Asset Pricing under Asymmetric Information
Rational Expectations Equilibrium

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A Classification of Market Microstructure Models

- simultaneous submission of demand schedules
  - competitive rational expectation models
  - strategic share auctions
- sequential move models
  - screening models in which the market maker submits a supply schedule first
    - static
      - ◇ uniform price setting
      - ◇ limit order book analysis
    - dynamic sequential trade models with multiple trading rounds
  - strategic market order models where the market maker sets prices ex-post

Screening Models à la Glosten

1. Uninformed (risk-neutral) market maker sets whole supply schedule
   - market making sector is *competitive*
   - oligopolistic market making sector
   - market maker is *monopolist*
2. Possibly informed trader submits
   - a single order which is executed at *uniform price*
   - many little orders in order to "walk along the limit order book" (*discriminatory prices*)

Uniform Price Setting - Glosten 1989

- Contrast competitive market maker sector with monopolistic market maker (specialist system NYSE).
- Model setup
  - market maker(s) set price (supply) schedule
  - single trader submits order
    - risk-averse with CARA utility function
    - endowment shock of u
    - private signal $S^i = v + \epsilon$
  - two-dimensional screening problem
    Glosten (1989) reduces it to a one-dimensional problem (see later)

Uniform Price Setting - Glosten 1989

- Competitive price schedule: $P^{\text{co}} = E[v|x]$
  - Perfect competition
    - $\Rightarrow$ expected profit for *any* order size x is ZERO.
    - prevents market makers from effectively screening orders
    - $\Rightarrow$ leads to instability
      formally, existence problem for certain parameters (Hellwig JET 1994 shows that this is due to unbounded support of type sapce and it existence problem is different to the one in Rothschild & Stiglitz)
- Monopolistic price schedule:

$$P^{\text{mo}} = \arg\max E[[P^{\text{mo}}(x^*(\cdot)) - v]x^*(\cdot)],$$

where $x^*(\cdot)$ is the optimal order size.

  - principal-agent problem
  - principal sets menu of contracts $(x, P^{\text{mo}}(x))$
  - Cross-subsidization: large profit from small trades
    small (-ve) profit from large trades
  - market with monopolistic setting stays open for larger trade sizes than a market with multiple market markers

Discrim. Pricing (Limit Order Book) Glosten 1994 - BRM 2000

- "upper tail" conditional expectations for next marginal order y
$$P^{CO}(y) = E[v|x \geq y]$$
- trader who buy only a tiny marginal quantity have to pay a higher (ask) price $\Rightarrow$ small trade spread
- competitive market makers do not know whether trader only buys first marginal unit or continues to buy further units.
- cross-subsidization from small orders to large orders
- limit order book is immune to "cream skimming" of orders by competing exchanges (no advantage of order splitting).

Discrim. Pricing - Biais, Rochet & Martimo
Oligopolistic Market Makers

- oligopolistic screening game (special cases $I = 1$, $I = \infty$)
- **Stage 1:** risk-neutral market maker(s)
  set supply schedule $p(x)$ (limit order book)
- **Stage 2:** informed trader buy $x = \sum_i^I x^i$ shares
  - x^i for market maker i
  - transfer to mm i: $t^i(x^i) = \int_0^{x^i} p(q)dq$, $T(x) = \sum_i t^i(x^i)$
  - trader's endowment shock u
  - trader's signal S, where $v = S + \varepsilon$.
    $\varepsilon \sim \mathcal{N}\left(0, \sigma^2\right)$
    u and S have bounded support.
  - trader's final wealth $W = v(u + x) - \sum_i t^i(x^i)$
    (conditional on u, S, wealth W is normally distributed with $E[v|S] = S$, $Var[v|S] = Var[\varepsilon]$)

BRM: One Dimensional Screening

- **Stage 2:** (ctd.) - "Glosten (1989)-trick"
  - with CARA utility function

$$E\left[W|u,S\right] - \frac{\rho}{2} V\left[W|u,S\right]$$

$$= (x+u)\,S - T(x) - \frac{\rho}{2}(x+u)^2 \underbrace{Var\left[v|S\right]}_{=\sigma^2}$$

$$= \underbrace{\left(uS - \frac{\rho\sigma^2}{2}u^2\right)}_{\text{independent of } x} + \underbrace{\left(\underbrace{xS - \rho\sigma^2 xu}_{\substack{\theta x \\ \text{i.e. } \theta := S - \rho\sigma^2 u}} - \frac{\rho\sigma^2}{2}x^2 - T(x)\right)}_{\substack{\text{depends on } x \\ \Longrightarrow \text{Info-Rent}}}$$

  - This reduces it to a one-dimensional screening problem
- function $v(\theta) = E\left[v|\theta\right]$ of (one-dimensional) type θ
  $1 \geq \dot{v}(\theta) \geq 0$

BRM: First Best Benchmark

- ex-ante

$$\text{optimal trading mechanism} \left\{ \overset{\text{transfers}}{\underset{\downarrow}{}}\tau(\theta)\ ,\quad \overset{\text{trading volume}}{\underset{\downarrow}{}} x(\theta) \right\}$$

$$\max_{\{\tau(\theta),x(\theta)\}} \int_{\underline{\theta}}^{\overline{\theta}} \left(\theta x(\theta) - \frac{\rho\sigma^2}{2} x(\theta)^2 - \tau(\theta)\right) f(\theta)\, d\theta$$

$$\text{s.t.} \quad \int_{\underline{\theta}}^{\overline{\theta}} (\tau(\theta) - v(\theta)\, x(\theta))\, f(\theta)\, d\theta = \Pi$$

- Π determines how surplus is distributed between P and A

$$\Longrightarrow \max \int_{\underline{\theta}}^{\overline{\theta}} (\underbrace{\theta x(\theta) - \frac{\rho\sigma^2}{2} x(\theta)^2 - v(\theta)\, x(\theta)}_{\text{surplus}} - \Pi) f(\theta)\, d\theta$$

BRM: First Best Benchmark

for a given θ

$$\begin{aligned} \theta - \rho\sigma^2 x(\theta) - v(\theta) &= 0 \\ x^*(\theta) &= \frac{\theta - v(\theta)}{\rho\sigma^2} \\ &= E[-u|\theta], \text{ since } u = -\frac{\theta - S}{\rho\sigma^2} \end{aligned}$$

- Assume $x^*(\underline{\theta}) < 0 < x^*(\overline{\theta})$
  $\Longrightarrow \exists\theta_0$ s.t. $x^*(\theta_0) = 0$
- almost all θ-types trade
  (see later that $\forall\theta > \theta_0 \Longrightarrow$ buy
  $\forall\theta < \theta_0 \Longrightarrow$ sell)

BRM: Monopolistic Screening $x^*(\theta)$ and $x_m(\theta)$

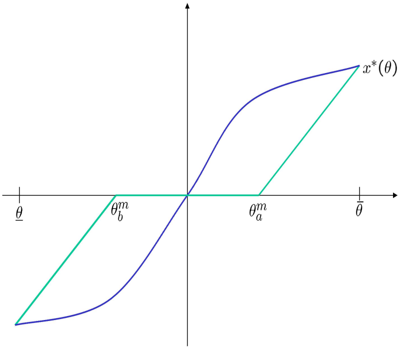


Figure: **xxx.** xx

BRM: Implementable Allocation under Adverse Selection

- social planner must elicit information
- Revelation Principle
  Any allocation that can be achieved with non-linear schedules $T(x)$ can also be achieved with a truthful direct mechanism $\{\tau(\cdot), x(\cdot)\}$.
- Incentive compatibility

$$\theta \in \arg\max_{\widehat{\theta}} \left(\theta x\left(\widehat{\theta}\right) - \frac{\rho\sigma^2}{2} x\left(\widehat{\theta}\right)^2 - \tau\left(\widehat{\theta}\right) \right)$$

$$\Longrightarrow U(\theta) = \max_{\widehat{\theta}} \left(\underbrace{\theta x\left(\widehat{\theta}\right) - \frac{\rho\sigma^2}{2} x\left(\widehat{\theta}\right)^2 - \tau\left(\widehat{\theta}\right)}_{\text{informational rent}} \right)$$

$\{\tau(\cdot), x(\cdot)\}$ transfers and allocation

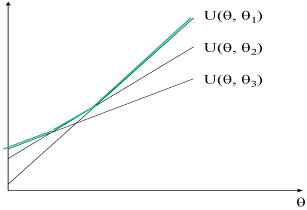
BRM: Dual (Mirrlees) Approach

$\{U(\cdot), x(\cdot)\}$ informational rent (see Fudenberg & Tirole Ch. 7)

Lemma 1:

A pair $\{U(\cdot), x(\cdot)\}$ is implementable iff $U(\cdot)$ is convex on $[\underline{\theta}, \overline{\theta}]$, and for a.e.$\theta$, $\dot{U}(\theta) = x(\theta)$,

$$\frac{dU\left(\theta, \widehat{\theta}(\theta)\right)}{d\theta} \underset{\substack{\uparrow \\ \text{envelope theorem}}}{=} \frac{\partial U}{\partial \theta} = x(\theta).$$



BRM: Monopolistic Screening

m.m.(principal) gets $\int_{\underline{\theta}}^{\overline{\theta}} \tau(x(\theta)) - v(\theta)x(\theta)$ replacing τ from information rent $U(\theta) = \theta x(\theta) - \frac{\rho\sigma^2}{2}x^2(\theta) - \tau(x(\theta))$, the m.m.'s objective becomes

$$\max_{\{U(\cdot),x(\cdot)\}} \int_{\underline{\theta}}^{\overline{\theta}} \{[\theta - v(\theta)]x(\theta) - \frac{\rho\sigma^2}{2}[x(\theta)]^2 - U(\theta)\} f(\theta) d\theta$$

subject to

$$\text{IC} \quad \begin{cases} U(\cdot) \text{ is convex on } [\underline{\theta}, \overline{\theta}] \\ \dot{U}(\theta) = x(\theta) \quad \forall\theta \text{ (almost everywhere)} \end{cases}$$

ex-post PC $\quad U(\theta) \geq 0 \quad$ **ex-post** participation constraints

(ex-post: since traders decide after knowing θ whether to participate)

BRM: Monopolistic Screening Dual Approach

(replace $x(\theta)$ with $\dot{U}(\theta)$)

$$\max_{U(\cdot)} B_m\left(U(\cdot), \dot{U}(\cdot)\right)$$

$$:= \int_{\underline{\theta}}^{\overline{\theta}} \left([\theta - v(\theta)]\, \dot{U}(\theta) - \frac{\rho\sigma^2}{2} \dot{U}(\theta)^2 - U(\theta) \right) f(\theta)\, d\theta$$

$$\text{s.t.} \quad \begin{array}{c} U(\cdot) \text{ convex} \\ U(\theta) \geq 0 \end{array}$$

Temporarily ignore convexity constraint and check ex-post. (Sufficient condition: $U(\cdot)$ is convex if

$$\forall \theta > \theta_0 \quad \frac{d}{d\theta}\left(\frac{1-F(\theta)}{f(\theta)}\right) < 0 \tag{18}$$

$$\forall \theta < \theta_0 \quad \frac{d}{d\theta}\left(\frac{F(\theta)}{f(\theta)}\right) > 0 \tag{19}$$

BRM: Monopolistic Screening

$$\mathcal{L}\left(U,\dot{U}\right) = B_m\left(U,\dot{U}\right) + \int_{\underline{\theta}}^{\overline{\theta}} U(\theta) \qquad \underset{\substack{\uparrow \\ \text{∞ many Lagrange multipliers} \\ \text{different from type to type} \\ \text{(ex-post constraint)}}}{d\Lambda(\theta)}$$

By complementary slackness condition, support of Λ be constrained in $(U_m)^{-1}(0)$, (θ-types which get zero info ret)
view $\Lambda(\theta)$ as c.d.f., i.e., $\exists$ a measure Λ

$$\Lambda(\theta) = \int_{\underline{\theta}}^{\theta} \frac{d\Lambda(s)}{\int_{\underline{\theta}}^{\overline{\theta}} d\Lambda(s)} \qquad \text{(slight abuse of notation)}$$

BRM: Monopolistic Screening

Aside: Integrating by parts

$$\int_{\underline{\theta}}^{\overline{\theta}} U(\theta)\, d\left[\Lambda(\theta) - F(\theta)\right] = -\int_{\underline{\theta}}^{\overline{\theta}} \dot{U}(\theta)\left(\Lambda(\theta) - F(\theta)\right) d\theta + U\left(\overline{\theta}\right)$$

Consequently, $\max \mathcal{L}\left(U, \dot{U}\right) =$

$$= \int_{\underline{\theta}}^{\overline{\theta}} \left(\left(\theta - v(\theta) + \frac{F(\theta) - \Lambda(\theta)}{f(\theta)}\right)\dot{U}(\theta) - \frac{\rho\sigma^2}{2}\dot{U}(\theta)^2\right) f(\theta)\, d\theta + U\left(\overline{\theta}\right)\left(\Lambda(\theta) - 1\right)$$

max only if $\Lambda(\theta) = 1$ (since $U\left(\overline{\theta}\right)$ is arbitrary)

pointwise maximization over $\dot{U}(\theta)$

$$\forall \theta \in \left[\underline{\theta}, \overline{\theta}\right], \; x_m(\theta) = \underbrace{\frac{\theta - v(\theta)}{\rho\sigma^2}}_{x^*(\theta)} + \frac{F(\theta) - \Lambda(\theta)}{f(\theta)\,\rho\sigma^2}$$

BRM: Monopolistic Screening

Complementary slackness condition ($d\Lambda = 0$ for some θ)

$$\begin{array}{ll} \forall\theta \in [\underline{\theta}, \theta_b^m] & \Lambda(\theta) = 0 \\ \forall\theta \in \left[\theta_a^m, \overline{\theta}\right] & \Lambda(\theta) = 1 \end{array}$$

$\Longrightarrow$ given (18) & (19), $U(\cdot)$ is convex and

Proposition 2

$\exists\theta_a^m > \theta_0$ and $\theta_b^m < \theta$ s.t.

$$\begin{array}{lll} \text{(i)} & \text{for all } \theta \in [\underline{\theta}, \theta_b^m), & x_m(\theta) = x^*(\theta) + \frac{F(\theta)}{\rho\sigma^2 f(\theta)} \\ \text{(ii)} & \text{for all } \theta \in [\theta_b^m, \theta_a^m], & x_m(\theta) = 0 \quad \text{(no info rent)} \\ \text{(iii)} & \text{for all } \theta \in \left(\theta_a^m, \overline{\theta}\right], & x_m(\theta) = x^*(\theta) - \frac{1-F(\theta)}{\rho\sigma^2 f(\theta)} \end{array}$$

BRM: Monopolistic Screening $x^*(\theta)$ and $x_m(\theta)$

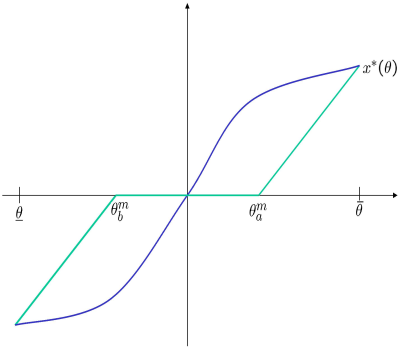


Figure: **xxx.** xx

BRM: Monopolistic Screening Price Schedule

for $\theta > \theta_a^m$ we know

$$(1)\ \theta \geq \theta_a^m \ U(\theta) = 0 + \int_{\theta_a^m}^{\theta} \dot{U}'(s)\,ds = \int_{\theta_a^m}^{\theta} x(s)\,ds$$

$$(2) \qquad U(\theta) = \theta x_m(\theta) - \frac{\rho\sigma^2 x_m(\theta)^2}{2} - T(x(\theta))$$

$$(1)=(2) \quad T(x(\theta)) = \theta x_m(\theta) - \frac{\rho\sigma^2 x_m(\theta)^2}{2} - \int_{\theta_a^m}^{\theta} x(s)\,ds$$

Differentiate w.r.t. θ

$$\begin{aligned} \frac{\partial T}{\partial x}\frac{\partial x}{\partial \theta} &= x_m(\theta) + \theta\frac{\partial x}{\partial \theta} - \rho\sigma^2 x_m(\theta)\frac{\partial x_m}{\partial \theta} - x_m(\theta) \\ \frac{\partial T}{\partial x} &= \theta - \rho\sigma^2 x_m(\theta) \end{aligned}$$

We have

$$x_m(\theta) = \underbrace{x^*(\theta)}_{\frac{\theta - v(\theta)}{\rho\sigma^2}} + \frac{F(\theta)}{\rho\sigma^2 f(\theta)}$$

BRM: Monopolistic Screening Price Schedule

$$\Longrightarrow \quad \frac{\partial T}{\partial x} = \theta - \theta + v(\theta) - \frac{F(\theta)}{f(\theta)}$$

$$t_m(x) \quad = \quad \frac{\partial T}{\partial x} = v(\theta) - \frac{F(\theta)}{f(\theta)}$$

for $\theta < \theta_b^m$ similar steps

$$\frac{\partial T}{\partial x} = v(\theta) + \frac{1 - F(\theta)}{f(\theta)}$$

Note that

$$t_m\left(x = 0^+\right) = \theta_a^m > \theta_b^m = t_m\left(x = 0^-\right)$$

"small trade spread"

BRM: Oligopolistic Screening

...

Limit Order Book vs. Uniform Pricing
Röell (1998)

- Model setup
  - order size of trader is *exogenous*
  - is double exponentially distributed $f(x) = \frac{1}{2}ae^{-a|x|}$
  - conditional expectations
    - $E[\cdot|x \geq y] \Rightarrow$ linear schedule in limit order book
    - $E[v|x] = v_0 + \gamma x$ *assumed* $\Rightarrow$ linear uniform price schedule
- $p^u(x) = v_0 + \frac{I-1}{I-2}\gamma x$ versus $p^d(x) = v_0 + \frac{I}{I-1}\frac{\gamma}{a} + \gamma x$

Limit Order Book vs. Uniform Pricing Röell (1998)

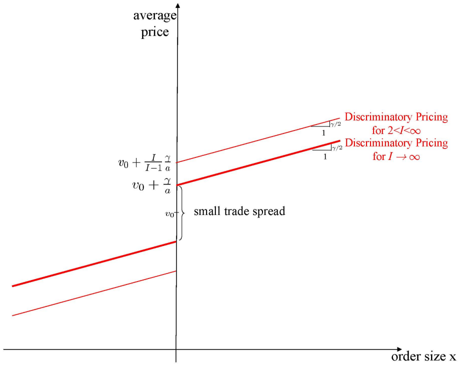


Figure: **Limit Order Book.**

Limit Order Book vs. Uniform Pricing Röell (1998)

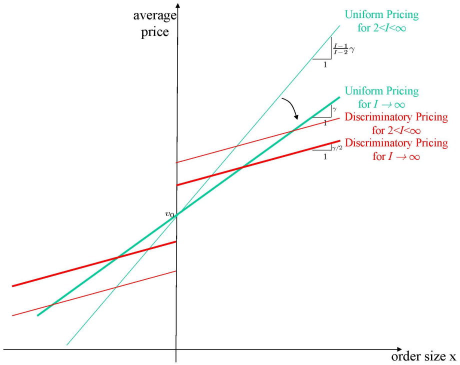


Figure: **Limit Order Book and Uniform Pricing.**

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Sequential Trade Models à la Glosten & Milgrom (1985)

- order size is restricted to $x \in \{-1, +1\}$

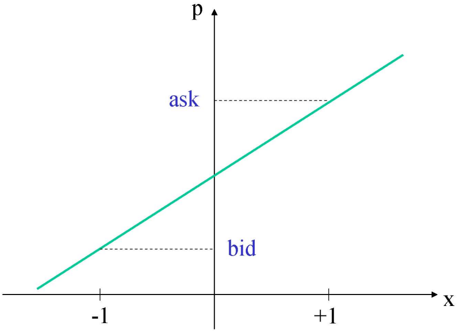

Figure: **Bid-Ask Spread.**

Sequential Trade Models à la Glosten & Milgrom (1985)

- Monopolistic Market Maker - Copeland & Galai (1983)
  - bid-ask spread is partially due to monopoly power partially due to adverse selection
  - difficult to handle in multi-period setting
- Competitive Market Makers - Glosten & Milgrom (1985)
  - bid-ask spread is only due to adverse selection
  - multi-period setting

Glosten & Milgrom (1985)

- Model Setup
  - value of the stock $\underline{v}$ and $\overline{v}$
  - with probability α an informed trader shows up
  - with probability $(1-\alpha)$ an uninformed trader shows up
  - all traders are chosen from a pool of a continuum of traders, i.e., the probability that they will trade a second time is zero (rule out strategic considerations as in Kyle's)
  - informed traders know true $\widetilde{v} \rightarrow$ buys if $v > a$ sells if $v < b$.
  - uninformed traders buy with probability μ and sell with probability $1-\mu$.
  - Note: Traders can only buy or sell 1 unit (No-Trade is also not allowed!)

Glosten & Milgrom (1985)

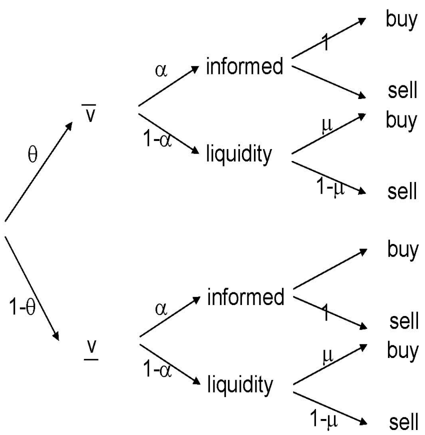

Figure: **Tree.**

Glosten & Milgrom (1985) Calculating Bid-Ask Spread

- Buy order

$$\begin{aligned} \mathrm{P}(\overline{v}) &= \theta \\ \mathrm{P}(\mathrm{buy}|\overline{v}) &= \alpha + (1-\alpha)\mu \\ \mathrm{P}(\mathrm{buy}|\underline{v}) &= (1-\alpha)\mu \end{aligned}$$

Bayes' Rule

$$\begin{aligned} \mathrm{P}(\overline{v}|\mathrm{buy}) &= \frac{(\alpha + (1-\alpha)\mu)\theta}{(\alpha + (1-\alpha)\mu)\theta + (1-\alpha)\mu(1-\theta)} \\ \mathrm{P}(\underline{v}|\mathrm{buy}) &= 1 - \mathrm{P}(\overline{v}|\mathrm{buy}) \end{aligned}$$

Glosten & Milgrom (1985) Calculating Bid-Ask Spread

- Sell order $\mathrm{P}(\overline{v}|\text{sell}) =$

$$= \frac{(1-\alpha)(1-\mu)\theta}{(1-\alpha)(1-\mu)\theta+[\alpha+(1-\alpha)(1-\mu)](1-\theta)}$$

$$\mathrm{P}(\overline{v}|\text{buy}) > \mathrm{P}(\overline{v}) > \mathrm{P}(\overline{v}|\text{sell})$$

$$\mathrm{P}(\underline{v}|\text{buy}) < \mathrm{P}(\underline{v}) < \mathrm{P}(\underline{v}|\text{sell})$$

- Market Maker makes zero expected profit (potential Bertrand competition)

$$\begin{aligned} b &= \text{bid} = E[v|\text{sell}] = \overline{v}\mathrm{P}(\overline{v}|\text{sell}) + \underline{v}\mathrm{P}(\underline{v}|\text{sell}) \\ a &= \text{ask} = E[v|\text{buy}] = \overline{v}\mathrm{P}(\overline{v}|\text{buy}) + \underline{v}\mathrm{P}(\underline{v}|\text{buy}) \end{aligned}$$

Remarks to Glosten & Milgrom (1985)

1. quotes are regret free
2. $\underline{v} < b < a < \overline{v}$
3. $(a-b) \rightarrow$ gain from liquidity traders $=$ loss to insider
4. bid-ask spread $(a-b)$ increases with α
5. over time price converge to true value
6. prices follow a martingale $E_t\left[p_{t+1}|I_t^I\right] = p_t$ (changes in prices are uncorrelated)
7. Simple setting price at t depends only on # buy orders − # sell orders (sequence of trades does not matter)
8. mid point of bid ask spread $\frac{a+b}{2}$ is <u>not</u> current market maker's expectation.

Extensions

- Easley and O'Hara (1987)
  - **'small and large' order size**
    - noise traders submit randomly a small or a large sized order
    - informed traders always prefer large order size (if bid and ask is the same for both order sizes)

    $\Rightarrow$ m.m. will set larger spread for large orders
  - Separating equilibrium
    - Informed traders' order size is 2
    - Uninformed traders' order size is 1 and 2 (exogenously given)

    $\Rightarrow$ Spread for small orders $= 0$
  - Pooling equilibrium
    - Informed traders' order size is 1 and 2
    - Uninformed traders' order size is 1 and 2 exogenously given)

    $\Rightarrow$ Larger spread for larger orders

Extensions

- **"event uncertainty"** (also in Easley & O'Hara (1992))
  - with prob γ info is like in Glosten & Milgrom
  - with prob $(1-\gamma)$ no news event occurs (nobody receives a signal)
- No-Trade $\rightarrow$ signals that nothing has occurred!
  $\Rightarrow$ quotes will pull towards $\frac{1}{2}$
  updating
  1. whether event has occurred AND
  2. about true value of the stock
- transaction price is still a Martingale
  but no longer Markov!

Herding - Avery & Zemsky (1998)

- Relates Glosten-Milgrom model to herding models (BHW 1992)
- Price adjustment eliminates herding and informational cascades if market maker learns at the same speed as other informed traders.
- Herding can still arise in a more general setting with event uncertainty and a more complicated information structure which guarantees that the market maker learns at a slower speed compared to other traders.

1987-Crash
Jacklin, Kleiden & Pfleiderer (1992)

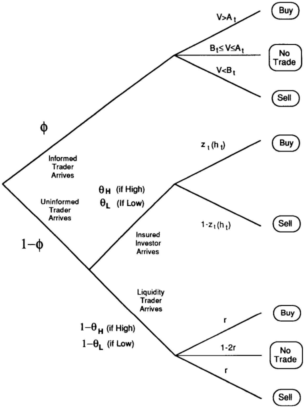

Figure: **Underestimating portfolio insurance traders θ.**