Asset Pricing under Asymmetric Information
Rational Expectations Equilibrium

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Overview

- Modeling Information
  - From Possibility Sets to Partitions
  - Knowledge Operators
  - Group Knowledge - Mutual/Common Knowledge
- No-Trade Theorem
  - Aumann's "Agreeing to Disagree"
  - Geanakoplos' generalization
  - No-Trade Theorems
    - Net trades are observable
    - Net trades are not observable
  - Allocative Efficiency (ex-ante, interim, ex-post)

From Possibility Sets to Partitions

- State Space - Example $\omega \in \Omega = \{\omega_1, \omega_2, \omega_3, \omega_4, \omega_5\}$
  - Five states: $\omega_1 = \{d_{\text{high}}, p_{\text{high}}\}$, $\omega_2 = \{d_{\text{high}}, p_{\text{low}}\}$, $\omega_3 = \{d_{\text{low}}, p_{\text{high}}\}$, $\omega_4 = \{d_{\text{low}}, p_{\text{low}}\}$ and $\omega_5 = \{d = 0, p = 0\}$.
  - event E: set of states, e.g. 'the dividend payment is high' $E = \{\omega_1, \omega_2\}$.
- Illustration: In ω_1 agent receives info that dividend is high agent can eliminate the states ω_3, ω_4 and ω_5.
  In state ω_1 she thinks that only ω_1, and ω_2 are possible $\rightarrow$ possibility sets.
- Example: possibility set $= \mathcal{P}^{i\prime\prime}(\omega_1) = \{\omega_1, \omega_2\}$ if the true state is ω_1 and $\mathcal{P}^{i\prime\prime}(\omega_2) = \{\omega_2, \omega_3\}$, $\mathcal{P}^{i\prime\prime}(\omega_3) = \{\omega_2, \omega_3\}$, $\mathcal{P}^{i\prime\prime}(\omega_4) = \{\omega_4, \omega_5\}$, $\mathcal{P}^{i\prime\prime}(\omega_5) = \{\omega_5\}$ for the other states. Individual i knows this information structure.

From Possibility Sets to Partitions

- **Axiom of truth (knowledge)**
  ensures that agent does not rule out the true state.

$$\omega \in \mathcal{P}^i(\omega) \text{ (axiom of truth).}$$

- **Positive introspection**
  - In ω_1, agent i thinks that ω_1 and ω_2 are both possible. However, by positive introspection she knows that in state ω_2 she would know that the true state of the world is either ω_2 or ω_3.
    Since ω_3 is not in her possibility set, she can exclude ω_2 and, hence, she knows the true state in ω_1.
  - Formally, after positive introspection

$$\omega' \in \mathcal{P}^i(\omega) \Rightarrow \mathcal{P}^i(\omega') \subseteq \mathcal{P}^i(\omega) \text{ (positive introspection).}$$

$$\Rightarrow \mathcal{P}^{i\prime}(\omega_1) = \{\omega_1\},\ \mathcal{P}^{i\prime}(\omega_2) = \{\omega_2, \omega_3\},$$
$$\mathcal{P}^{i\prime}(\omega_3) = \{\omega_2, \omega_3\},\ \mathcal{P}^{i\prime}(\omega_4) = \{\omega_4, \omega_5\},\ \mathcal{P}^{i\prime}(\omega_5) = \{\omega_5\}.$$

From Possibility Sets to Partitions

- **Negative introspection**
  - In state ω_4, i holds ω_4 and ω_5 as possible.
  - However, in state ω_5 she knows that the true state of the world is *not* in $\{\omega_1, \omega_2, \omega_3, \omega_4\} = \Omega \backslash \{\omega_5\}$.
  - can infer that she must be in state ω_4 because she does not know whether the true state is in $\Omega \backslash \{\omega_5\}$ or not.
  - Formally after negative introspection

$$\omega' \in \mathcal{P}^i(\omega) \Rightarrow \mathcal{P}^i(\omega') \supseteq \mathcal{P}^i(\omega) \text{ (negative introspection).}$$

- After making use of positive and negative introspection, individual i has the following information structure: $\mathcal{P}^i(\omega_1) = \{\omega_1\}$, $\mathcal{P}^i(\omega_2) = \{\omega_2, \omega_3\}$, $\mathcal{P}^i(\omega_3) = \{\omega_2, \omega_3\}$, $\mathcal{P}^i(\omega_4) = \{\omega_4\}$, $\mathcal{P}^i(\omega_5) = \{\omega_5\}$. $=$ partition
- *In general:* Information structure becomes partition of Ω a collection of subsets that are mutually disjoint and have a union Ω.

Knowledge Operator

$$\mathcal{K}^i(E) = \{\omega \in \Omega : \mathcal{P}^i(\omega) \subseteq E\}$$

- *possibility set* $\mathcal{P}^i(\cdot)$ reports all states of the world individual i considers as possible given true state,
- the *knowledge operator* reports all the states of the world, i.e. an event, in which agent i considers a certain event E possible.
  (That is, it reports the set of all states in which agent i knows that the true state of the world is in the event $E \subseteq \Omega$.)
- In our example, individual i knows event $E' = \{$dividend is high$\} = \{\omega_1, \omega_2\}$ only in state ω_1, i.e. $\mathcal{K}^i(E') = \omega_1$.

3 Properties of Knowledge Operator

1. Agent i always knows that one of the states $\omega \in \Omega$ is true.

$$\mathcal{K}^i(\Omega) = \Omega.$$

2. If i knows that the true state of the world is in event E_1 then she also knows that the true state is in any E_2 containing E_1, i.e.

$$\mathcal{K}^i(E_1) \subseteq \mathcal{K}^i(E_2) \text{ for } E_1 \subseteq E_2$$

3. If i knows that the true state of the world is in event E_1 and she knows that it is also in event E_2, then she also knows that the true state is in event $E_1 \cap E_2$.

$$\mathcal{K}^i(E_1) \cap \mathcal{K}^i(E_2) = \mathcal{K}^i(E_1 \cap E_2).$$

Restatement of Axiom

- Axiom of Truth

$$\mathcal{K}^i(E) \subseteq E$$

  That is, if i knows E (e.g. dividend is high) then E is true, i.e. the true state $\omega \in E$.

- Positive introspection $\Leftrightarrow$ *'knowing that you know'* (KTYK) axiom

$$\mathcal{K}^i(E) \subseteq \mathcal{K}^i\left(\mathcal{K}^i(E)\right) \text{ (KTYK).}$$

  This says that in all states in which individual i knows E, she also knows that she knows E. This refers to higher knowledge, since it is a knowledge statement about her knowledge.

Restatement of Axiom

- Negative introspection $\Leftrightarrow$ *'knowing that you do not know'* (KTYNK).

$$\Omega \backslash \mathcal{K}^i(E) \subseteq \mathcal{K}^i\left(\Omega \backslash \mathcal{K}^i(E)\right) \text{ (KTYNK).}$$

For any state in which individual i does not know whether the true state is in E or not, she knows that she does not know whether the true state is in E or not.
requires a high degree of rationality. It is the most demanding axiom of the three axioms.
Adding the last three axioms allows one to represent information in partitions.

Group Knowledge & Common Knowledge

- Intersection of all events reported by the individual knowledge operators gives us the states in which all members of the group G know an event E.

$$\mathcal{K}^{G}(E) := \cap_{i \in G} \mathcal{K}^{i}(E).$$

- Mutual knowledge does not guarantee that all members of the group know that all the others know it too. Knowledge about knowledge, i.e. second order knowledge can easily be analyzed by applying the knowledge operator again, e.g. $\mathcal{K}^{i_1}\left(\mathcal{K}^{i_2}(E)\right)$.
- An event is second order mutual knowledge if everybody knows that everybody knows event E.

$$\mathcal{K}^{G(2)}(E) := \cap_{i \in G}\left(\cap_{-i \in G \setminus \{i\}} \mathcal{K}^{i}\left(\mathcal{K}^{-i}(E)\right)\right) \cap \mathcal{K}^{G}(E).$$

If the axiom of truth holds, the second order mutual knowledge operator simplifies to

$$\mathcal{K}^{G(2)}(E) = K^{G}(K^{G}(E)).$$

Group Knowledge & Common Knowledge

- n^{th} order mutual knowledge, $\mathcal{K}^{G(n)}(E)$. Given the axiom of truth

$$\mathcal{K}^{G(n)}(E) = \underbrace{\mathcal{K}^{G}(\mathcal{K}^{G}(...(\mathcal{K}^{G}(}_{n\text{-times}}E))))$$

- E is common knowledge if everybody knows that everybody knows that everybody knows and so on ad infinitum that event E is true.

$$\mathcal{CK}(E) := \cap_{n=1}^{\infty}\mathcal{K}^{G(n)}(E),$$

Note that as long as the axiom of truth holds $\mathcal{CK}(E) = \mathcal{K}^{G(\infty)}(E)$.

Physical and Epistemic Part of State Space - Depth of Knowledge -

- in *complete* model state space and each individuals' partitions are "common knowledge" (outside the model)
- to analyze higher order uncertainty (knowledge) state of the world describes not only
  - the physical world (fundamentals) but also
  - the epistemic world, i.e. what each agent knows about fundamentals or others' knowledge.
- Simple Example:
  - Individual 1 knows whether interest rate r will be high or low. Individual 2 does not know it.
  - Standard model: Ω', $\omega'_1 = \{r_{high}\}$, $\omega'_2 = \{r_{low}\}$
    Individual 1's partition: $\{\{\omega'_1\}, \{\omega'_2\}\}$
    Individual 2's partition: $\{\omega'_1, \omega'_2\}$.
    Since partitions are common knowledge $\Rightarrow$ '1' knows that '2' does not know whether the interest rate is high or low and '2' knows that '1' knows it.
    $\Rightarrow$ second order knowledge is common knowledge (any event which is mutually known is also common knowledge).

Group Knowledge & Common Knowledge

- Simple Example (ctd.)
  - Model second order uncertainty:
    extended state space Ω with $\omega_1 = \{r_{\text{high}}, 2 \text{ knows } r_{\text{high}}\}$,
    $\omega_2 = \{r_{\text{high}}, 2 \text{ does not know } r_{\text{high}}\}$,
    $\omega_3 = \{r_{\text{low}}, 2 \text{ knows } r_{\text{low}}\}$, $\omega_4 = \{r_{\text{low}}, 2 \text{ does not know } r_{\text{low}}\}$.
    If agent 1 does not know whether agent 2 knows the interest rate, his partition is $\{\{\omega_1, \omega_2\}, \{\omega_3, \omega_4\}\}$. Agent 2's partition is $\{\{\omega_1\}, \{\omega_3\}, \{\omega_2, \omega_4\}\}$ since he knows whether he knows the interest rate or not.
  - Note that the description of a state also needs to contain knowledge statements in order to model higher order uncertainty. These statements can also be in indirect form, e.g. agent i received a message m.

Group Knowledge & Common Knowledge

- Simple Example (ctd.)
  - state space
    - fundamentals partition: $\Omega = \left\{E_{r_{\text{high}}}, E_{r_{\text{low}}}\right\}$ into two events, $E_{r_{\text{high}}} = \{\omega_1, \omega_2\}$ and $E_{r_{\text{low}}} = \{\omega_3, \omega_4\}$.
    - epistemic (knowledge) partition: $\Omega = \{E_{2 \text{ knows } r}, E_{2 \text{ does not know } r}\}$ into $E_{2 \text{ knows } r} = \{\omega_1, \omega_3\}$ and $E_{2 \text{ does not know } r} = \{\omega_2, \omega_4\}$.
  - depth of knowledge (of state space)
    - $= 0$ state description only specifies first order knowledge
    - > 0 state description contains higher order knowledge statements.

Tractable Notion of Common Knowledge

- Some Definitions
  - An event E is self evident for agent i if E is a union of i's partition cells $\mathcal{P}^i(\omega)$, i.e. $\mathcal{P}^i(E) = E$. In other words, E is self-evident if for all $\omega \in E$, $\mathcal{P}^i(\omega) \subseteq E$.
  - Event E is a public event if it is simultaneously self-evident for all agents $i \in \mathbb{I}$.
  - A partition consisting of public events is called common coarsening. The meet $\mathcal{M} := \bigwedge_i^I \mathcal{P}^i$ is the finest common coarsening, i.e. a partition whose cells are the smallest public events $\mathcal{M}(\omega)$. The meet reflects the information which is common knowledge among all agents.
  - The join $\mathcal{J} := \bigvee_i^I \mathcal{P}^i$ is the partition which reflects the pooled information of all individuals in the economy.

Tractable Notion of Common Knowledge

- Aumann (1976)
  A public event $\mathcal{M}(\omega) \ni \omega$ is common knowledge at ω. Obviously, at this $\omega \in \mathcal{M}(\omega)$, any event $E' \supseteq \mathcal{M}(\omega)$ is also common knowledge.
  assume that ω' is the true state of the world.
- Reachability
  A public event $\mathcal{M}(\omega)$ can also be viewed as a set of states which are reachable from the true ω.

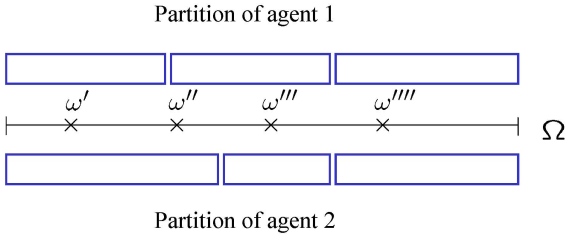

Figure: Suppose ω' is the true state of the world

Tractable Notion of Common Knowledge

- $\Rightarrow$ '1' thinks that any $\omega \in \mathcal{P}^1(\omega')$ is possible.
  He knows that ω'' is not the true state of the world, but he also knows that agent 2 thinks that ω'' is possible.
  Therefore, the event $\omega \in \mathcal{P}^1(\omega')$ is surely not common knowledge since ω'' is reachable through the partition cell $\omega \in \mathcal{P}^2(\omega')$ of agent 2.
- Is event $\mathcal{P}^2(\omega')$ common knowledge?
  Take state ω'''. '1' and '2' know that ω''' is not the true state, i.e. the event $\mathcal{P}^2(\omega')$ is mutual knowledge in ω'.
  However, a state ω''' is still reachable.
  Therefore, $\mathcal{P}^2(\omega')$ is not common knowledge.
  The public event $\mathcal{M}(\omega') = \mathcal{P}^1(\omega') \cup \mathcal{P}^1(\omega''))$ is common knowledge since any ω outside this event is not reachable.
- meet $\mathcal{M}$ for this example is given by $\{\mathcal{P}^1(\omega') \cup \mathcal{P}^1(\omega''), \mathcal{P}^1(\omega'''')\}$.

Agreeing to Disagree

- common priors
- event $E^i_{P^i}$ groups all states ω with same posterior P^i for 'i' about event D
- Since the posteriors of all agents $\{P^i\}_{i \in \mathbb{I}}$ are common knowledge, the true state ω must lie in a public event $E^{public} \subseteq \bigcap_i E^i_{P^i}$.
- conditional probability of D conditional on any union of $\mathcal{P}^i(\omega) \subseteq E^{public}$ including on the public event E^{public} is also the same. (sure thing principle)
- Note that posterior conditioning on the join $\mathcal{J} := \bigvee_i^I \mathcal{P}^i$ might be different.

Agreeing to Disagree

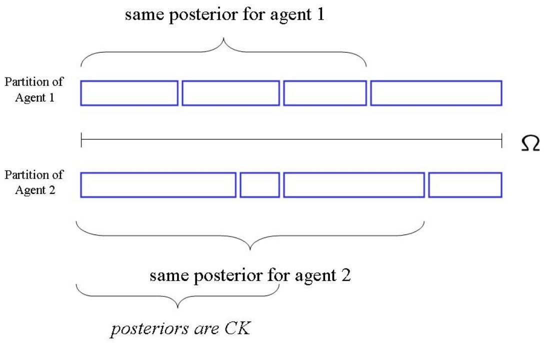

Figure: **Agreeing to Disagree.**

Geanakoplos' Generalization

- Aumann: commonly known *posteriors*
- Geanakoplos: commonly known *action rules* (mapping from partition cells into action space)
- Theorem "common knowledge of actions negates asymmetric information about events"
  If the actions chosen by players based on their private information are common knowledge, then there exists an environment with symmetric information which would lead to the same actions.
- *Special case:* All follow same action rule & actions are common knowledge, then the chosen action has to be the same for all players.
- Net-Trade vector is observable $\Rightarrow$ No-Trade Theorem I

Allocative Efficiency

- allocation $\{\{x^i(\omega)\}_{\omega\in\Omega}\}_{i\in\mathbb{I}}$
  (each node along the decision tree)
- (allocative) Pareto efficient if there is no other allocation which makes at least one agent strictly better off without making somebody else worse off.
- Problem: "better off" and "worse off" - depend on information.
  - ex-ante: $E[U^i(\cdot)]$
  - interim: $E[U^i(\cdot)|S^i(\omega)]$
  - ex-post: $E[U^i(\cdot)|\omega]$

Allocative Efficiency

- Intuitive reasoning using negations:
  allocation is interim inefficient, i.e. an interim Pareto improvement is possible, then an ex-ante Pareto improvement is also possible.
  Similarly, if an allocation is ex-post inefficient it is also interim inefficient. Intuitively, ex-ante Pareto efficiency does not only require that the allocation is Pareto efficient for each state ω but also that the allocation optimally insures all risk-averse agents over the different states of the world.
- representation via measurability restrictions on individual weights $\lambda^i(\omega) \in \mathbb{R}$ of a social welfare function:

$$W(\{\{x^i(\omega)\}_{\omega\in\Omega}\}_{i\in\mathbb{I}}) = \sum_{i\in\mathbb{I}}\sum_{\omega\in\Omega} \lambda^i(\omega)\Pr(\omega)U^i(x^i(\omega),\omega).$$

Allocative Efficiency

- Find
  - arbitrary constant λ^i's for the welfare function, such that this allocation maximizes $W(\cdot)$, then this allocation is ex-ante efficient.
  - $\lambda^i(\omega)$s which are measurable only on the partitions associated $\mathcal{S}^i$, then this allocation is interim efficient.
  - $\lambda^i(\omega)$s which depend on ω then the allocation is ex-post efficient.
- ex-ante efficiency $\Rightarrow$ interim efficiency $\Rightarrow$ ex-post efficiency
- restrict feasible set of implementable allocations:
  An allocation is only incentive compatible or individually rational if the individuals are willing to report their information, i.e their types. One can define ex-ante, interim and ex-post incentive compatible efficiency as above by restricting attention to the set of incentive compatible allocations.
- In sum, in a world with asymmetric information, there are six notions of allocative efficiency.

No Trade Theorem II

- No-Trade Theorem (Milgrom & Stokey 1982):
  If it is common knowledge that all traders are rational and the current allocation is ex-ante Pareto efficient, then new asymmetric information will not lead to trade, provided traders are strictly risk averse and hold concordant beliefs.
  - Holmström & Myerson (1983) proof:
    ex-ante $\Rightarrow$ interim $\Rightarrow$ ex-post efficiency
  - Kreps (1977), Tirole (1982) proof:
    zero sum-game argument
- Market Breakdowns due to Asymmetric Information
  - related to Akerlof's markets for lemons
  - restrict attention to individually rational allocations
  - see also Morris (1994)