Asset Pricing under Asymmetric Information Rational Expectations Equilibrium

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A Classification of Market Microstructure Models

- simultaneous submission of demand schedules
  - competitive rational expectation models
  - strategic share auctions
- sequential move models
  - screening models in which the market maker submits a supply schedule first
    - static
      - ◇ uniform price setting
      - ◇ limit order book analysis
    - dynamic sequential trade models with multiple trading rounds
  - strategic market order models where the market maker sets prices ex-post

Strategic Market Order Models - Overview

- Kyle (1985) model
  - static version
  - dynamic version (in discrete time)
    - Refresher in Dynamic Programming
  - continuous time version (Back 1992)
- Multi-insider Kyle (1985) version
- Other strategic market order models

Kyle 1985 Model

- Model Setup
  - asset return $v \sim \mathcal{N}(p_0, \Sigma_0)$
  - Agents (risk neutral)
    - Insider who knows v and submit market order of size x
    - Noise trader who submit market orders of exogenous aggregate size $u \sim \mathcal{N}(0, \sigma_u^2)$
    - Market maker sets competitive price after observing **net** order flow $X = x + u$
  - Timing (order of moves)
    - Stage 1: Insider & liquidity traders submit market orders
    - Stage 2: Market Maker sets the execution price
  - Repeated trading in dynamic version

Kyle 1985 Model — Static Version

Single informed trader	(Competitive) Market Maker
0) *Information*	**0)** *Information*
$v :=$ asset's payoff	$X = x + u$ batch net order flow
1) *Conjecture (price-rule)*	**1)** *Conjecture (insider trading rule)*
$p = \mu + \lambda(x + u)$	$x = \alpha + \beta v$
2) *No Updating*	**2)** *Updating* $E[v \mid x + u]$
3) *Optimal Demand*	**3)** *Price Setting Rule*
$\max_x E[(v - p) \mid v] x$	$p = E[v \mid x + u]$
$\max_x E[v - \mu - \lambda x \mid v] x$	$p = E[v] + \frac{Cov[v, x+u]}{Var[x+u]} \{x + u - E[x + u]\}$
FOC: $x = -\frac{\mu}{2\lambda} + \frac{1}{2\lambda} v$	$p = p_0 + \frac{\beta \Sigma_0}{\beta^2 \Sigma_0 + \sigma_u^2} \{x + u - \alpha + \beta E[v]\}$
SOC: $\lambda > 0$	
4) *Correct Beliefs*	**4)** *Correct Beleifs*
$\alpha = -\frac{\mu}{2\lambda}$, $\beta = \frac{1}{2\lambda}$	$\mu = p_0$ Martingale, $\lambda = \frac{\beta \Sigma_0}{\beta^2 \Sigma_0 + \sigma_u^2}$

Kyle 1985 Model — Static Version

- solve for unknown coefficients
  - 4 unknown Greeks
  - 4 equations
- $\lambda = \frac{1}{2}\sqrt{\frac{\Sigma_0}{\sigma_u^2}}$
  - λ (illiquidity) decreases with noise trading, σ_u^2
  - Σ_0 reflect info advantage of insider

Dynamic Programming - A Refresher

- The Problem:
  max for several periods $t = 1, ..., T$ (discrete time)

$$\max_{u_t} E_t \left[\sum_{s=1}^{T} v_s \left(\tilde{x}_s, u_s \right) \right] \qquad \forall t \quad \Rightarrow \text{(sequential rationality)}$$

  under the following law of motion

$$\tilde{x}_{t+1} = f_t \left(\tilde{x}_t, u_t, \tilde{\varepsilon}_t \right)$$

  $\tilde{x}_t$: vector of <u>state</u> variables (sufficient state space)
  u_t: vector of <u>control</u> variables
  ε_t: vector of random <u>shocks</u>

- Method
  - Backward Induction
  - Dynamic Programming

Dynamic Programming - A Refresher

- Define Value Function

$$V_t(x_t) := E_t[\sum_{s=t}^{T} v_s(\tilde{x}_s, \underset{\substack{\uparrow \\ \text{optimal values}}}{u_s^*})]$$

- $\Rightarrow$ Bellman Equation

$$\max_{u_t} E_t\left[v_t(x_t, u_t) + V_{t+1}(x_{t+1})\right]$$

- Start at final **date** T

$$V_{T+1}(\cdot) := 0$$

$\Rightarrow$ in $t = T$

$$\max_{u_T} E_T\left[v_T(x_T, u_T)\right]$$

$$\overset{\text{FOC}}{\Rightarrow} u_T^* = g_T(x_T)$$

$$\Rightarrow V_T(x_T) = E_T\left[v_T(x_T, u_T^*)\right]$$

Dynamic Programming - A Refresher

- at **date** $T-1$

$$\max_{u_{T-1},u_T} E_{T-1}\left[\sum_{s=T-1}^{T} v_s\left(\widetilde{x}_s, u_s\right)\right]$$

given $V_T\left(x_T\right)$

$\Leftrightarrow$ $\max_{u_{T-1}} E_{T-1}\left[v_{T-1}\left(x_{T-1}, u_{T-1}\right) + V_T\left(x_T\right)\right]$
given law of motion

$\Leftrightarrow$

$\max_{u_{T-1}} E_{T-1}\left[v_{T-1}\left(x_{T-1}, u_{T-1}\right) + V_T\left(f_{T-1}\left(x_{T-1}, u_{T-1}, \widetilde{\varepsilon}_{T-1}\right)\right)\right]$
$\Rightarrow u^*_{T-1} = g_{T-1}\left(x_{T-1}\right)$
$\Rightarrow V_{T-1} =$
$E_{T-1}\left[v_{T-1}\left(x_{T-1}, u^*_{T-1}\right) + V_T\left(f_{T-1}\left(x_{T-1}, u^*_{T-1}, \widetilde{\varepsilon}_{T-1}\right)\right)\right]$

- and so on for **date** $T-2$ etc. (and if they didn't die in the uncertainties they are still solving ...)
- This process is quite time consuming.

Dynamic Programming - A Refresher

Alternative way:

- **Step 1:** "Guess" the general form of the value function

$$V_{t+1}(x_{t+1}) = \underbrace{H_{t+1}(x_{t+1})}_{\text{e.g. } H_{t+1}(x_{t+1})=\alpha_{t+1}x_{t+1}^2}$$

- **Step 2:** Derive optimal level of current control

$$\max_{u_t} E_t\left[v_t(x_t, u_t) + H_{t+1}(\tilde{x}_{t+1})\right]$$

$$\max_{u_t} E_t\left[v_t(x_t, u_t) + H_{t+1}(f_t(x_t, u_t, \varepsilon_t))\right]$$

$$\Rightarrow \quad u_t^* = \cdots$$

- **Step 3:** Derive value function and check whether it coincides with general value function

$$\begin{aligned} V_t(x_t) &= E_t\left[v_t(x_t, u_t^*) + H_{t+1}(f_t(x_t, u_t^*, \tilde{\varepsilon}_t))\right] \\ \stackrel{?}{=} H_t(x_t) &= \alpha_t x_t^2 \end{aligned}$$

Kyle (1985) — Dynamic Version

Insider

- **Step 1:** Conjectured price setting strategy (pricing rule)

$$\begin{aligned} p_n &= p_{n-1} + \lambda_n \Delta X_n \\ &= p_{n-1} + \lambda_n \left(\Delta x_n + \Delta u_n\right) \qquad \left(\frac{1}{\lambda_t} \cong \text{Liquidity}\right) \end{aligned}$$

- **Step 2:** 'Guess' Value function for insider's profit pricing rule is linear → guess quadratic value function)

$$E[\pi_{n+1} | \underbrace{\widetilde{p}_1, \cdots, p_n, v}_{\substack{\text{Information set} \\ \text{up to } n}}] = \alpha_n \left(v - p_n\right)^2 + \delta_n$$

(expected profit from time $n+1$ onwards)

$$\pi_n = E_n \left[\pi_{n+1} + \left(v - p_n\right) \Delta x_n^i\right]$$

Kyle (1985) — Dynamic Version

Insider ctd.

- **Step 3:** Write Bellman Equation

$$\max_{\Delta x_n^i} E\left[(v - p_n)\,\Delta x_n^i + \alpha_n (v - p_n)^2 + \delta_n \middle| \underbrace{p_1, \cdots, p_{n-1}, v}_{I_{n-1}} \right]$$

- **Step 4:** Given insider's beliefs $p_n = p_{n-1} + \lambda_n \Delta X_n$

$$\max_{\Delta x_n^i} E\left[\begin{array}{c} \left(v - p_{n-1} - \lambda_n \Delta x_n^i - \lambda_n \Delta u_n\right) \Delta x_n^i \\ +\alpha_n \left(v - p_{n-1} - \lambda_n \Delta x_n^i - \lambda_n u_n\right)^2 + \delta_n \end{array} \middle| I_n \right]$$

Take expectations

$$\max_{\Delta x_n^i} \left[\begin{array}{c} \left(v - p_{n-1} - \lambda_n \Delta x_n^i\right) \Delta x_n^i + \alpha_n (\underbrace{v - p_{n-1}}_{\text{state variable}} - \lambda_n \underbrace{\Delta x_n^i}_{\text{control}})^2 \\ +\delta_n + \alpha_n \lambda_n^2 \underbrace{\sigma_u^2}_{u \Rightarrow p_n \text{ noisy}} \Delta t_n \end{array} \right]$$

Kyle (1985) — Dynamic Version

Insider ctd.

- **Step 5:** maximize

$$\text{FOC:} \quad (v - p_{n-1}) - 2\lambda_n \Delta x_n^i - 2\alpha_n \lambda_n (v - p_{n-1}) + 2\alpha_n \lambda_n^2 \Delta x_n^i =$$

$$\Delta x_n^i = \underbrace{\frac{1 - 2\alpha_n \lambda_n}{2\lambda_n (1 - \alpha_n \lambda_n)}}_{:=\beta_n \Delta t_n} (v - p_{n-1})$$

$$\text{SOC:} \quad \lambda_n (1 - \alpha_n \lambda_n) > 0$$

- **Step 6:** Check whether 'guessed' value fcn is correct

$$\begin{aligned} E[\pi | I_{n-1}] &= \max_{\Delta x_n^i} E\left[(v - p_n) \Delta x_n^i + \alpha_n (\widetilde{v} - \widetilde{p}_n)^2 + \delta_n | I_{n-1} \right. \\ &= \alpha_{n-1} (v - p_{n-1})^2 + \delta_{n-1}, \text{where} \\ \alpha_{n-1} &= \frac{1}{4\lambda_n (1 - \alpha_n \lambda_n)}, \ \delta_{n-1} = \delta_n + \alpha_n (\lambda_n)^2 \sigma_u^2 \Delta t_n \end{aligned}$$

Kyle (1985) — Dynamic Version

Market Maker (Filtering Problem)

- **Step 1:** MM's belief about insider's strategy

$$\Delta x_n^i = \beta_n \Delta t_n (v - p_{n-1})$$
$$\Delta X_n = \beta_n \Delta t_n (v - p_{n-1}) + \underbrace{\Delta u_n}_{Var[\Delta u_n]=\sigma_u^2 \Delta t_n}$$

- **Step 2:** MM's filtering problem
  By definition:

$$p_{n-1} \quad : \quad = E[v|\Delta X_1, \cdots, \Delta X_{n-1}]$$
$$\Sigma_{n-1} \quad : \quad = Var[v|\Delta X_1, \cdots, \Delta X_{n-1}]$$

$$E[\Delta X_n|\Delta X_1, \cdots, \Delta X_{n-1}] = \beta_n \Delta t_n E[(v - p_{n-1}) + \Delta u_n| \cdot$$
$$Var[\Delta X_n| \cdots] = (\beta_n \Delta t_n)^2 \Sigma_{n-1} + \sigma_u^2 \Delta t_n$$
$$Cov[v, \Delta X_n| \cdots] = E[v(\beta_n \Delta t_n (v - p_{n-1}) + \Delta u_r$$
$$= \beta_n \Delta t_n \Sigma_{n-1}$$

Kyle (1985) — Dynamic Version

Now we have all ingredients for the Projection Theorem

$$p_n = p_{n-1} + \underbrace{\frac{\beta_n \Delta t_n \Sigma_{n-1}}{(\beta_n \Delta t_n)^2 \Sigma_{n-1} + \Delta t \sigma_u^2}}_{:=\lambda_n} \Delta X_n$$

$$\begin{aligned} \sum\nolimits_n &= V[v| \cdots \Delta X_n] = \Sigma_{n-1} - \frac{(\beta_n \Delta t_n)^2 \Sigma_{n-1}^2}{(\beta_n \Delta t_n)^2 \Sigma_{n-1} + \Delta t \sigma_u^2} \\ &= \frac{\sigma_u^2 \Sigma_{n-1}}{(\beta_n)^2 \Delta t_n \Sigma_{n-1} + \sigma_u^2} \end{aligned}$$

$\Rightarrow$

$$\begin{aligned} \lambda_n &= \frac{\beta_n \Delta t_n \sum_{n-1}}{(\beta_n \Delta t_n)^2 \Sigma_{n-1} + \Delta t \sigma_u^2} \\ \sum\nolimits_n &= (1 - \lambda_n \beta_n \Delta t_n) \Sigma_{n-1} = \frac{\sigma_u^2 \lambda_n}{\beta_n} \\ &\Rightarrow \lambda_n = \beta_n \Sigma_n / \sigma_u^2 \end{aligned}$$

Kyle (1985) — Dynamic Version

- **Step 3:** Equate coefficients α_n,β_n,δ_n,$\sum_n$

$$\left.\begin{array}{c} \beta_n \Delta t_n = \frac{1-2\alpha_n\lambda_n}{2\lambda_n(1-\alpha_n\lambda_n)} \\ \alpha_{n-1} = \frac{1}{4\lambda_n(1-\alpha_n\lambda_n)} \\ \delta_{n-1} = \delta_n + \alpha_n \lambda_n^2 \sigma_u^2 \Delta t_n \\ \left.\begin{array}{c} \sum_n = \sigma_u^2 \Sigma_{n-1} \\ \lambda_n = \ldots \end{array}\right\} \text{as above} \end{array}\right\} \text{Solve recursive system of equations.}$$

- Interpretation of Equilibrium
  - restrain from aggressive trading
    - price impact in current trading round
    - price impact in all future trading rounds
  - ...

Generalizations of Kyle (1985)

- Multiple Insiders
  - all have same information
  - all hold different information
  - information is correlated
    $\Rightarrow$ see Foster & Viswanathan, JF 51, 1437-1478
- Risk averse insiders
  - CARA utility
- etc. etc.