting, monetary policy redistributes wealth and thereby mitigates debt overhang problems. In other words, instead of emphasizing the substitution effect of interest rate changes, the I Theory stresses wealth/income effects of interest rate changes. Tobin (1982) and Auclert (2016) stress the role of monetary policy in aggregate demand management by redistributing wealth from households with low to households with high marginal propensity to consume. In our setting all agents have the same propensity to consume and monetary policy redistributes wealth towards balance sheet impaired sectors and affects endogenous risk and risk premia. Also, like in Woodford (1990) and Kehoe, Levine and Woodford (1992) optimal monetary policy deviates from the Friedman Rule in our model.

Like in monetarism (see e.g. Friedman and Schwartz (1963)), an endogenous reduction of the money multiplier (given a fixed monetary base) leads to disinflation in our setting. While inside and outside money have identical risk-return profiles and so are perfect substitutes for individual investors, they are *not* the same for the economy as a whole. Inside money serves a special function. By creating inside money, intermediaries diversify risks and foster economic growth. Hence, in our setting a monetary intervention should aim to recapitalize undercapitalized borrowers rather than simply increase the money supply across the board. A key characteristic of our approach is that we focus more on the role of money as a store of value instead of the transaction role of money. The latter plays an important role in the "new monetarist economics" as outlined in Williamson and Wright (2011) and references therein.

Instead of the "money view" our approach is closer in spirit to the "credit view" à la Gurley and Shaw (1955), Patinkin (1965), Tobin (1969, 1970), Bernanke (1983) Bernanke and Blinder (1988) and Bernanke, Gertler and Gilchrist (1999).[1]

[1] The literature on credit channels distinguishes between the bank lending channel and the balance sheet channel (financial accelerator), depending on whether banks or corporates/households are capital constrained. Strictly speaking our setting refers to the former, but we are agnostic about it and prefer the broader credit channel interpretation.

As in Samuelson (1958) and Bewley (1980), money is essential in our model in the sense of Hahn (1973). In Samuelson (1958) households cannot borrow from future not yet born generations. In Bewley (1980) and Scheinkman and Weiss (1986) households face explicit borrowing limits and cannot insure themselves against idiosyncratic shocks. The motive to self-insure through precautionary savings creates a demand for the single asset, money. In our model households can hold money and physical capital. The return on capital is risky and its risk profile differs from the endogenous risk profile of money. Financial institutions create inside money and mitigate financial frictions. In Kiyotaki and Moore (2008) money and capital coexist. Money is desirable as it does not suffer from a resellability constraint as physical capital does. Lippi and Trachter (2012) characterize the trade-off between insurance and production incentives of liquidity provision. Levin (1991) shows that monetary policy is more effective than fiscal policy if the government does not know which agents are productive. The finance papers by Diamond and Rajan (2006) and Stein (2012) also address the role of monetary policy as a tool to achieve financial stability.

More generally, there is a large macro literature that also investigates how macro shocks that affect the balance sheets of intermediaries or end-borrowers become amplified and affect the amount of lending and the real economy. These papers include Bernanke and Gertler (1989), Kiyotaki and Moore (1997) and Bernanke, Gertler and Gilchrist (1999), who study financial frictions using models that are log-linearized near steady state. In these models shocks to intermediary/end-borrower net worths affect the efficiency of capital allocation and asset prices. However, log-linearized solutions preclude volatility effects and lead to stable system dynamics. Brunnermeier and Sannikov (2014) study the full equilibrium dynamics, focusing on the differences in system behavior near the steady state, and away from it. They find that the system is stable to small shocks near the steady state, but large shocks make the system unstable and generate systemic endogenous risk. Thus, system dynamics are highly nonlinear. Large shocks have much more serious effects on the real economy than small shocks. Also, Brunnermeier and Sannikov (2014) highlight an important discontinuity. In the limit as risk goes to zero the steady state does not converge to one that arises in traditional models in which shocks only occur with zero probability. He and Krishnamurthy (2013) also study the full equilibrium dynamics and focus in particular on intermediary asset pricing. In Mendoza and Smith's (2006) international setting the initial shock is also amplified through a Fisher debt-disinflation that arises from the interaction between domestic agents and foreign traders in the equity market. In our paper debt disinflation is due to the appreciation of inside money. Recently, there is revival of monetary economics, now set frequently in continuous time models. Like in our model, in Drechsler, Savov and Schnabl (2016) monetary policy also works through the risk premium. Less risk averse bankers lever up, but also hold more liquid assets in form of reserves as well as a long-term bond, whose returns are affected by monetary policy. Di Tella and Kurlat (2016) argue that the financial sector chooses to be exposed to interest rate risk due to dynamic hedging demand. Capital gains caused by an interest rate cut are a good hedge against the lower future net interest rate margins. A lower policy rate reduces the supply of (cheap) deposits, as holding currency at zero interest is less costly. In contrast, in our setting appropriate monetary policy provides intermediaries a hedge against other balance sheet shocks. For a more detailed review of the traditional literature we refer to Brunnermeier et al. (2013) and for continuous time models to Brunnermeier and Sannikov (2016).

This paper is organized as follows. Section 2 sets up the model and derives equilibrium conditions without policy intervention. Section 3 characterizes equilibrium without intermediaries in closed form, and discusses idiosyncratic risk as a determinant of the value of money, as well as motivation for macroprudential policy that distorts money holdings. Section 4 presents computed examples and discusses equilibrium properties, including capital and money value dynamics, the amount of lending through intermediaries, and the money multiplier for various parameter values. Section 5 introduces long-term bonds and studies the effect of interest-rate policies as well as open-market operations. It also demonstrates that in the absence of macroprudential tools, monetary policy alone cannot control asset risk separately from risk-taking. Section 6 concludes.

2 The Baseline Model Absent Policy Intervention

The economy is populated by two types of agents: households and intermediaries. Each household can use capital to produce either good a or good b, but can only be active in one sector at a time. Production carries both idiosyncratic and aggregate sector-specific risk. The two goods are then combined into an aggregate good that can be consumed or invested.

Incomplete markets hinder households' ability to issue risky claims. Specifically, households can transfer only some of the risk by selling claims to intermediaries. There is asymmetry, as households in sector b are able to share risk with intermediaries more easily than those in sector a, so intermediaries end up over-exposed to sector-b aggregate risk. Otherwise, intermediaries can fully diversify household idiosyncratic risk by pooling risky claims, while households must keep some of their idiosyncratic risk. Intermediaries obtain funding for the claims they hold by accepting money deposits.

To protect themselves against idiosyncratic risk, households are able to hold money. In our model, what is money? Money is infinitely divisible asset available in fixed supply with no intrinsic value, but with value in equilibrium because households demand it for self insurance (as in Bewley (1980)). Money could be gold (except for the fact that gold is not intrinsically worthless as it can be used in jewelry) or currency printed by the monetary authority. This is outside money - its nominal supply is fixed in the absence of monetary policy in Sections 2, 3 and 4. Intermediaries can also create inside money, promises to pay back outside money at a later date, by taking deposits. While the nominal supply of outside money is fixed, the real value of money (and hence the price level) is determined endogenously in equilibrium.

The dynamic evolution of the economy is driven by the effect of shocks on the agents' wealth distribution through their portfolio choice. The model is solved using standard portfolio choice theory, except that asset prices - including the price of money - are endogenous.

Technologies. All physical capital K_t in the world is allocated between the two technologies. If capital share $\psi_t \in [0,1]$ is devoted to produce good b, then goods a and b combined make $A(\psi)K_t$ of the aggregate good. Function $A(\psi)$ is concave and has an interior maximum, an example is the standard technology with constant elasticity of substitution s,[2]

$$A(\psi) = \mathcal{A}\left(\frac{1}{2}\psi^{\frac{s-1}{s}} + \frac{1}{2}(1-\psi)^{\frac{s-1}{s}}\right)^{\frac{s}{s-1}}.$$

In competitive markets, prices of goods a and b reflect their marginal contributions to the aggregate good. Prices must be such that a unit of capital employed in each sector produces output valued at

$$A^a(\psi) = -\psi A'(\psi) + A(\psi) \quad \text{and} \quad A^b(\psi) = (1-\psi)A'(\psi) + A(\psi),$$

respectively.[3]

[2]For $s = \infty$ the outputs are perfect substitutes, for $s = 0$ there is no substitutability at all, while for $s = 1$ the substitutability corresponds to that of a Cobb-Douglas production function.

[3]If total output is $A(\psi)K$, then an ϵ amount of capital devoted to technology a would change total productivity to

$$A\left(\frac{\psi K}{K+\epsilon}\right)(K+\epsilon).$$

Differentiating with respect to ϵ at $\epsilon = 0$, we obtain

$$\frac{-\psi K}{(K+\epsilon)^2}A'(\psi)(K+\epsilon) + A(\psi)\bigg|_{\epsilon=0} = -\psi A'(\psi) + A(\psi).$$

Physical capital k_t is subject to shocks that depend on the technology in which it is employed. If used in technology a capital follows

$$\frac{dk_t}{k_t} = (\Phi(\iota_t) - \delta)\, dt + \sigma^a\, dZ_t^a + \tilde{\sigma}^a\, d\tilde{Z}_t, \tag{2.1}$$

where dZ_t^a are the sector-wide Brownian shocks and $d\tilde{Z}_t$ are project-specific shocks, independent across agents, which cancel out in the aggregate. A similar equation applies if capital is used in technology b. Sector-wide shocks dZ_t^a and dZ_t^b are independent of each other. The investment function Φ has the standard properties $\Phi' > 0$ and $\Phi'' \leq 0$, and the input for investment ι_t is the aggregate good.

Preferences. All agents have identical logarithmic preferences with a common discount rate ρ. That is, any agent maximizes the expected utility of

$$E\left[\int_0^\infty e^{-\rho t} \log c_t\, dt\right],$$

subject to individual budget constraints, where c_t is the consumption of the aggregate good at time t.

Financing Constraints. Each household can hold money and invest in *either* technology a *or* technology b. Households can issue risky claims only towards the intermediary sector (not to each other). However, the amount of risk they can offload to the intermediary sector is bounded above, with bounds $\bar{\chi}^a$ and $\bar{\chi}^b$ satisfying $0 \leq \bar{\chi}^a < \bar{\chi}^b \leq 1$.[4] For simplicity, we set in our baseline model $\bar{\chi}^a = 0$, and then denote $\bar{\chi} \equiv \bar{\chi}^b$, with $\bar{\chi}$ near 1. Intermediaries finance their risky holdings (households' outside equity) by issuing claims (nominal IOUs) with return identical to the return on money. These claims, or *inside money*, are as safe as currency, *outside money*. In the baseline model, there is a fixed amount of outside fiat money in the economy that pays zero interest. Figure 1 provides a schematic representation

Likewise, the marginal contribution of capital devoted to technology b would be $(1-\psi)A'(\psi) + A(\psi)$. The weighted sum of the two terms is $A(\psi)$ since the production technology is homogenous of degree 1.

[4]Notice that if $\bar{\chi}^a = \bar{\chi}^b$, then by holding this maximum fraction of equity of each sector, intermediaries guarantee that the fundamental risk of their assets is proportional to the risk of the economy as a whole. In this case, intermediaries end up perfectly hedged, as the risk of money is also proportional to the risk of the whole economy and the intermediaries' wealth share follows a deterministic path. In contrast, if $\bar{\chi}^a < \bar{\chi}^b$, then intermediaries are always overexposed to the risk of sector b. In this case, they hold the maximum amount $\bar{\chi}^a$ of equity of sector a, as this helps them hedge and also helps households in sector a offload aggregate risk. They also hold more than fraction $\bar{\chi}^a$ of equity of sector b, as the risk premium they demand is initially second-order, and households in sector b demand insurance.

of the basic financing structure of the model.[5]

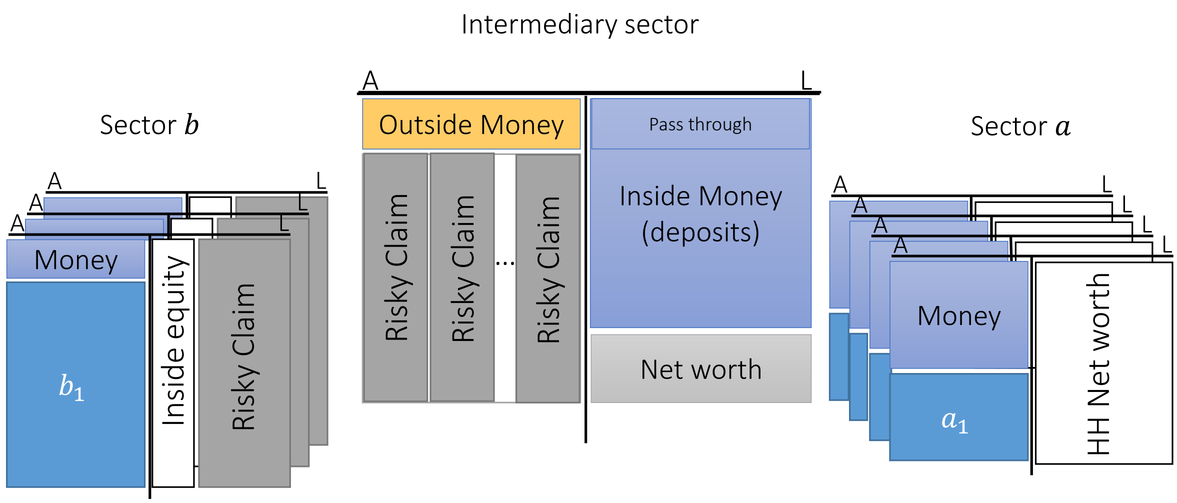

Figure 1: Schematic Balance Sheet Representation.

Finally let us offer some additional brief remarks on model interpretation. First, since outside money and inside money have the same return and risk profile, it is equivalent whether households hold outside money or the intermediary/financial sector holds outside money and issues a corresponding amount of inside money. Second, we interpret our intermediary/financial sector as a sector that includes traditional banking, but also shadow banking and other forms of intermediation and risk mitigation. And third, as all households have some money balances, the model has no clear borrowing or lending sectors. This feature distinguishes our model from more conventional loanable funds models.

Assets, Returns and Portfolios. Let q_t denote the price of physical capital per unit relative to the numeraire, the aggregate consumption good. Then the value of all capital in the economy is $q_t K_t$. Likewise denote by $p_t K_t$ the *real* value of outside money, where K_t reflects the fact that money is worth more in a bigger economy, and p_t reflects the way that wealth distribution affects the value of money. Then the total wealth of all agents is given by $(q_t + p_t)K_t$. Since inside money is a liability for the intermediary sector and an asset for the household sector, it nets out overall.

[5]The model could be easily enriched to allow intermediaries to sell off part of the equity claims up to a limit $\bar{\chi}^I < 1$. This would not alter the qualitative results of the model.

First, let us discuss return on capital, and later, return on money. We do not consider equilibria with jumps, so let us postulate for now that q_t follows a Brownian process of the form

$$\frac{dq_t}{q_t} = \mu_t^q \, dt + (\sigma_t^q)^T \, dZ_t, \tag{2.2}$$

where $dZ_t = [dZ_t^a, dZ_t^b]^T$ is the vector of aggregate shocks. Then the capital gains component of the return on capital, $d(k_t q_t)/(k_t q_t)$, can be found using Ito's lemma. The dividend yield is $(A^a(\psi) - \iota_t)/q_t$ for technology a and $(A^b(\psi) - \iota_t)/q_t$ for technology b.

The total (real) return of an individual project in technology a is

$$dr_t^a = \frac{A^a(\psi_t) - \iota_t}{q_t} \, dt + \left(\Phi(\iota_t) - \delta + \mu_t^q + (\sigma_t^q)^T \sigma^a 1^a\right) dt + (\sigma_t^q + \sigma^a 1^a)^T \, dZ_t + \tilde{\sigma}^a \, d\tilde{Z}_t,$$

where 1^a is the column coordinate vector with a single 1 in position a. The (real) return in technology b is written analogously. The optimal investment rate ι_t, which maximizes the return of any technology, is given by the first-order condition $1/q_t = \Phi'(\iota_t)$. We denote the investment rate that satisfies this condition by $\iota(q_t)$.

The return on technology b is split between households who hold inside equity and earn dr_t^{bH} and intermediaries who hold outside equity and earn dr_t^{bI}, so

$$dr_t^b = (1 - \chi_t) \, dr_t^{bH} + \chi_t \, dr_t^{bI},$$

where $\chi_t \leq \bar{\chi}$ is the fraction of outside equity issued by households and held by intermediaries. The two types of equity have identical risks, but potentially different returns. The required return on inside equity may be higher if households would like to issue more outside equity but cannot due to the constraint. That is, in equilibrium we have we have $dr_t^{bH} \geq dr_t^{bI}$, with equality if $\chi_t < \bar{\chi}$.

To write down the return on money, let us postulate that p_t follows a Brownian process of the form

$$\frac{dp_t}{p_t} = \mu_t^p \, dt + (\sigma_t^p)^T \, dZ_t. \tag{2.3}$$

The law of motion of aggregate capital is

$$\frac{dK_t}{K_t} = (\Phi(\iota_t) - \delta) \, dt + \underbrace{\psi_t \sigma^a \, dZ_t^a + (1 - \psi_t)\sigma^b \, dZ_t^b}_{(\sigma_t^K)^T \, dZ_t}. \tag{2.4}$$

Since all outside money in the world is worth $p_t K_t$, the return on money, the real interest rate, is given just by the capital gains rate

$$dr_t^M = \frac{d(p_t K_t)}{p_t K_t} = \left(\Phi(\iota_t) - \delta + \mu_t^p + (\sigma_t^p)^T \sigma_t^K\right) dt + \underbrace{(\sigma_t^K + \sigma_t^p)^T \, dZ_t}_{(\sigma_t^M)^T \, dZ_t}.$$

When a household chooses to produce good a, its net worth follows

$$\frac{dn_t}{n_t} = x_t^a \, dr_t^a + (1 - x_t^a) \, dr_t^M - \zeta_t^a \, dt, \tag{2.5}$$

where x_t^a is the portfolio weight on capital and ζ_t^a is its propensity to consume (i.e. consumption per unit of net worth).

The net worth of a household who produces good b follows

$$\frac{dn_t}{n_t} = x_t^b \, dr_t^{bH} + (1 - x_t^b) \, dr_t^M - \zeta_t^b \, dt. \tag{2.6}$$

Households can choose whether to work in sector a or b, that is, in equilibrium they must be indifferent with respect to this choice. Denote by α_t the net worth of households who specialize in sector a, as a fraction of total household net worth.

The net worth of an intermediary follows

$$\frac{dn_t}{n_t} = x_t \, d\bar{r}_t^{bI} + (1 - x_t) \, dr_t^M - \zeta_t \, dt, \tag{2.7}$$

where $\bar{r}_t^{bI}$ denotes the return on households' outside equity dr_t^{bI} with idiosyncratic risk diversified away, i.e. removed. If intermediaries use leverage, i.e. issue inside money, then of course $x_t > 1$.

Equilibrium Definition. The agents start initially with some endowments of capital and money. Over time, they trade - they choose how to allocate their wealth between the assets available to them. That is, they solve their individual optimal consumption and portfolio choice problems to maximize utility, subject to the budget constraints (2.5), (2.6) and (2.7). Individual agents take prices as given. Given prices, markets for capital, money and consumption goods have to clear.

If the net worth of intermediaries is N_t, then given the world wealth of $(q_t + p_t)K_t$, the intermediaries' net worth share is denoted by

$$\eta_t = \frac{N_t}{(q_t + p_t)K_t}. \tag{2.8}$$

Definition. *Given any initial allocation of capital and money among the agents, an equilibrium is a map from histories $\{Z_s,\ s \in [0,t]\}$ to prices p_t and q_t, return differential $dr_t^{bH} - dr_t^{bI} \geq 0$, the households' wealth allocation α_t, equity allocation $\chi_t \leq \bar{\chi}$, portfolio weights (x_t^a, x_t^b, x) and consumption propensities $(\zeta_t^a, \zeta_t^b, \zeta_t)$, such that*

(i) all markets, for capital, equity, money and consumption goods, clear,

(ii) all agents choose technologies, portfolios and consumption rates to maximize utility (households who produce good b also choose χ_t).

One important choice here is that of households: each household can run only one project either in technology a or b. They must be indifferent between the two choices. Households who choose to produce good b must also choose how much equity to issue. If outside equity earns less than the return of technology b, these households would want to issue the maximal amount of outside equity of $\chi_t = \bar{\chi}$. This happens in equilibrium only if intermediaries are willing to accept this supply of equity at a return discount, i.e. $dr_t^{bI} < dr_t^b$, so that inside equity earns a premium. This is the case only if the intermediaries are well-capitalized. Otherwise, $dr_t^{bH} = dr_t^{bI} = dr_t^b$, i.e. inside and outside equity of technology b earns the same return as technology b. In this case, households are indifferent with respect to the amount of equity they issue, so the equity issuance constraint does not bind.

2.1 Equilibrium Conditions

Logarithmic utility has two well-known tractability properties. First, an agent with logarithmic utility and discount rate ρ consumes at the rate given by ρ times net worth. Thus, $\zeta_t = \zeta_t^a = \zeta_t^b = \rho$ and the market-clearing condition for consumption goods is

$$\rho(q_t + p_t)K_t = (A(\psi_t) - \iota_t)K_t. \tag{2.9}$$

Second, the excess return of any risky asset over any other risky asset is explained by the covariance between the difference in returns and the agent's wealth.

From (2.5) and (2.6), the wealth of households in sectors a and b is exposed to real aggregate risk of

$$\sigma_t^{Na} = x_t^a \underbrace{\left(\sigma^a 1^a + \sigma_t^q - \sigma_t^M\right)}_{\nu_t^a} + \sigma_t^M \quad \text{and} \quad \sigma_t^{Nb} = x_t^b \underbrace{\left(\sigma^b 1^b + \sigma_t^q - \sigma_t^M\right)}_{\nu_t^b} + \sigma_t^M,$$

and real idiosyncratic risk of $x_t^a \tilde{\sigma}^a$ and $x_t^b \tilde{\sigma}^b$, respectively. Note that ν^a and ν^b are measures of "nominal" risk, since the negative of σ^M is inflation risk. Hence, the difference between expected returns of technology a and money is given by

$$\frac{E_t[dr_t^a - dr_t^M]}{dt} = (\nu_t^a)^T \sigma_t^{Na} + x_t^a (\tilde{\sigma}^a)^2, \tag{2.10}$$

where the right-hand side is the covariance of the net worth of a household in sector a with the excess risk of technology a over money.

To write an analogous condition for technology b, we have to take into account the split of risk between households and intermediaries. Note that the net worth of intermediaries is exposed to risk

$$\sigma_t^N = x_t \nu_t^b + \sigma_t^M.$$

Therefore, the expected excess return of technology b must satisfy

$$\frac{E_t[dr_t^b - dr_t^M]}{dt} = (1 - \chi_t)((\nu_t^b)^T \sigma_t^{Nb} + x_t^b (\tilde{\sigma}^b)^2) + \chi_t (\nu_t^b)^T \sigma_t^N \tag{2.11}$$

The difference in return of inside and outside equity of households in sector b is then

$$\frac{dr_t^{bH} - dr_t^{bI}}{dt} = (\nu_t^b)^T \sigma_t^{Nb} + x_t^b (\tilde{\sigma}^b)^2 - (\nu_t^b)^T \sigma_t^N \geq 0, \tag{2.12}$$

with equality if $\chi_t < \bar{\chi}$.

Households must be indifferent between investing in technologies a and b. The following proposition summarizes the relevant condition

Proposition 1. *In equilibrium*

$$(x_t^a)^2 (|\nu_t^a|^2 + (\tilde{\sigma}^a)^2) = (x_t^b)^2 (|\nu_t^b|^2 + (\tilde{\sigma}^b)^2). \tag{2.13}$$

Proof. See Appendix. □

Market clearing for capital implies that portfolio weights, given the net worth shares of intermediaries and households, have to be consistent with the allocation of the fraction ψ_t of capital to technology b. Denote by

$$\vartheta_t = \frac{p_t}{q_t + p_t} \tag{2.14}$$

the fraction of the world wealth that is in the form of money. Then

$$x_t = \frac{\chi_t \psi_t (1 - \vartheta_t)}{\eta_t}. \tag{2.15}$$

Furthermore, the net worth of households who employ technologies a and b, together, must add up to $1 - \eta_t$, i.e.,

$$\frac{(1 - \psi_t)(1 - \vartheta_t)}{x_t^a} + \frac{\psi_t (1 - \chi_t)(1 - \vartheta_t)}{x_t^b} = 1 - \eta_t, \tag{2.16}$$

and the fraction household wealth in sector a is given by

$$\alpha_t = \frac{(1 - \psi_t)(1 - \vartheta_t)}{x_t^a (1 - \eta_t)}.$$

2.2 Evolution of the State Variable

Finally, we have to describe how the state variable η_t, which determines prices of capital and money p_t and q_t, evolves over time. The law of motion of η_t, together with the specification of prices and allocations as functions of η_t, constitute the full description of equilibrium: i.e. the map from any initial allocation and a history of shocks $\{Z_s\, s \in [0, t]\}$ into the description of the economy at time t after that history. The following proposition characterizes the equilibrium law of motion of η_t.

Proposition 2. *The equilibrium law of motion of η_t is given by*

$$\frac{d\eta_t}{\eta_t} = (1 - \eta_t) \left(x_t^2 |\nu_t^b|^2 - (x_t^a)^2 (|\nu_t^a|^2 + (\tilde{\sigma}^a)^2) \right) dt + (x_t \nu_t^b + \sigma_t^\vartheta)^T (\sigma_t^\vartheta \, dt + dZ_t). \tag{2.17}$$

The law of motion of η_t is so simple because the earnings of intermediaries and households can be expressed in terms of risks they take and the required equilibrium risk premia. The first term on the right-hand side reflects the relative earnings of intermediaries and households. The second term on the right-hand side of (2.17) reflects mainly the volatility of η_t, due to the imperfect risk sharing between intermediaries and households.

Proof. The law of motion of total net worth of intermediaries, given the risks that they take, must be

$$\frac{dN_t}{N_t} = dr_t^M - \rho\, dt + x_t(\nu_t^b)^T(\underbrace{(x_t\nu_t^b + \sigma_t^M)}_{\sigma_t^N}\, dt + \, dZ_t). \tag{2.18}$$

The law of motion of world wealth $(q_t + p_t)K_t$, the denominator of (2.8), can be found from the total return on world wealth, after subtracting the dividend yield of ρ (i.e., aggregate consumption). To find the returns, we take into account the risk premia that various agents earn. We have

$$\frac{d((q_t+p_t)K_t)}{(q_t+p_t)K_t} = dr_t^M - \rho\, dt + (1-\vartheta_t)\underbrace{(\sigma_t^K + \sigma_t^q - \sigma_t^M)^T}_{(\sigma_t^q - \sigma_t^p)^T}\, dZ_t +$$

$$(1-\vartheta_t)((1-\psi_t)\underbrace{((\nu_t^a)^T\sigma_t^{Na} + x_t^a(\tilde{\sigma}^a)^2)}_{\frac{E_t[dr_t^a - dr_t^M]}{dt}} + \psi_t\underbrace{\left(\chi_t(\nu_t^b)^T\sigma_t^N + (1-\chi_t)((\nu_t^b)^T\sigma_t^{Nb} + x_t^b(\tilde{\sigma}^b)^2)\right)}_{\frac{E_t[dr_t^b - dr_t^M]}{dt}})\, dt.$$

Recall that

$$\sigma_t^N = x_t\nu_t^b + \sigma_t^M, \quad \sigma_t^{Na} = x_t^a\nu_t + \sigma_t^M \quad \text{and} \quad \sigma_t^{Nb} = x_t^b\nu_t^b + \sigma_t^M$$

and note that

$$(1-\psi_t)\nu_t^a + \psi_t\nu_t^b = \sigma_t^q - \sigma_t^p.$$

Therefore, the law of motion of aggregate wealth can be written as[6]

$$\frac{d((q_t+p_t)K_t)}{(q_t+p_t)K_t} = dr_t^M - \rho\, dt + \underbrace{(1-\vartheta_t)(\sigma_t^q - \sigma_t^p)^T}_{-(\sigma_t^\vartheta)^T}(\sigma_t^M\, dt + dZ_t) +$$

$$(1-\vartheta_t)\left((1-\psi_t)x_t^a(|\nu_t^a|^2 + (\tilde{\sigma}^a)^2) + \psi_t\left(\chi_t x_t|\nu_t^b|^2 + (1-\chi_t)x_t^b(|\nu_t^b|^2 + (\tilde{\sigma}^b)^2)\right)\right)\, dt =$$

[6] Ito's lemma implies that $\sigma_t^\vartheta = (1-\vartheta)(\sigma_t^p - \sigma_t^q)$ and $\mu_t^\vartheta = (1-\vartheta)(\mu_t^p - \mu_t^q) - \sigma^\vartheta\sigma^p + (\sigma^\vartheta)^2$.

$$dr_t^M - \rho\, dt - (\sigma_t^\vartheta)^T\, (\sigma_t^M\, dt + dZ_t) + \eta_t\, x_t^2 |\nu_t^b|^2\, dt + (1-\eta_t)(x_t^a)^2(|\nu_t^a|^2 + (\tilde{\sigma}^a)^2)\, dt, \quad (2.19)$$

where we used (2.16) and the indifference condition of Proposition 13.

Thus, using Ito's lemma, we obtain (2.17).[7] □

3 Model without Intermediaries

The goal of this section is to understand the determinants of the value of money in a model without intermediaries. The key determinant of the value of money is, of course, the level of idiosyncratic risk.

We can anticipate properties of full equilibrium dynamics through our understanding of the economy without intermediaries. Since intermediaries reduce the amount of idiosyncratic risk in the economy, the presence of a healthy intermediary sector is akin to a reduction in idiosyncratic risk parameters in the model without intermediaries.

3.1 Value and Risk of Money

Assume that $\eta = 0$. Suppose for the sake of simplicity that $\sigma^a = \sigma^b = \sigma$, $\tilde{\sigma}^a = \tilde{\sigma}^b = \tilde{\sigma}$ and that $\max_\psi A(\psi) = \bar{A}$ is maximized at $\psi = 1/2$. Then half of all households produce good a, and the rest, good b. Aggregate capital in the economy follows

$$\frac{dK_t}{K_t} = (\Phi(\iota_t) - \delta)\, dt + \frac{\sigma}{2}\, dZ_t^a + \frac{\sigma}{2}\, dZ_t^b.$$

Prices p and q are constant. The volatility of the money (or the whole economy) and the incremental risk of a project in either sector (orthogonal to the risk of money) are

$$\bar{\sigma} \equiv \sqrt{\sigma^2/2} \quad \text{and} \quad \hat{\sigma} \equiv \sqrt{\tilde{\sigma}^2 + \sigma^2/2},$$

[7] If processes X_t and Y_t follow

$$dX_t/X_t = \mu_t^X\, dt + \sigma_t^X\, dZ_t \quad \text{and} \quad dY_t/Y_t = \mu_t^Y\, dt + \sigma_t^Y\, dZ_t,$$

then

$$\frac{d(X_t/Y_t)}{X_t/Y_t} = (\mu_t^X - \mu_t^Y)\, dt + (\sigma_t^X - \sigma_t^Y)^T (dZ_t - \sigma_t^Y\, dt).$$

respectively. Note that the total risk of technology a or b is $\sqrt{\bar{\sigma}^2 + \hat{\sigma}^2} = \sqrt{\sigma^2 + \tilde{\sigma}^2}$.

Effectively, the economy is equivalent to a single-good economy with aggregate risk $\bar{\sigma}$ and project-specific risk $\hat{\sigma}$. In this economy, the market-clearing condition for output (2.9) becomes

$$\bar{A} - \iota(q) = \rho \underbrace{(p + q)}_{q/(1-\vartheta)}. \tag{3.1}$$

Each household chooses a portfolio share of risky capital that is equal to the expected excess return on capital over money, which equals the dividend yield $(\bar{A} - \iota(q))/q$, since the capital gains rates are the same, divided by covariance of this excess return with household's net worth, which equals by $\hat{\sigma}^2$. Capital markets clearing implies that the portfolio weight demand equal $\frac{q}{p+q}$, that is $1 - \vartheta$. Hence, each household's net worth is exposed to aggregate risk $\bar{\sigma}$ and project-specific risk $(1 - \vartheta)\hat{\sigma}$. In other words, the asset-pricing condition of capital relative to money is

$$\frac{\bar{A} - \iota(q)}{q} = (1 - \vartheta)\hat{\sigma}^2 \quad \Rightarrow \quad \vartheta = 1 - \sqrt{\rho}/\hat{\sigma}. \tag{3.2}$$

Equilibrium in which money has positive value exists only if $\hat{\sigma}^2 > \rho$. As $\hat{\sigma}$ increases, the value of money relative to capital rises.

For a special form of the investment function $\Phi(\iota) = \log(\kappa\iota + 1)/\kappa$, we can also get closed-form expressions for the equilibrium prices of money and capital.[8] Then (3.1) implies that

$$q = \frac{\kappa\bar{A} + 1}{\kappa\sqrt{\rho}\hat{\sigma} + 1} \qquad \text{and} \quad p = \frac{\hat{\sigma} - \sqrt{\rho}}{\sqrt{\rho}}q. \tag{3.3}$$

There is always an equilibrium in which money has no value. In that equilibrium the price of capital satisfies $\bar{A} - \iota(q) = \rho q$, so that

$$q = \frac{\kappa\bar{A} + 1}{\kappa\rho + 1}. \tag{3.4}$$

Then the dividend yield on capital is $(\bar{A} - \iota_t)/q = \rho$ and expected return on capital is $\rho + \Phi(\iota_t) - \delta$. Subtracting the idiosyncratic risk premium of $\hat{\sigma}^2$ the required return on an

[8]When the investment adjustment cost parameter κ is close to 0, i.e. $\Phi(\iota)$ is close to 1, then the price of capital q is goes to 1 (this is Tobin's q). As κ becomes large, the price of capital depends on dividend yield $\bar{A}$ relative to the discount rate ρ and the level of idiosyncratic risk that affects the value of money.

asset that carries the same risk as the whole economy, or K_t, is

$$\rho - \hat{\sigma}^2 + \Phi(\iota_t) - \delta.$$

If this rate is lower than the growth rate of the economy, i.e. $\Phi(\iota_t) - \delta$, then an equilibrium in which money has positive value exists. Lemma 1 in the Appendix generalizes these results to the case when $\sigma^a \neq \sigma^b$ and $\tilde{\sigma}^a \neq \tilde{\sigma}^b$.

These closed-form solutions allow us to anticipate how the value of money may fluctuate in an economy with intermediaries. When η_t approaches 0, households face high idiosyncratic risk in both sectors, leading to a high value of money. In contrast, when η_t is large enough, then most of idiosyncratic risk is concentrated in sector a, as households in sector b pass on the idiosyncratic risk to intermediaries. This leads to a lower value of money.

Intermediary net worth and the value of money will generally fluctuate due to aggregate shocks Z^a and Z^b. Relative to total aggregate wealth - recall that η_t measures the intermediary net worth relative to total wealth - intermediaries are long shocks Z^b and short shocks Z^a when they invest in equity of households who produce good b. A fundamental assumption of our model is that intermediaries cannot hedge this aggregate risk exposure. Due to this, they may suffer losses, and losses force them to stop investing in equity of households who use technology b. The intermediary sector may become undercapitalized.

Impossibility of "As If" Representative Agent Economies. Note that it is impossible to construct an "as if" representative agent economy with the same aggregate output and investment streams and same asset prices that mimics the equilibrium outcome of our heterogeneous agents economy. In any representative agent economy, absence of individual-level idiosyncratic risk, capital returns strictly dominate money and hence money could never have some positive value.

3.2 Welfare Analysis

We start with a general result, which allows us to compute welfare of agents with logarithmic utility. Expression (3.6) below is valid for an arbitrary process (3.5), regardless of whether it arises from a feasible equilibrium trading strategy or not.[9]

[9] For example, we can use (3.5) to evaluate welfare of a hypothetical *representative agent*, who consumes a portion of world output, to estimate welfare that could be attained without idiosyncratic risk.

Proposition 3. *Consider an agent who consumes at rate ρn_t where n_t follows*

$$\frac{dn_t}{n_t} = \mu_t^n \, dt + \sigma_t^n \, dZ_t \tag{3.5}$$

Then the agent's expected future utility at time t takes the form

$$E_t \left[\int_t^\infty e^{-\rho(s-t)} \log(\rho n_s) \, ds \right] = \frac{\log(\rho n_t)}{\rho} + \frac{1}{\rho} E_t \left[\int_t^\infty e^{-\rho(s-t)} \left(\mu_s^n - \frac{|\sigma_s^n|^2}{2} \right) ds \right]. \tag{3.6}$$

Proof. See Appendix. □

Without intermediaries, drift and volatility of wealth for all households are time-invariant. In general, given portfolio weights $1 - \vartheta$ on capital and ϑ on money, we have

$$\mu^n = (1 - \vartheta) \frac{\bar{A} - \iota(q)}{q} + \Phi(\iota(q)) - \delta - \rho, \quad \sigma^n = \sqrt{(1 - \vartheta)^2 \hat{\sigma}^2 + \bar{\sigma}^2}. \tag{3.7}$$

For the equilibrium value of ϑ given by (3.2), we have

$$\mu^n = \Phi(\iota(q)) - \delta \quad \text{and} \quad \sigma^n = \sqrt{\rho + \bar{\sigma}^2}. \tag{3.8}$$

Combining (3) with (3.8), we get the following proposition

Proposition 4. *Suppose $\hat{\sigma}^2 > \rho$, so that monetary equilibrium exists in the economy without intermediaries. Then in this equilibrium, the welfare of a household with initial wealth $n_0 = 1$ is*

$$U^H = \frac{\log(\rho)}{\rho} + \frac{\Phi(\iota(q)) - \delta - (\rho + \bar{\sigma}^2)/2}{\rho^2}.$$

Macro-prudential regulation. How does welfare in equilibrium with money compare to welfare in the money-less equilibrium? If the regulator can control the value of money by specifying a money holding requirement of the agents, will the money under optimal policy have greater value than in equilibrium, or lower value? Note that higher value of money allows agents to reduce their idiosyncratic risk exposure, but creates a distortion on the investment front, since the value of capital becomes lower.

What if the regulator can control ϑ by forcing the agents to hold specific amounts of money? As it turns out, under some mild restrictions on ϑ, it will be optimal for the planner to force agents to hold more money. Our results are summarized in the following proposition.

Proposition 5. *Assume that $\Phi(\iota) = \log(\kappa\iota + 1)/\kappa$. Then if money can have positive equilibrium value, welfare in equilibrium with money is always greater than that in the moneyless equilibrium. Furthermore, relative to the value of ϑ in the equilibrium with money, optimal policy raises ϑ if and only if*

$$\hat{\sigma}(1 - \kappa\rho) < 2\sqrt{\rho}. \tag{3.9}$$

Proof. See Appendix. □

Condition (3.9) reflects the trade-off between the role of money as an insurance asset, and the distortionary effect of rising money value on investment. On the one hand, the returns to money are free of idiosyncratic risk, so individual households have less exposure to their own individual-specific shocks, improving welfare. On the other hand, in the money equilibrium, the price of capital is lower, so investment is lower, so overall growth is lower. When adjustment costs κ are large enough, these distortions are minimal, so the diversification benefit dominates, as we see in condition (3.9).

4 Analysis with Intermediaries

In this section, we analyze the full model economy with intermediaries. Intermediaries are diversifiers, allowing households that invest in technology b to offload some of their idiosyncratic risk. The capacity of intermediaries to act as "diversifiers" depends on their capitalization, and so it is not surprising that aggregate economic activity also depends on intermediary capitalization. Since intermediaries are exposed (in a levered way) to the risk of sector b, their wealth share moves over time, as different a-shocks and different b-shocks hit the economy.

In the previous section, we considered the extreme polar case when intermediary capitalization is 0. In that case, in the money equilibrium, the value of money is high – it is an attractive insurance vehicle for households invested in either of the two technologies. In contrast, with a functioning intermediary sector, households that invest in technology b can offload some of their idiosyncratic risk, so there is less demand for insurance vehicles. As a result money is less attractive and so its real value is low. At the other end of the spectrum, η_t can, however, also be too high: When η_t is close to 1 there is too much focus on the sector

b good and so aggregate economic activity declines.

The rest of this section proceeds as follows. First, we provide a full characterization of the equilibrium of our economy. Second, we conduct welfare analysis.

4.1 Equilibrium

The computational procedure we employ, both with and without monetary policy, is described in Appendix A. Consider parameter values $\rho = 0.05$, $A = 0.5$ $\sigma^a = \sigma^b = 0.1$, $\tilde{\sigma}^a = 0.6$, $\tilde{\sigma}^b = 1.2$, $s = 0.8$, $\Phi(\iota) = \log(\kappa\iota + 1)/\kappa$ with $\kappa = 2$, and $\bar{\chi} \to 1$. That is, in this sector sector b households face no fraction in selling off their risk to the (well-capitalized) intermediary sector.

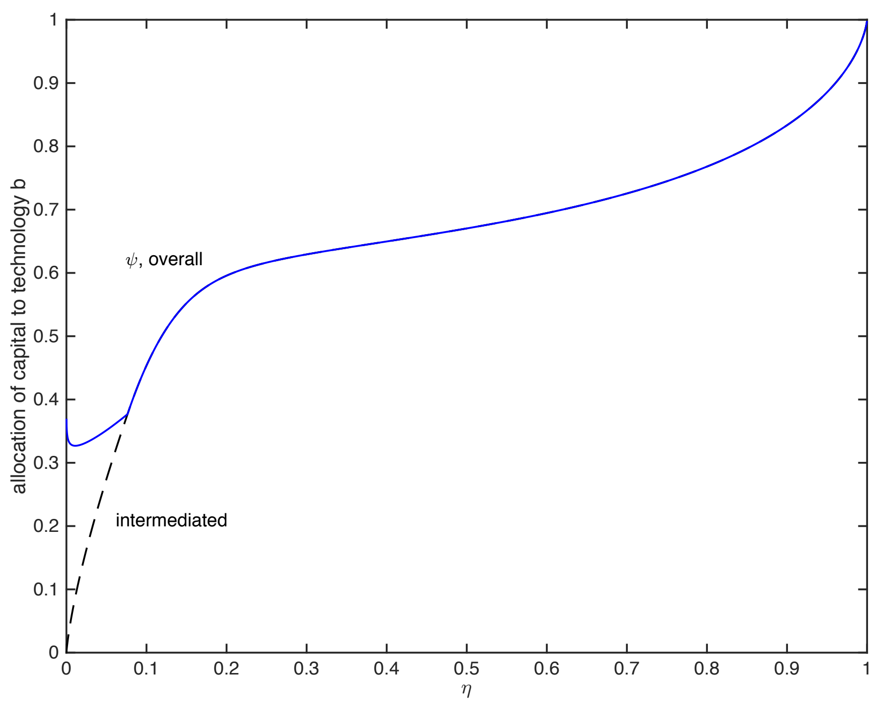

Figure 2: Equilibrium allocations.

We start by looking at the allocation of capital. The production of good b depends on intermediaries, it increases in the net worth share of the intermediary sector η. When η drops, the risk premia that intermediaries demand for equity stakes in projects of households in sector b rise, to the point that the households may be willing to sell less than fraction $\bar{\chi}$ of outside equity. See Figure 2.

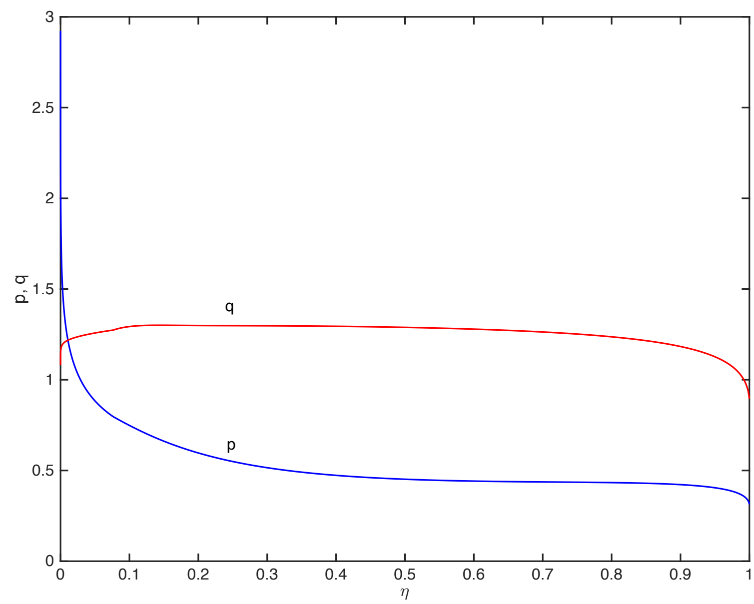

Figure 3: Equilibrium prices of capital and money.

Figure 3 shows the prices $p(\eta)$ and $q(\eta)$ of money and capital in equilibrium. At $\eta = 0$, the values of p and q converge to those under the benchmark without intermediaries, $q = 1.0532$ and $p = 3.4151$. As η rises, the price of capital rises and the price of money drops (although the price of capital drops again near $\eta = 1$). Money becomes less valuable as η rises mainly because intermediaries create money. The inside money on the liabilities sides of the intermediaries' balance sheets is a perfect substitute to outside money.

The Volatility of η, Liquidity and Disinflationary Spirals. Figure 4 illustrates the equilibrium dynamics through the drift and volatility of the state variable η. From Proposition 2,

$$\sigma_t^\eta = \underbrace{x_t(\sigma^b 1^b - \sigma_t^K)}_{\text{fundamental volatility}} + \underbrace{\sigma_t^\vartheta \left(1 - \frac{x_t}{1-\vartheta_t}\right)}_{\text{amplification}} \tag{4.1}$$

Variable η_t has volatility for two reasons: from the mismatch between the fundamental risk of assets that intermediaries hold, $\sigma^b dZ_t^b$, and the overall fundamental risk in the economy $\sigma_t^K \, dZ_t$ and from amplification. Amplification results from the changes in the price of money relative to capital, $\vartheta(\eta_t)$. As long as the intermediaries' portfolio share of households' equity x_t is greater than $1 - \vartheta_t$, the world capital share, and as long as $\vartheta'(\eta) < 0$, amplification exists.

Note that $\sigma^{\vartheta} = (1-\vartheta_t)(\sigma_t^p - \sigma_t^q)$. Amplification arises from two spirals: changes in the price of capital q_t, i.e. the *liquidity* spiral, and changes in the value of money p_t, the *disinflationary* spiral. In the region where intermediaries are undercapitalized (i.e. η is low), negative shocks are amplified both on the asset sides of intermediary balance sheets, as the price of physical capital $q(\eta)$ drops following a negative shock, and on the liability sided, through the Fisher disinflationary spiral, as the value of money $p(\eta)$ rises. These effects can be seen for $\eta \in (0, 0.1)$ in Figure 3. Both effects impair the intermediaries' net worth. Intermediaries' response to these losses is to shrink their balance sheets, leading to fire-sales (lowering the price q) and reduction in inside money (increasing the value of liabilities p). In other words, intermediaries take fewer deposits, create less inside money, and the money multiplier collapses.[10] This again reduces their net worth, and so on. The "Paradox of Prudence" emerges. Each individual intermediary micro-prudent behavior to scale back his risk is macro-imprudent, as it raises endogenous risk.

Specifically, this feedback effects lead to a geometric series, which can be summed up by rewriting equation (4.1) as

$$\sigma_t^{\eta} = \frac{x_t(\sigma^b 1^b - \sigma_t^K)}{1 + \frac{\vartheta'(\eta_t)}{\vartheta(\eta_t)}\left(\frac{x_t \eta_t}{1-\vartheta_t} - \eta_t\right)}.$$

Amplification becomes greater as $\vartheta'(\eta)$ becomes more negative, and as intermediary leverage x_t rises. How large can amplification be in this model?

Figure 4 shows both the fundamental portion of the volatility of η_t and total volatility that includes the effects of amplification. Amplification becomes prominent when intermediaries are undercapitalized. While the left panel illustrates dynamics for our baseline parameters, the right panel reduces fundamental risk parameters to $\sigma_a = \sigma_b = 0.03$. The right panel illustrates the volatility paradox: endogenous risk persists due to amplification even as fundamental risk declines. We see that the maximal volatility of η below the steady state stays roughly constant as fundamental risk declines, i.e. amplification in this model can be very large.

Drift of η. The drift of η_t given in Proposition 2 is

$$\mu_t^{\eta}\eta = \eta(1-\eta)\left(x_t^2|\nu_t^b|^2 - (x_t^a)^2(|\nu_t^a|^2 + (\tilde{\sigma}^a)^2)\right) + \eta_t(x_t\nu_t^b + \sigma_t^{\vartheta})^T\sigma_t^{\vartheta} \tag{4.2}$$

[10] In reality, rather than turning savers away, financial intermediaries might still issue demand deposits and simply park the proceeds with the central bank as excess reserves.

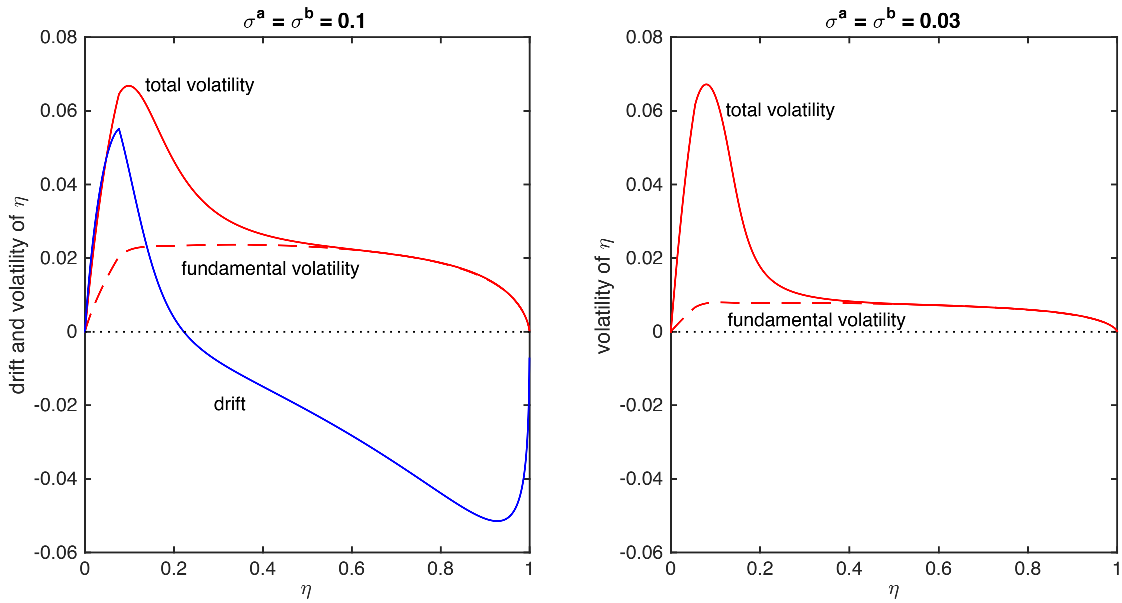

Figure 4: Equilibrium dynamics.

The first term captures the relative risk premia that intermediaries and households earn on their portfolios relative to money. As intermediaries become undercapitalized, the price of and return from producing good b rises, leading intermediaries to take on more risk. The opposite happens when intermediaries are overcapitalized - then the price of good a and the households' rate of earnings rises. The stochastic steady state of η_t is the point where the drift of η_t equals zero - at that point the earnings rates of intermediaries and households balance each other out. See the left panel of Figure 4.

4.2 Inefficiencies and Welfare

In this section, we calculate welfare in our model. Before we proceed, let us briefly describe the sources of inefficiency. In the process, we would like to emphasize relevant trade-offs with the intention of preparing ground for thinking about policy. First, there is inefficient sharing of idiosyncratic risk. Some of it can be mitigated through the use of intermediaries who can hold equity of households producing good b and diversify some of idiosyncratic risk. Consequently, cycles that can cause intermediaries to be undercapitalized can be harmful. Inefficiencies connected with idiosyncratic risks are also mitigated with the use of money - both inside and outside. Money allows households to diversify their wealth, but high value of money results in lower price of capital and potential inefficiency due to underinvestment.

Second, there is inefficient sharing of aggregate risk, which can cause whole sectors to become undercapitalized, e.g. intermediaries. If intermediaries become undercapitalized, barriers to entry into the intermediary sector help the intermediaries: the price of good b rises when η_t is low, mitigating the intermediaries risk exposures and allowing the intermediaries to recapitalize themselves. Thus, the limited competition in the intermediary sector creates a *terms-of-trade* hedge, which depends on the extent to which intermediaries cut back the financing of households in sector b, the extent to which those households are willing to self-finance, and the substitutability s among the intermediate goods.

Finally, there is productive inefficiency: when intermediaries or households are undercapitalized, then production may be inefficiently skewed towards good a or good b. Even at the steady state production can be inefficient due to financial frictions, e.g. imperfect sharing of idiosyncratic risks.

To understand the cumulative effect of all these inefficiencies, one needs a proper welfare measure. Welfare analysis is complicated by heterogeneity. We cannot focus on a representative household, since different households are exposed to different idiosyncratic risks. Some households become richer, while others become poorer.

Welfare Calculation. Recall that, according to Proposition 3, for a general wealth process welfare is given by (3.6). We will use this expression to calculate the welfare of intermediaries, households, as well as a fictitious "*representative agent*" who consumes a fixed portion of aggregate output. Intermediaries and households are the focus of our analysis, while the representative agent is a useful auxiliary construct.

Proposition 6. *Welfare of a representative agent with net worth is given by* $\log(\rho n_t)/\rho + U^R(\eta_t)$, *where*

$$U^R(\eta_t) = -\frac{\log(p_t + q_t)}{\rho} + E_t\left[\int_t^{\infty} e^{-\rho(s-t)}\left(\log(p_s + q_s) + \frac{\Phi(\iota_s) - \delta}{\rho} - \frac{|\sigma_s^K|^2}{2\rho}\right)\, ds\right]. \quad (4.3)$$

Proof. See Appendix. □

Besides being an interesting benchmark, as a welfare measure that excludes the effects of idiosyncratic risk, measure (4.3) can be adjusted to quantify the welfare of intermediaries and households.

Proposition 7. *The welfare of an intermediary with wealth n_t^I is $\log(\rho n_t^I)/\rho + U^I(\eta_t)$, where*

$$U^I(\eta_t) = U^R(\eta_t) - \frac{\log(\eta_t)}{\rho} + E_t\left[\int_t^\infty e^{-\rho(s-t)}\log(\eta_s)\,ds\right]. \tag{4.4}$$

The welfare of a household with net worth n_t^H is $\log(\rho n_t^H)/\rho + U^H(\eta)$, where

$$U^H(\eta_t) = U^R(\eta_t) + \tag{4.5}$$

$$\frac{1}{\rho}E_t\left[\int_t^\infty e^{-\rho(s-t)}\left(\eta_s\left((x_s^a)^2(|\nu_s^a|^2+\tilde{\sigma}_a^2) - x_s^2|\nu_s^b|^2\right) + \frac{|\sigma_s^\vartheta|^2 - (x_s^a)^2(|\nu_s^a|^2+\tilde{\sigma}_a^2)}{2}\right)\,ds\right].$$

To actually compute intermediary and household welfare, it suffices to note that all included quantities are functions of the single state variable η_t, and that in general

$$g(\eta_t) = E_t\left[\int_t^\infty e^{-\rho(s-t)}y(\eta_s)\,ds\right] \quad\Rightarrow\quad \rho g(\eta) = y(\eta) + g'(\eta)\mu_t^\eta\eta + \frac{g''(\eta)|\eta\sigma_t^\eta|^2}{2}$$

The actual computation of welfare levels thus merely requires us to solve an ordinary differential equation.

Welfare in equilibrium and preliminary thoughts on policy. Figure 5 shows welfare for parameter values we described at the beginning of this section, for an economy with K_0 normalized to 1. The welfare of a representative intermediary is given by $\log(\rho n_0)/\rho + U^I(\eta_0) = \log(\rho\eta_0(p_0+q_0))/\rho + U^I(\eta_0)$. The welfare of a representative household is $\log(\rho(1-\eta_0)(p_0+q_0))/\rho + U^H(\eta_0)$.

The welfare of each agent type tends to increase in its wealth share, but only to a certain point. At the extreme, one class of agents becomes so severely undercapitalized that productive inefficiency makes everybody worse off. At those extremes redistribution towards the undercapitalized sector would be Pareto improving.

In the next section we discuss policy. Our primary focus is monetary policy, but we also look at combinations of monetary and macroprudential policies. Before proceeding, let us reiterate the inefficiencies present in our model, and discuss how policies may affect these inefficiencies. First, as in the benchmark without intermediaries, the value of money affects welfare - higher value of money helps hedge idiosyncratic risk but creates investment distortions. Of course, monetary policy alone affects the value of money only endogenously, while macroprudential policy can influence the value of money directly. Second, there are

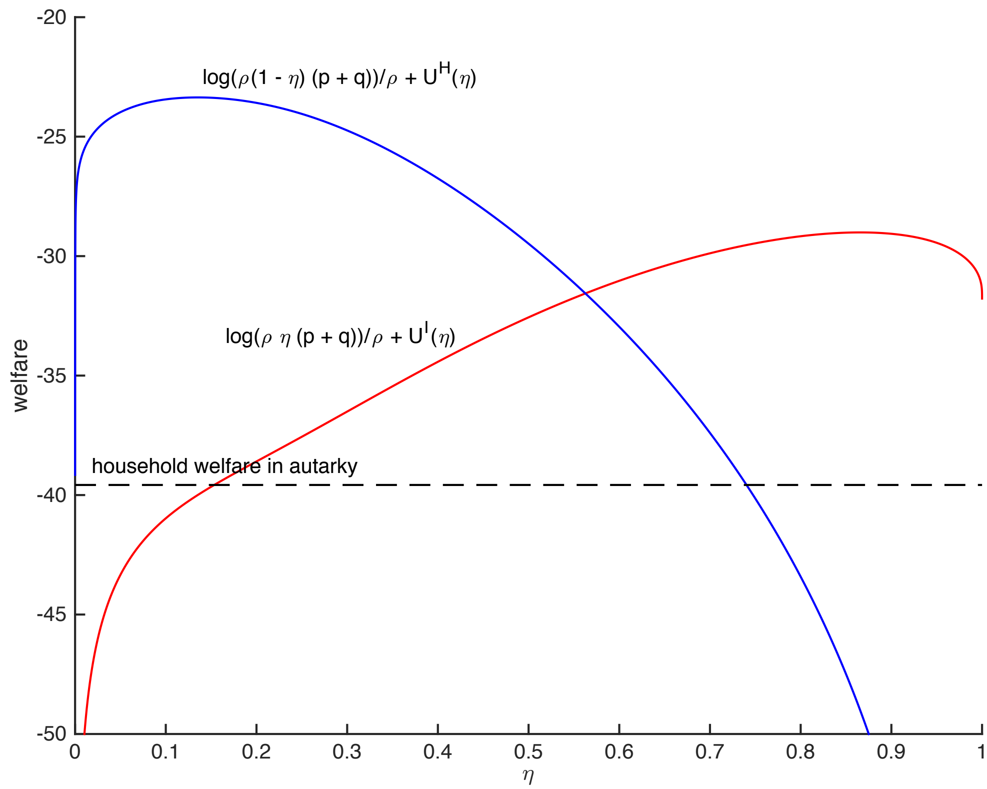

Figure 5: Equilibrium welfare.

inefficiencies with respect to the sharing of aggregate risk - inefficiencies accompanied by production and investment distortions when one of the sectors is undercapitalized. Monetary policy can redistribute risk, and so it can help in this regard. Also, with monetary policy alone, risk premia, which determine earnings, are determined by the concentration of risk. Thus, monetary policy cannot be used to target risk premia separately from risk taking. In contrast, macroprudential policy, through its control of quantities, can affect risk premia independently of risk-taking.

5 Monetary and Macro-prudential Policy

Policy has the potential to mitigate some of the inefficiencies that arise in equilibrium. It can undo some of the endogenous risk by redistributing wealth towards compromised sectors. It can control the path of deleveraging in crisis times and prevent the build-up of systemic risk in booms.

In general, policy is a broad notion, so it is important to make several distinctions. One is the distinction between *ex-post* and *ex-ante* policy. There are important questions related to crisis management - what are the effective ways to recover if the initial state is in crisis. Traditional analysis, which applies policy after an unanticipated shock that pushes the system away from the steady state, is ex-post as ex-ante agents do not anticipate the shock or the policy. In our setting, ex-post monetary policy operates by redistributing wealth - a "helicopter drop" of money has real effects only to the extent that it affects the value of η. In contrast, nominal effects are determined by the value of η as well as change in the money supply. A drop to intermediaries has different effects from a drop to households, both in nominal and real terms, even if the increase in money supply is the same, because the effects on η are different.

Ex-post monetary policy can be thought of as redistributing risk by affecting the values of assets directly controlled by the policy. For example, monetary policy can provide insurance by making certain assets, such as bonds, appreciate in value at times when intermediaries become undercapitalized. We focus in this paper mostly on ex-post policy.

While monetary policy affects risk profiles of assets, asset allocation, risk taking and risk premia remain endogenous. For example, monetary policy that becomes accommodative in downturns can improve aggregate risk sharing and stabilize the price of money by making intermediaries more functional in downturns, but it has side effects. Intermediary leverage rises in booms, as intermediaries anticipate insurance, and the value of money drops, as states of the "flight to safety" - where households demand money for self insurance because insurance through intermediaries is too expensive - become less likely. In contrast, macroprudential policy can affect risk-taking independently of risk profiles of assets. This has broad potential implications. For example, in a broader class of models where loose monetary policy can lead to inflated asset prices and stimulate the formation of bubbles, macroprudential policy can work against these effects. In our model specifically, monetary policy that provides insurance to intermediaries can lead to a shortage of money, and macroprudential policy that boosts the value of money can be beneficial, as it allows households to better self-insure against idiosyncratic risk. Thus, the effects of macroprudential policy that we discussed in Section 3.2 in the context without intermediaries extend in general.

Finally, from the point of view of the dual objective of central banks of maintaining price stability and financial stability, it is interesting to observe not only real, but also nominal effects of monetary policy. Different policies that have the same real effects can have different nominal implications. However, generally there is a strong force that the lack of financial stability poses a threat to price stability, as we saw in the context of the disinflationary spiral in the baseline model.

To commence discussing policy, we extend the baseline model to allow the central bank to control money supply. Specifically, we allow the central bank to set the short-term interest rate i_t on money. For example, the central bank pays interest rate on reserves (outside money) held by the intermediary sector. It funds these expenses simply by "printing money" in order to avoid any fiscal implications.

The following proposition demonstrates that this alone has no real effects on the economy, because intermediaries simply pass on the interest earned on reserves to depositors. Policy has real effects only if there are other assets, e.g. long-term bonds, whose values are affected by interest rate policy.

Proposition 8. (Super-Neutrality of Money) *If the central bank allows the nominal supply of outside money to grow at rate i_t by paying interest to holders of outside money, then the analysis of Section 4 is unaffected. That is, the law of motion of η_t, all real returns and asset allocation remain unchanged.*

Proof. If the outstanding nominal supply of outside money is M_t units at time t, then

$$\frac{dM_t}{M_t} = i_t \, dt.$$

Given the value of outside of $p_t K_t$, the return on outside money is given by $d(p_t K_t)/(p_t K_t)$. Inside money has to earn the same return as outside money - otherwise intermediaries can earn infinite profit by borrowing inside money and investing in outside money/reserves. Hence, all equations that characterize equilibrium in Section 4 remain unchanged, and since none of those equations contain the nominal interest rate i_t, interest rate policy has no real effects. □

While the interest rate policy alone has no real effects, it does affect inflation. Indeed, from the basic Fisher equation,[11]

$$dr_t^M = i_t \, dt - d\pi_t.$$

Since i_t does not affect the return on money dr_t^M, a rise in the interest rate leads to an identical rise in inflation.

[11]We write $d\pi$ instead of $\pi \, dt$ because the return on money dr_t^M has a Brownian component.

5.1 Introducing Nominal Long-term Bonds

We now extend the model to allow for a realistic monetary policy with redistributive effects that matter for real quantities. Specifically, we introduce nominal perpetual bonds, which pay a fixed interest rate i^B in money. The monetary authority sets the total outstanding quantity of these bonds L_t through open market operations (quantitative easing, or QE in short). We restrict both interest-rate and QE policies to be revenue neutral – the monetary authority pays interest and/or performs QE in a way that has no fiscal implications. In other words, the central bank does not alter its seignorage income when changing its monetary policy.

If B_t is the price in money of long-term bonds, per unit of interest, then the quantities of outstanding long-term bonds and money are affected by interest rate and QE policies as follows. We have

$$dM_t = i_t M_t \, dt + i^B L_t \, dt - (i^B B_t) \, dL_t.$$

That is, the outstanding nominal quantity of money is enhanced by "printing" to pay interest on money and long-term bonds, and decreases when long-term bonds are sold for money.

Analytically, rather than counting the number of nominal bonds outstanding, it is useful to work with real values of outstanding bonds and money. Denote by $p_t K_t$ the real value of all outstanding nominal (safe) assets, outside money and perpetual bonds, and by $b_t K_t$ the real value of all outstanding perpetual bonds, so that

$$\frac{b_t}{p_t} = \frac{i^B B_t L_t}{i^B B_t L_t + M_t},$$

since the ratio must be the same regardless of whether quantities are measured in real or nominal terms. The central bank controls the pair (i_t, L_t), or, equivalently, the pair (i_t, b_t) since the relationship between L_t and b_t is one-to-one given the equilibrium bond price B_t.

Given the nominal money supply M_t and the real value of money $(p_t - b_t)K_t$, the price level is given by

$$\frac{M_t}{(p_t - b_t)K_t} = \frac{i^B B_t L_t + M_t}{p_t K_t} \tag{5.1}$$

Returns. The expressions for the return on capital from Section 2 do not change, but money earns the return that depends on policy. To derive the returns on money and bonds and the asset-pricing condition for bonds, we postulate that B_t follows the following endogenous equilibrium process

$$\frac{dB_t}{B_t} = \mu_t^B\, dt + (\sigma_t^B)^T\, dZ_t. \tag{5.2}$$

When intermediaries hold bonds, using them as a hedge against their net worth risk, then the difference between expected returns on bonds dr_t^B and money dr_t^M can be priced according to

$$\frac{E_t[dr_t^B - dr_t^M]}{dt} = (\sigma_t^B)^T \sigma_t^N, \quad \sigma_t^N = \sigma_t^M + x_t \nu_t + x_t^B \sigma_t^B, \tag{5.3}$$

where σ_t^B is the incremental risk of bonds over money and x_t^B is the intermediary portfolio weight on bonds.

The return on the world portfolio of bonds and money is

$$\frac{d(p_t K_t)}{p_t K_t} = \left(\Phi(\iota_t) - \delta + \mu_t^p + (\sigma_t^p)^T \sigma_t^K\right) dt + (\sigma_t^K + \sigma_t^p)^T\, dZ_t = \frac{b_t}{p_t}\, dr_t^B - \left(1 - \frac{b_t}{p_t}\right)\, dr_t^M,$$

b_t/p_t and $1 - b_t/p_t$ are the portfolio weights on bonds and money. Using (5.3), we find that the return and risk of money, which enters the capital-pricing equations (2.10) and (2.11) as well as the expressions for ν_t^a and ν_t^b, are given by

$$dr_t^M = \left(\Phi(\iota) - \delta + \mu_t^p + (\sigma_t^p)^T \sigma_t^K\right) dt - \frac{b_t}{p_t} (\sigma_t^B)^T \sigma_t^N\, dt + \underbrace{\left(\sigma_t^K + \sigma_t^p - \frac{b_t}{p_t} \sigma_t^B\right)}_{\sigma_t^M}\, dZ_t. \tag{5.4}$$

In all policies we compute as examples, bonds are negatively correlated to the risk that intermediaries face and intermediaries hold all the bonds using them as a hedge. Then the intermediaries' portfolio weight on bonds is $x_t^B = \vartheta_t/\eta_t\, b_t/p_t$. For this to be the case, intermediaries must value the insurance that bonds provide the most, i.e.

$$(\sigma_t^B)^T \sigma_t^N \le (\sigma_t^B)^T \underbrace{(\sigma_t^M + x_t^a \nu_t)}_{\sigma_t^{Na}},\ (\sigma_t^B)^T \underbrace{(\sigma_t^M + x_t^b \nu_t^b)}_{\sigma_t^{Nb}}.$$

In general, however, households who use technology b may also choose to hold bonds, but to a lesser extent. All the formulas can be easily generalized to the case when some households hold bonds.

The law of motion of η_t has to be adjusted for the hedge that the intermediaries receive from bonds. The following proposition provides the relevant expression.

Proposition 9. *The equilibrium law of motion of η_t is given by*

$$\frac{d\eta_t}{\eta_t} = (1-\eta_t)\left(|x_t\nu_t^b + x_t^B\sigma_t^B|^2 - (x_t^a)^2(|\nu_t^a|^2 + (\tilde{\sigma}^a)^2)\right)\, dt + \tag{5.5}$$

$$\left(x_t\nu_t^b + \sigma_t^\vartheta + (1-\eta_t)x_t^B\sigma_t^B\right)^T \left(dZ_t + \left(\sigma_t^\vartheta - \eta_t x_t^B\sigma_t^B\right)\, dt\right).$$

Proof. See Appendix. □

We see that the real impact of policy on equilibrium is fully summarized by the risk transfer term $(b_t/p_t)\sigma_t^B$, since this term alone enters all the equilibrium conditions. We summarize this result in a proposition.

Proposition 10. *The real effect of monetary policy on equilibrium is fully summarized by the process $(b_t/p_t)\sigma_t^B$.*

Of course, the values of $(b_t/p_t)\sigma_t^B$ depend on the policy (i_t, b_t), and we characterize the relationship in Proposition 14 in the Appendix. Since two tools determine a single process, there are multiple ways to produce the same real effect on equilibrium dynamics, although of course different policies can have different nominal effects. We study the impact of policy on equilibrium next. In particular, we highlight that while monetary policy can provide insurance, it cannot control risk from risk-taking and risk premia separately.

Mitigated Liquidity and Disinflationary Spiral. Let us consider policies that set the short-term interest rate i_t as well as the level of b_t as functions of η_t, lowering the interest rate i_t when η_t drops. Then the bond price risk σ_t^B exactly opposite from the risk exposure of intermediaries $\sigma^b 1^b - \sigma_t^K$ or σ_t^η. Intermediaries can use bonds as a hedge. Monetary policy can be used implement more efficient sharing of aggregate risk, e.g. undo endogenous risk.

Using (5.5), $x_t^B = (\vartheta_t/\eta_t)b_t/p_t$ and Ito's lemma, the volatility of η_t, which can be rewritten as

$$\sigma_t^\eta = \frac{x_t(\sigma^b 1^b - \sigma_t^K)}{1 + \underbrace{\frac{\vartheta'(\eta)}{\vartheta(\eta)}(\psi_t\chi_t - \eta_t)}_{\text{amplification spirals}} - \underbrace{\frac{b_t}{p_t}\frac{B'(\eta)}{B(\eta)}(x_t\eta_t + (1-\eta_t)\vartheta_t)}_{\text{mitigation}}}. \tag{5.6}$$

The numerator reflects the incremental risk of technology b relative to average risk in the economy multiplied by the intermediaries' exposure to this risk (i.e. portfolio weight x). If the relative prices of money, capital and bonds were fully stable, then the volatility of

η_t would equal $x_t(\sigma^b 1^b - \sigma_t^K)$. The denominator of (5.6) contains a term that reflects the amplification of aggregate risk: $\vartheta'(\eta) < 0$ when, following a drop in η_t, the price of money p_t rises relative to the price of capital q_t. The denominator also contains a mitigating term as bonds appreciate when η_t falls. As the mitigating effect $-(b_t/p_t)\ B'(\eta)/B(\eta)$ rises, σ^η declines and goes to 0 in the limit (i.e. the law of motion of η becomes deterministic).

Prices of bonds relative to money also affect the incremental risk that agents face when they add exposure to capital b, given by

$$\nu_t^a = \sigma^a 1^a - \sigma_t^K - \frac{\sigma_t^\vartheta}{1-\vartheta} + \frac{b_t}{p_t}\sigma_t^B \quad \text{and} \quad \nu_t^b = \sigma^b 1^b - \sigma_t^K - \frac{\sigma_t^\vartheta}{1-\vartheta} + \frac{b_t}{p_t}\sigma_t^B \tag{5.7}$$

In this equation $-\sigma_t^\vartheta/(1-\vartheta)$ reflects the nominal price of capital, positively correlated to $1^b\sigma^b - \sigma_t^K$, which adds to the risk that intermediaries face. In contrast, bonds stabilize the value of money, and hence the term $\frac{b_t}{p_t}\sigma_t^B$ mitigates the risk that intermediaries face.

In the following section, we provide an example that illustrates the risk transfer effects of monetary policy by focusing on the mitigating term in (5.6). The one-dimensional function

$$\frac{b(\eta)}{p(\eta)}\frac{B'(\eta)}{B(\eta)}$$

of η summarizes the effects of two policy tools i_t and b_t, with which any such function can be implemented in multiple ways.

Which policy is most natural to focus on, out of the many possibilities? In the next subsection we illustrate a policy that completely removes amplification in the law of motion of η_t, so that (5.6) becomes reduced to

$$\sigma_t^\eta = x_t(\sigma^b 1^b - \sigma_t^K). \tag{5.8}$$

This effect is achieved by setting $b_t/p_t\,\sigma_t^B$ appropriately. As a result, endogenous risk in ν_t^b is offset partially, so that the remaining endogenous risk of capital holdings on the asset sides of intermediary balance sheets is exactly offset by the hedge that the bonds provide.

We also discuss the theoretical possibility of what happens in the limit when monetary policy allows for perfect sharing of aggregate risk. It is natural to ask the question of optimal welfare that can be attained with monetary policy alone. We do not provide an answer to this question under the excuse that welfare can be significantly improved if monetary policy is used in combination with macroprudential policy. The reason is that monetary policy cannot control risk separately from risk taking. We discuss optimal macroprudential policy at the end of this section. While we do not want the prescriptions to be taken literally, as our model is still too stylized, we learn valuable lessons about avenues in which macrorpudential policy can operate to improve welfare.

5.2 An Example: Removing Amplification

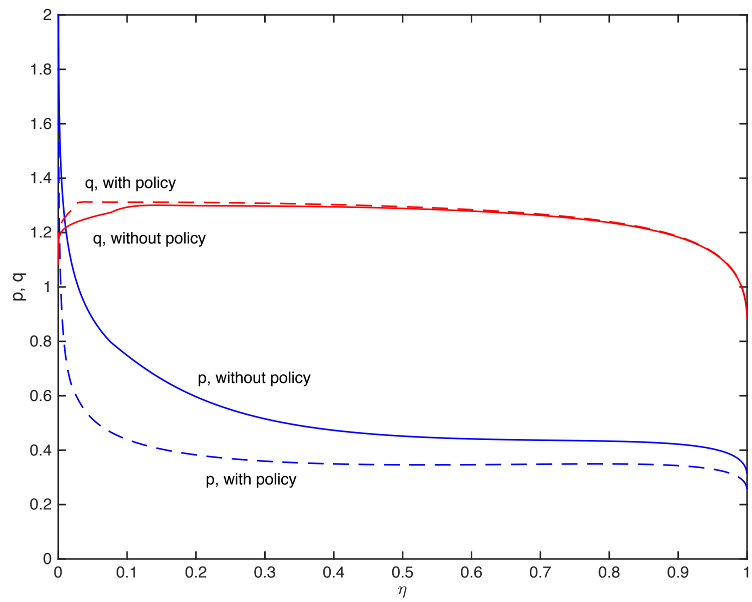

Figure 6: Equilibrium prices of capital and money without policy (solid) and with (dashed).

Consider a policy that sets $b_t/p_t\,\sigma_t^B$ to remove amplification from the law of motion of η_t, so that the volatility of η_t is given by (5.8). Here we illustrate what this policy does to our numerical example of Section 4, i.e. for parameter values $\rho = 0.05$, $A = 0.5$ $\sigma^a = \sigma^b = 0.1$, $\tilde{\sigma}^a = 0.6$, $\tilde{\sigma}^b = 1.2$, $s = 0.8$, $\Phi(\iota) = \log(\kappa\iota + 1)/\kappa$ with $\kappa = 2$, and $\bar{\chi} \to 1$. Figure 6 shows the effect of policy on prices. The price of money falls since the intermediary sector creates more inside money: it does not need to absorb as much aggregate risk to do that. As a consequence, the price of capital rises - there is more demand for capital from the sector producing good b. As Figure 7 illustrates, capital is shifted to sector b with policy.

Finally 8 shows the drift and volatility of η with and without policy. With policy, the intermediary net worth is lower at the steady state. Consequently, their leverage is higher.

Ultimately, monetary policy affects the degree of market incompleteness with respect to

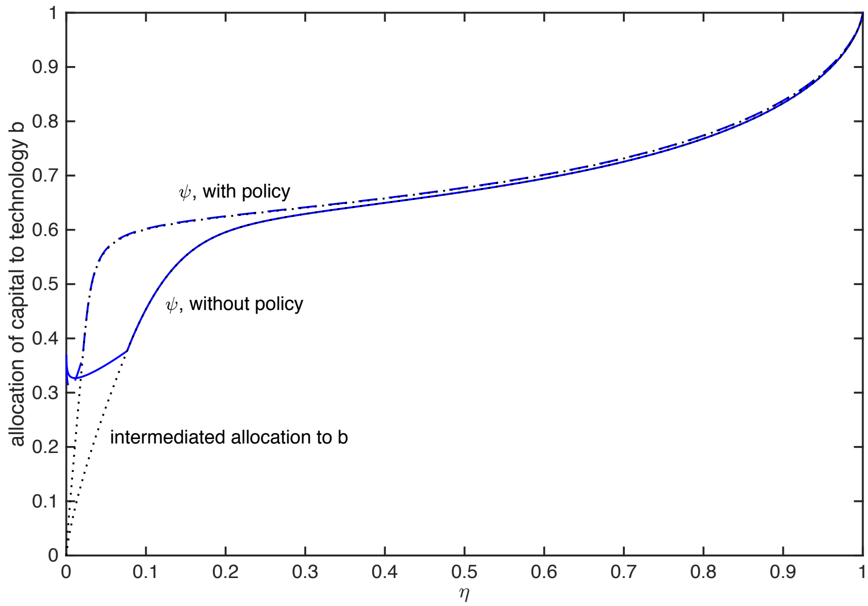

Figure 7: Equilibrium allocations without policy (solid) and with (dashed).

sharing of aggregate risk, but it cannot disentangle risk and risk-taking. The allocation of capital, the value of money relative to capital, and earnings rates of sectors a and b as well as intermediaries are endogenously determined by the risk profiles of available assets.

5.3 Economy with Perfect Sharing of Aggregate Risk

If the mitigation term in (5.6) goes to infinity, then $\sigma_t^\eta \to 0$ and we obtain an economy with perfect sharing of aggregate risk. Households in sector b also hold bonds to offset the risk of technology b. This is exactly the outcome we would see if intermediaries and households could trade contracts based on systemic risk, i.e. risk of the form

$$(\sigma^b 1^b - \sigma^a 1^a)^T \, dZ_t.$$

In this case the aggregate risk exposures of all households and intermediaries is proportional to σ_t^K, and η_t, p_t and q_t have no volatility. Also, since intermediaries can trade aggregate risk freely, households in sector b issue maximal equity shares $\bar{\chi}$ to intermediaries.

The following proposition characterizes the function $\vartheta(\eta)$ through a first-order differential equation, together with ψ_t, household leverage x_t^a and x_t^b, price q_t and the dynamics of η.

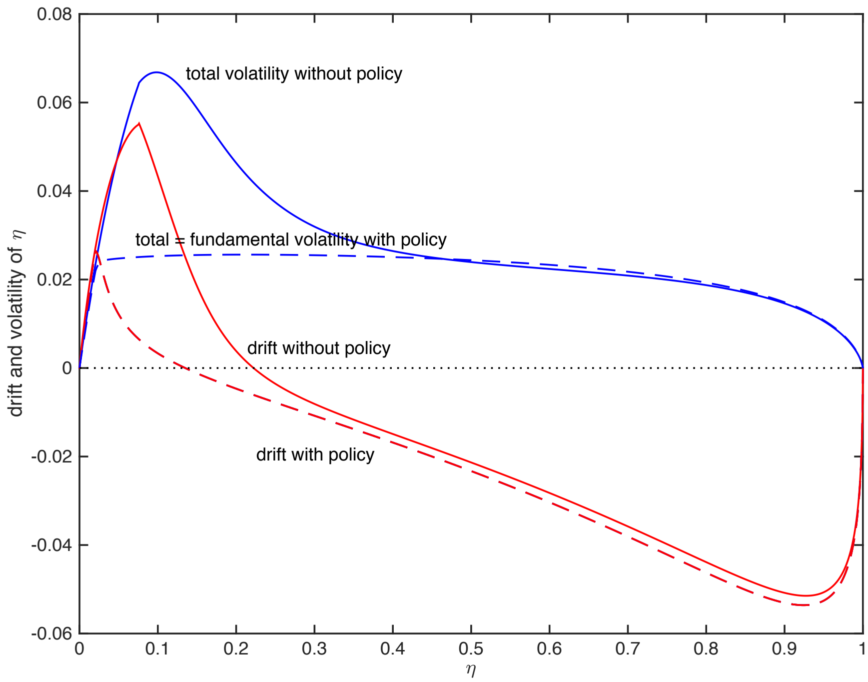

Figure 8: Drift and volatility of η without policy (solid) and with (dashed).

Proposition 11. *The function $\vartheta(\eta)$ satisfies the first-order differential equation*

$$\mu_t^{\vartheta} = \frac{\vartheta'(\eta)}{\vartheta(\eta)}\eta\mu_t^{\eta}, \tag{5.9}$$

where

$$\mu_t^{\eta} = -(1-\eta)(x_t^b)^2(\tilde{\sigma}^b)^2, \quad \mu_t^{\vartheta} = \rho + \mu_t^{\eta}, \tag{5.10}$$

and ψ_t, x_t^a, x_t^b and q_t satisfy

$$A(\psi_t) - \iota(q_t) = \frac{\rho q_t}{1-\vartheta_t}, \quad (1-\bar{\chi})\psi_t + (1-\psi_t)\frac{\tilde{\sigma}^a}{\tilde{\sigma}^b} = x_t^b\frac{1-\eta_t}{1-\vartheta_t}, \quad x_t^a\tilde{\sigma}^a = x_t^b\tilde{\sigma}^b \quad \text{and} \tag{5.11}$$

$$\frac{A^b(\psi_t) - A^a(\psi_t)}{q_t} = \psi_t(\sigma^b)^2 - (1-\psi_t)(\sigma^a)^2 + (1-\bar{\chi})\, x_t^b\tilde{\sigma}_b^2 - x_t^a\tilde{\sigma}_a^2. \tag{5.12}$$

Proof. See Appendix. □

Figure 9 compares prices, allocations and dynamics in the baseline model, under policy

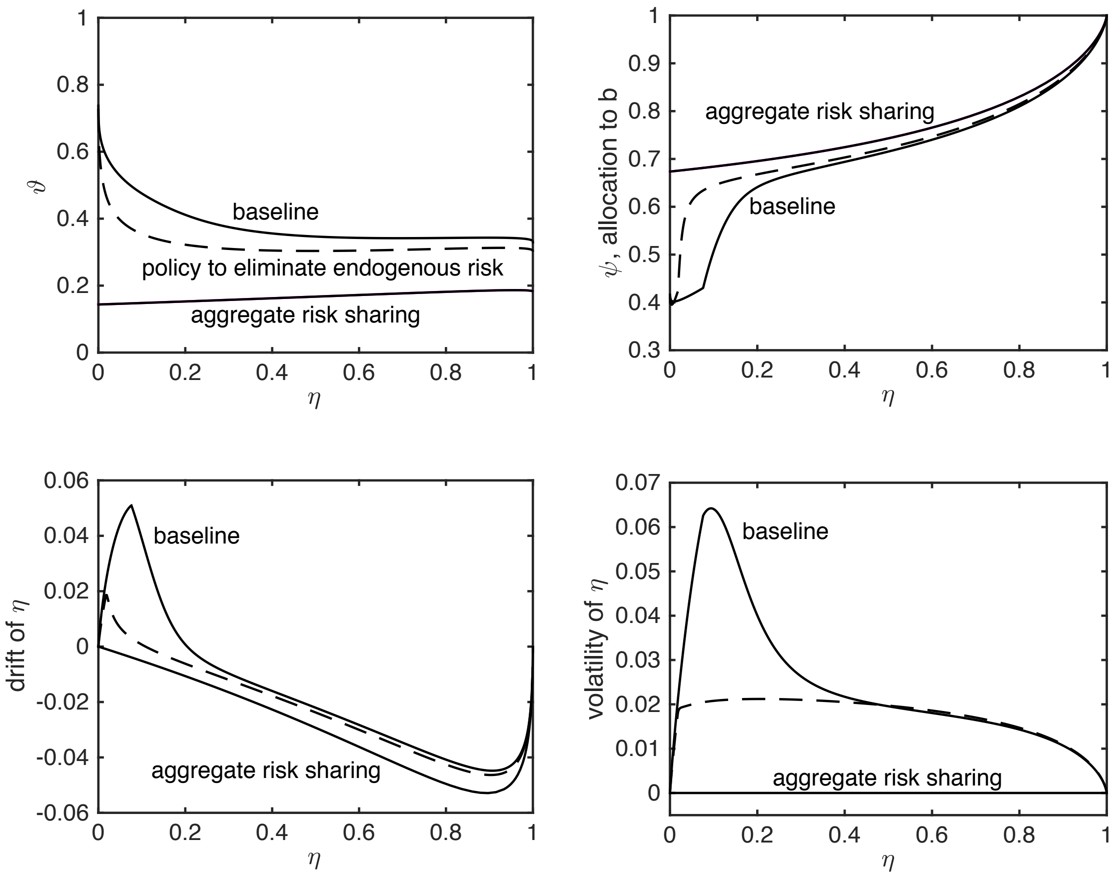

Figure 9: Comparison on the degree of aggregate risk sharing.

that eliminates endogenous risk, and with perfect risk sharing, in an economy with parameters $\rho = 5\%$, $A = 0.5$, $\sigma^a = \sigma^b = 0.1$, $\tilde{\sigma}^a = 0.8$, $\tilde{\sigma}^b = 1.2$, $s = 0.8$, $\Phi(\iota) = \log(\kappa\iota + 1)/\kappa$ with $\kappa = 2$, and $\underline{\chi} \to 1$. Equilibrium moves further in the direction that it took with the application of policy that removes endogenous risk. Specifically, the value of money falls, the allocation of capital becomes more skewed to technology b that intermediaries can facilitate, the steady state of η_t goes to 0 since intermediaries can fully hedge risks, and the volatility of η_t becomes 0.

Qualitatively, what makes perfect aggregate risk sharing different is the fact that the boundary condition without intermediaries no longer plays a role at $\eta = 0$. The absence of crisis dynamics contributes to the significant is the drop in the relative value of money $\vartheta(\eta)$.[12] Also, leverage of intermediaries rises without bound approaching $\eta = 0$ - in normal

[12]In fact, we raised the idiosyncratic volatility of good b to 0.8 in this example, because otherwise money

circumstances this would be impossible due to the rise of endogenous risk, since endogenous risk is generated by the increase in leverage even in environment when exogenous shocks are small (but not zero).

It is important to highlight one more time the observation that monetary policy cannot provide insurance and control risk-taking at the same time. Leverage rises endogenously the more risk sharing becomes possible. Asset allocation, together with asset prices and risk premia, are also endogenous and dependent on the insurance that monetary policy provides. Hence, the value of money ϑ falls with perfect risk sharing, which may be detrimental to welfare as we observed in the model without intermediaries.

These links, which cannot be broken without macroprudential policy, have implications beyond the stylized elements of our model. In particular, loose monetary policy can lead to excessive leverage in some sectors, reduced risk premia and, consequently, bubbles in some asset classes. These can pose significant threat to financial stability. Also, with incomplete markets, improving risk sharing along some dimensions does not necessarily lead to higher welfare.

5.4 Welfare

We can extend our welfare calculation to allow for policy as follows.

Proposition 12. *The welfare of an intermediary with wealth* n_t^I *is* $\log(\rho n_t^I)/\rho + U^I(\eta_t)$*, where* $U^I(\eta_t)$ *is given by (4.5) taking into account the law of motion of* η_t *under policy. The welfare of a household with net worth* n_t^H *is given by* $\log(\rho n_t^H)/\rho + U^H(\eta)$*, with* $U^H(\eta)$ *given by a generalized version of (4.5),*

$$U^H(\eta_t) = U^R(\eta_t) + \tag{5.13}$$

$$\frac{1}{\rho}E_t\left[\int_t^\infty e^{-\rho(s-t)}\left(\eta_s\left((x_s^a)^2(|\nu_s^a|^2+\tilde{\sigma}_a^2) - |x_s\nu_s^b + x_s^B\sigma_s^B|^2\right) + \frac{|\eta_s x_s^B\sigma_s^B - \sigma_s^\vartheta|^2 - (x_s^a)^2(|\nu_s^a|^2+\tilde{\sigma}_a^2)}{2}\right)\,ds\right].$$

Proof. See Appendix. □

Figure 10 shows the welfare frontiers that are attainable in equilibrium with various amounts of aggregate risk sharing. Better sharing of aggregate risk improves the welfare of households. For the policy that removes endogenous risk, household welfare reaches a

in the equilibrium with perfect aggregate risk sharing would be worthless.

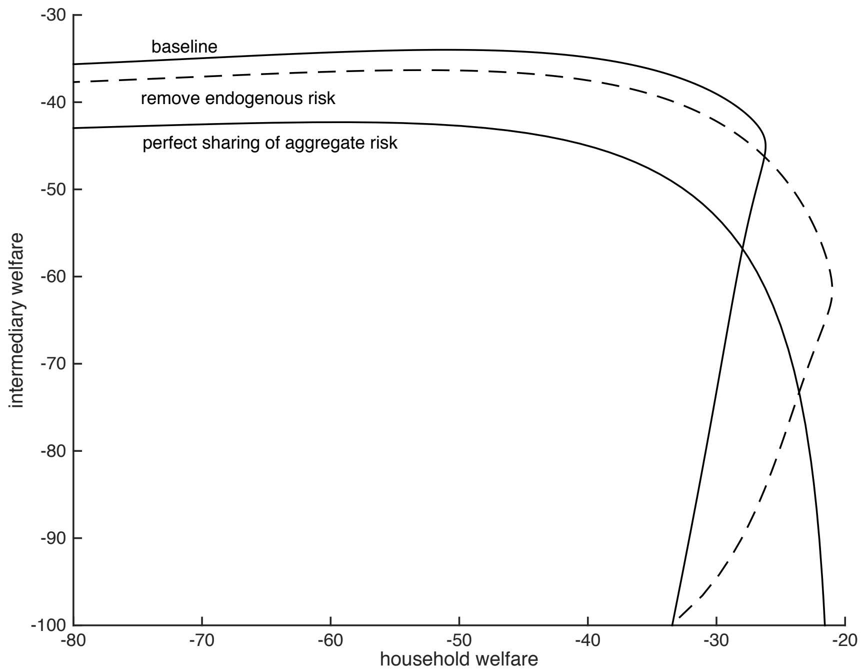

Figure 10: Welfare for different degrees of aggregate risk sharing.

higher level before the "back-bending" portion in the lower right corner, which corresponds to the crisis region where intermediaries are undercapitalized and a simple transfer from intermediaries to households is Pareto improving. Under perfect risk sharing, household welfare is higher only slightly relative to the policy that just removes aggregate risk.

In contrast, intermediary welfare goes down due to the fact that risk premia, which drive intermediary earnings in this model, become lower with greater risk sharing. Better sharing of aggregate risk reduce costs of intermediation, which reduce intermediary profits here due to perfect competition among intermediaries. Of course, in reality this may not be the case, depending on the degree of competition in the intermediary sector.[13] If we imagine that agents can self-select whether to become households or intermediaries, then monetary policy that allows for better sharing of aggregate risk can lead to a smaller and more efficient intermediary sector, and a welfare improvement.

[13]Higher competition may not be desirable from a policy perspective, as it leads to greater risk-taking by intermediaries.

5.5 Optimal Macroprudential Policy

Macroprudential policies can achieve significantly higher welfare. Macroprudential policies can control quantities and affect the allocation of resources independently of the allocation of risk.

Here we study the theoretical limit that can be attained when markets for sharing of aggregate risk are open and the policy maker can control the asset allocation, portfolios and returns. The regulator cannot, however, control consumption or investment.

One question that comes up immediately is whether the policy maker should control the allocation of resources between sectors a and b by forcing some households specialize in either of these two sectors against their will. The following proposition shows that this is not so.

Proposition 13. *To maximize welfare, the policy maker must expose households in sectors a and b to the same amounts of idiosyncratic risk. It is also welfare-maximizing for households in the two sectors to earn the same expected returns, and with this, households are indifferent between specializing in sectors a and b.*

Proof. Fix the allocation ψ_t of capital to technology b and the total earnings of the household sector, so that the aggregate net worth of households N_t^H follows

$$dN_t^H/N_t^H = \mu_t^H\,dt + (\sigma_t^K)^T\,dZ_t.$$

For these fixed ψ_t and μ_t^H, consider the problem of choosing the net worth of households in each sector, such that $N_t^a + N_t^b = N_t^H$, wealth accumulation in each sector μ_t^a and μ_t^b such that

$$\mu_t^H N_t^H = \mu_t^a N_t^a + \mu_t^b N_t^b,$$

to maximize average household welfare. Then leverage x_t^a and x_t^b in each sector is given by

$$N_t^a x_t^a = (1-\psi_t)(1-\vartheta_t) \quad \text{and} \quad N_t^b x_t^b = \psi_t(1-\vartheta_t)(1-\bar{\chi}),$$

since households in sector b must issue the maximal amount of outside equity to minimize idiosyncratic risk exposure.

The effect of these choices on the average welfare of households in sectors a and b, from (3.6), is proportional to

$$N_t^a\left(\mu_t^a - \frac{(x_t^a)^2\tilde{\sigma}_a^2 + |\sigma_t^K|^2}{2}\right) + N_t^b\left(\mu_t^b - \frac{(x_t^b)^2\tilde{\sigma}_b^2 + |\sigma_t^K|^2}{2}\right) =$$

$$N_t^H\left(\mu_t^H-\frac{|\sigma_t^K|^2}{2}\right)-\frac{((1-\psi_t)(1-\vartheta_t))^2\tilde{\sigma}_a^2}{2N_t^a}-\frac{(\psi_t(1-\vartheta_t)(1-\bar{\chi}))^2\tilde{\sigma}_b^2}{2(N_t^H-N_t^a)}.$$

The first-order condition with respect to N_t^a is

$$0=\frac{((1-\psi_t)(1-\vartheta_t))^2\tilde{\sigma}_a^2}{2(N_t^a)^2}-\frac{(\psi_t(1-\vartheta_t)(1-\bar{\chi}))^2\tilde{\sigma}_b^2}{2(N_t^H-N_t^a)^2}\quad\Rightarrow\quad(x_t^a)^2\tilde{\sigma}_a^2=(x_t^b)^2\tilde{\sigma}_b^2.$$

Thus, the policy maker should expose households in the two sectors to the same amounts of idiosyncratic risk. Notice also that $\mu_t^a=\mu_t^b=\mu_t^H$ maximizes household welfare, and with this, households are indifferent between specializing in sectors a and b at any moment of time.[14] □

Furthermore, notice that the welfare of intermediaries is given by $\log(\rho\eta_0(p_0+q_0)K_0)/\rho+U^I(\eta_0)$, where $U^I(\eta_0)$ is (4.4), whereas the welfare of a hypothetical agent who consumes a portion of total household net worth is

$$\frac{\log(\rho(p_0+q_0)K_0)}{\rho}+U^R(\eta_0)+E_0\left[\int_0^\infty e^{-\rho t}\log(1-\eta_t)\,dt\right]. \tag{5.14}$$

Accounting for idiosyncratic risk, the welfare of each household is that minus

$$E_0\left[\int_0^\infty\frac{\left((1-\psi_t)\tilde{\sigma}^a+(1-\bar{\chi})\psi_t\tilde{\sigma}^b\right)^2}{2\rho}\frac{(1-\vartheta_t)^2}{(1-\eta_t)^2}\,dt\right],$$

since the households' idiosyncratic risk exposure is

$$x_t^a\tilde{\sigma}^a=x_t^b\tilde{\sigma}^b=((1-\psi_t)\tilde{\sigma}^a+(1-\bar{\chi})\psi_t\tilde{\sigma}^b)\frac{1-\vartheta_t}{1-\eta_t}.$$

Hence the problem of maximizing welfare, with weights λ and $1-\lambda$ on intermediaries and households, reduces static problems of choosing η_t, ψ_t and q_t to maximize

$$\log(A(\psi_t)-\iota_t)+\frac{\Phi(\iota_t)-\delta}{\rho}-\frac{|\sigma_t^K|^2}{2\rho}$$

$$+\lambda\log(\eta_t)+(1-\lambda)\left(\log(1-\eta_t)+\frac{\left((1-\psi_t)\tilde{\sigma}^a+(1-\bar{\chi})\psi_t\tilde{\sigma}^b\right)^2}{2\rho}\frac{(1-\vartheta_t)^2}{(1-\eta_t)^2}\right),$$

[14] Strictly speaking, any other distribution of returns is also welfare maximizing, since average return is always μ_t^H by the law of large numbers when the household spends a fraction N_t^a/N_t^H of time in sector a and N_t^b/N_t^H in sector b.

where

$$\iota_t = \iota(q_t), \qquad |\sigma_t^K|^2 = \psi_t^2\sigma_b^2 + (1-\psi_t)^2\sigma_a^2, \quad \text{and} \quad \vartheta_t = \frac{q_t}{p_t + q_t} \quad \text{with} \quad p_t + q_t = \frac{A(\psi_t) - \iota_t}{\rho}.$$

Notice that the problem is separable across time points, and results in the identical values of η_t, ψ_t and q_t for all times t.[15]

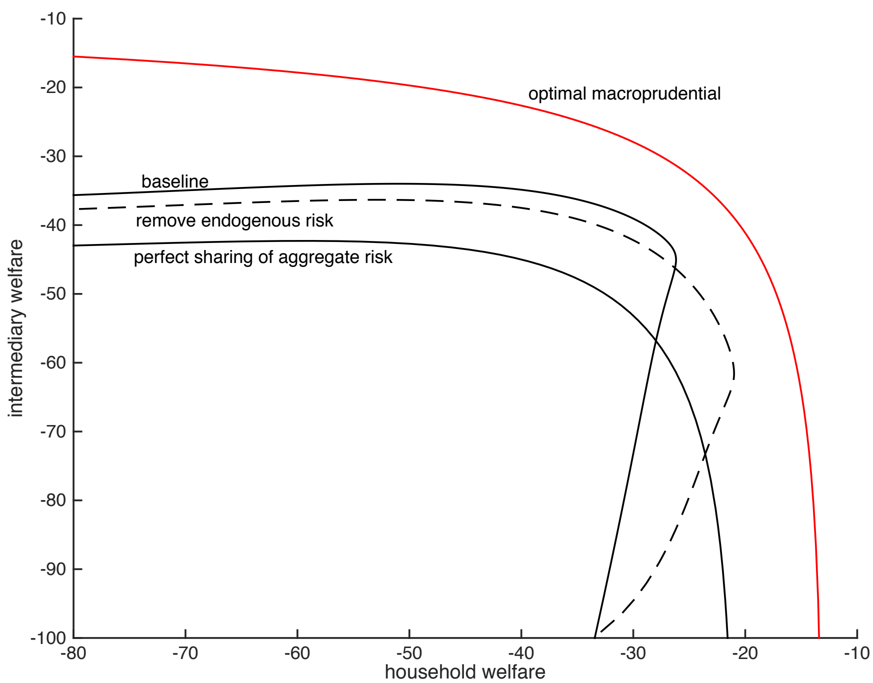

Figure 11: Monetary and Macruprudential Policy.

This policy can be implemented by imposing these portfolio weight constraints on households, in addition to taxes/subsidies on goods a and b to achieve an appropriate allocation ψ_t. The regulator does not need to control the households' choice between sectors a and b or the market for aggregate risk. Figure 11 illustrates the Pareto frontier for intermediaries' and households' welfare that can be obtained under optimal macroprudential policy. Welfare is significantly improved relative to Figure 10 that illustrates monetary policy alone.

[15]The reader may be wondering why η_t is constant over time, even though households face idiosyncratic risk while intermediaries do not. Since average marginal utility of households rises relative to that of intermediaries, due to idiosyncratic risk, it is tempting to conjecture that the planner must raise the households' wealth share over time. However, notice that any redistribution towards households, such as that implemented by raising the households' share of return, has to be proportional to individual households' wealth. As a result, households with higher marginal utility receive a smaller share of wealth redistribution, and it is this effect that prevents redistribution of this sort from raising welfare.

We obtain an extreme policy which is not very realistic, but, nevertheless, the exercise leads to important takeaways. Monetary policy can alter the risk profile of assets and provide natural hedges in incomplete markets, but it cannot control risk taking/risk premia separately from risk itself. In our model, while monetary policy improves the sharing of aggregate risk, it stimulates the price of capital relative to money so that households are overexposed to idiosyncratic risk. As intermediaries are less likely to become undercapitalized, they provide better insurance to households to offset idiosyncratic risk as the supply of inside money rises. It seems like households should become better-insured, but they are not as the value of outside money falls. Macroprudential policy, which limits the households' portfolio weights on capital is welfare improving, because it reduces the households' exposure to idiosyncratic risk. The cost of this insurance is investment distortion, as we discussed in the Section 3 without intermediaries.

Going beyond our model, we can make the following more realistic interpretations. Monetary policy can provide some insurance to the economy, but it is a crude redistributive tool that can only target some of the aggregate risks. Individual portfolio choices are completely endogenous with monetary policy alone, and loose monetary policy can easily be accompanied by the excessive leverage, bubbles in prices in some asset classes, and overexposure to risk of these assets on individual level. This can create motivation for macroprudential tools that control households portfolio choices, such as loan-to-value ratios for household borrowing against some of the assets. These tools push down the prices of these assets, and reduce idiosyncratic risk exposure at individual level.

6 Conclusion

In our economy household entrepreneurs and intermediaries make investment decisions. Household entrepreneurs can invest only in a single real production technology at a time. Intermediaries are "diversifiers" as they can provide risky funding across a number of household entrepreneurs. Intermediaries scale up their activity by issuing demand deposits, *inside money*, held by household entrepreneurs. In addition, households and intermediaries can hold *outside* money provided by the government. Intermediaries are leveraged and assume liquidity mismatch. Intermediaries' assets are long-dated and have low market liquidity - after an adverse shock the price can drop - while their debt financing is redeemable, i.e. short-term. Endogenous risk emerges through an amplification mechanism in form of two spirals. First, the liquidity spiral: a shock to intermediaries causes them to shrink balance sheets and "fire sale some of their assets." This depresses the price of their assets which induces further fire-sales and so on. Second, the disinflationary spiral: as intermediaries shrink their balance sheet, they also create less inside money; such a shock leads to a rising demand for outside money, i.e. disinflation. This disinflationary spiral amplifies shocks, as it hurts borrowers who owe nominal debt. It works on the liabilities side of the intermediary balance sheets, while the liquidity spiral that hurts the price of capital works on the asset side. Importantly, intermediaries' response to shrink their balance sheet, i.e. to act microprodent, leads to higher endogenous risk in the economy, i.e. is macro-imprudent. We coined this inconsistency as "Paradox of Prudence", as it resembles Keynes' Paradox of thrift, just in terms of risk instead of savings.

Monetary policy can mitigate the adverse effects due to both spirals in a world with (default-free) long-term government bonds. Conventional monetary policy changes the path of interest rate earned on short-term "money" and consequently impacts the relative value of long-term government bonds and money. For example, interest rate cuts in downturns, which are expected to persist for a while, enable intermediaries to refinance their long-bond holding more cheaply. This recapitalizes institutions that hold these assets and also increases the (nominal) supply of the safe asset. This reduces endogenous risk, and also enhances competition among banks, which lowers their rents. Of course, any policy that provides insurance against downturns could potentially create moral hazard. While intermediaries do take on higher leverage in the presence of monetary policy, moral hazard is nevertheless limited. It is not a policy that saves the weakest institutions, thus creating most perverse incentives ex-ante. Rather, "stealth recapitalization" through a persistent interest rate cut recapitalizes specifically the institutions that took precaution to hold long-term bonds as a hedge. The finding that moral hazard is limited might change if one were to include intermediaries with negative net worth. Including such zombie banks is one fruitful direction to push this line of research further.

Combining macruprudential policy with monetary policy can achieve strictly higher welfare. The reason is that, while monetary policy can transfer risk between intermediaries and households, risk-taking (i.e. portfolio choices) and risk premia are still endogenous. Macro-prudential instruments allow policy makers to also impact risk taking. Already in an economy without intermediaries, macroprudential policy can improve welfare of households by controlling the externality related to money holdings. When households demand too much money, the price of capital falls, leading to underinvestment. With intermediaries, macroprudential policy also affects intermediary leverage and their earnings in equilibrium.

We have not attempted to disentangle these effects, but rather we considered an extreme problem of optimal macroprudential policy with perfect sharing of aggregate risk. This gives us a theoretical upper bound on welfare improvement through a cocktail of policies. Our analysis shows that the potential welfare improvement from a combination of policies can be significant. In other words, there are large potential welfare gains from controlling risk and risk-taking separately.

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A Computing Equilibria: Numerical Details

In this section, we describe computation of equilibria in our model. Essentially, equilibrium characterization reduces to a single second-order differential equation for $\vartheta(\eta)$, which we refer to as the "return equation," but with a number of variables that have to satisfy a separate set of equations, which we call "asset allocation" equations. In this appendix, we first describe these equations without and with monetary policy, second, we break them down to simple algebra, and last, we provide some essential details of numerics.

Without Policy. Let us collect the *asset allocation* equations - for seven variables p, q, ψ, x, x^a, x^b and χ, we have six equations from (2.9), (2.12), (2.13), (2.14), (2.15), (2.16) and an additional 7th equation

$$\frac{A^b(\psi) - A^a(\psi)}{q_t} = \tag{A.1}$$

$$(1-\chi_t)x_t^b(|\nu_t^b|^2 + (\tilde{\sigma}^b)^2) + \chi_t x_t |\nu_t^b|^2 - x_t^a(|\nu_t^a|^2 + (\tilde{\sigma}^a)^2) + (\sigma^b 1^b - \sigma^a 1^a)^T \left(\frac{\sigma^\vartheta}{1-\vartheta} + \sigma^K\right),$$

obtained by subtracting (2.10) from (2.11). These equations contain expressions that depend on $\vartheta(\eta)$ and $\vartheta'(\eta)$,

$$\nu^a = \psi(\sigma^a 1^a - \sigma^b 1^b) \underbrace{-\frac{\sigma^\vartheta}{1-\vartheta}}_{\sigma^q - \sigma^p} \quad \text{and} \quad \nu^b = (1-\psi)(\sigma^b 1^b - \sigma^a 1^a) \underbrace{-\frac{\sigma^\vartheta}{1-\vartheta}}_{\sigma^q - \sigma^p},$$

where, using Ito's lemma and (2.17),

$$\sigma^\theta = \frac{\vartheta'(\eta)}{\vartheta(\eta)} \eta \underbrace{\left(x\left((1-\psi)(\sigma^b 1^b - \sigma^a 1^a) - \frac{\sigma^\vartheta}{1-\vartheta}\right) + \sigma^\vartheta\right)}_{\sigma^\eta} \quad \Rightarrow$$

$$\sigma^\theta = \frac{\vartheta'(\eta)}{\vartheta(\eta)} \eta \underbrace{\frac{x(1-\psi)(\sigma^b 1^b - \sigma^a 1^a)}{1 + \frac{\vartheta'(\eta)}{\vartheta(\eta)}(\psi\chi - \eta)}}_{\sigma^\eta}. \tag{A.2}$$

This expression captures endogenous risk, given the sensitivity of the price of money $\vartheta'(\eta)/\vartheta$ relative to capital.

We obtain a convenient *return equation* by adding (2.10) and (2.11) with weights $1-\psi$

and ψ, to obtain[16]

$$\rho = (1-\eta_t)(x_t^a)^2(|\nu_t^a|^2 + (\tilde{\sigma}^a)^2) + \eta_t x_t^2 |\nu_t^b|^2 + \mu^\vartheta - |\sigma^\vartheta|^2, \tag{A.3}$$

where we used the identity $\mu^\vartheta - |\sigma^\vartheta|^2 = (1-\vartheta)(\mu^p - \mu^q + (\sigma^q - \sigma^p)\sigma^p)$. Equation (A.3) determines μ^ϑ, which we can use to find $\vartheta''(\eta)$ using Ito's lemma,

$$\vartheta''(\eta) = \frac{\mu^\vartheta \vartheta(\eta) - \vartheta'(\eta)\mu_t^\eta \eta}{\eta^2|\sigma^\eta|^2/2}. \tag{A.4}$$

This equation can be solved as a second-order ordinary differential equation using the "shooting method" - by starting at $\eta_0 \sim 0$ near the autarky solution (i.e. the solution without intermediaries), and choosing an initial slope $\vartheta'(\eta_0)$ such that the resulting solution is non-explosive and converges as $\eta \to 1$.

An alternative "iterative" method of finding the equilibrium involves a partial differential equation in time, solved backwards from a terminal condition $\vartheta(\eta, T)$. The iterative method is attractive because it resembles the familiar discrete-time value function iteration, because it readily extends to problems that contain a system of second-order differential equations for several functions, and it avoids the analysis of explosive solutions that arise when employing the shooting method.

Adding the time dimension, using Ito's lemma, we obtain

$$\vartheta_t(\eta,t) + \vartheta_\eta(\eta,t)\mu_t^\eta \eta + \frac{\eta^2|\sigma^\eta|^2}{2}\vartheta_{\eta\eta}(\eta,t) = \mu^\vartheta \vartheta(\eta,t), \tag{A.5}$$

In this equation, the time derivative $\vartheta_t(\eta,t)$ is the key unknown, and we solve for $\vartheta(\eta,t)$ through a parabolic equation backward time on $[0,T]$.

With Monetary Policy. Here we describe what happens when only intermediaries hold bonds (either because only they can hold bonds, or because households in sector b, whose aggregate risk exposure is similar to but less than that of intermediaries, have lower need

[16]We have

$$\frac{(1-\psi)A^a(\psi) + \psi A^b(\psi) - \iota}{q} + \mu_t^q - \mu_t^p + (\sigma_t^q - \sigma_t^p)^T \sigma^K =$$

$$\psi(1-\chi_t)((\nu_t^b)^T \sigma_t^{Nb} + x_t^b(\tilde{\sigma}^b)^2) + \psi\chi_t(\nu_t^b)^T\sigma_t^N + (1-\psi)((\nu_t^a)^T\sigma_t^{Na} + x_t^a(\tilde{\sigma}^a)^2) =$$

$$\frac{\psi(1-\chi_t)}{x_t^b}(x_t^b)^2(|\nu_t^b|^2 + (\tilde{\sigma}^b)^2) + \frac{\psi\chi_t}{x_t}x_t^2|\nu_t^b|^2 + \frac{1-\psi}{x_t^a}(x_t^a)^2(|\nu_t^a|^2 + (\tilde{\sigma}^a)^2) + (\sigma^q - \sigma^p)^T\sigma_t^M$$

Hence, using (2.9), (2.13), (2.14), (2.15) and (2.16), we obtain (A.3).

for insurance than intermediaries).[17]

With policy we are still solving a second-order "return equation" for $\vartheta(\eta)$ together with a set of "asset allocation equations" for 7 additional variables, p, q, ψ, x, x^a, x^b and χ. Policy affects the law of motion (5.5) of η_t, so we have

$$\sigma_t^\eta = x_t \nu_t^b + \sigma_t^\vartheta + (1-\eta_t)\frac{\vartheta_t}{\eta_t}\frac{b_t}{p_t}\sigma_t^B, \quad \text{where} \quad \nu_t^b = (1-\psi)(\sigma^b 1^b - \sigma^a 1^a)\underbrace{-\frac{\sigma_t^\vartheta}{1-\vartheta} + \frac{b_t}{p_t}\sigma_t^B}_{\sigma^q - \sigma^p + \frac{b_t}{p_t}\sigma_t^B}.$$

Hence,

$$\sigma_t^\eta = x_t(1-\psi_t)(\sigma^b 1^b - \sigma^a 1^a) - \frac{\vartheta'(\eta)}{\vartheta(\eta)}(\psi_t\chi_t - \eta_t)\sigma^\eta + \frac{b_t}{p_t}\frac{B'(\eta)}{B(\eta)}\left(x_t\eta_t + (1-\eta_t)\vartheta_t\right)\sigma^\eta \quad \Rightarrow$$

$$\sigma_t^\eta = \frac{x_t(1-\psi_t)(\sigma^b 1^b - \sigma^a 1^a)}{1 + \frac{\vartheta'(\eta)}{\vartheta(\eta)}(\psi_t\chi_t - \eta_t) - \frac{b_t}{p_t}\frac{B'(\eta)}{B(\eta)}\left(x_t\eta_t + (1-\eta_t)\vartheta_t\right)}, \quad \sigma_t^\theta = \frac{\vartheta'(\eta)}{\vartheta(\eta)}\eta\sigma_t^\eta, \quad \sigma_t^B = \frac{B'(\eta)}{B(\eta)}\eta\sigma_t^\eta.$$

Otherwise, equations (2.9) through (2.16) remain valid, except that σ_t^M and σ_t^N are now given by the expressions

$$\sigma_t^M = \sigma_t^p + \sigma_t^K - \frac{b_t}{p_t}\sigma_t^B \quad \text{and} \quad \sigma_t^N = x_t\nu_t^b + x_t^B\sigma_t^B + \sigma_t^M. \tag{A.6}$$

Notice also that the form of σ_t^M affects $\nu^a = \sigma^a 1^a + \sigma_t^q - \sigma_t^M$ and $\nu^b = \sigma^b 1^b + \sigma_t^q - \sigma_t^M$. Accounting for this, equations (A.1) and (A.3) have to be modified to

$$\frac{A^b(\psi_t) - A^a(\psi_t)}{q_t} = (1-\chi_t)x_t^b(|\nu_t^b|^2 + (\tilde\sigma^b)^2) + \chi_t(\nu_t^b)^T(x_t\nu_t^b + x_t^B\sigma_t^B) - x_t^a(|\nu_t^a|^2 + (\tilde\sigma^a)^2)$$
$$+(\sigma^b 1^b - \sigma^a 1^a)^T\left(\frac{\sigma_t^\vartheta}{1-\vartheta} + \sigma_t^K - \frac{b_t}{p_t}\sigma_t^B\right)$$

and

$$\rho + (1-\vartheta)\frac{b_t}{p_t}(\sigma_t^B)^T\left(x_t\nu_t^b + x_t^B\sigma_t^B - \frac{\sigma_t^\vartheta}{1-\vartheta}\right) =$$

[17] If households in sector b want to hold bonds as well, we have modify the procedure slightly to solve for the households' bond portfolios.

$$\eta_t x_t(\nu_t^b)^T(x_t\nu_t^b + x_t^B\sigma_t^B) + (1-\eta_t)(x_t^a)^2(|\nu_t^a|^2 + (\tilde{\sigma}^a)^2) + \mu_t^\vartheta - |\sigma_t^\vartheta|^2,$$

where we used the following expression for the expected return on money

$$E[dr_t^M]/dt = \left(\Phi(\iota) - \delta + \mu_t^p + (\sigma_t^p)^T\sigma_t^K\right)dt - \frac{b_t}{p_t}\underbrace{\left(\frac{1}{B_t} - i_t + \mu_t^B + (\sigma_t^B)^T\sigma_t^M\right)}_{(\sigma_t^B)^T\sigma_t^N}$$

The Algebra of Asset Allocation Equations. Here we simplify the equations further into a convenient numerical form. First, letting

$$X_1 = 1 - \frac{\vartheta'(\eta)}{\vartheta(\eta)}\eta_t - \frac{b_t}{p_t}\frac{B'(\eta)}{B(\eta)}(1-\eta_t)\vartheta_t, \quad X_3 = \eta_t\left(\frac{b_t}{p_t}\frac{B'(\eta)}{B(\eta)} - \frac{\vartheta'(\eta)}{\vartheta(\eta)(1-\vartheta(\eta))}\right),$$

we have

$$\sigma_t^\eta = \frac{(1-\psi_t)x_t}{X_1 - x_tX_3}(\sigma^b 1^b - \sigma^a 1^a),$$

$$\nu_t^b = y(\sigma^b 1^b - \sigma^a 1^a), \quad \nu_t^a = (y-1)(\sigma^b 1^b - \sigma^a 1^a) \quad \text{where } y = \frac{(1-\psi)X_1}{X_1 - x_tX_3}.$$

Also, letting

$$X_2 = 1 - \frac{\vartheta'(\eta)}{\vartheta(\eta)}\eta_t + \frac{b_t}{p_t}\frac{B'(\eta)}{B(\eta)}\eta_t\vartheta_t = X_1 + \frac{b_t}{p_t}\frac{B'(\eta)}{B(\eta)}\vartheta_t,$$

we have

$$x_t\nu_t^b + x_t^B\sigma_t^B = x_t y\frac{X_2}{X_1}(\sigma^b 1^b - \sigma^a 1^a).$$

Given these definitions, we can reduce the *asset allocation* equations to the following five equations for $z = (y, \psi, x, x^b, x^b/x^a)$.

$$X_1(1-\psi) = y(X_1 - xX_3), \quad \left(\frac{x^b}{x^a}\right)^2(y^2\sigma^2 + \tilde{\sigma}_b^2) = ((y-1)^2\sigma^2 + \tilde{\sigma}_a^2),$$

$$\frac{(1-\eta)x^b + x\eta}{1-\vartheta} - \psi - (1-\psi)\frac{x^b}{x^a} = 0$$

$$\underbrace{\frac{-A'(\psi)}{\kappa A(\psi)+1}\frac{\kappa\rho+1-\vartheta}{1-\vartheta}}_{\frac{A^b(\psi)-A^a(\psi)}{q}} = \left(1-\bar{\chi}-\frac{x^b}{x^a}\right)x^b(y^2\sigma^2+\tilde{\sigma}_b^2)+\bar{\chi}xy^2\frac{X_2}{X_1}\sigma^2+\sigma_b^2-y\sigma^2$$

$$\text{and} \qquad x = \min\left(x^b\left(1+\frac{\tilde{\sigma}_b^2}{y^2\sigma^2}\right)\frac{X_1}{X_2}, \frac{(1-\vartheta)\psi\bar{\chi}}{\eta}\right), \tag{A.7}$$

where $\sigma^2 = \sigma_a^2 + \sigma_b^2$. Notice that we have added variable y and removed q, p and χ_t from the set described above. Denote this system by $F(z) = 0$.

We can write this set of five equations as $F_1(z) = 0$ in the region where the equity issuance constraint is binding, i.e. $\chi_t = \bar{\chi}$, and $F_2(z) = 0$ in the region where $\chi_t < \bar{\chi}$. If z solves the equations approximately, then we can find a nearly exact solution using the Newton method. That is, given z, compare values on the right-hand side of (A.7) to determine if the equity issuance constraint is binding. If it binds, then $z - \left(\frac{\partial F}{\partial z}\right)^{-1} F(z)$ approximates the solution with error of $O((z-z^*)^2)$, where z^* is the true solution. This procedure of solving the system $F(z) = 0$ is useful when solving for $\vartheta(\eta)$ through the shooting method or the iterative method, because once we have a solution at (η, t) we can use it to find the solution at $(\eta+\epsilon, t)$ or at $(\eta, t-\epsilon)$. Typically, one step of the Newton method is sufficient because we consider the problem at a nearby point in space or time.

The PDE for $\vartheta(\eta, t)$. Given the "allocation vector" z, the time derivative $\vartheta_t(\eta, t)$ can be found from equation (A.5), where

$$|\sigma^\eta| = \frac{(1-\psi)x\sigma}{X_1 - xX_3}, \qquad \mu^\eta = (1-\eta)\left(x^2y^2\frac{X_2^2}{X_1^2}\sigma^2 - (x^b)^2(y^2\sigma^2+\tilde{\sigma}_b^2)\right) + (1-X_2)|\sigma^\eta|^2,$$

$$|\sigma^\vartheta| = \frac{\vartheta'(\eta)}{\vartheta(\eta)}\eta|\sigma^\eta| \quad \text{and}$$

$$\mu_t^\vartheta = \rho + \frac{X_2 - X_1}{\vartheta}\eta|\sigma^\eta|\left((1-\vartheta)xy\frac{X_2}{X_1}\sigma - |\sigma^\vartheta|\right) - \eta x^2y^2\frac{X_2}{X_1}\sigma^2 - (1-\eta)(x^b)^2(y^2\sigma^2+\tilde{\sigma}_b^2) + |\sigma^\vartheta|^2.$$

For the purposes of numeric stability in (A.5) left derivative of $\vartheta(\eta)$ must be used if $\mu^\eta < 0$, and right derivative if $\mu^\eta > 0$.

Numerical Implementation. We solve the PDE (A.5) backwards in time using the finite difference method, until convergence. For date T, we set ϑ at $\eta = 0$ according to the autarky solution of (1) and at $\eta = 1$ according to the asymptotic solution realized when the intermediary sector overwhelms the economy. We then interpolate ϑ linearly on a grid on $[0, 1]$. We choose an unevenly-spaced grid of $N + 1$ points, with $\eta(n) = 3n^2/N^2 - 2n^3/N^3$, for $n = 0, \ldots N$. The spacing in this grid is on the order of $\min(\eta, 1 - \eta)$, i.e. spacing becomes finer near 0 and 1 reflecting the decline in the equilibrium volatility of η towards these endpoints.

Once we have the grid $\eta(n)$ and the terminal condition $\vartheta(\eta(n), t)$, we compute $\vartheta_t(\eta(n), t)$ using (A.5) for each n. In order to do that, we need to compute $z(\eta(n), t)$. For $t = T$, we find $z(\eta(n), T)$ using one step of the Newton method from the guess $z(\eta(n-1), T)$. The initial solution $z(\eta(0), T)$ is the autarky solution of (1), with $z = (1 - \psi, \psi, x, x^b, x^b/x^a)$, with $x = x^b(1 + \tilde{\sigma}_b^2/((1-\psi)^2\sigma^2))$. For $t < T$, we find $z(\eta(n), t)$ using one step of the Newton method from the guess $z(\eta(n), t + \epsilon)$.

We solve the PDE using the Euler method, by setting $\vartheta(\eta(n), t - \epsilon) = \vartheta(\eta(n), t) - \epsilon\vartheta_t(\eta(n), t)$ for time step ϵ, which has to be chosen to be sufficiently small for the equation to be stable. We do this for $n = 1 \ldots N - 1$, keeping the endpoints $n = 0, N$ fixed. As mentioned earlier, the Euler method is not as precise as higher-order methods for solving systems of ODEs, but it is transparent and easy to implement numerically. We chose the Euler method because we are not looking for the precise time solution, but rather for the stationary equilibrium - the fixed point at which all time derivatives are 0. For this goal, the Euler method is as good as any other method.

We need to evaluate derivatives of $\vartheta(\eta, t)$ with respect to η numerically to implement this method. The left, right and centered derivatives of ϑ, and the second derivative, are given by

$$\vartheta_\eta^L(\eta(n), t) = \frac{\vartheta(\eta(n), t) - \vartheta(\eta(n-1), t)}{\eta(n) - \eta(n-1)}, \quad \vartheta_\eta^R(\eta(n), t) = \frac{\vartheta(\eta(n+1), t) - \vartheta(\eta(n), t)}{\eta(n+1) - \eta(n)},$$

$$\vartheta_\eta^C(\eta(n), t) = \frac{\vartheta_\eta^R(\eta(n), t) + \vartheta_\eta^L(\eta(n), t)}{2}, \quad \text{and} \quad \vartheta_{\eta\eta}(\eta(n), t) = 2\frac{\vartheta_\eta^R(\eta(n), t) - \vartheta_\eta^L(\eta(n), t)}{\eta(n+1) - \eta(n-1)}.$$

We use centered derivative of ϑ to evaluate X_1, X_2 and X_3, and appropriate directional derivative in (A.5).

B Proofs

Proof of Proposition 13. Consider the law of motion of net worth

$$\frac{dn_t}{n_t} = \mu_t^n\, dt + \sigma_t^n\, dZ_t = dr_t^M - \rho dt + \begin{cases} x_t^a(\nu_t^a)^T((x_t^a\nu_t^a + \sigma_t^M)\, dt + dZ_t) + x_t^a\tilde{\sigma}^a(x_t^a\tilde{\sigma}^a\, dt + d\tilde{Z}_t) \\ x_t^b(\nu_t^b)^T((x_t^b\nu_t^b + \sigma_t^M)\, dt + dZ_t) + x_t^b\tilde{\sigma}^b(x_t^b\tilde{\sigma}^b\, dt + d\tilde{Z}_t), \end{cases}$$

depending on whether the household employs technology a or b.

According to (3.6), the household gets the same utility from any choice over these two technologies if and only if $\mu_t^n - |\sigma_t^n|^2/2$ is the same for both technologies. For technology a, this is

$$\frac{E[dr_t^M]}{dt} - \rho + (x_t^a)^2(|\nu_t^a| + (\tilde{\sigma}^a)^2) + x_t^a(\nu_t^a)^T\sigma_t^M - \frac{|x_t^a\nu_t^a + \sigma_t^M|^2 + (x_t^a\tilde{\sigma}^a)^2}{2} =$$

$$\frac{E[dr_t^M]}{dt} - \rho + \frac{(x_t^a)^2(|\nu_t^a| + (\tilde{\sigma}^a)^2) - |\sigma_t^M|^2}{2}.$$

Equating this for technologies a and b, we obtain the indifference condition (2.13). □

Lemma 1. *Suppose that* $\eta = 0$*, i.e. there are no intermediaries. The equilibrium is characterized by a single equation for the allocation* ψ *of capital to technology* b

$$\frac{A^a(\psi) - A^b(\psi)}{q} = \frac{\rho}{x^a} - \frac{\rho}{x^b} + (1-\psi)(\sigma^a)^2 - \psi(\sigma^b)^2, \tag{B.1}$$

with the remaining quantities expressed as

$$x^a = \sqrt{\frac{\rho}{\psi^2((\sigma^a)^2 + (\sigma^b)^2) + (\tilde{\sigma}^a)^2}}, \quad x^b = \sqrt{\frac{\rho}{(1-\psi)^2((\sigma^a)^2 + (\sigma^b)^2) + (\tilde{\sigma}^b)^2}}, \tag{B.2}$$

$$1 - \vartheta = \frac{x^a x^b}{(1-\psi)x^b + \psi x^a} \quad \frac{\rho q}{1-\vartheta} = A(\psi) - \iota(q) \quad \text{and} \quad p = \frac{\vartheta}{1-\vartheta}q. \tag{B.3}$$

Household welfare in autarky is characterized by $\log(\rho n_t)/\rho + U^H(0)$*, where*

$$U^H(0) = \frac{\Phi(\iota) - \delta}{\rho^2} - \frac{\psi^2\sigma_B^2 + (1-\psi)^2\sigma_A^2}{2\rho^2} - \frac{1}{2\rho}. \tag{B.4}$$

Proof. The aggregate risk of capital is $\sigma^K\, dZ_t = (1-\psi)\sigma^a 1^a\, dZ_t^a + \psi\sigma^b 1^b\, dZ_t^b$, incremental aggregate risks from exposures to technologies a and b are

$$\nu_t^a = \psi(1^a\sigma^a - 1^b\sigma^b)\,dZ_t, \quad \nu_t^b = (1-\psi)(1^b\sigma^b - 1^a\sigma^a)\,dZ_t,$$

and the risk exposure of households in sectors a and b are

$$x^a\nu_t^a\,dZ_t + \sigma^K\,dZ_t + x^a\tilde{\sigma}^a\,d\tilde{Z}_t \quad \text{and} \quad x^b\nu_t^b\,dZ_t + \sigma^K\,dZ_t + x^b\tilde{\sigma}^b\,d\tilde{Z}_t,$$

respectively. Hence, household indifference condition is

$$X \equiv (x^a)^2(\underbrace{\psi^2((\sigma^a)^2+(\sigma^b)^2)}_{|\nu_t^a|^2}+(\tilde{\sigma}^a)^2) = (x^b)^2(\underbrace{(1-\psi)^2((\sigma^a)^2+(\sigma^b)^2)}_{|\nu_t^b|^2}+(\tilde{\sigma}^b)^2). \tag{B.5}$$

The asset-pricing conditions for capital employed in sectors a and b are

$$\frac{A^a(\psi)-\iota}{q} = x^a(|\nu_t^a|^2+(\tilde{\sigma}^a)^2) + \psi((1-\psi)(\sigma^a)^2 - \psi(\sigma^b)^2) \quad \text{and}$$

$$\frac{A^b(\psi)-\iota}{q} = x^b(|\nu_t^b|^2+(\tilde{\sigma}^b)^2) + (1-\psi)(\psi(\sigma^b)^2 - (1-\psi)(\sigma^a)^2).$$

Adding up these two equations with coefficients $1-\psi$ and ψ, and using the market-clearing condition for capital, we obtain

$$\frac{\rho}{1-\vartheta} = \underbrace{\left(\frac{1-\psi}{x^a}+\frac{\psi}{x^b}\right)}_{1/(1-\vartheta)\text{ by (2.16)}} X \quad \Rightarrow \quad X = \rho.$$

This, together with (B.5), implies (B.2).

Equations in (B.3) follow from (2.16), market clearing condition for output and the definition of ϑ.

Finally, the difference between the asset-pricing conditions is

$$\frac{A^a(\psi)-A^b(\psi)}{q} = \underbrace{x^a(|\nu_t^a|^2+(\tilde{\sigma}^a)^2)}_{\rho/x^a} - \underbrace{x^b(|\nu_t^b|^2+(\tilde{\sigma}^b)^2)}_{\rho/x^b} + (1-\psi)(\sigma^a)^2 - \psi(\sigma^b)^2.$$

This yields (B.1).

To characterize household welfare, notice that by (4.3),

$$U^R(\eta_t) = \frac{\Phi(\iota) - \delta}{\rho^2} - \frac{\psi^2 \sigma_B^2 + (1-\psi)^2 \sigma_A^2}{2\rho^2},$$

since $|\sigma^K|^2 = \psi^2\sigma_B^2 + (1-\psi)^2\sigma_A^2$. Furthermore, by (4.5), $U^H(\eta_t) = U^R(\eta_t) - X/(2\rho^2)$, which implies (B.4). □

Proof of Proposition 3. Notice that

$$n_s = n_t \exp\left(\int_t^s \left(\mu^n_{s'} - \frac{|\sigma^n_{s'}|^2}{2}\right) ds' + \int_t^s \sigma^n_{s'}\, dZ_t\right),$$

since Ito's lemma implies that then process n_s satisfies (3.5) as required.

Hence,

$$E_t[\log(\rho n_s)] = \log(\rho n_t) + E_t\left[\int_t^s \left(\mu^n_{s'} - \frac{|\sigma^n_{s'}|^2}{2}\right) ds'\right].$$

Integrating over $[t, \infty)$ and discounting, we obtain

$$E_t\left[\int_t^\infty e^{-\rho(s-t)} \log(\rho n_s)\, ds\right] = \frac{\log(\rho n_t)}{\rho} + E_t\left[\int_t^\infty e^{-\rho(s-t)} \int_t^s \left(\mu^n_{s'} - \frac{|\sigma^n_{s'}|^2}{2}\right) ds'\, ds\right],$$

which yields (3.6) after changing the order of integration. □

Proof of Proposition 5. Let us normalize $K_0 = 1$. Consider an economy, in which households are required to allocate fraction $\vartheta \in [0, 1)$ of their wealth to money. Then, from the market-clearing condition for consumption goods, if $\Phi(\iota) = \log(\kappa\iota + 1)/\kappa$, then

$$\bar{A} - \iota(q) = \rho \underbrace{(p+q)}_{q/(1-\vartheta)} \quad \Rightarrow \quad q = \frac{(\kappa\bar{A} + 1)(1-\vartheta)}{\kappa\rho + 1 - \vartheta} \quad \text{and} \quad \Phi(\iota) = \frac{\log(q)}{\kappa}.$$

Then (3.7) implies that, given policy,

$$\mu^n = \Phi(\iota) - \delta, \quad |\sigma^n|^2 = (1-\vartheta)^2\hat{\sigma}^2 + \bar{\sigma}^2,$$

hence welfare (3.6) is

$$\frac{\log(\rho(p+q))}{\rho} + \frac{\mu^n - |\sigma^n|^2/2}{\rho^2} = \frac{\log(\rho) + \log(q/(1-\vartheta))}{\rho} + \frac{\log(q)/\kappa - \delta - (1-\vartheta)^2\hat{\sigma}^2/2 - \bar{\sigma}^2/2}{\rho^2}$$

$$= \frac{\log(\rho)}{\rho} - \frac{\delta + \bar{\sigma}^2/2}{\rho^2} + \frac{\kappa\rho + 1}{\kappa\rho}\left(\frac{\log(\kappa\bar{A} + 1)}{\rho} - \frac{\log(\kappa\rho + 1 - \vartheta)}{\rho}\right) + \frac{\log(1-\vartheta)}{\kappa\rho^2} - \frac{(1-\vartheta)^2\hat{\sigma}^2}{2\rho^2}$$

Let us show that welfare in the equilibrium with money is greater than in that without money. In the equilibrium with money $1 - \vartheta = \sqrt{\rho}/\hat{\sigma}$, so we need to show that

$$-\frac{\kappa\rho + 1}{\kappa\rho}\log(\kappa\rho + \sqrt{\rho}/\hat{\sigma}) + \frac{1}{\kappa\rho}\log(\sqrt{\rho}/\hat{\sigma}) - \frac{1}{2} \geq -\frac{\kappa\rho + 1}{\kappa\rho}\log(\kappa\rho + 1) - \frac{\hat{\sigma}^2}{2\rho} \quad \Leftrightarrow$$

$$-\frac{\kappa\rho + 1}{\kappa\rho}\log(\kappa\rho x + 1) + \log x + \frac{x^2}{2} \geq -\frac{\kappa\rho + 1}{\kappa\rho}\log(\kappa\rho + 1) + \frac{1}{2} \tag{B.6}$$

where $x = \hat{\sigma}/\sqrt{\rho} > 1$. If we set $x = 1$, the two sides are equal. Differentiating the left-hand side with respect to x we obtain

$$\frac{1-x}{(\kappa\rho x + 1)x} + x \geq \frac{1-x}{x} + x = \frac{1 - x + x^2}{x} > 0.$$

Hence (B.6) holds for all $x > 1$, i.e. the equilibrium with money is strictly better.

Now, consider the optimal policy. Differentiating welfare with respect to ϑ we get

$$\frac{\kappa\rho + 1}{\kappa\rho^2}\frac{1}{\kappa\rho + 1 - \vartheta} - \frac{1}{\kappa\rho^2(1-\vartheta)} + \frac{(1-\vartheta)\hat{\sigma}^2}{\rho^2} =$$

$$-\frac{1}{\rho}\frac{\vartheta}{(\kappa\rho + 1 - \vartheta)(1-\vartheta)} + \frac{(1-\vartheta)\hat{\sigma}^2}{\rho^2} = \frac{1}{\rho(1-\vartheta)}\left(\frac{-\vartheta}{\kappa\rho + 1 - \vartheta} + \frac{(\vartheta - 1)^2\hat{\sigma}^2}{\rho}\right),$$

where the term in parentheses is increasing in ϑ. For the equilibrium level of $\vartheta = 1 - \sqrt{\rho}/\hat{\sigma}$, this term becomes

$$\frac{\sqrt{\rho}/\hat{\sigma} - 1}{\kappa\rho + \sqrt{\rho}/\hat{\sigma}} + 1 = \frac{2\sqrt{\rho}/\hat{\sigma} - 1 + \kappa\rho}{\kappa\rho + \sqrt{\rho}/\hat{\sigma}},$$

positive if and only if $2\sqrt{\rho}/\hat{\sigma} > 1 - \kappa\rho$. Thus, the welfare-maximizing policy raises ϑ over the equilibrium level if and only if condition (3.9) holds. □

Proof of Proposition 6. By (3.6), the welfare of an agent who consumes ρK_t, where K_t is aggregate capital, is given by

$$E_t\left[\int_t^\infty e^{-\rho(s-t)}\log(\rho K_s)\,ds\right] = \frac{\log(\rho K_t)}{\rho} + E_t\left[\int_t^\infty e^{-\rho(s-t)}\left(\frac{\Phi(\iota_s) - \delta}{\rho} - \frac{|\sigma_s^K|^2}{2\rho}\right)ds\right]. \tag{B.7}$$

Since we are dealing with an agent who consumes $\log(\rho(p_s + q_s)K_s) = \log(\rho K_s) + \log(p_s + q_s)$, with net worth $N_t = (p_t + q_t)K_t$, the welfare of this agent can be found by adding

$$E\left[\int_t^\infty e^{-\rho(s-t)} \log(p_s + q_s)\, ds\right]$$

to (B.7) and noting that $\log(\rho K_t)/\rho = \log(\rho n_t)/\rho - \log(p_t + q_t)/\rho$. This implies that the agent's utility is $\log(\rho n_t)/\rho + U^R(\eta_t)$, with $U^R(\eta_t)$ is given by (4.3). □

Proof of Proposition 7. Since intermediary with net worth $n_t^I = \eta_t(p_t + q_t)K_t$ consumes

$$\log(\rho\eta_s(p_s + q_s)K_s) = \log(\eta_s) + \log(\rho(p_s + q_s)K_s),$$

receiving the same utility flow as a representative household plus $\log(\eta_s)$, we find the welfare of an intermediary from (4.3) to be

$$\underbrace{\frac{\log(\rho(p_t + q_t)K_t)}{\rho}}_{\log(\rho n_t^I)/\rho - \log(\eta_t)/\rho} + U^R(\eta) + E_t\left[\int_t^\infty e^{-\rho(s-t)} \log(\eta_s)\, ds\right].$$

Hence, we obtain the desired expression.

To compute the welfare of households, notice that by Proposition 3, if two agents have wealth processes

$$\frac{dn_t}{n_t} = \mu_t^n\, dt + \sigma_t^n\, dZ_t \quad \text{and} \quad \frac{dn_t'}{n_t'} = \mu_t^{n'}\, dt + \sigma_t^{n'}\, dZ_t,$$

then the difference in their utility is

$$\frac{\log(\rho n_t') - \log(\rho n_t)}{\rho} + \frac{1}{\rho}\mathbb{E}_t\left[\int_t^\infty e^{-\rho(s-t)} \left(\mu_s^{n'} - \mu_s^n + \frac{|\sigma_s^n|^2 - |\sigma_s^{n'}|^2}{2}\right) ds\right] \tag{B.8}$$

We can now obtain household utility by adjusting the utility of a representative agent. Recall that households are indifferent between technologies a and b, so we can focus on households who use technology a without loss of generality. According to (2.19), world wealth follows

$$\frac{dn_t}{n_t} = dr_t^M - \rho\, dt - (\sigma_t^\vartheta)^T\, (\sigma_t^M\, dt + dZ_t) + \eta_t\, x_t^2|\nu_t^b|^2\, dt + (1 - \eta_t)(x_t^a)^2(|\nu_t^a|^2 + \tilde{\sigma}_a^2)\, dt,$$

while the net worth of a household that uses technology a follows

$$\frac{dn_t^H}{n_t^H} = dr_t^M - \rho\,dt + \underbrace{x_t^a((\nu_t^a)^T(\sigma_t^{Na}\,dt + dZ_t) + x_t^a\tilde{\sigma}_a^2\,dt + \tilde{\sigma}_a\,d\tilde{Z}_t)}_{x_t^a((\nu_t^a)^T(\sigma_t^M\,dt+dZ_t)+\tilde{\sigma}_a\,d\tilde{Z}_t)+(x_t^a)^2(|\nu_t^a|^2+\tilde{\sigma}_a^2)\,dt} \tag{B.9}$$

Hence,

$$\mu_t^{n^H} - \mu_t^n = \eta_t\left((x_t^a)^2(|\nu_t^a|^2 + \tilde{\sigma}_a^2) - x_t^2|\nu_t^b|^2\right) + (x_t^a\nu_t^a + \sigma_t^\vartheta)^T\sigma_t^M,$$

$$\frac{|\sigma_s^n|^2 - |\sigma_s^{n^H}|^2}{2} = \frac{|\sigma_t^M - \sigma_t^\vartheta|^2 - |\sigma_t^M + x_t^a\nu_t^a|^2 - (x_t^a)^2\tilde{\sigma}_a^2}{2} \quad \text{and so}$$

$$\mu_t^{n^H} - \mu_t^n + \frac{|\sigma_s^n|^2 - |\sigma_s^{n^H}|^2}{2} = \eta_t\left((x_t^a)^2(|\nu_t^a|^2 + \tilde{\sigma}_a^2) - x_t^2|\nu_t^b|^2\right) + \frac{|\sigma_t^\vartheta|^2 - (x_t^a)^2(|\nu_t^a|^2 + \tilde{\sigma}_a^2)}{2}$$

Thus, using (4.3) and (B.8) to value the welfare of households, we obtain the desired expression. □

Proof of Proposition 9. Money has risk $\sigma_t^M = \sigma_t^K + \sigma_t^p - b_t/p_t\,\sigma_t^B$. Effectively, risk $b_t/p_t\,\sigma_t^B$ is subtracted from money and is allowed to be traded separately, carrying the risk premium of $b_t/p_t\,(\sigma_t^B)^T\sigma_t^N$, since it is the intermediaries that hold bonds. The net worth of intermediaries follows

$$\frac{dN_t}{N_t} = dr_t^M - \rho\,dt + (x_t\nu_t^b + x_t^B\sigma_t^B)^T(\sigma_t^N\,dt + \,dZ_t), \quad \text{where } x_t^B = \frac{b_t}{\eta_t(p_t + q_t)}.$$

World wealth follows

$$\frac{d((q_t + p_t)K_t)}{(q_t + p_t)K_t} = dr_t^M - \rho\,dt - (\sigma_t^\vartheta)^T\,(\sigma_t^M\,dt + dZ_t) + \underbrace{\eta_t(x_t^B\sigma_t^B)^T(\sigma_t^N\,dt + dZ_t)}_{\text{bonds in the world portfolio}} \tag{B.10}$$

$$+\eta_t\,x_t(\nu_t^b)^T(x_t\nu_t^b + \underbrace{x_t^B\sigma_t^B)}_{\text{risk premium adjustment}}\,dt + (1-\eta_t)(x_t^a)^2(|\nu_t^a|^2 + (\tilde{\sigma}^a)^2)\,dt,$$

where we adjusted equation (2.19) for the presence of bonds in the world portfolio, as well as for the effect of bonds on the risk premium demanded by intermediaries.

Therefore, using Ito's lemma,

$$\frac{d\eta_t}{\eta_t} = (1-\eta_t)(|x_t \nu_t^b + x_t^B \sigma_t^B|^2 - (x_t^a)^2(|\nu_t^a|^2 + (\tilde{\sigma}^a)^2))\, dt +$$

$$(x_t \nu_t^b + \sigma_t^\vartheta + (1-\eta_t) x_t^B \sigma_t^B)^T\, (dZ_t + (\sigma_t^\theta - \eta_t x_t^B \sigma_t^B)\, dt).$$

$\square$

Proposition 14. *The relationship between the policy (i_t, b_t) and bond prices as well as the risk transfer process $(b_t/p_t)\sigma_t^B$ is given by the equation*

$$\underbrace{\frac{1}{B_t} - i_t + \mu_t^B + (\sigma_t^B)^T \sigma_t^M}_{E[dr_t^B - dr_t^M]/dt} = (\sigma_t^B)^T \sigma_t^N \tag{B.11}$$

Proof. Since the equilibrium excess return on bonds over money is given by (5.3), it remains to be shown that the left-hand side of (B.11) is this excess return. Denote by $\hat{p}_t$ the law of motion of the real value of money given by $d\hat{p}_t/\hat{p}_t = r_t^M\, dt - i_t\, dt$. Then the real capital gains rate on bonds is given by $d(B_t\hat{p}_t)/(B_t\hat{p}_t)$. Hence, using Ito's lemma, the total return on bonds is

$$dr_t^B = \underbrace{\frac{1}{B_t}\, dt}_{\text{dividend yield}} + \frac{d(B_t\hat{p}_t)}{B_t\hat{p}_t} = \frac{1}{B_t}\, dt + \mu_t^B\, dt + (\sigma_t^B)^T \sigma_t^M\, dt + \frac{d\hat{p}_t}{\hat{p}_t} + (\sigma_t^B)^T\, dZ_t.$$

It follows immediately that the excess return on bonds over money is given by (B.11). $\square$

Proof of Proposition 11. Equation (5.9) follows directly from Ito's lemma. Let us justify the remaining six equations.

Relative to money, capital devoted to the production of good a earns the return of

$$dr_t^a - dr_t^M = \frac{A^a(\psi_t) - \iota_t}{q_t}\, dt + (\mu_t^q - \mu_t^p)\, dt + (\sigma^a 1^a - \sigma_t^K)^T\, dZ_t + \tilde{\sigma}^a\, d\tilde{Z}_t.$$

Likewise,

$$dr_t^b - dr_t^M = \frac{A^b(\psi_t) - \iota_t}{q_t}\, dt + (\mu_t^q - \mu_t^p)\, dt + (\sigma^b 1^b - \sigma_t^K)^T\, dZ_t + \tilde{\sigma}^b\, d\tilde{Z}_t.$$

Fraction $\bar{\chi}$ of the risk of good b is borne by intermediaries, who are exposed to aggregate risk σ_t^K, and fraction $1-\bar{\chi}$, by households, who are exposed to aggregate risk σ_t^K and idiosyncratic risk $x_t^b\tilde{\sigma}^b$. Thus,

$$\frac{A^b(\psi_t)-\iota_t}{q_t}+\mu_t^q-\mu_t^p=(\sigma^b 1^b-\sigma^K)^T\sigma^K+(1-\bar{\chi})\,x_t^b(\tilde{\sigma}^b)^2, \tag{B.12}$$

where $(\sigma^b 1^b-\sigma^K)^T\sigma^K$ is the risk premium for aggregate risk of this investment, and $(1-\bar{\chi})\,x_t^b(\tilde{\sigma}^b)^2$ is the price of idiosyncratic risk. For good a,

$$\frac{A^a(\psi_t)-\iota_t}{q_t}+\mu_t^q-\mu_t^p=(\sigma^a 1^a-\sigma^K)^T\sigma^K+x_t^a(\tilde{\sigma}^a)^2. \tag{B.13}$$

Now, to the six equations. Since any investment in capital includes a hedge for the aggregate risk component, $\nu_t^a=\nu_t^b=0$ including this hedge, so the indifference condition of households (2.13) becomes, one,

$$(x_t^a)^2(\tilde{\sigma}^a)^2=(x_t^b)^2(\tilde{\sigma}^b)^2 \quad \Leftrightarrow \quad \frac{x_t^b}{x_t^a}=\frac{\tilde{\sigma}^a}{\tilde{\sigma}^b}, \tag{B.14}$$

and the law of motion of η_t is, two,

$$\frac{d\eta_t}{\eta_t}=-(1-\eta_t)(x_t^a)^2(\tilde{\sigma}^a)^2\,dt, \tag{B.15}$$

From (2.16), we have, three,

$$\frac{(1-\bar{\chi})\psi_t}{x_t^b}+\frac{1-\psi_t}{x_t^a}=\frac{1-\eta_t}{1-\vartheta_t} \quad \Rightarrow \quad (1-\bar{\chi})\psi_t+(1-\psi_t)\frac{\tilde{\sigma}^a}{\tilde{\sigma}^b}=x^b\frac{1-\eta_t}{1-\vartheta_t}. \tag{B.16}$$

Subtracting (B.13) from (B.12), we get, four,

$$\frac{A^b(\psi_t)-A^a(\psi_t)}{q_t}=(\sigma^b 1^b-\sigma^a 1^a)^T\sigma^K+(1-\bar{\chi})x_t^b(\tilde{\sigma}^b)^2-x_t^a(\tilde{\sigma}^a)^2. \tag{B.17}$$

The market-clearing condition for consumption goods is, five,

$$A(\psi_t)-\iota(q_t)=\frac{\rho q_t}{1-\vartheta_t}.$$

Finally, taking a weighted average of (B.12) and (B.13), with weights ψ and $1-\psi$, and using

(B.14) to eliminate x_t^a on the right-hand side, we have

$$\underbrace{\frac{A(\psi)-\iota_t}{q_t}}_{\rho/(1-\vartheta_t)} + \mu_t^q - \mu_t^p = \left((1-\bar{\chi})\psi + (1-\psi)\frac{\tilde{\sigma}^a}{\tilde{\sigma}^b}\right) x_t^b (\tilde{\sigma}^b)^2.$$

This, in combination with (B.16), and the identity $\mu_t^\vartheta = (1-\vartheta_t)(\mu_t^p - \mu_t^q) - \sigma^\vartheta \sigma^p + (\sigma^\vartheta)^2$, leads to the last equation, (5.10), six. □

Proof of Proposition 12. The derivation of intermediary welfare remains unchanged from the proof of Proposition 7. For households, we use equation (B.8), but take into account that the law of motion of world wealth becomes modified to

$$\frac{dn_t}{n_t} = dr_t^M - \rho\, dt + (\eta_t x_t^B \sigma_t^B - \sigma_t^\vartheta)^T \left(\sigma_t^M\, dt + dZ_t\right)$$

$$+\eta_t\, |x_t \nu_t^b + x_t^B \sigma_t^B|^2\, dt + (1-\eta_t)(x_t^a)^2(|\nu_t^a|^2 + (\tilde{\sigma}^a)^2)\, dt,$$

as in (B.10). The law of motion of household net worth (B.9) remains unchanged. Combining these, we obtain

$$\mu_t^{n^H} - \mu_t^n + \frac{|\sigma_s^n|^2 - |\sigma_s^{n^H}|^2}{2} = \eta_t((x_t^a)^2(|\nu_t^a|^2 + \tilde{\sigma}_a^2) - |x_t\nu_t^b + x_t^B\sigma_t^B|^2) + \frac{|\eta_t x_t^B \sigma_t^B - \sigma_t^\vartheta|^2 - (x_t^a)^2(|\nu_t^a|^2 + \tilde{\sigma}_a^2)}{2}.$$

Hence, using (B.8), we obtain the desired expression for household welfare. □

C “As If ” Representative Agent Model

Salient features of the various equilibria discussed above can also be achieved through closely related representative agent economies. Suppose the representative agent is endowed with initial capital $K_0 \equiv \int_0^1 k_0^i di$ and that he faces technology-specific, but no purely idiosyncratic risk. Let us furthermore assume that this representative agent also has log preferences, but now with discount factor $\tilde{\rho}$. For now, suppose that he can only invest in technologies a and b (no money is available). In that case his portfolio problem is simple: He will consume a constant fraction $\tilde{\rho}$ of his wealth, he will invest equal fractions of his capital at each instant in technologies a and b, and the price of capital will be time-invariant and satisfy

$$q^R = \frac{\kappa \bar{A} + 1}{\kappa \tilde{\rho} + 1}$$

For $\tilde{\rho} = \rho$ we exactly recover the no-money equilibrium above, with the sole difference that welfare of the representative households exceeds welfare of the atomistic households above (since the representative household need not bear any idiosyncratic risk). It is also instructive to consider the case $\tilde{\rho} = \sqrt{\rho}\hat{\sigma}$. In that case, the equilibrium with the representative agent is, as far as real quantities and the price of capital are concerned, *exactly* the same as the money equilibrium above. Notice that q^R is equal to the price of capital in the economy with money and discount rate ρ, hence investment rate is the same in both economies. Since $\tilde{\rho} > \rho$, we see that the money equilibrium effectively amounts to an decrease in patience. Despite the similarity in the investment rate, the welfare is again different in the two economies.

Finally, it is interesting to consider the question of whether money can have value in a representative agent economy. Evidently, the answer is no – an asset that never generates any real payoff cannot be held in positive quantities forever without violating the investor's transversality condition. However, let us for the moment assume that the transversality condition is allowed to be violated. In that case, as long as the representative household has no purely idiosyncratic risk, there still could be no equilibrium with constant $p, q \gg 0$, for money would be a strictly dominated asset. We could of course deal with strict dominance by re-introducing idiosyncratic risk, but then the aggregate law of motion for capital would be different (since now idiosyncratic cannot cancel in the aggregate). Only if the representative agent would *perceive* the threat of idiosyncratic risk at every period, but without this risk every materializing, would we recover allocations and prices from the money equilibrium above – and all of course subject to the proviso of ignoring the representative household's transversality condition.